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SOLVABILITY CRITERION FOR MULTIPLE INTERPOLATION PROBLEM IN PREIMAGE OF CONVOLUTION OPERATOR

V.V. NAPALKOV (JR.), A.A. NUYATOV

Abstract. In this paper we find a solvability criterion for the multiple interpolation problem in the preimage of a convolution operator and, consequently, a criterion for the solvability of Abel — Goncharov problem in the same space. When the kernel of the operator serves as the preimage, uniqueness of the solution to these problems holds provided the set of interpolation nodes is the uniqueness set in the kernel of the convolution operator.

Keywords: multiple interpolation, Abel problem, convolution operator, entire functions.

Mathematics Subject Classification: 46A13, 30D20

1. Introduction

Let $H(\mathbb{C})$ be the space of entire functions with the topology of uniform convergence on compact sets; we denote the dual of the space $H(\mathbb{C})$ by $H^*(\mathbb{C})$. We also introduce the notation

$$P_{\mathbb{C}} = \{f(z) \in H(\mathbb{C}) : \exists\, c_1, c_2 > 0, |f(z)| \leqslant c_1 e^{c_2|z|}\}.$$

With a function $\varphi(z) \in P_{\mathbb{C}}$ we associate a functional $F \in H^*(\mathbb{C})$ such that $\widehat{F}(z) = \varphi(z)$, where $\widehat{F}(z) = \langle F_\lambda, e^{\lambda z}\rangle$ is the Laplace transform of the functional F.

The convolution operator acting from $H(\mathbb{C})$ to $H(\mathbb{C})$ with the characteristic function $\varphi(z) \in P_{\mathbb{C}}$ reads

$$M_\varphi[f](z) = \frac{1}{2\pi i}\int\limits_C f(z+t)\gamma(t)dt, \tag{1.1}$$

where C is a closed contour, $\gamma(t)$ is an analytic function on C and outside C, $\gamma(\infty) = 0$.

We take and fix an arbitrary function $g_0(z) \in H(\mathbb{C})$ and consider the convolution equation

$$M_\varphi[f](z) = g_0(z),$$

We denote by $\operatorname{Im}^{-1} M_\varphi[f]$ the preimage of the convolution operator. As $g_0(z) \equiv 0$, the preimage of the convolution operator becomes the kernel $\operatorname{Ker} M_\varphi[f]$.

For an arbitrary function $\psi(z) \in H(\mathbb{C})$ we define the ideal in $H(\mathbb{C})$

$$(\psi) = \{\psi(z) \cdot R(z) : \ R(z) \in H(\mathbb{C})\}.$$

Definition 1.1. *The identities*

$$H(\mathbb{C}) = (\psi) + \operatorname{Im}^{-1} M_\varphi, \tag{1.2}$$

$$H(\mathbb{C}) = (\psi) \oplus \operatorname{Im}^{-1} M_\varphi, \tag{1.3}$$

are called the Fisher representation and Fisher decomposition for the preimage of M_φ in $H(\mathbb{C})$, respectively.

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When identities (1.2) hold, each entire function can be represented, generally speaking, non–uniquely as

$$f(z) = h(z) + g(z), \qquad g(z) \in \operatorname{Im}^{-1} M_\varphi, \qquad h(z) \in (\psi).$$

Let us provide an example when the characteristic function $\varphi(z)$ of the operator M_φ is the polynomial $P(z) = z^2 + 1$, then the convolution operator becomes a differential operator with constant coefficients

$$P\left(\frac{d}{dz}\right) = \frac{d^2}{d\,z^2} + 1,$$

and as a function from the ideal we take the polynomial $Q(z) = z + 1$.

We consider the differential equation

$$\frac{d^2 f(z)}{d\,z^2} + f(z) = e^z,$$

the function from the preimage is

$$f(z) = c_1 e^{i\,z} + c_2 e^{-i\,z} + \frac{1}{2} e^z.$$

Then the Fisher representation reads

$$H(\mathbb{C}) = (Q) + c_1 e^{i\,z} + c_2 e^{-i\,z} + \frac{1}{2} e^z.$$

This means that for each function $f(z) \in H(\mathbb{C})$ we have

$$\frac{f(z) - (c_1 e^{i\,z} + c_2 e^{-i\,z} + \frac{1}{2} e^z)}{z+1} \in H(\mathbb{C}),$$

hence, as the zero of the polynomial $z + 1$ we have the identity

$$f(-1) = c_1 e^{-i} + c_2 e^{i} + \frac{1}{2} e^{-1}$$

Hence, the function $f(z)$ is defined but not uniquely.

As identities (1.3) hold, then such a representation of the entire function is unique.

The multiple interpolation problem in $\operatorname{Im}^{-1} M_\varphi$ with nodes $\mu_j \in \mathbb{C}$ being the zeroes of $\psi \in H(\mathbb{C})$, with multiplicities q_j, $j = 0, 1, 2, \dots$ is formulated as follows: given an arbitrary sequence of complex numbers a_j^k, $j = 0, 1, 2, \dots;$, $k = 0, 1, \dots, q_k - 1$, whether there exists a function $y \in \operatorname{Im}^{-1} M_\varphi$ such that

$$y^{(k)}(\mu_j) = a_j^k, \quad j = 0, 1, 2, \dots; \quad k = 0, 1, \dots, q_k - 1.$$

Lemma 1.1. *The solvability of the multiple interpolation problem is equivalent to the validity of the Fisher representation*

$$H(\mathbb{C}) = (\psi) + \operatorname{Im}^{-1} M_\varphi.$$

Proof. 1. Let Fisher representation (1.2) hold. Mittag — Leffler theorem and the Weierstrass theorem on the existence of entire functions with given zeros imply the existence of a function $h(z) \in H(\mathbb{C})$ such that $h^{(k)}(\mu_j) = a_j^k$, $j = 0, 1, 2, \dots$; $k = 0, 1, \dots, q_k - 1$. By (1.2)

$$h(z) = y(z) + \psi(z) \cdot g(z), \qquad y(z) \in \operatorname{Im}^{-1} M_\varphi, \quad g(z) \in H(\mathbb{C}),$$

and this implies the identities $y^{(k)}(\mu_j) = a_j^k$, $j = 0, 1, 2, \dots$; $k = 0, 1, \dots, q_k - 1$. We note that if (1.3) holds, then the function $y(z)$ is unique.

2. Let the multiple interpolation problem be solvable. The validity of the Fisher representation holds can be written in an another way: for each function $h(z) \in H(\mathbb{C})$, there exists a function $y(z) \in \operatorname{Im}^{-1} M_\varphi$ such that

$$\frac{h(z) - y(z)}{\psi(z)} \in H(\mathbb{C}).$$

The latter holds if $h(z) - y(z)$ vanishes at the zeros of the function $\psi(z)$ (otherwise the function $\dfrac{h(z) - y(z)}{\psi(z)}$ would be meromorphic).

Since by the assumption the problem of multiple interpolation is solvable in the preimage of convolution operator and there exists a function $h(z) \in H(\mathbb{C})$ such that $h^{(k)}(\mu_j) = a_j^k$, $j = 0, 1, 2, \dots$; $k = 0, 1, \dots, q_k - 1$, we have

$$h^{(k)}(\mu_j) = a_j^k = y^{(k)}(\mu_j), \quad j = 0, 1, 2, \dots; \quad k = 0, 1, \dots, q_k - 1,$$

where μ_j, $j = 0, 1, 2, \dots$ the zeros of the function $\psi(z)$. Thus, Fisher representation (1.2) holds. Note that if the problem of multiple interpolation in the preimage of convolution operator has a unique solution, then (1.3) holds. The proof is complete. □

A particular case of the considered problem under is

$$y^{(k)}(\mu_j) = a_j^k, \quad j = 0, 1, 2, \dots; \quad k = \overline{0, j}.$$

Hence, the preimage of the convolution operator contains a function $y(z)$, which for the sequence of complex numbers a_0^0, a_1^1, …, a_n^n, … satisfies

$$y^{(k)}(\mu_k) = a_k^k.$$

Thus, we obtain the Abel problem [4] in the preimage of convolution operator.

When the characteristic function of convolution operator is a polynomial, the convolution operator becomes a finite order linear differential operator with constant coefficients. Therefore, an important particular consequence is that for a finite order homogeneous linear differential equation with constant coefficients the multiple interpolation problem and the Abel — Goncharov problem are solvable. Furthermore, a differential–difference operator, an integro–differential operator, and a linear differential operator of infinite order with constant coefficients are also particular cases of the convolution operator, and, as a consequence, these problems are solvable for the corresponding homogeneous equations.

In [9], the interpolation problem was solved for the case where the nodes are simple and lie on the real axis. In [10], the interpolation problem in the kernel of the convolution operator was solved with complex nodes. In [11], the problem of multiple interpolation in the kernel of convolution operator was solved. The aim of this paper is to find conditions under which the problem of multiple interpolation in the preimage of convolution operator is solvable.

2. Preliminaries

Along with the operator $M_\varphi[f](z)$, we introduce a linear operator continuous in the topology of space $H(\mathbb{C})$

$$M_\varphi[\psi(z) \cdot y(z)] + g_0(z) : \; H(\mathbb{C}) \to H(\mathbb{C}), \tag{2.1}$$

as above, $g_0(z)$ is a function from the image of operator $M_\varphi[f](z)$.

Theorem 2.1. *Fisher representation in $H(\mathbb{C})$ holds if and only if operator (2.1) is surjective.*

Proof. 1. Let identity (1.2) hold and $g(z) \in H(\mathbb{C})$. Since convolution operator (1.1) is surjective [6], for $g(z)$ there exists $f(z) \in H(\mathbb{C}) : \; M_\varphi[f] = g(z)$. According to (1.2), the function $f(z)$ can be represented as

$$f(z) = \psi(z) \cdot h(z) + u(z), \; h(z) \in H(\mathbb{C}), \; u(z) \in \operatorname{Im}^{-1} M_\varphi,$$

this is why

$$M_\varphi[f] = M_\varphi[\psi(z) \cdot h(z)] + M_\varphi[u(z)] = M_\varphi[\psi(z) \cdot h(z)] + g_0(z) = g(z),$$

and we see that the operator $M_\varphi[\psi(z) \cdot h(z)] + g_0(z)$ is surjective.

2. Let operator (2.1) be surjective. We need to prove identity (1.2). Since operator (2.1) and the convolution operator are surjective, then for each $f(z) \in H(\mathbb{C})$ there exist functions $w_1(z) \in \operatorname{Im}^{-1} M_\varphi, y_1(z) \in H(\mathbb{C})$ satisfying the identity

$$M_\varphi[f(z)] = M_\varphi[\psi(z) \cdot y_1(z)] + g_0(z) = M_\varphi[\psi(z) \cdot y_1(z)] + M_\varphi[w_1(z)],$$

and this is why

$$f(z) = w_1(z) + \psi(z) \cdot y_1(z).$$

This proves Fisher representation (1.2). The proof is complete. □

We denote by N_φ and N_ψ the zero sets of functions $\varphi(z)$ and $\psi(z)$. Let us formulate the injectivity conditions for the operator $M_\varphi[\psi\,\cdot\,]+g_0(z)$.

Theorem 2.2. *If two conditions hold*

1. $\exists\, h_0(z)\in H(\mathbb{C}) : h_0(z)\in(\psi)\bigcap\{f\in H(\mathbb{C}) : M_\varphi[f]=g_0\}$;
2. N_ψ *is the uniqueness set in* $\operatorname{Ker} M_\varphi$,

then the operator $M_\varphi[\psi\cdot]+g_0(z)$ *is injective.*

Proof. According to Assertion 1, $\exists\hat{h}_0\in H(\mathbb{C}) : h_0(z)=\hat{h}_0(z)\cdot\psi(z)$, and

$$M_\varphi[\psi\cdot f]+g_0(z)=M_\varphi[\psi\cdot f]+M_\varphi[h_0]=M_\varphi[\psi(f+\hat{h}_0)].$$

We consider the equation

$$M_\varphi[\psi(z)(f(z)+\hat{h}_0(z))]=0.$$

The function $u(z)=\psi(z)(f(z)+\hat{h}_0(z))$ vanishes at the points of the set N_ψ, which by assumption is the uniqueness set in $\operatorname{Ker} M_\varphi$, therefore $u(z)\equiv 0$. Since for a linear operator the injectivity is equivalent to the triviality of its kernel, the operator $M_\varphi[\psi\,\cdot\,]+g_0(z)$ is injective. The proof is complete. □

Corollary 2.1. *If* N_ψ *is the uniqueness set in* $\operatorname{Ker} M_\varphi$*, then the operator* $M_\varphi[\psi\,\cdot\,]$ *is injective.*

By Theorem 2.1, the surjectivity of operator $M_\varphi[\psi\,\cdot\,]+g_0(z)$ is equivalent to the Fisher representation

$$H(\mathbb{C})=(\psi)+\operatorname{Im}^{-1}M_\varphi.$$

It is obvious that for the Fisher decomposition requites the injectivity of the operator $M_\varphi[\psi\,\cdot\,]+g_0(z)$, and this injectivity is equivalent to

$$(\psi)\cap\{f\in H(\mathbb{C}) : M_\varphi[f]=g_0\}=\{0\}$$

since the direct sum of two linear subspaces is possible only if the subspaces intersect at zero. Therefore, by Theorem 2.2, generally speaking, the Fisher decomposition does not exist, but the Fisher representation exists, i.e., the interpolation problem in the preimage of convolution operator is solvable but not uniquely. Note that by Corollary 2.1, in the case where N_ψ is the uniqueness set in $\operatorname{Ker} M_\varphi$ the decomposition exists

$$H(\mathbb{C})=(\psi)\oplus\operatorname{Ker} M_\varphi.$$

3. Convolution operator in space $P_{\mathbb{C}}$

According to the results of [15], [19] the function $\psi\in H(\mathbb{C})$ generates a surjective convolution operator $M_\psi : P_{\mathbb{C}}\to P_{\mathbb{C}}$ in the space $P_{\mathbb{C}}$

$$M_\psi[f](z)=\frac{1}{2\pi i}\int\limits_C \psi(t)\gamma(t)e^{zt}\,dt, \tag{3.1}$$

where $\gamma(t)$ is the function associated with $f(z)$ in the Borel sense and C is a closed contour enclosing all singular points of $\gamma(t)$.

We arbitrarily choose and fix the function $G_0(z)\in P_{\mathbb{C}}$, and consider the equation

$$M_\psi[f]=G_0(z).$$

Let $(\varphi)=\{\varphi(z)\cdot G(z) : G(z)\in P_{\mathbb{C}}\}$ be an ideal in the space $P_{\mathbb{C}}$, and $\operatorname{Im}^{-1}M_\psi[f]$ be the preimage of operator (3.1).

Definition 3.1. *Identities*

$$P_{\mathbb{C}}=(\varphi)+\operatorname{Im}^{-1}M_\psi,$$
$$P_{\mathbb{C}}=(\varphi)\oplus\operatorname{Im}^{-1}M_\psi,$$

are call the Fisher representation and Fisher decomposition for the preimage of M_ψ *in* $P_{\mathbb{C}}$*, respectively.*

We introduce the linear continuous operator

$$M_\psi[\varphi(z)\cdot y(z)] + G_0(z): \ P_{\mathbb{C}} \to P_{\mathbb{C}}. \tag{3.2}$$

Let us formulate an analogue of Theorem 2.1 for operator (3.2).

Theorem 3.1. *Fisher representation holds in $P_{\mathbb{C}}$ if and only if operator (3.2) is surjective.*

The proof of Theorem 3.1 is similar to the proof of Theorem 2.1 since for each $g(z) \in P_{\mathbb{C}}$ there exists $f(z) \in P_{\mathbb{C}} : M_\psi[f](z) = g(z)$.

Theorem 3.2. *If two conditions hold*

1. $\exists\, h_0(z) \in P_{\mathbb{C}} : h_0(z) \in (\varphi) \bigcap \{f \in P_{\mathbb{C}} : M_\psi[f] = G_0\}$;
2. N_φ *is the uniqueness set in* $\operatorname{Ker} M_\psi$,

then the operator $M_\psi[\varphi\cdot] + G_0(z)$ is injective.

The proof is similar to the proof of Theorem 2.2.

Corollary 3.1. *If N_φ is a uniqueness set in $\operatorname{Ker} M_\psi$, then the operator $M_\psi[\varphi\,\cdot\,]$ is injective.*

By analogy with the space $H(\mathbb{C})$, generally speaking, there is uniqueness in the interpolation problem in the preimage of operator M_ψ, but under certain conditions the decomposition

$$P_{\mathbb{C}} = (\varphi) \oplus \operatorname{Ker} M_\psi$$

holds. We rewrite operator (2.1) as

$$M_\varphi[\psi\cdot f + f_0].$$

Here ψ and f_0 are fixed entire functions, and f "ranges" over the entire space $H(\mathbb{C})$. Since the operator $M_\varphi[\psi\cdot f + f_0]$ linearly and continuously maps $H(\mathbb{C})$ into $H(\mathbb{C})$, the adjoint operator $M^*_\varphi[\psi\cdot f + f_0]$ linearly and continuously maps the space $H^*(\mathbb{C})$ into $H^*(\mathbb{C})$. Since the spaces $H^*(\mathbb{C})$ and $P_{\mathbb{C}}$ are topologically isomorphic [6], the operator $M^*_\varphi[\psi\cdot f + f_0]$ generates a linear continuous operator acting from $P_{\mathbb{C}}$ to $P_{\mathbb{C}}$ by rule: for $G(z) \in P_{\mathbb{C}}$

$$\widehat{M^*_\varphi}[G] = M_\psi[\varphi\cdot G], \tag{3.3}$$

where M_ψ is an operator of form (3.1). Since the space $H(\mathbb{C})$ is a Fréchet space, we apply the result of [3] (or [1, Thm. 1.2]) and obtain the following theorem.

Theorem 3.3. *The statements hold*

1. $\operatorname{Im}(M_\varphi[\psi\,\cdot\,] + g_0)$ *is closed in* $H(\mathbb{C})$ *if and only if* $\operatorname{Im} M_\psi[\varphi\,\cdot\,]$ *is closed in* $P_{\mathbb{C}}$;
2. $\operatorname{Im}(M_\varphi[\psi\,\cdot\,] + g_0)$ *is everywhere dense in* $H(\mathbb{C})$ *if and only if* $M_\psi[\varphi\,\cdot\,]$ *is injective;*
3. $M_\varphi[\psi\,\cdot\,] + g_0$ *is injective if and only if* $\operatorname{Im} M_\psi[\varphi\,\cdot\,]$ *is dense in* $P_{\mathbb{C}}$.

By Theorems 2.1, 3.1 and 3.3 we obtain the next statement.

Corollary 3.2.

$$H(\mathbb{C}) = (\psi) \oplus \operatorname{Ker} M_\varphi \Leftrightarrow P_{\mathbb{C}} = (\varphi) \oplus \operatorname{Ker} M_\psi,$$

where $\operatorname{Ker} M_\psi$ *is the kernel of operator (3.1).*

4. Sufficiency sets

The topology $\tau_{\mathbb{C}}$ of the space $P_{\mathbb{C}}$ is defined as the inductive limit of normed weighted spaces

$$B_n = \{\varphi(\lambda) \in P_{\mathbb{C}} : \|\varphi\|_n = \sup_{\lambda\in\mathbb{C}} |\varphi(\lambda)| e^{-n|\lambda|} < \infty\}, \quad n \in \mathbf{N}.$$

Let $S \subset \mathbb{C}$ be the uniqueness set in $P_{\mathbb{C}}$. Then in $P_{\mathbb{C}}$ we can introduce the topology τ_S of the inductive limit of spaces

$$B_{n,S} = \{\varphi(\lambda) \in P_{\mathbb{C}} : \|\varphi\|_{n,S} = \sup_{\lambda\in S} |\varphi(\lambda)| e^{-n|\lambda|} < \infty\}, \quad n \in \mathbf{N}.$$

In what follows, we need convergence to zero in the topology $\tau_{\mathbb{C}}$ [12]: let f_m be a countable sequence of functions from $P_{\mathbb{C}}$, then $f_m \to 0$ for $m \to \infty$ in the topology $\tau_{\mathbb{C}}$ if and only if there exist numbers $\sigma > 0$ and $M > 0$ such that

(a.1) $|f_m(z)| \leqslant Me^{\sigma|z|}$, $\forall m \in \mathbb{N}$, $\forall z \in \mathbb{C}$;

(b.1) for each compact set $K_{\mathbb{C}} \subset \mathbb{C}$ we have $|f_m(z)| \rightrightarrows 0$ as $m \to \infty$, $z \in K_{\mathbb{C}}$.

Let us introduce the concept of sufficiency of a set $S \subset \mathbb{C}$ in $U \subset P_{\mathbb{C}}$ with the topology induced from $P_{\mathbb{C}}$.

Definition 4.1. *We say that S is a sufficient set on U the conditions*

(a.2) *for each sequence of functions $q_k(z) \in U$ there exist numbers $\sigma > 0$ and $M > 0$ such that $|q_k(z)| \leqslant Me^{\sigma|z|}$, $\forall k \in \mathbb{N}$, $\forall z \in S$;*

(b.2) *for each compact set $K_S \subset S$ we have $|q_k(z)| \rightrightarrows 0$ as $k \to \infty$, $z \in K_S$;*

imply the convergence of this sequence on U.

Conditions (a.2) and (b.2) define convergence to zero in the topology τ_S.

The next theorem was proved in [9, Thm. 6].

Theorem 4.1. *If N_φ is a sufficient set in $\operatorname{Ker} M_\psi$, then the operator $M_\psi[\varphi\,\cdot\,]$ is injective and $\operatorname{Im} M_\psi[\varphi\,\cdot\,]$ is closed in $P_{\mathbb{C}}$.*

Since by Theorem 3.3 the two conditions: $M_\psi[\varphi\,\cdot\,]$ is injective and $\operatorname{Im} M_\psi[\varphi\,\cdot\,]$ is closed in $P_{\mathbb{C}}$ are equivalent to the surjectivity of operator $M_\varphi[\psi\,\cdot\,] + g_0$, we arrive at the next theorem.

Theorem 4.2. *Let $\varphi(z) \in P_{\mathbb{C}}, \psi(z) \in H(\mathbb{C})$. If N_φ is a sufficient set in $\operatorname{Ker} M_\psi$, then the Fisher representation*

$$H(\mathbb{C}) = (\psi) + \operatorname{Im}^{-1} M_\varphi$$

holds. Moreover, if N_ψ is a uniqueness set in $\operatorname{Ker} M_\varphi$, then the Fisher decomposition

$$H(\mathbb{C}) = (\psi) \oplus \operatorname{Ker} M_\varphi$$

holds.

Let $N_\varphi = \{\lambda_k\}_{k=1}^{+\infty}$ be the zero set of function $\varphi \in P_{\mathbb{C}}$, each zero is repeated as many times as its multiplicity (to avoid cumbersome notations, in what follows, by $\lambda_{\tilde{k}}$, $\tilde{k} = 1, 2\ldots$ we mean some subsequence of the sequence $\{\lambda_k\}_{k=1}^{+\infty}$); $N_\psi = \{\mu_k\}_{k=1}^{+\infty}$ is the set of zeros of the function $\psi \in H(\mathbb{C})$, each zero is repeated as many times as its multiplicity; by q_j denote the multiplicity of zero μ_j; $\tilde{N}_\psi$ is an infinite set, which consists of all distinct zeros of the function $\psi \in H(\mathbb{C})$.

Theorem 4.3 ([11])**.** *Let for some fixed $\alpha \in [0, +\infty)$ there exists a number $\beta \in [0, +\infty)$ such that $\alpha \cdot \beta < 1$ and the following conditions are satisfied:*

(a) *$N_\varphi \subset D_\alpha = \{z \in \mathbb{C} : |\operatorname{Im} z| \leqslant \alpha \operatorname{Re} z\}$ and there exists a subsequence $\lambda_{\tilde{k}}$ such that*

$$\operatorname{Re}(\lambda_{\tilde{k}}) < \operatorname{Re}(\lambda_{\tilde{k}+1}), \qquad \tilde{k} \in \mathbb{N}.$$

(b) *$N_\psi \subset D_\beta = \{z \in \mathbb{C} : |\operatorname{Im} z| \leqslant \beta \operatorname{Re} z\}$, and the elements of set $\tilde{N}_\psi$ satisfy*

$$\operatorname{Re}(\mu_k) < \frac{1 - \alpha\beta}{1 + \alpha\beta} \operatorname{Re}(\mu_{k+1}), \quad k \in \mathbb{N}.$$

Then the set N_φ is sufficient in $\operatorname{Ker} M_\psi$.

In [5] there is an example, which shows that there exist convolution operators in the considered class such that the problem of interpolation by functions from the kernel of the convolution operator with arbitrary complex interpolation nodes μ_j, $j = 1, 2, \ldots$, is, generally speaking, unsolvable. Suppose that the set of interpolation nodes contains the points $\mu_1 \in \mathbb{R}$, $\mu_2 = \mu_1 + i \in \mathbb{C}$, and $\varphi(z) = 1 - e^{iz}$. Then the problem of simple interpolation by entire functions from $\operatorname{Ker} M_\varphi = \{f \in H(\mathbb{C}) : f(z) = f(z + i)\}$ is unsolvable.

In this example, $\lambda_n = 2\pi n \in \mathbb{R}$. All functions from the $\operatorname{Ker} M_\varphi$ kernel are periodic with period i, therefore, it is not possible to specify arbitrary interpolation data at the nodes $\mu_1 \in \mathbb{R}$, $\mu_2 = \mu_1 + i \in \mathbb{C}$.

5. Main result

Let $N_\varphi = \{\lambda_k\}_{k=1}^{s\leqslant\infty}$ be the zero set of $\varphi \in P_{\mathbb{C}}$, $N_\psi = \{\mu_k\}_{k=1}^{m\leqslant\infty}$ be the zero set of $\psi(z) \in H(\mathbb{C})$, each zero is repeated taking into account its multiplicity. Using the result of [15], [19], we note that $\operatorname{Ker} M_\psi$ consists of quasi–polynomials with exponents from the set N_ψ, i.e., for each $r(z) \in \operatorname{Ker} M_\psi$ we can write

$$r(z) = \sum_{i=1}^{n} \sum_{j=0}^{q_i-1} C_{ij} z^j e^{\mu_i \cdot z}, \tag{5.1}$$

where q_i is the multiplicity of zero μ_i, all coefficients C_{ij}, $i = \overline{1,n}$, $j = \overline{0, q_i - 1}$ are non–zero.

Remark 5.1. *It is important that identity (5.1) involves only monomials with the exponents $\mu_i \in N_\psi$, $i = \overline{1,n}$, belonging to the conjugate diagram $r(z)$.*

We arrange μ_i, $i = \overline{1,n}$ in order of increasing absolute values. Since there are finitely many terms in (5.1), we have

$$|\mu_i| \leqslant C_\mu, \quad \exists C_\mu > 0, \quad i = \overline{1,n}.$$

Theorem 5.1. *Let a finite set of complex numbers $\widetilde{N}_\psi = \{\mu_i\}_{i=1}^n$ be the set of zeros $\psi(z)$ located in the conjugate diagram of function $r(z) \in \operatorname{Ker} M_\psi[\varphi\,\cdot\,]$, i.e.*

$$r(z) = \sum_{i=1}^{n} \sum_{j=0}^{q_i-1} C_{ij} z^j e^{\mu_i \cdot z}.$$

The set N_φ is sufficient in $\operatorname{Ker} M_\psi[\varphi\,\cdot\,]$ if and only if there exist $Q = \sum_{i=1}^{n} q_i$ elements of the set N_φ such that

$$D_{\mu\lambda} = \det\left(\lambda_p^j e^{\mu_i \lambda_p}\right) \neq 0, \quad i = \overline{1,n}, \quad j = \overline{0, q_i - 1}, \quad p = 1, \ldots, Q.$$

Proof. 1. Let $D_{\mu\lambda} \neq 0$. We take a sequence in the space $\operatorname{Ker} M_\psi$

$$r_m(z) = \sum_{i=1}^{n} \sum_{j=0}^{q_i-1} C_{ij}(m) z^j e^{\mu_i \cdot z}, \quad m = 1, 2, \ldots$$

According to the assumptions of theorem, there exist Q elements of the set N_φ such that

$$D_{\mu\lambda} = \det\left(\lambda_p^j e^{\mu_i \lambda_p}\right) \neq 0, \quad i = \overline{1,n}, \quad j = \overline{0, q_i - 1}, \quad p = \overline{1,Q},$$

then the coefficients $C_{ij}(m)$ solve the system

$$\sum_{i=1}^{n} \sum_{j=0}^{q_i-1} C_{ij}(m) \lambda_p^j e^{\mu_i \cdot \lambda_p} = r_m(\lambda_p), \quad p = \overline{1,Q},$$

and by Cramer rule they are expressed by formulas

$$C_{ij}(m) = \frac{\Delta_{l(j,i)}}{D_{\mu\lambda}},$$

where $\Delta_{l(j,i)}$ is the determinant of the matrix obtained from the matrix $A = \left(\lambda_p^j e^{\mu_i \lambda_p}\right)$ by replacing the $l(j,i)$th column by a column of free terms, while

$$\Delta_{l(j,i)} = \sum_{p=1}^{Q} (-1)^{l(j,i)+p} \det A_{l(j,i),p} r_m(\lambda_p).$$

We arrange λ_p, $p=\overline{1,Q}$ in order of increasing absolute values, then

$$|\lambda_p| \leqslant C_\lambda, \quad \exists\, C_\lambda > 0, \quad p=\overline{1,Q}.$$

Let us show that all coefficients $C_{ij}(m)$, $i=\overline{1,n}$, $j=\overline{0,q_i-1}$, are bounded. We are going to estimate the determinants $\Delta_{l(j,i)}$; in order to do this, by using (a.2), we estimate $r_m(\lambda_p)$, $p=\overline{1,Q}$:

$$|r_m(\lambda_p)| \leqslant Me^{\sigma|\lambda_p|} \leqslant Me^{\sigma\cdot C_\lambda},$$

and also estimate $\det A_{l(j,i),p}$ from above

$$\begin{aligned}\left|\det A_{l(j,i),p}\right| &\leqslant (Q-1)!\, |\lambda_{Q-1}|^{Q-1}\, e^{(Q-1)\operatorname{Re}(\mu_n\cdot\lambda_{Q-1})}\\ &\leqslant (Q-1)!\, |C_\lambda|^{Q-1}\, e^{(Q-1)C_\mu C_\lambda}.\end{aligned}$$

Then $C_{ij}(m)$ satisfies the inequality

$$|C_{ij}(m)| \leqslant C := \frac{Q!\cdot M}{D_{\mu\lambda}}\cdot |C_\lambda|^{Q-1}\, e^{(Q-1)C_\mu C_\lambda + \sigma C_\lambda}.$$

Let us prove that N_φ is a uniqueness set in $\operatorname{Ker} M_\psi$. In order to do this, we need to show that the identity $r_m(\lambda_p)=0$ implies the identity $r_m(z)\equiv 0$. We consider the system

$$\sum_{i=1}^{n}\sum_{j=0}^{q_i-1} C_{ij}(m)\lambda_p^j e^{\mu_i\cdot\lambda_p} = 0.$$

According to the assumptions of theorem, the determinant of this system is nonzero. Expressing the coefficients $C_{ij}(m)$ by the Cramer rule, as it has been done above, and replacing the $l(j,i)$ column in the determinant $\Delta_{l(j,i)}$ with the column of constant terms, we obtain $C_{ij}(m)=0$, and therefore $r_m(z)\equiv 0$.

Let us prove that

$$\lim_{m\to\infty} C_{ij}(m) = 0, \qquad i=\overline{1,n}, \qquad j=\overline{0,q_i-1}.$$

The above arguing implies that for $C_{ij}(m)$, by the relation $|r_m(\lambda_p)|\to 0$, $m\to\infty$, the estimate holds

$$|C_{ij}(m)| \leqslant \frac{Q!}{D_{\mu\lambda}} C_\lambda^{Q-1} e^{(Q-1)C_\mu C_\lambda}\cdot |r_m(\lambda_p)| \to 0, \quad m\to\infty, \quad \forall p\in\mathbb{N}.$$

Therefore, $C_{ij}(m)\to 0$, $m\to\infty$, $i=\overline{1,n}$, $j=\overline{0,q_i-1}$, that is, $\max\limits_{i=\overline{1,n},j=\overline{0,q_i-1}} |C_{ij}(m)|\to 0$, and this means that for each compact set $K_{\mathbb{C}}$ we have

$$|r_m(z)| \leqslant Q\cdot \max_{i=\overline{1,n},j=\overline{0,q_i-1}} |C_{ij}(m)| \max_{z\in K_{\mathbb{C}}} |z|^{\max\limits_{i=\overline{1,n}} q_i}\, e^{C_\mu\cdot \max\limits_{z\in K_{\mathbb{C}}}|z|} \to 0, \quad m\to\infty, \quad z\in K_{\mathbb{C}}.$$

We also have the estimate for $z\in\mathbb{C}$

$$|r_m(z)| \leqslant Q\cdot |C_{ij}(m)| |z|^{\max\limits_{i=\overline{1,n}} q_i}\, e^{\operatorname{Re}(\mu_i\cdot z)} \leqslant Q\cdot C\cdot e^{\left(\max\limits_{i=\overline{1,n}} q_i\right)\ln|z| + C_\mu |z|} \leqslant Q\cdot C\cdot e^{2\max\left(\max\limits_{i=\overline{1,n}} q_i, C_\mu\right)|z|}.$$

We obtain that $r_m(z)\to 0$ in the $\tau_{\mathbb{C}}$ topology, and the set N_φ is sufficient in $\operatorname{Ker} M_\psi$.

2. Let N_φ be a sufficient set in $\operatorname{Ker} M_\psi[\varphi\,\cdot\,]$. This means that N_φ is a uniqueness set in $\operatorname{Ker} M_\psi[\varphi\,\cdot\,]$, i.e., for each function $r(z)\in \operatorname{Ker} M_\psi[\varphi\,\cdot\,]$ vanishing on each element of the set N_φ we have $r(z)\equiv 0$.

We substitute Q elements of the set N_φ into the function

$$r(z) = \sum_{i=1}^{n}\sum_{j=0}^{q_i-1} C_{ij} z^j e^{\mu_i\cdot z}$$

and obtain the homogeneous system

$$\sum_{i=1}^{n}\sum_{j=0}^{q_i-1} C_{ij}\lambda_p^j e^{\mu_i\cdot\lambda_p} = 0, \qquad p=1,\ldots,Q.$$

Since this homogeneous system has only the trivial solution ($r(z) \equiv 0$), the determinant of this system is nonzero. Therefore,

$$D_{\mu\lambda} = \det\left(\lambda_p^j e^{\mu_i \lambda_p}\right) \neq 0, \quad i = \overline{1,n}, \quad j = \overline{0, q_i - 1}, \quad p = 1, \dots, Q.$$

□

Corollary 5.1. *Let the set N_ψ consist only of simple zeros of the function ψ, where $\widetilde{N}_\psi = \{\mu_i\}_{i=1}^n$ is the set of zeros of $\psi(z)$ located in the conjugate diagram of function $r(z) \in \operatorname{Ker} M_\psi[\varphi\,\cdot]$, that is,*

$$r(z) = \sum_{i=1}^{n} C_{ij} e^{\mu_i \cdot z}.$$

The set N_φ is sufficient in $\operatorname{Ker} M_\psi[\varphi\,\cdot]$ if and only if there exist n elements of the set N_φ obeying

$$D_{\mu\lambda} = \det\left(e^{\mu_i \lambda_p}\right) \neq 0, \qquad i, p = \overline{1,n}.$$

Remark 5.2. *The injectivity of a linear operator is equivalent to the triviality of its kernel, meaning that if the operator $M_\psi[\varphi\,\cdot]$ is injective, then*

$$D_{\mu\lambda} = \det\left(\lambda_p^j e^{\mu_i \lambda_p}\right) \neq 0, \quad i = \overline{1,n}, \quad j = \overline{0, q_i - 1}, \quad p = 1, \dots, Q.$$

The proof of this fact is similar to the proof of Statement 2 of Theorem 5.1.

In the case of simple interpolation nodes, we can specify conditions, under which the determinant $D_{\mu\lambda}$ is nonzero. In order to do this, we use the result in [16].

Theorem 5.2. *Let p be a prime number, and ε be the p root of unity in a field of characteristics zero. Suppose that $a_1, \dots, a_n \in \mathbb{Z}$ are pairwise incomparable modulo p; similarly for $b_1, \dots, b_n \in \mathbb{Z}$. Then*

$$\det\left(\varepsilon^{a_i \cdot b_j}\right) \neq 0.$$

Since the space of complex numbers is a field of characteristic zero, we can formulate the following statement.

Corollary 5.2. *Let p be a prime number, and we fix a natural number $k \in \{1, 2, \dots, p-1\}$. Suppose that $a_1, \dots, a_n \in \mathbb{Z}$ are pairwise incomparable modulo p; similarly for $b_1, \dots, b_n \in \mathbb{Z}$. Then*

$$\det\left(e^{\frac{2\pi i k}{p}(a_i + i \cdot A)\cdot(b_j + i \cdot B)}\right) \neq 0, \quad \forall A, B \in \mathbb{R}. \tag{5.2}$$

Proof. By the properties of the determinants and (5.2) for all $A, B \in \mathbb{R}$ we have

$$\det\left(e^{\frac{2\pi i k}{p}(a_i + i \cdot A)\cdot(b_j + i \cdot B)}\right) = e^{-\frac{2\pi k B}{p}\sum\limits_{i=1}^{n} a_i} \cdot e^{-\frac{2\pi k A}{p}\sum\limits_{j=1}^{n} b_j} \cdot e^{-\frac{2\pi i k n}{p} A \cdot B} \cdot \det\left(e^{\frac{2\pi i k}{p} a_i \cdot b_j}\right) \neq 0.$$

The proof is complete. □

We are in position to formulate our main results.

Theorem 5.3. *The following conditions are equivalent.*

(1) *The multiple interpolation problem in the preimage of convolution operator is solvable.*

(2) *The Fischer representation $H(\mathbb{C}) = \operatorname{Im}^{-1} M_\varphi + (\psi)$ holds.*

(3) *The operator $M_\varphi[\psi\,\cdot] + g_0(z)$ is surjective.*

(4) *The operator $M_\psi[\varphi\,\cdot]$ is injective and $\operatorname{Im} M_\psi[\varphi\,\cdot]$ is closed in $P_{\mathbb{C}}$.*

(5) *The set N_φ is sufficient in $\operatorname{Ker} M_\psi$.*

(6) *We have $D_{\mu\lambda} = \det\left(\lambda_p^j e^{\mu_i \lambda_p}\right) \neq 0$, $i = \overline{1,n}$, $j = \overline{0, q_i - 1}$, $p = 1, \dots, Q = \sum\limits_{i=1}^{n} q_i$.*

Proof. 1. (1) ⇔ (2) was proved in Lemma 1.1.

2. (2) ⇔ (3) was proved in Theorem 2.1.

3. (3) $\Leftrightarrow$ (4) is implied by Theorem 3.3.
4. (4) $\Rightarrow$ (6) is implied by Remark 5.2.
5. (5) $\Leftrightarrow$ (6) was proved in Theorem 5.1.
6. (5) $\Rightarrow$ (2) is implied by Theorem 4.2.

The proof is complete. $\square$

6. Particular cases

6.1. Linear differential equations with constant coefficients. If the characteristic function of a convolution operator is a polynomial, then the convolution operator becomes a linear differential operator with constant coefficients of finite order. We take two polynomials $P(z)$ and $Q(z)$ over the field of complex numbers and consider the equation

$$P\left(\frac{d}{dz}\right) f_0(z) = g_0(z).$$

Here $g_0(z) \in H(\mathbb{C})$ is chosen arbitrarily and is fixed. For the introduced polynomials $P(z)$, $Q(z)$ and the function $g_0(z)$, we define the operator

$$T[f] = P\left(\frac{d}{dz}\right)(Q \cdot f) + g_0(z),$$

which all acts in the space of entire functions.

All theorems of Sections 2 and 3 of this article for the functions $\psi(z) \in H(\mathbb{C})$, $\varphi(z) \in P_{\mathbb{C}}$ hold for the case of polynomials $P(z)$ and $Q(z)$ over the field of complex numbers. By Corollary 3.2 we obtain the next theorem.

Theorem 6.1.

$$H(\mathbb{C}) = (Q) \oplus \operatorname{Ker} P\left(\frac{d}{dz}\right) \Leftrightarrow P_{\mathbb{C}} = (P) \oplus \operatorname{Ker} Q\left(\frac{d}{dz}\right).$$

Theorem 6.1 was also obtained in [17]. In [18] the following theorem was proved.

Theorem 6.2. *A pair of polynomials (P, Q) in $\mathbb{C}$ satisfies*

$$H(\mathbb{C}) = (Q) \oplus \operatorname{Ker} P\left(\frac{d}{dz}\right)$$

if and only if the degrees of the polynomials $P(z)$ and $Q(z)$ are equal and the operator $P\left(\frac{d}{dz}\right)(Q\,\cdot)$ is injective.

In particular, when the zeros of the polynomials $P(z)$ and $Q(z)$ are simple, we denote them by λ_k and μ_k, $k = \deg P(z) = \deg Q(z)$, and the injectivity means

$$\det \left| e^{\lambda_k \mu_k} \right| \neq 0. \tag{6.1}$$

Moreover, since there is no Fisher decomposition in $H(\mathbb{C})$ for the functions (φ, ψ), there is no Fisher decomposition for the pair of polynomials (P, Q)

$$H(\mathbb{C}) = (Q) \oplus \operatorname{Im}^{-1} P\left(\frac{d}{dz}\right),$$

but the Fisher representation

$$H(\mathbb{C}) = (Q) + \operatorname{Im}^{-1} P\left(\frac{d}{dz}\right)$$

holds.

6.2. Fisher representation with conjugate zeros. For a function $f(z) \in H(\mathbb{C})$, the function $f^*(z) := \overline{f(\overline{z})} \in H(\mathbb{C})$ has complex conjugate zeros. For a pair of polynomials $(P(z), P^*(z))$ in one variable, it is known [18]) that the decomposition

$$H(\mathbb{C}) = (P^*) \oplus \operatorname{Ker} P\left(\frac{d}{dz}\right)$$

holds. Let us give another way of proving

$$H(\mathbb{C}) = (P^*) + \operatorname{Im}^{-1} P\left(\frac{d}{dz}\right)$$

for the case when all zeros of the polynomial $P(z)$ are simple, namely, by using condition (6.1). Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be a set of points from $\mathbb{C}$. The matrix A reads

$$A = \begin{pmatrix} e^{\lambda_1\overline{\lambda_1}} & e^{\lambda_1\overline{\lambda_2}} & \ldots & e^{\lambda_1\overline{\lambda_m}} \\ e^{\lambda_2\overline{\lambda_1}} & e^{\lambda_2\overline{\lambda_2}} & \ldots & e^{\lambda_2\overline{\lambda_m}} \\ \ldots & \ldots & \ldots & \ldots \\ e^{\lambda_m\overline{\lambda_1}} & e^{\lambda_m\overline{\lambda_2}} & \ldots & e^{\lambda_m\overline{\lambda_m}} \end{pmatrix}. \tag{6.2}$$

Proposition 6.1. *If the numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ are pairwise distinct, then the determinant of matrix A is nonzero. If the set of numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ contains coinciding numbers, then*

$$\det A = 0.$$

Proof. Let us construct the following finite–dimensional unitary space R (see definition in [2]). We take a set of functions of the variable $z \in \mathbb{C}$ generated by the numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$

$$\mathcal{E} = \{e^{\lambda_1 z}, e^{\lambda_2 z}, \ldots, e^{\lambda_m z}\}.$$

All functions from the set $\mathcal{E}$ are entire and, moreover, $\mathcal{E} \subset F$, where F is a Fock space, i.e.

$$F = \{f \in H(\mathbb{C})\colon \|f\|_F^2 = \frac{1}{\pi}\int\limits_{\mathbb{C}} |f(z)|^2 e^{-|z|^2}\, dv(z) < \infty\}.$$

Let $R \overset{def}{=} \operatorname{span}\mathcal{E}$, then the elements of the space R are entire functions, which are represented as finite linear combinations of functions from $\mathcal{E}$. For functions $p(z), q(z) \in R$, $z \in \mathbb{C}$, we introduce the scalar product

$$(p,q)_R \overset{def}{=} (p,q)_F = \frac{1}{\pi}\int\limits_{\mathbb{C}} p(z)\cdot\overline{q(z)}\, e^{-|z|^2}\, dv(z).$$

We obtain that R is a finite–dimensional closed subspace of the Fock Hilbert space. It is also easy to verify that for each $z_0 \in \mathbb{C}$ the estimate holds

$$|p(z_0)| \leqslant C_{z_0}\|p\|_R, \quad \forall p \in R.$$

where C_{z_0} is a constant depending only on z_0. This means that the delta functional $\delta_{z_0}\colon p \to p(z_0)$ is a linear continuous functional on R. This means that R is a space with a reproducing kernel [13].

It is easy to calculate

$$(e^{\lambda_j z}, e^{\lambda_k z})_R \overset{def}{=} \frac{1}{\pi}\int\limits_{\mathbb{C}} e^{\lambda_j z}\cdot\overline{e^{\lambda_k z}}\, e^{-|z|^2}\, dv(z) = e^{\lambda_j\overline{\lambda_k}}, \quad j,k = 1,\ldots,m. \tag{6.3}$$

In [2], there is a criterion for the linear independence of vectors $x_1, x_2, \ldots, x_m$ in an abstract finite–dimensional unitary space R. It is proved there that a system of vectors $x_1, x_2, \ldots, x_m$ is linearly independent if and only if the determinant of the Gram matrix of this system

$$\det\Gamma(x_1, x_2, \ldots, x_m) = \det\begin{pmatrix} (x_1,x_1)_R & (x_1,x_2)_R & \ldots & (x_1,x_m)_R \\ (x_2,x_1)_R & (x_2,x_2)_R & \ldots & (x_2,x_m)_R \\ \ldots & \ldots & \ldots & \ldots \\ (x_m,x_1)_R & (x_m,x_2)_R & \ldots & (x_m,x_m)_R \end{pmatrix}$$

is nonzero. The same reasoning applies to R. As vectors x_1, x_2, $\dots$, x_m, we take the functions

$$e^{\lambda_1 z}, \quad e^{\lambda_2 z}, \quad \dots, \quad e^{\lambda_m z}.$$

Linear independence of this system means that the condition

$$c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z} + \dots + c_m e^{\lambda_m z} = \varnothing, \tag{6.4}$$

where $\varnothing$ is the zero element in the space R, and c_1, c_2, $\dots$, c_m are complex numbers, implies $c_j = 0$, $j = 1, 2, \dots, m$.

This is why, by [13] the linear independence of a system of functions

$$e^{\lambda_1 z}, \quad e^{\lambda_2 z}, \quad \dots, \quad e^{\lambda_m z}$$

means that the condition

$$c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z} + \dots + c_m e^{\lambda_m z} \equiv 0, \quad z \in \mathbb{C}$$

implies $c_j = 0$, $j = 1, 2, \dots, m$. In our case the Gram matrix is the matrix A (see (6.2), (6.3)).

We are going to prove that if the set λ_1, λ_2, $\dots$, λ_m contains coinciding numbers, then the system of functions

$$e^{\lambda_1 z}, \quad e^{\lambda_2 z}, \quad \dots, \quad e^{\lambda_m z}$$

is linearly dependent in R.

If all the numbers λ_1, λ_2, $\dots$, λ_m are pairwise distinct, then the system of functions

$$e^{\lambda_1 z}, \quad e^{\lambda_2 z}, \quad \dots, \quad e^{\lambda_m z}$$

is linearly independent in R.

Indeed, let us suppose the contrary and there exist distinct complex numbers λ_1, λ_2, $\dots$, λ_m such that the system of functions

$$e^{\lambda_1 z}, \quad e^{\lambda_2 z}, \quad \dots, \quad e^{\lambda_m z}$$

is linearly dependent in R. This means that there exists a set of complex numbers c_1, c_2, $\dots$, c_l, $l \leqslant m$, $c_j \neq 0$ such that

$$\sum_{j=1}^{l} c_j e^{\lambda_j z} = \varnothing,$$

see (6.4).

We consider the space $\widehat{F}$

$$\widehat{F} = \{\widehat{f} \colon \widehat{f}(\lambda) = (e^{\lambda z}, f)_F,\ f \in F\}$$

Laplace transforms of functionals on F, $R \subset F$. For each function $f \in F$ we have

$$0 = (\varnothing, f)_F = \left(\sum_{j=1}^{l} c_j e^{\lambda_j z}, f(z)\right)_F = \sum_{k=1}^{l} c_k \widehat{f}(\lambda_k). \tag{6.5}$$

We have found that there are nonzero constants c_1, $\dots$, c_l such that for each function $\widehat{f} \in \widehat{F}$ identity (6.5) holds. It is well–known that the space $\widehat{F}$ coincides with the space F [14].

Therefore, the space $\widehat{F}$ contains the same functions as the space F. This implies that for each function $g \in F$ the identity

$$\sum_{k=1}^{l} c_k g(\lambda_k) = 0 \tag{6.6}$$

holds and $c_1, \dots, c_l \neq 0$. Relation (6.6) is obviously impossible for each function $g \in F$.

Hence, the system of functions

$$e^{\lambda_1 z}, \quad e^{\lambda_2 z}, \quad \dots, \quad e^{\lambda_m z}$$

is linearly independent in R. By [2] the Gram determinant of this system is nonzero, and therefore the determinant of the matrix A is nonzero. The proof is complete. $\square$

It is obvious that Proposition 6.1 is also valid for an arbitrary finite set of points from $\mathbb{C}^n$, $n \in \mathbb{N}$.

Proposition 6.2. *Let $\lambda_1, \lambda_2, \dots, \lambda_m$ be a set of points from $\mathbb{C}^n$, $n \in \mathbb{N}$. The matrix A reads*

$$A = \begin{pmatrix} e^{\langle \lambda_1, \overline{\lambda_1}\rangle} & e^{\langle \lambda_1, \overline{\lambda_2}\rangle} & \dots & e^{\langle \lambda_1, \overline{\lambda_m}\rangle} \\ e^{\langle \lambda_2, \overline{\lambda_1}\rangle} & e^{\langle \lambda_2, \overline{\lambda_2}\rangle} & \dots & e^{\langle \lambda_2, \overline{\lambda_m}\rangle} \\ \dots & \dots & \dots & \dots \\ e^{\langle \lambda_m, \overline{\lambda_1}\rangle} & e^{\langle \lambda_m, \overline{\lambda_2}\rangle} & \dots & e^{\langle \lambda_m, \overline{\lambda_m}\rangle} \end{pmatrix}.$$

If the points $\lambda_1, \lambda_2, \dots, \lambda_m$ from $\mathbb{C}^n$ are pairwise distinct, then the determinant of matrix A is nonzero. If the set of points $\lambda_1, \lambda_2, \dots, \lambda_m$ from $\mathbb{C}^n$ contains identical points, then $\det A = 0$.

The proof of this proposition is almost literally reproduces the proof of Proposition 6.1.

In conclusion of this section, we note that, according to [8, Thm. 37] and Theorem 4.2 of this paper, the next theorem is true.

Theorem 6.3. *If $\varphi(z) \in P_{\mathbb{C}}$, $\psi(z) \equiv \varphi^*(z)$, then the Fisher representation*

$$H(\mathbb{C}) = (\varphi^*) + \operatorname{Im}^{-1} M_\varphi$$

holds.

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Valerii Valentinovich Napalkov,
Institute of Mathematics,
Ufa Federal Research Center, RAS
Chernyshevsky str. 112,
450008, Ufa, Russia
E-mail: `vnap@mail.ru`

Andrey Alexandrovich Nuyatov,
Nizhny Novgorod State Technical University
named after R.E. Alekseev,
Minin str. 24,
603155, Nizhny Novgorod, Russia
E-mail: `nuyatov1aa@rambler.ru`