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ON CONTROL PROBLEMS FOR FRACTIONAL SYSTEMS WITH AFTEREFFECT

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Abstract. We consider dynamical models with aftereffect described by functional–differential equations with fractional derivatives. These models encompass processes, in which the system state may change abruptly at certain points in time, which is interpreted as the result of impulse effects (shocks). The trajectories of such systems may have discontinuities at certain points in time, and between these points the behavior of system is described by differentiable functions, which satisfy the equation in the usual sense. We pose a general control problem for a given system. We formulate solvability conditions for this problem in the class of impulse controls, L_2–controls, and their hybrids. The proposed approach to studying systems with fractional derivatives is based on the systematic use of abstract functional–differential equation theory and offers certain advantages for studying systems and processes with aftereffects.

Keywords: functional–differential systems with fractional derivatives, systems with aftereffect, impulse systems, control problems.

Mathematics Subject Classification: 34K06, 34K30, 34K35, 34K45

1. Introduction

Despite the abundance and diversity of results obtained over the past 10–15 years, equations and systems with fractional derivatives continue attracting the attention of both Russian and international scientists. The total number of papers in this area has exceeded one hundred thousands. Furthermore, the range of applied problems, for which the use of fractional derivatives is not only justified but also expedient, continues to expand. Along with the traditional areas of application of fractional derivative models mentioned in well–known review [2], which are relaxation processes in dielectrics, the behavior of electrochemical media, semiconductor physics, plasma physics, processes in micro– and nanostructured media, signal processing, new application areas emerge, in particular, ones related to dynamic economic models [23], [12], [20], [21].

We mention only relatively new works related to control problems. In [13], there were studied models of multi–agent systems, in which the agents dynamics are described by a variable–order Caputo fractional derivative. There was considered the case, in which information exchange between agents occurs only at an initially specified time. Two cases of multi–agent systems were studied, leaderless and with a leader. The right hand sides of the equations in both cases were described by locally defined operators (Nemytskii operators) without memory. We also mention works devoted to obtaining analogs of Pontryagin maximum principle for optimal control problems in systems with fractional derivatives, see, for example, [10].

We separately mention a series of studies devoted to the development of methods for constructing optimal strategies based on the feedback principle in differential games and optimal control problems, see [16] and the references therein.

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In contrast to most studies devoted to impulse control problems (see review [24]), we consider systems with fractional derivatives, in which not only the fractional differentiation operator, but also the operator defining the right hand side of the equation possesses memory, which seems more natural. Furthermore, when formulating the control problem, we use a rather general target functional, including cases of multi–point and integral targets, as well as an operator implementing programmatic controls, which also possesses memory.

2. PRELIMINARIES

Let $\mathbf{D}$ and $\mathbf{B}$ be Banach spaces, and $\mathbf{D}$ be isomorphic to the direct product $\mathbf{B}\times\mathbb{R}^{\mathbf{n}}$ ($\mathbf{D}\simeq\mathbf{B}\times\mathbb{R}^{\mathbf{n}}$). Hereinafter we suppose that the norm of an element $\{z,a\}\in\mathbf{B}\times\mathbb{R}^{\mathbf{n}}$ is defined by the identity

$$\|\{z,a\}\|_{\mathbf{B}\times\mathbb{R}^{\mathbf{n}}}=\|z\|_{\mathbf{B}}+|a|_{\mathbb{R}^n},$$

where $|\cdot|_{\mathbb{R}^n}$ is a norm in $\mathbb{R}^n$.

The equation

$$\mathcal{L}x=f \tag{2.1}$$

with a linear bounded operator $\mathcal{L}:\mathbf{D}\to\mathbf{B}$ is called a linear abstract functional–differential equation (AFDE). The theory of equation (2.1) was systematically presented in [15], [1]. We fix an isomoprhism $J=\{\Lambda,Y\}:\mathbf{B}\times\mathbb{R}^{\mathbf{n}}\to\mathbf{D}$ and denote the inverse operator $J^{-1}=[\delta,r]$. Here $\Lambda:\mathbf{B}\to\mathbf{D}$, $Y:\mathbb{R}^n\to\mathbf{D}$ and $\delta:\mathbf{D}\to\mathbf{B}$, $r:\mathbf{D}\to\mathbb{R}^{\mathbf{n}}$ are the corresponding components of operators J J^{-1}:

$$J\{z,a\}=\Lambda z+Ya\in\mathbf{D},\quad z\in\mathbf{B},\quad a\in\mathbb{R}^n,$$
$$J^{-1}x=\{\delta x,rx\}\in\mathbf{B}\times\mathbb{R}^n,\quad x\in\mathbf{D}.$$

For each $\{z,a\}\in\mathbf{B}\times\mathbb{R}^{\mathbf{n}}$ the system

$$\delta x=z,\qquad rx=a$$

has a unique solution

$$x=\Lambda z+Ya. \tag{2.2}$$

This identity implies the representation for the operator $\mathcal{L}$:

$$\mathcal{L}x=\mathcal{L}(\Lambda z+Ya)=\mathcal{L}\Lambda z+\mathcal{L}Ya=Qz+Aa,$$

where the so–called principal part of operator $\mathcal{L}$, an operator $Q:\mathbf{B}\to\mathbf{B}$ and a finite–dimensional operator $A:\mathbb{R}^n\to\mathbf{D}$ are defined by the identities $Q=\mathcal{L}\Lambda$ and $A=\mathcal{L}Y$. The general theory of equation (2.1) was constructed under the assumption Q is a Fredholm operator, which is the sum of an invertible operator and a compact operator [4].

The general boundary value problem is the system

$$\mathcal{L}x=f,\qquad \ell x=\beta, \tag{2.3}$$

where $\ell=(\ell^1,\dots,\ell^N):\mathbf{D}\to\mathbb{R}^N$ is a linear bounded vector functional with linearly independent components. The boundary value problem (2.3) is one of the central objects of the theory of AFDEs. In the case, when $N=n$ and problem (2.3) is uniquely solvable for each $f\in\mathbf{B}$ and $\beta\in\mathbb{R}^n$, the solution can be represented as

$$x=G_\ell f+X\beta. \tag{2.4}$$

The operator $G_\ell:\mathbf{B}\to\mathbf{D}$ is called the Green operator of problem (2.3), and the operator $X:\mathbb{R}^n\to\mathbf{D}$ is called the fundamental vector. A necessary and sufficient condition for the unique solvability of problem (2.3) is the invertibility of the matrix ℓX. In the special case $\ell=r$ problem (2.3) is called the principal boundary value problem.

We consider the abstract control problem

$$\mathcal{L}x = Fu + f, \qquad rx = a, \qquad \ell x = \beta, \tag{2.5}$$

where the control u belongs to the Hilbert space $\mathbf{H}$, $F : \mathbf{H} \to \mathbf{B}$ is a linear bounded operator, $\ell = (\ell_1, \dots \ell_N)$ is an a target vector functional that determines the target, $\ell x = \beta$. We assume that the principle boundary value problem

$$\mathcal{L}x = f, \qquad rx = a \tag{2.6}$$

is uniquely solvable. To formulate a theorem on unique solvability of problem (2.5), we define a linear bounded functional $\lambda_i : \mathbf{H} \to \mathbb{R}, i = 1, \dots, N$, by the identity $\lambda_i u = \ell_i G_r F u$, where G_r is the Green operator of problem (2.6). Obviously, the value of $\lambda_i u$ can be written in the form $\lambda_i u = \langle \nu_i, u \rangle$, where the brackets $\langle \cdot, \cdot \rangle$ denote the scalar product in $\mathbf{H}$, and ν_i is the element of the space $\mathbf{H}$ generating the functional $\lambda_i : \mathbf{H} \to \mathbb{R}$.

Theorem 2.1 ([18]). *Control problem* (2.5) *is solvable for all* $f \in \mathbf{B}$ *and* $a \in \mathbb{R}^n$, $\beta \in \mathbb{R}^N$ *if and only if the matrix* $W \overset{def}{=} \{\langle \nu_i, \nu_j \rangle\}_{i,j=1,\dots,N}$ *is invertible. The control* $u_0 = \sum_{i=1}^{N} \nu_i c_i$, *where* $\operatorname{col}(c_i, \dots, c_N) = W^{-1}[\beta - \ell G_r f - \ell X a]$, *resolves problem* (2.5).

3. Impulse systems

A permanent interest to impulsive systems with integer order derivatives arose in the middle of 20th century. Their systematic study is associated with many renowned names; references to fundamental results in this area can be found in [1, Ch. 3]. The trajectories of such systems can have discontinuities at certain moments in time, and between these times the behavior of system is described by differentiable functions, which satisfy an equation in the usual sense. The modern theory of impulsive systems is based on the theory of generalized functions (distributions), the foundations of which were laid by S.L. Sobolev and L. Schwartz. Another approach to the study of differential equations with discontinuous solutions is associated with so-called “generalized ordinary differential equations,” the study of which was initiated by J. Kurzweil. Nowadays this theory is widely developed. According to admitted approaches, impulsive equations are considered in the class of functions of bounded variation. In this case, the solution is understood as a function of bounded variation satisfying an integral equation with a Lebesgue — Stieltjes or Perron — Stieltjes integral. Integral equations in the space of functions of bounded variation became a subject of special interest and were studied in detail in [22] and subsequent works. We recall that a function of bounded variation can be represented as the sum of an absolutely continuous function, a jump function, and a singular component (a continuous function, the derivative of which vanishes almost everywhere). The solutions of equations with impulsive action, which are considered below, do not contain a singular component and can have discontinuities only at a finitely many given points.

We follow the approach to impulsive systems proposed in [14], which is based on considering equations in a space being a finite–dimensional extension of the traditional space of absolutely continuous functions. This approach does not employ a complicated theory of generalized functions, it proved to be rich in content, and it finds many applications in cases, where the question of the singular component does not arise, in particular, in certain problems of economic dynamics [15]. From the point of view of AFDE theory, the considered impulsive systems are one of the specific representatives of equations considered on finite–dimensional extensions of traditional spaces. Moreover, the features of such equations is determined not only by the features of these spaces, but also by the specific properties of the isomorphism $J : \mathbf{D} \to \mathbf{B} \times \mathbb{R}^n$. For equations with integer derivatives, various examples of such equations can be found in [1, Ch. 3].

We first recall the construction of the basic trajectory space for impulsive systems with integer order derivatives.

Let $\mathbf{L}$ be the space of Lebesgue summable functions $z : [0, T] \to \mathbb{R}^n$ with the norm

$$\|z\|_{\mathbf{L}} = \int_0^T |z(t)|_{\mathbb{R}^n}\, dt;$$

$\mathbf{AC}$ be the space of absolutely continuous functions $y : [0, T] \to \mathbb{R}^n$ with the norm

$$\|y\|_{\mathbf{AC}} = \|\dot{y}\|_{\mathbf{L}} + |y(0)|_{\mathbb{R}^n};$$

$\mathbf{L}_\infty$ be the space of measurable and essentially bounded functions $z : [0, T] \to \mathbb{R}^n$ with the norm

$$\|z\|_{\mathbf{L}_\infty} = \operatorname{vraisup}(|z(t)|_{\mathbb{R}^n} : t \in [0, T]);$$

$\mathbf{AC}_\infty$ be the space of absolutely continuous functions $y : [0, T] \to \mathbb{R}^n$ with essentially bounded derivative $\dot{y}$ and the norm

$$\|y\|_{\mathbf{AC}_\infty} = \|\dot{y}\|_{\mathbf{L}_\infty} + |y(0)|_{\mathbb{R}^n}.$$

We fix a discrete set of points $t_k \in (0, T)$, $0 < t_1 < \ldots < t_m < T$. We consider the space $\mathbf{D} = \mathbf{DS}(m)$ of functions $x : [0, T] \to \mathbb{R}^n$, which can be represented as

$$x(t) = \int_0^t z(s)\, ds + x(0) + \sum_{k=1}^m \chi_{[t_k,T]}(t)\Delta x(t_k),$$

where $z \in \mathbf{L}$, $\Delta x(t_k) = x(t_k) - x(t_k - 0)$, $\chi_{[t_k,T]}(t)$ is the characteristic function of the interval $[t_k, T]$. In this case we have $\mathbf{D} \simeq \mathbf{L} \times \mathbb{R}^{n+mn}$ with the isomorphism $J = \{\Lambda, Y\}$,

$$(\Lambda z)(t) = \int_0^t z(s)\, ds; \qquad (Ya)(t) = a^0 + \sum_{k=1}^m \chi_{[t_k,T]}(t)a^k; \qquad a = \operatorname{col}(a^0, \ldots, a^m).$$

At the same time, for $J^{-1} = \{\delta, r\}$ we have

$$\delta x = \dot{x}, \qquad rx = \operatorname{col}(x(0), \Delta x(t_1), \ldots, \Delta x(t_m)).$$

The results of studying control problems for a wide class of impulse systems with an ordinary derivative and a trajectory space $\mathbf{D} = \mathbf{DS}(m)$ we presented in [5], [18].

Let us now turn to the definition of the underlying spaces and isomorphisms that allow us to consider impulse systems from the point of view of AFDE theory in the case of fractional derivatives. We shall employ two well–known operators, the Caputo fractional differentiation operator of order $\alpha \in (0, 1)$ [11], [17],

$$(\mathcal{D}^\alpha x)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(s)}{(t-s)^\alpha}\, ds,$$

and the Riemann — Liouville fractional integration operator of the same order [11], [17],

$$(\mathcal{J}^\alpha z)(t) = (\Lambda z)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} z(s)\, ds.$$

Here $\Gamma(\cdot)$ is the Euler Gamma function.

The discrete set of points $t_k \in (0, T)$, $0 < t_1 < \ldots < t_m < T$ (“moments of impulse action”) is still supposed to be fixed.

We consider the space $\mathbf{D} = \mathbf{DS}_\infty(m)$ of functions $x : [0, T] \to \mathbb{R}^n$ represented as

$$x(t) = (\mathcal{J}^\alpha z)(t) + x(0) + \sum_{k=1}^{m} \chi_{[t_k,T]}(t)\Delta x(t_k),$$

where $z \in \mathbf{L}_\infty$, $\Delta x(t_k) = x(t_k) - x(t_k - 0)$, $\chi_{[t_k,T]}(t)$ is the characteristic function of segment $[t_k, T]$. In this case we have $\mathbf{D} \simeq \mathbf{L}_\infty \times \mathbb{R}^{n+mn}$,

$$(\Lambda z)(t) = (\mathcal{J}^\alpha z)(t); \qquad (Ya)(t) = a^0 + \sum_{k=1}^{m} \chi_{[t_k,T]}(t)a^k; \qquad a = \operatorname{col}(a^0, \dots, a^m),$$
$$\delta x = \mathcal{D}^\alpha x; \qquad rx = \operatorname{col}(x(0), \Delta x(t_1), \dots, \Delta x(t_m)).$$

We consider the impulse system

$$(\mathcal{D}^\alpha x)(t) = (\mathcal{T} x)(t) + f(t), \qquad t \in [0, T], \tag{3.1}$$

where $\mathcal{T} : \mathbf{DS}_\infty(m) \to \mathbf{L}_\infty$ is a linear bounded Volterra operator [1] such that there exists $\rho > 0$, which ensures the inequality

$$|(\mathcal{T} y)(t)| \leqslant \rho \max_{s\in[0,t]} |y(s)|, \qquad t \in [0, T],$$

for all $y \in \mathbf{AC}^\alpha_\infty$. Here $\mathbf{AC}^\alpha_\infty$ is the space of functions $y : [0, T] \to \mathbb{R}^n$, which can be represented as

$$y(t) = y(0) + (\mathcal{J}^\alpha z)(t),$$

where $z \in \mathbf{L}_\infty$, with the norm

$$\|y\|_{\mathbf{AC}^\alpha_\infty} = \|\mathcal{D}^\alpha y\|_{\mathbf{L}_\infty} + |y(0)|_{\mathbb{R}^n}.$$

Hereafter we assume that the norm $| \cdot |$ in $\mathbb{R}^n$ has the monotonicity property: for all $a = \operatorname{col}(a_1, \dots, a_n)$, $b = \operatorname{col}(b_1, \dots, b_n) \in \mathbb{R}^n$ such that $|a_i| \leqslant |b_i|$, $i = 1, \dots, n$, we have $|a| \leqslant |b|$.

We also suppose that the operator $K = \mathcal{T} J^\alpha$ is a regular Volterra integral operator [3, Ch. 5]

$$(Kz)(t) = \int_0^t K(t, s)z(s)\, ds.$$

It was shown in [9] that the operator $(I - K)^{-1}$ is well–defined and can be represented as a convergent Neumann series; hereinafter I is the identity operator. As it is known [3, Thm 2.2], in this case $(I - K)^{-1} = I + R$, where R is the resolvent operator, which is also a Volterra integral operator

$$(Rf)(t) = \int_0^t R(t, s)f(s)\, ds$$

with the resolvent kernel $R(t, s)$.

As an example of the operator $\mathcal{T}$ obeying the above formulated conditions, the following operator serves

$$(\mathcal{T} x)(t) = P(t)x_h(t), \qquad h(t) \leqslant t, \tag{3.2}$$

where the columns of $(n \times n)$ matrix P belong to the space $\mathbf{L}_\infty$,

$$x_h(t) = \begin{cases} x[h(t)], & t \in [0, T], \\ 0, & t \notin [0, T], \end{cases} \tag{3.3}$$

and the function $h : [0,\ T] \to R$ is measurable.

The principal boundary value problem for system (3.1) reads

$$\mathcal{D}^\alpha x = \mathcal{T} x + f; \qquad x(0) = a^0, \qquad \Delta x(t_1) = a^1, \quad \dots, \quad \Delta x(t_m) = a^m. \tag{3.4}$$

It was shown in [9] that for $a^k = 0$, $k = 0, \dots, m$, this problem is uniquely solvable for each $f \in \mathbf{L}_\infty$ and its solution can be represented as

$$x(t) = (Cf)(t) = \int_0^t C(t,s) f(s)\, ds, \tag{3.5}$$

where $C(t,s)$ is the Cauchy matrix defined by the identity

$$C(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} E + \int_s^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} R(\tau, s)\, d\tau; \tag{3.6}$$

hereinafter E is the unit $(n \times n)$ matrix.

We introduce the notation

$$\mathcal{A}^0(t) = (\mathcal{T}E)(t), \qquad \mathcal{A}^k(t) = (\mathcal{T}(E\chi_{[t_k,T]}))(t), \qquad k = 1, \dots, m.$$

We note that for an $(n \times n)$ matrix X with columns X_i in $\mathbf{DS}_\infty(m)$, the notation $(\mathcal{T}X)(t)$ means the matrix $((\mathcal{T}X_1)(t), \dots, (\mathcal{T}X_n)(t))$.

We define matrices $X^0(t)$, $X^1(t)$, …, $X^m(t)$ by the identities

$$X^0(t) = E + \int_0^t C(t,s)\mathcal{A}^0(s)\, ds, \tag{3.7}$$

$$X^k(t) = \chi_{[t_k,T]}(t)E + \int_0^t C(t,s)\mathcal{A}^k(s)\, ds, \quad k = 1, \dots, m.$$

Let us obtain the representation for solution to problem (3.4) for all a^k, $k = 0, \dots, m$.

Theorem 3.1. *Problem (3.4) is uniquely solvable for all $f \in \mathbf{L}_\infty$ and $a^k \in \mathbb{R}^n$, $k = 0, 1, \dots, m$, and its solution $x \in \mathbf{DS}_\infty(m)$ admits the representation*

$$x(t) = (X^0(t), X^1(t), \dots, X^m(t)) \begin{pmatrix} a^0 \\ a^1 \\ \dots \\ a^m \end{pmatrix} + \int_0^t C(t,s) f(s)\, ds. \tag{3.8}$$

Proof. We need to prove only the representation for solution of homogeneous system (3.1) (the case $f = 0$). We consider the problem

$$\mathcal{D}^\alpha x = \mathcal{T}x; \qquad x(0) = a^0, \qquad \Delta x(t_1) = 0, \quad \dots, \quad \Delta x(t_m) = 0. \tag{3.9}$$

The change $y(t) = x(t) - a^0$ reduces it to

$$\mathcal{D}^\alpha y = \mathcal{T}y + \mathcal{T}a^0; \qquad y(0) = 0, \qquad \Delta y(t_1) = 0, \quad \dots, \quad \Delta y(t_m) = 0. \tag{3.10}$$

For the solution of this problem we have

$$y(t) = \int_0^t C(t,s)(\mathcal{T}a^0)(s)\, ds = \int_0^t C(t,s)(\mathcal{T}E)(s)\, ds\, a^0 = \int_0^t C(t,s)\mathcal{A}^0(s)\, ds\, a^0. \tag{3.11}$$

Returning back to the original phase variable and denoting by x^0 the solution to problem (3.9), we obtain

$$x^0(t) = a^0 + \int_0^t C(t,s)\mathcal{A}^0(s)\, ds\, a^0 = X^0(t)\, a^0. \tag{3.12}$$

Repeating this arguing for the problem

$$\mathcal{D}^\alpha x = \mathcal{T}x; \quad x(0) = 0, \qquad \Delta x(t_1) = a^1, \qquad \Delta x(t_2) = 0, \quad \dots, \quad \Delta x(t_m) = 0 \tag{3.13}$$

with the change $y(t) = x(t) - \chi_{[t_1,T]}(t)a^1$, we obtain the representation for the solution of problem (3.13)

$$x^1(t) = \chi_{[t_1,T]}(t)a^1 + \int\limits_0^t C(t,s)\mathcal{A}^1(s)\,ds\,a^1 = X^1(t)\,a^1. \tag{3.14}$$

We obtain a similar representation for the solution of problem

$$\mathcal{D}^\alpha x = \mathcal{T}x; \qquad x(0) = 0, \qquad \Delta x(t_1) = 0, \dots, \Delta x(t_k) = a^k, \dots, \Delta x(t_m) = 0: \tag{3.15}$$

$$x^k(t) = \chi_{[t_k,T]}(t)a^k + \int\limits_0^t C(t,s)\mathcal{A}^k(s)\,ds\,a^k = X^k(t)\,a^k, \quad k = 2, \dots, m.$$

It remains to note that $x(t) = \sum_{k=0}^m x^k(t)$. The proof is complete. □

4. IMPULSE CONTROL PROBLEM

We impose control problem for system (3.1) treating the jumps of trajectories $\Delta x(t_k)$ at the time moments t_k, $k = 1, \dots, m$, as control.

We suppose that the initial state of system is given

$$x(0) = a^0, \tag{4.1}$$

while the target of control is defined by the identity

$$\ell x \equiv \sum_{j=1}^{\mu} A_j x(\tau_j) + \int\limits_0^T B(\tau)x(\tau)\,d\tau + \sum_{k=1}^m H_k \Delta x(t_k) = \beta \in \mathbb{R}^N, \tag{4.2}$$

where τ_j, $j = 1, \dots, \mu$, are fixed points in $[0, T]$, generally speaking, these points are not related with the points t_k, $k = 1, \dots, m$, introduced above; A_j and H_k are constant $(N \times n)$ matrices, $B(\cdot)$ is an $(N \times n)$ matrix with summable entries, β is a given constant of target values.

We rewrite representation (3.8) as

$$x(t) = X^0(t)a^0 + \sum_{k=1}^m X^k(t)a^k + \int\limits_0^t C(t,s)f(s)\,ds. \tag{4.3}$$

In this representation, the vectors $a^k \in \mathbb{R}^n$, $k = 1, \dots, m$ are free and are to be determined to attain target (4.2).

To formulate the solvability conditions for e problem (3.1), (4.1), (4.2) we introduce the notation

$$\Xi^k = \sum_{j=1}^{\mu} A_j X^k(\tau_j) + \int\limits_0^T B(\tau)X^k(\tau)\,d\tau + H^k, \quad k = 1, \dots, m, \tag{4.4}$$

$$\beta_1 = \left(\sum_{j=1}^{\mu} A_j X^k(\tau_j)\right) a^0 + \int\limits_0^T B(\tau)X^0(\tau)\,d\tau\,a^0, \tag{4.5}$$

$$\beta_2 = \sum_{j=1}^{\mu} A_j \int\limits_0^{\tau_j} C(\tau_j, s)f(s)\,ds + \int\limits_0^T \int\limits_s^T B(\tau)C(\tau,s)\,d\tau\,f(s)\,ds. \tag{4.6}$$

Theorem 4.1. *The solvability of problem* (3.1), (4.1), (4.2) *is equivalent to the solvability of linear system*

$$\sum_{k=1}^{m} \Xi^k \, a^k = \beta - \beta_1 - \beta_2, \tag{4.7}$$

where $(N \times n)$ *matrices* Ξ^k *and the vectors* β_1, $\beta_2 \in \mathbb{R}^N$ *are defined by identities* (4.4), (4.5), (4.6). *Each solution* $(a_0^1, \dots, a_0^m)$, $a_0^k \in \mathbb{R}^n$ *of this system generates a solution* $x_0 \in \mathbf{DS}_\infty(m)$ *of problem* (3.1), (4.1), (4.2), *and* $\Delta x_0(t_k) = a_0^k$.

Proof. By means of representation (4.3) we find the value of the target vector functional ℓ (4.2) on the right hand side of this identity. In order to do this, we sequentially calculate ℓ on each of its three terms, denoting these components by $\ell_1 x$, $\ell_2 x$, $\ell_3 x$:

$$\ell_1 x = \sum_{j=1}^{\mu} A_j X^0(\tau_j) a^0 + \sum_{j=1}^{\mu} \sum_{k=1}^{m} A_j X^k(\tau_j) a^k + \sum_{j=1}^{\mu} A_j \int_0^{\tau_j} C(\tau_j, s) f(s)\, ds,$$

$$\ell_2 x = \int_0^T B(\tau) X^0(\tau)\, d\tau \, a^0 + \int_0^T B(\tau) \sum_{k=1}^{m} X^k(\tau) d\tau \, a^k$$

$$+ \int_0^T B(\tau) \int_0^{\tau} C(\tau, s) f(s) ds \, d\tau,$$

$$\ell_3 x = \sum_{k=1}^{m} H^k a^k.$$

Equating the like terms in the expression $\ell x = \ell_1 x + \ell_2 x + \ell_3 x$ with respect to the factors a^0, a^k and the integral operators acting on $f(\cdot)$, interchanging the summation order in the double sums and the integration order in double integrals, we express target identity (4.2) in form (4.7). The proof is complete. □

We note that in the case $nm > N$, the freedom in choosing the impulse control variables is excessive. In this case, the target conditions can include additional conditions requiring the jumps of individual trajectory components to be zero ($\Delta x_i(t_k) = 0$).

We do not discuss the practical implementation of impulse effects, i.e., the corresponding trajectory jumps. We note only that in economic dynamics problems, such jumps have a natural interpretation. For example, a negative jump in some trajectory component means an one–time transfer of production assets (capital) from the corresponding industry to another industry, for which the trajectory component jump at the same time will be positive. If the absolute values of the aforementioned jumps are equal, such a transfer of assets does not require additional resources. For systems with integer derivatives, these issues were discussed in detail in [5], [6, Part 2, Lect. 1], [7, Ch. 3].

5. MIXED CONTROL PROBLEM

We consider the control functional–differential system with fractional derivative

$$\mathcal{D}^\alpha x = \mathcal{T} x + F u + f, \tag{5.1}$$

where $F : \mathbf{L}_2 \to \mathbf{L}_\infty$ is a linear bounded Volterra operator implementing control actions $u \in \mathbf{L}_2$. Here $\mathbf{L}_2$ is the space of square–summable functions $v : [0, T] \to \mathbb{R}^q$ with the scalar product $\langle u, v \rangle = \int_0^T u'(t) v(t)\, dt$, and $(\cdot)'$ denotes the transposition.

We still continue to suppose that the initial state of the system is given by (4.1) and the control target is given by (4.2).

In the case under consideration, the entire set of trajectories of the system (5.1) is described by the identity

$$x(t)=X^0(t)a^0+\sum_{k=1}^{m}X^k(t)a^k+\int_0^t C(t,s)(Fu)(s)\,ds+\int_0^t C(t,s)f(s)\,ds. \tag{5.2}$$

In this notation, the mixed control action contains an impulse finite–dimensional component $(a^1,\dots,a^m)'\in\mathbb{R}^{nm}$ and a component $u\in\mathbf{L}_2$.

To formulate the solvability conditions for control problem (5.1),(4.1),(4.2), we introduce an additional notation. By Θ_0 we denote the $(N\times q)$ matrix, which defines the linear bounded vector function $\Theta_0:\mathbf{L}_2\to\mathbb{R}^N$:

$$\Theta_0 u=\int_0^T\Theta_0(s)u(s)\,ds=\int_0^T\int_s^T B(\tau)C(\tau,s)\,d\tau(Fu)(s)\,ds. \tag{5.3}$$

Let Θ_j — $(N\times q)$ be a matrix, which defines a linear bounded vector functional $\Theta_j:\mathbf{L}_2\to\mathbb{R}^N$:

$$\Theta_j u=\int_0^T\Theta_j(s)u(s)\,ds=\int_0^T\chi_j(s)C(\tau_j,s)(Fu)(s)\,ds, \tag{5.4}$$

where $\chi_j(\cdot)$ is the characteristic function of segment $[0,\tau_j]$, $j=1,\dots,\mu$,

$$M(s)=\Theta_0(s)+\sum_{j=1}^{\mu}A_j\Theta_j(s);\qquad W=\int_0^T M(s)M'(s)\,ds. \tag{5.5}$$

Theorem 5.1. *Control problem (5.1),(4.1),(4.2) is solvable if and only if the linear algebraic system*

$$\sum_{k=1}^{m}\Xi^k a^k+Wd=\beta-\beta_1-\beta_2, \tag{5.6}$$

where Ξ^k, $k=1,\dots,m$ are $(N\times n)$ matrices, and the vectors β_1, $\beta_2\in\mathbb{R}^N$ are defined by identities (4.4), (4.5), (4.6), respectively, is solvable with respect to the $(nm+N)$–dimensional vector $(a^1,\dots,a^m,d)'$. Each solution $(a_0^1,\dots,a_0^m,d_0)'$ of system (5.6) defines a mixed control that solves the control problem (5.1), (4.1), (4.2): $\Delta x(t_k)=a_0^k$, $k=1,\dots,m$, $u(t)=M'(t)d_0$.

Proof. The introduction of $u\in\mathbf{L}_2$ into the control system leads to an additional term $\int_0^t C(t,s)(Fu)(s)\,ds$ in description (5.2) of the set of all trajectories and the need to calculate the value of the target vector functional on this term. Such a calculation gives two additional terms relative to the case of impulse control (Theorem 4.1):

$$\ell_4 x=\int_0^T B(\tau)\int_0^\tau C(\tau,s)(Fu)(s)\,ds d\tau, \tag{5.7}$$

$$\ell_5 x=\sum_{j=1}^{\mu}A_j\int_0^{\tau_j}C(\tau_j,s)(Fu)(s)\,ds. \tag{5.8}$$

In view of the introduced notations Θ_0 and Θ_j we obtain

$$\ell_4 x + \ell_5 x = \int_0^T [\Theta_0(s) + \sum_{j=1}^{\mu} A_j \Theta_j(s)] u(s)\, ds = \int_0^T M(s) u(s)\, ds. \tag{5.9}$$

Let L be the linear manifold spanned by the columns of matrix M', and $L^{\perp}$ be its orthogonal complement in $\mathbf{L}_2$. We take an arbitrary linear combination of the columns of M': $g = M'd$, $d \in \mathbb{R}^N$. Clearly, $g \in L$. By the theorem on orthogonal decomposition in Hilbert space, each control u can be represented as $u = g + h$, where $h \in L^{\perp}$. Moreover, $\int_0^T M(s)h(s)\, ds = 0$, and the term h makes zero contribution to the value of target vector functional. Thus, the term $\int_0^T M(s)u(s)\, ds$ in the values of the target vector functional can be replaced by the term Wd without loss of generality, and we obtain the system (5.6). The proof is complete. ☐

We note that in the case of invertible matrix W the control

$$u(t) = M'(t) W^{-1} (\beta - \beta_1 - \beta_2 - \sum_{k=1}^{m} \Xi^k a^k)$$

solves problem (5.1), (4.1), (4.2) for all $a^k = \Delta x(t_k)$, $k = 1, \ldots, m$.

Reduction of control problem (5.1), (4.1), (4.2) to linear algebraic system (5.6) gives an opportunity to apply constructive methods developed for control problems of functional–differential systems with integer order derivatives (see [5]; [1, Ch. 6]), and implemented by using modern computer technology, reliable computer experiment (RCE). Note that the key point of this reduction is the usage of a representation of the set of all trajectories of the control system using the Cauchy matrix. In the general case, the absence of an explicit representation of the Cauchy matrix leads to the need to use its approximations with a guaranteed error estimate within the framework of RCE. Methods and algorithms for such construction for some very broad classes of functional–differential systems were described in [8], [19].

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