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EXISTENCE CRITERION FOR PERIODIC SOLUTIONS TO ONE CLASS OF SYSTEMS OF SECOND ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this paper we study the existence of periodic solutions for a class of systems of second order ordinary differential equations with a separated main nonlinear part. Taking into consideration the structure of zero set for the main nonlinear part, we find new conditions ensuring an apriori estimate for periodic solutions. Under the apriori estimate, we formulate and prove a criterion for the existence of periodic solutions under any perturbation from a given class. The proof is made by using methods for calculating mapping degree of the vector field and employing the invariance of the existence of periodic solutions under a continuous varying of the main nonlinear part.

Keywords: main nonlinear part, perturbation, periodic solution, apriori estimate, guiding function, mapping degree of the vector field.

Mathematics Subject Classification: 34B15, 45G15

1. Introduction

In this paper we study the existence of ω-periodic solutions for systems of nonlinear ordinary differential equations

$$x'' = \prod_{j=1}^{q} \varphi_j(x' - B_j(x)) Q(x' - B(x)) + f(t, x, x'), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n. \tag{1.1}$$

Here n, q are natural numbers $n \geqslant 2$, $\mathbb{R} = (-\infty, +\infty)$, the mappings $\varphi_j : \mathbb{R}^n \mapsto \mathbb{R}$, $B_j, Q, B : \mathbb{R}^n \mapsto \mathbb{R}^n$, $f : \mathbb{R}^{1+2n} \mapsto \mathbb{R}^n$ are continuous and for some positive ω, α, α_j, $j = \overline{1,q}$, where $\alpha_1 + \ldots + \alpha_q + \alpha = m > 1$, they obey the following conditions

1) $\varphi_j(\lambda y) \equiv \lambda^{\alpha_j} \varphi_j(y)$, $B_j(\lambda y) \equiv \lambda B_j(y)$, $j = \overline{1,q}$, $B(\lambda y) \equiv \lambda B(y)$, $Q(\lambda y) \equiv \lambda^{\alpha} Q(y)$ for all $\lambda > 0$ (positive homogeneity condition);
2) $\varphi_j(y) > 0$, $j = \overline{1,q}$ and $Q(y) \neq 0$ for $y \neq 0$;
3) $f(t + \omega, y_1, y_2) \equiv f(t, y_1, y_2)$;
4) the growth order of $|f(t, y_1, y_2)|$ for large $|y_1| + |y_2|$ is bounded by the limit

$$\lim_{|y_1|+|y_2| \to \infty} (|y_1| + |y_2|)^{-m} \max_{t \in [0,\omega]} |f(t, y_1, y_2)| = 0.$$

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The term

$$\prod_{j=1}^{q} \varphi_j(x' - B_j(x)) Q(x' - B(x))$$

in the right hand side of system (1.1), which is positive homogeneous in x and x', is called the main nonlinear part, while $f(t, x, x')$ is the perturbation. A solution $x \in C^2(\mathbb{R}; \mathbb{R}^n)$ of system of equations (1.1) is called ω-periodic if $x(t + \omega) \equiv x(t)$.

In [5], [6], [7] the existence of ω-periodic solutions for systems of equations of the form

$$x'' = P(t, x, x') + f(t, x, x'), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \tag{1.2}$$

was investigated when the homogeneous nonlinear part P and the perturbation f satisfy Conditions 3, 4. The study was made in two steps. At the first step, there were found conditions, which ensured an apriori estimate of ω-periodic solutions x, i.e., the existence of a number $M_1 > 0$ independent of x such that

$$\|x\| + \|x'\| < M_1, \tag{1.3}$$

where $\|x\| = \max\{|x(t)| : t \in [0, \omega]\}$.

At the second step, under conditions ensuring the apriori estimate, by the methods for calculating the mapping degree of the vector field [2], the existence of ω-periodic solutions was established.

Conditions ensuring apriori estimate of (1.3) were studied subject to the structure of zero set of main nonlinear part $P(t, y_1, y_2)$. There were considered cases, where the zero set consists of

a) a single surface $y_2 = B(t, y_1)$, and the system of equations $x' = B(t, x)$ has no non–zero ω-periodic solutions;

b) a finitely many surfaces $y_2 = B_j(t, y_1)$, $j = \overline{1, q}$, between which there are no so–called switchings, which means that the autonomous system $y' = P(t_0, x_0, y)$, $y \in \mathbb{R}^n$, for each fixed t_0 and x_0, has no non–stationary bounded solution;

c) finitely many surfaces $y_2 = B_j(t, y_1)$, $j = \overline{1, q}$, between which the aforementioned switchings are possible.

Cases a) and b) were studied in [5], [6], and Case c) for $n = 2$ was considered in one example in [7].

System (1.1) belongs to Case c). By developing the approach from [7], we find conditions ensuring estimate (1.3) for any perturbation of f. To derive the apriori estimate, we apply the guiding function method and take into consideration the explicit structure of the set of zeros of main nonlinear part. Under the apriori estimate, we formulate and prove a criterion for the existence of ω-periodic solutions for any perturbation of f. The proof applies and develops the methods of [6], [8] and employs the invariance property of the existence of ω-periodic solutions under continuous varying of the main nonlinear part.

In [1], the existence of a periodic solution to nonlinear ordinary differential equations of higher orders was investigated by means of the guiding function method.

2. MAIN RESULTS

Together with Conditions 1–4 we consider the following conditions:

5) the system of equations $y' = Q(y)$, $y \in \mathbb{R}^n$ has no bounded non-zero solutions;

6) $B \in C^1(\mathbb{R}^n \setminus \{0\}; \mathbb{R}^n)$ and there exists a function $W \in C^1(\mathbb{R}^n \setminus \{0\}; \mathbb{R})$ such that for any $y \in \mathbb{R}^n \setminus \{0\}$ the inequalities
$$\langle B(y), W'(y)\rangle > 0, \qquad \langle B_j(y), W'(y)\rangle > 0, \qquad j = \overline{1, q},$$
hold, where $W'(y)$ is the derivative (gradient) of $W(y)$, $\langle y, z\rangle = y_1 z_1 + \ldots + y_n z_n$ is the scalar product in $\mathbb{R}^n$;

7) $B_j(y) = A_j y$, $j = \overline{1, q}$ and $B(y) = Ay$, where A, A_j, $j = \overline{1, q}$ are mutually commuting matrices and $\det(\exp(\omega D) - I) \neq 0$ for each convex combination $D = \mu_0 A + \mu_1 A_1 + \ldots + \mu_q A_q$.

We introduce the family of mappings
$$P_\lambda(y_1, y_2) = \prod_{j=1}^{q} \varphi_j(y_2 - B_{j,\lambda}(y_1)) Q(y_2 - B(y_1)), \quad y_1, y_2 \in \mathbb{R}^n, \quad \lambda \in [0, 1],$$
where $B_{j,\lambda}(y) = (1-\lambda)B_j(y) + \lambda B(y)$. Apriori estimate (1.3) for ω-periodic solutions to system (1.1) is implied by the following theorem.

Theorem 2.1. *Let Conditions 1, 2, 5 and one of Conditions 6 or 7 be satisfied. Then there exist positive numbers σ and M such that for each ω-periodic vector-function $x \in C^2(\mathbb{R}; \mathbb{R}^n)$ satisfying the inequality $\|x\| + \|x'\| > M$, the estimate*
$$\|x'' - P_\lambda(x, x')\| \geqslant \sigma\left(\|x\| + \|x'\|\right)^m \tag{2.1}$$
holds for each $\lambda \in [0, 1]$.

We denote by $\gamma(Q)$, $\gamma(B)$ the mappings degrees of the vector fields $Q, B : \mathbb{R}^n \mapsto \mathbb{R}^n$ on the unit sphere $|y| = 1$ of the space $\mathbb{R}^n$ [2].

Theorem 2.2. *Under the assumptions of Theorem 2.1, there exist ω-periodic solutions to system (1.1) for any perturbation f if and only if $\gamma(Q)\gamma(B) \neq 0$.*

In the proof of Theorem 2.2, we apply and develop the methods from [6], [8] and employ the invariance of the existence of ω-periodic solutions to the system of equations
$$x'' = P_\lambda(x, x') + f(t, x, x'), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n,$$
under continuous varying of $\lambda \in [0, 1]$.

Examples of positive homogeneous mappings Q satisfying Condition 5 were provided in [4]. For these mappings, an efficient algorithm for computing $\gamma(Q)$ was proposed.

3. PROOF OF THEOREM 2.1

Suppose that estimate (2.1) fails. Then there exist sequences $\lambda_k \in [0, 1]$, $x_k \in C^2(\mathbb{R}; \mathbb{R}^n)$, $k = 1, 2, \ldots$, such that
$$x_k(t + \omega) \equiv x_k(t), \qquad r_k := \|x_k\| + \|x_k'\| \to \infty, \qquad k \to \infty,$$
$$\|x_k'' - P_{\lambda_k}(x_k, x_k')\| < \frac{1}{k}\Big(\|x_k\| + \|x_k'\|\Big)^m.$$
We consider the vector functions $y_k(t) = r_k^{-1} x_k(t)$, $t \in \mathbb{R}$, $k = 1, 2, \ldots$. For these vector functions we have
$$y_k \in C^2(\mathbb{R}; \mathbb{R}^n), \quad y_k(t + \omega) \equiv y_k(t), \quad \|y_k\| + \|y_k'\| = 1,$$
$$r_k^{1-m} y_k''(t) = P_{\lambda_k}(y_k(t), y_k'(t)) + o(1), \quad t \in \mathbb{R}. \tag{3.1}$$
Without loss of generality, we can assume that $\lambda_k \to \lambda_0$ and $\|y_k - y_0\| \to 0$ as $k \to \infty$.

If $y_0(t) \equiv 0$, then for the sequence of vector functions $z_k(t) = y_k'(t_k + tr_k^{1-m})$, $k = 1, 2, \ldots$, where $|y_k'(t_k)| = \|y_k'\| \to 1$ as $k \to \infty$, we have

$$|z_k(t)| \leqslant |z_k(0)|, \qquad z_k'(t) = P_{\lambda_0}(0, z_k(t)) + o(1), \quad t \in \mathbb{R},$$
$$|z_k(0)| \to 1, \qquad k \to \infty.$$

Passing to the limit, we obtain a non–zero bounded solution to the system of equations

$$z' = P_{\lambda_0}(0, z(t)), \quad z \in \mathbb{R}^n. \tag{3.2}$$

On the other hand, one can directly verify that, due to Conditions 2 and 5, system (3.2) cannot have nonzero bounded solutions. Therefore, $y_0(t) \not\equiv 0$. Proceeding then as in the proof of Theorem 1 in [5], we get

$$y_0(t) \neq 0, \quad t \in \mathbb{R}. \tag{3.3}$$

In what follows we can suppose that $|y_k(t)| > \frac{1}{2}|y_0(t)|$ for all t and k.

Now we employ the fact that, due to Condition 5 and [3, Thms. 2.2, 2.3], for Q there exists a so–called guiding function $V \in C^1(\mathbb{R}^n; \mathbb{R})$ with the properties

$$V(\lambda y) \equiv \lambda^2 V(y), \qquad \langle Q(y), V'(y) \rangle > c_1 |y|^{\alpha+1} > 0, \qquad y \in \mathbb{R}^n \setminus \{0\},$$

where $V'(y)$ is the derivative (gradient) of $V(y)$. We calculate the scalar product of both sides of (3.1) with $V'(y_k'(t) - B(y_k(t)))$ and integrate from 0 to ω

$$r_k^{1-m} \int_0^\omega \langle y_k''(t), V'(y_k'(t) - B(y_k(t))) \rangle dt$$
$$= \int_0^\omega \langle P_{\lambda_0}(y_k(t), y_k'(t)), V'(y_k'(t) - B(y_k(t))) \rangle dt + o(1).$$

We estimate the right hand side from below

$$\int_0^\omega \langle P_{\lambda_0}(y_k(t), y_k'(t)), V'(y_k'(t) - B(y_k(t))) \rangle dt$$
$$\geqslant c_1 \int_0^\omega \prod_{j=1}^q \varphi_j(y_k'(t) - B_{j,\lambda_0}(y_k(t))) |y_k'(t) - B(y_k(t))|^{\alpha+1} dt,$$

and represent the integrand in the left hand side as

$$\langle y_k'', V'(y_k' - B(y_k)) \rangle = (V(y_k' - B(y_k)))' + \langle B'(y_k) y_k', V'(y_k' - B(y_k)) \rangle.$$

As a result, taking into consideration the boundedness of y_k, y_k' and passing to the limit, we obtain

$$\int_0^\omega \prod_{j=1}^q \varphi_j(y_k'(t) - B_{j,\lambda_0}(y_k(t))) |y_k'(t) - B(y_k(t))|^{\alpha+1} dt \to 0, \quad k \to \infty.$$

Without loss of generality, we can suppose that for all $t \in [0, \omega]$

$$\prod_{j=1}^q \varphi_j(y_k'(t) - B_{j,\lambda_0}(y_k(t))) |y_k'(t) - B(y_k(t))|^{\alpha+1} \to 0, \quad k \to \infty.$$

For each $\varepsilon > 0$ for the measure of set

$$E_{k,\varepsilon} = \{t \in [0, \omega] : |y_k'(t) - B(y_k(t))| \geqslant \varepsilon, |y_k'(t) - B_{j,\lambda_0}(y_k(t))| \geqslant \varepsilon, j = \overline{1, q}\}$$

we have $\operatorname{mes}(E_{k,\varepsilon}) \to 0$, $k \to \infty$.

We fix $\varepsilon > 0$ and introduce the sets

$$E^0_{k,\varepsilon} = \{t \in [0,\omega] : |y'_k(t) - B_{0,\lambda_0}(y_k(t))| < \varepsilon\}, \qquad B_{0,\lambda_0} = B,$$

$$E^l_{k,\varepsilon} = \{t \in [0,\omega] : |y'_k(t) - B_{l,\lambda_0}(y_k(t))| < \varepsilon, |y'_k(t) - B_{j,\lambda_0}(y_k(t))| \geqslant \varepsilon, j = \overline{0, l-1}\}, \quad l = \overline{1,q}.$$

The sets $E_{k,\varepsilon}$, $E^0_{k,\varepsilon}$, …, $E^q_{k,\varepsilon}$ are mutually disjoint and their union coincides with $[0,\omega]$.

If Condition 6 holds, we choose a positive c_2 satisfying the inequalities

$$\langle B(y), W'(y)\rangle \geqslant c_2, \quad \langle B_j(y), W'(y)\rangle \geqslant c_2, \quad j = \overline{1,q}$$

for each $y \in \mathbb{R}^n$, $a_0 \leqslant |y| \leqslant 1$, where $a_0 = 0,5 \min\{|y_0(t)| : t \in [0,\omega]\}$. We then have

$$\begin{aligned} 0 =& W(y_k(\omega)) - W(y_k(0)) = \int_0^\omega \Big(W(y_k(t))\Big)' dt = \int_0^\omega \langle y'_k(t), W'(y_k(t))\rangle dt \\ =& \sum_{j=0}^q \int_{E^j_{k,\varepsilon}} \langle B_{j,\lambda_0}(y_k(t)), W'(y_k(t)\rangle dt + \sum_{j=0}^q \int_{E^j_{k,\varepsilon}} \langle y'_k(t) - B_{j,\lambda_0}(y_k(t)), W'(y_k(t)\rangle dt \\ &+ \int_{E_{k,\varepsilon}} \langle y'_k(t), W'(y_k(t)\rangle dt > c_2 (\omega - \mathrm{mes}\,(E_{k,\varepsilon})) - \Big(\varepsilon\omega + \mathrm{mes}\,(E_{k,\varepsilon})\Big) \max_{a_0 \leqslant |y| \leqslant 1} |W'(y)|. \end{aligned}$$

Taking into consideration the limit $\mathrm{mes}\,(E_{k,\varepsilon}) \to 0$ as $k \to \infty$, we can choose the number $\varepsilon > 0$ and the index k so that

$$c_2 (\omega - \mathrm{mes}\,(E_{k,\varepsilon})) - \Big(\varepsilon\omega + \mathrm{mes}\,(E_{k,\varepsilon})\Big) \max_{a_0 \leqslant |y| \leqslant 1} |W'(y)| > 0,$$

which is the contradiction.

If Condition 7 holds, we define

$$D_{k,\varepsilon}(t) = \begin{cases} A, & t \in E_{k,\varepsilon} \cup E^0_{k,\varepsilon}, \\ A_l, & t \in E^l_{k,\varepsilon}, l \geqslant 1, \end{cases}$$

$$h_{k,\varepsilon}(t) = y'_k(t) - D_{k,\varepsilon}(t) y_k(t).$$

It is easy to verify that

$$|h_{k,\varepsilon}(t)| < \varepsilon, \qquad t \in E^l_{k,\varepsilon}, \qquad l = \overline{0,q},$$

$$D_{k,\varepsilon}(t) \int_0^s D_{k,\varepsilon}(\tau) d\tau = \int_0^s D_{k,\varepsilon}(\tau) d\tau D_{k,\varepsilon}(t), \qquad t, s \in [0,\omega].$$

In view of these properties for y_k we have

$$y_k(\omega) = e^{\int_0^\omega D_{k,\varepsilon}(\tau)d\tau} y_k(0) + \int_0^\omega e^{\int_t^\omega D_{k,\varepsilon}(\tau)d\tau} h_{k,\varepsilon}(t) dt,$$

$$\left| \left(e^{\int_0^\omega D_{k,\varepsilon}(\tau)d\tau} - I \right) y_k(0) \right| < N\varepsilon \quad \text{for} \quad k > k_0(\varepsilon),$$

where N is independent of k and ε,

$$\left| \left(e^{\mu_{k,\varepsilon} A + \mu^{(0)}_{k,\varepsilon} A + \mu^{(1)}_{k,\varepsilon} A_1 + \ldots + \mu^{(q)}_{k,\varepsilon} A_q} - I \right) y_k(0) \right| < N\varepsilon, \quad k > k_0(\varepsilon).$$

Here

$$\mu_{k,\varepsilon} = \mathrm{mes}\,(E_{k,\varepsilon}), \qquad \mu^{(l)}_{k,\varepsilon} = \mathrm{mes}\,\big(E^l_{k,\varepsilon}\big), \qquad \mu_{k,\varepsilon} + \mu^{(0)}_{k,\varepsilon} + \ldots + \mu^{(q)}_{k,\varepsilon} = \omega, \qquad \mu_{k,\varepsilon} \to 0, \quad k \to \infty.$$

Passing to the limit as $k \to \infty$, we obtain

$$\left|\left(e^{\mu_\varepsilon^{(0)}A+\mu_\varepsilon^{(1)}A_1+\ldots+\mu_\varepsilon^{(q)}A_q} - I\right) y_0(0)\right| \leqslant N\varepsilon,$$

where $\mu_\varepsilon^{(0)} + \ldots + \mu_\varepsilon^{(q)} = \omega$. Tending ε to zero, we arrive at the identities

$$\left(e^{\mu_0^{(0)}A+\mu_0^{(1)}A_1+\ldots+\mu_0^{(q)}A_q} - I\right) y_0(0) = 0, \qquad \mu_0^{(0)} + \ldots + \mu_0^{(q)} = \omega.$$

Since $y_0(0) \neq 0$ (by (3.3)), we find

$$\det\left(e^{\mu_0^{(0)}A+\mu_0^{(1)}A_1+\ldots+\mu_0^{(q)}A_q} - I\right) = 0,$$

and this contradicts Condition 7. The proof of Theorem 2.1 is complete.

Let us verify that under the assumptions of Theorem 2.1, estimate (1.3) is implied by (2.1). Indeed, for each ω-periodic solution of system (1.1), either $\|x\| + \|x'\| \leqslant M$, or $\|x\| + \|x'\| > M$, and it follows from (2.1) that

$$\sigma\left(\|x\| + \|x'\|\right)^m \leqslant \|x'' - P_0(x, x')\| = \|f(\cdot, x, x')\|.$$

By Condition 4,

$$\|f(\cdot, x, x')\| < \frac{\sigma}{2}\left(\|x\| + \|x'\|\right)^m + M_{f,\sigma}.$$

Therefore, in the case $\|x\| + \|x'\| > M$ the estimate

$$\left(\|x\| + \|x'\|\right)^m < \frac{2}{\sigma} M_{f,\sigma}$$

holds, while in the general case

$$\left(\|x\| + \|x'\|\right)^m < M^m + \frac{2}{\sigma} M_{f,\sigma}.$$

4. Proof of Theorem 2.2

Following the lines of [5, Thm. 3], we first prove the invariance of the existence of ω-periodic solutions to the system of equations

$$x'' = P_\lambda(x, x') + f(t, x, x'), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \tag{4.1}$$

under a continuous varying of $\lambda \in [0, 1]$.

Lemma 4.1. *The system (4.1) has ω-periodic solutions for $\lambda = 0$ and any perturbation f if and only if has ω-periodic solutions for $\lambda = 1$ and any perturbation f.*

Proof. Let $\lambda_1, \lambda_2 \in [0, 1]$ and each vector function $y \in C^1([0, \omega]; \mathbb{R}^n)$ satisfies the inequality

$$\|P_{\lambda_1}(y, y') - P_{\lambda_2}(y, y')\| \leqslant \frac{1}{4}\sigma\Big(\|y\| + \|y'\|\Big)^m.$$

Using estimate (2.1), we are going to show that for system (4.1) with $\lambda = \lambda_1$ replaced by $\lambda = \lambda_2$ the existence of ω-periodic solution is preserved for any perturbation f. This will prove the lemma.

We take an arbitrary perturbation $\tilde{f}$. Each vector function $y \in C^1([0, \omega]; \mathbb{R}^n)$ satisfies the inequality

$$\|\tilde{f}(\cdot, y, y')\| \leqslant \frac{1}{4}\sigma\Big(\|y\| + \|y'\|\Big)^m + M_{\tilde{f},\sigma},$$

where $M_{\tilde{f},\sigma}$ is a positive number depending only on $\tilde{f}$ and σ. We choose a number

$$L > \max\left(M, \left(2\sigma^{-1} M_{\tilde{f},\sigma}\right)^{\frac{1}{m}}\right)$$

and define the perturbation

$$g_L(t,y_1,y_2)=\tilde f(t,y_1,y_2)+\eta(|y_1|+|y_2|)\left(P_{\lambda_2}(y_1,y_2)-P_{\lambda_1}(y_1,y_2)\right),$$

where $\eta(s)\in C(\mathbb{R})$, $0\leqslant\eta(s)\leqslant 1$ for all $s\in\mathbb{R}$, $\eta(s)=1$ for $|s|\leqslant L$ and $\eta(s)=0$ for $|s|\geqslant L+1$.

Let x be a ω-periodic solution to system (4.1) for $\lambda=\lambda_1$ and perturbation $f=g_L$. We are going to verify that $\|x\|+\|x'\|\leqslant L$; then it implies that x is also a ω-periodic solution to system (4.1) for $\lambda=\lambda_2$ and the perturbation $f=\tilde f$. Indeed, if $\|x\|+\|x'\|>L$, then, according to estimate (2.1) and the choice of the number L, we have

$$\sigma\Big(\|x\|+\|x'\|\Big)^m<\|x''-P_{\lambda_1}(x,x')\|\leqslant\|\tilde f(\cdot,x,x')\|$$
$$+\|P_{\lambda_1}(x,x')-P_{\lambda_2}(x,x')\|\leqslant 0,5\sigma\Big(\|x\|+\|x'\|\Big)^m+M_{\tilde f,\sigma}.$$

This leads us to the contradiction

$$\|x\|+\|x'\|\leqslant\max\left(2\sigma^{-1}M_{\tilde f,\sigma}\right)^{\frac{1}{m}}<L.$$

The proof is complete. □

Thus, system (1.1) has ω-periodic solutions for any perturbation f if and only if the system of equations

$$x''=Q_1(x'-B(x))+f(t,x,x'),\quad t\in\mathbb{R},\quad x\in\mathbb{R}^n,\tag{4.2}$$

with $Q_1(y)=\varphi_1(y)\cdot\ldots\cdot\varphi_q(y)Q(y)$, has ω-periodic solutions for any perturbation f. At the same time, the condition holds

8) the system of equations $y'=Q_1(y)$, $y\in\mathbb{R}^n$ has no bounded non–zero solutions.

Under Condition 6 the estimate $\langle B(y),W'(y)\rangle>0$, $y\in\mathbb{R}^n\setminus\{0\}$, implies

9) the system of equations $y'=B(y)$, $y\in\mathbb{R}^n$ has no bounded non–zero solutions.

If Condition 7 is satisfied, then $B(y)=Ay$. In this case, the mapping $B(y)$ by the formula $Ay+\lambda\varepsilon y$, $\lambda\in[0,1]$ is reduced to the mapping $Ay+\varepsilon y$ with matrix $A+\varepsilon I$, which has no purely imaginary eigenvalues. Under this change, according to [5, Thm. 3], the existence of ω-periodic solutions to system (4.2) under any perturbation f is preserved. This is why, in what follows, without loss of generality, we can suppose that Condition 9 is satisfied under Condition 7. Now Theorem 2.2 is implied by the next lemma.

Lemma 4.2. *System (4.2) possesses ω-periodic solutions for any perturbation f if and only if $\gamma(Q)\gamma(B)\neq 0$.*

Proof. Necessity. Let $\gamma(Q)\gamma(B)=0$. We are going to prove that for some perturbation f, system (4.2) has no ω-periodic solutions. In order to do this, we follow the lines of [8, Thm. 2, Necessity].

By the known properties of mapping degree of vector fields we have

$$\gamma(Q_1(-B))=\gamma(Q)\gamma(-B)=(-1)^n\gamma(Q)\gamma(B)=0.$$

Then by [2, Thm. 5.2] the mapping $Q_1(-B)$ can be continuously continued inside the ball $|y|<1$ without zeros

$$F(y)=Q_1(-B(y))\quad\text{for}\quad|y|=1\quad\text{and}\quad F(y)\neq 0\quad\text{for}\quad|y|<1.$$

We let $g(y)=F(y)-Q_1(-B(y))$ for $|y|\leqslant 1$ and $g(y)=0$ for $|y|>1$. It is easy to verify that

$$|y_2|+|Q_1(y_2-B(y_1))+g(y_1)|>0\quad\text{for all}\quad y_1,y_2\in\mathbb{R}^n.\tag{4.3}$$

Let us show that for some $\omega_0>0$ the system of equations

$$x''=Q_1(x'-B(x))+g(x),\quad x\in\mathbb{R}^n,\tag{4.4}$$

has no ω_0-periodic solutions. Indeed, otherwise there exists a sequence of $\left(\frac{1}{k}\right)$-periodic solutions x_k, $k = 1, 2, \dots$ to system (4.4). Following lines of the proof of Theorem 2.1, it can be shown that

$$\sup_{k\geqslant 1}\sup_{t\in\mathbb{R}}\Big(|x_k(t)| + |x_k'(t)|\Big) < \infty.$$

Taking into consideration (4.4), we can assume that the sequences of vector functions x_k, x_k' converge uniformly on each interval $[a, b]$

$$\sup_{a\leqslant t\leqslant b}\Big(|x_k(t) - x_0(t)| + |x_k'(t) - x_0'(t)|\Big) \to 0, \qquad k \to \infty.$$

Moreover, due to the periodicity of x_k, $\left(\frac{1}{k}\right)$ for all $z_1, z_2 \in \mathbb{R}^n$, $k = 1, 2, \dots$ we have

$$\frac{1}{k}\int\limits_0^{\frac{1}{k}}\Big(\langle x_k'(t), z_1\rangle + \langle Q(x_k(t), x_k'(t)) + g(x_k(t)), z_2\rangle\Big)dt = 0.$$

Passing to the limit, we get

$$\langle x_0'(0), z_1\rangle + \langle Q(x_0(0), x_0'(0)) + g(x_0(0)), z_2\rangle = 0$$

for all $z_1, z_2 \in \mathbb{R}^n$. This contradicts (4.3).

In system (4.4) we make the change $x(t) = r^\beta y(r^\nu t)$, where $r = \left(\frac{\omega}{\omega_0}\right)^{\frac{1}{\nu}}$, $\beta = 0$, $\nu = 1$ for $m = 2$ and $\beta = 1$, $\nu = \frac{1-m}{m-2}$ for $m \neq 2$. This gives the system of equations

$$y'' = Q_1(y' - r^{-\nu}B(y)) + r^{-(\beta+\nu)m}g(r^\beta y), \quad y \in \mathbb{R}^n. \tag{4.5}$$

If a solution $y(t)$ of system (4.5) is ω-periodic, then the solution $x(t) = r^\beta y(r^\nu t)$ of system (4.4) is ω_0-periodic. By the above proven facts, this implies that system (4.5) has no ω-periodic solutions.

By Conditions 8 and 9, the family of mappings $Q_1(y_2 - ((1-\lambda)r^{-\nu} + \lambda)B(y_1))$, $\lambda \in [0, 1]$ satisfies the assumptions of Theorem 3 in [5]. According to this theorem, since system (4.5) has no ω-periodic solutions, then for some perturbation f system (4.2) also has no ω-periodic solutions.

Sufficiency. Let $\gamma(Q)\gamma(B) \neq 0$. Then $\gamma(Q_1(-B)) \neq 0$ and all assumptions of Theorem 2 in [6] are satisfied by system (4.2). According to this theorem, system (4.2) has ω-periodic solutions for any perturbation f. The proof is complete. □

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