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INFINITE ORDER EULER OPERATORS IN SPACES OF HOLOMORPHIC FUNCTIONS AND STIRLING NUMBERS

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Abstract. We study infinite order Euler operators in the space $H(\Omega)$ of all functions holomorphic on an open set Ω in $\mathbb{C}^N$, with the topology of uniform convergence on compact sets in Ω. In terms of their characteristic functions, we prove necessary and sufficient conditions for the applicability of these operators to $H(\Omega)$. The special case $\Omega = (\mathbb{C}\backslash\{0\})^N$ is considered. We study the relationship between the two representations for the Euler operator, in which the Stirling numbers of the first and second kinds play a significant role. It is expressed by means of associated functions, one of which is the sum of the Newton interpolation series. The obtained results imply that each entire function of exponential type 0 on $\mathbb{C}^N$ can be expanded into a multidimensional Newton interpolation series. A multidimensional version of the Wigert — Leau theorem is proved. We show that in the space $H(\mathbb{C}^N)$ of all integer functions in $\mathbb{C}^N$, each Hadamard type operator in $H(\mathbb{C}^N)$ is an Euler operator, that is, each continuous linear operator in $H(\mathbb{C}^N)$, for which each monomial is an eigenvector.

Keywords: holomorphic function, infinite order Euler operator, Stirling numbers of the first and second kind.

Mathematics Subject Classification: 46E10, 47B99, 11B73

1. Introduction

In this paper, we study infinite order Euler operators acting in the space $H(\Omega)$ of all functions holomorphic on an open set Ω in $\mathbb{C}^N$, with the topology of uniform convergence on compact sets in Ω, which can be represented as

$$\mathcal{E}_a(f)(t) := \sum_{\beta\in\mathbb{N}_0^N} a_\beta t^\beta f^{(\beta)}(t) \tag{1.1}$$

or

$$\mathcal{E}_{\theta,a}(f)(t) := \sum_{\beta\in\mathbb{N}_0^N} a_\beta \theta^\beta(f)(t), \tag{1.2}$$

where

$$\theta^\beta := \theta_1^{\beta_1}\cdots\theta_N^{\beta_N}, \qquad \theta_j(f)(t) := t_j\frac{\partial f}{\partial t_j}(t), \qquad a_\beta\in\mathbb{C}, \qquad \beta\in\mathbb{N}_0^N, \qquad \mathbb{N}_0 := \mathbb{N}\cup\{0\}.$$

These operators in $H(\Omega)$ were studied in a lot of papers. The problem of well–definedness and surjectivity of Euler operators was mostly addressed. In [9], [11], [15], [22] (for $N=1$) and in [29], [30] (for $N \geqslant 1$), operators of form (1.1) were studied. In [6], [31], Euler operators in

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$H(\Omega)$ with the representation (1.2) for $N = 1$ were investigated. Infinite order Euler operators in spaces of real analytic functions of one and several variables obeying representation (1.2) were studied in [23]–[27].

The main goals of this paper are to study the convergence of series defining Euler operators and the relationship between their representations. The problem on the action of $\mathcal{E}_a$ and $\mathcal{E}_{\theta,a}$ in $H(\Omega)$ is resolved in Section 2. We follow the definition of a summable and absolutely summable family in a locally convex space from [16, Ch. 1, Sect. 1.3]. For a countable set of indices Λ, instead of the term "the family $\{x_\beta : \beta \in \Lambda\}$ is absolutely summable", we use the term "the series $\sum\limits_{\beta\in\Lambda} x_\beta$ converges absolutely". Following Yu.F. Korobeynik, we call the operators defined by identities (1.1) and (1.2) applicable to $H(\Omega)$ if the series in their definition absolutely converges at each point $t \in \Omega$ for each function $f \in H(\Omega)$. A sufficient condition for both (1.1) and (1.2) to be applicable to $H(\Omega)$ for all open sets Ω in $\mathbb{C}^N$ is the identity

$$\lim_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} = 0.$$

For a wide class of open sets $\Omega \subset \mathbb{C}^N$, this identity is necessary. A necessary and sufficient condition for the applicability of both types of operators to $H(\mathbb{C}^N)$ is the relation

$$\limsup_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} < +\infty.$$

The mentioned conditions ensure the absolute convergence of series (1.1) and (1.2) in $H(\Omega)$, and hence the continuity of $\mathcal{E}_a$ and $\mathcal{E}_{\theta,a}$ in $H(\Omega)$. We note that the aforementioned results on applicability for $N = 1$ were known; only the question of whether the condition

$$\limsup_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} < +\infty$$

is necessary for the applicability of $\mathcal{E}_{\theta,a}$ to $H(\mathbb{C})$ remained open; this condition was formulated in [31, Prop. 6.2]. We consider the special case $\Omega = (\mathbb{C}\backslash\{0\})^N$, which was not investigated in this direction even for $N = 1$. For this domain Ω, the criterion for applicability to $H(\Omega)$ of the operator $\mathcal{E}_a$ is the condition

$$\limsup_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} < 1,$$

and $\mathcal{E}_{\theta,a}$ is the relation

$$\limsup_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} < +\infty.$$

When proving necessary conditions on the coefficients a_β, $\beta \in \mathbb{N}_0^N$, sets that are Köthe duals to special spaces of (multi)sequences are used. Information about them is given, for example, in [10, Sect. 2].

Section 3 is devoted to the relations between representations (1.1) and (1.2). It seems that this problem was not systematically studied for infinite order Euler operators. We study this problem by using Stirling numbers of the first and second kind. When the order of $\mathcal{E}_a$ and $\mathcal{E}_{\theta,a}$ is finite, by means of these numbers, each of these operators can be expressed in terms of the other. This follows easily from the well–known one–dimensional identities relating the operators $f \mapsto t^\beta f^{(\beta)}(t)$ and θ^β. If the orders of $\mathcal{E}_a$ and $\mathcal{E}_{\theta,a}$ are infinite, then the connection between the representations can be established due to the availability of the needed upper bounds for the Stirling numbers of the second kind and for the moduluses of the Stirling numbers of the first kind, which ensure the absolute convergence of the corresponding series and the validity of the needed identities. It is described by means of functions associated with a, one of which is the sum of the Newton interpolation series. We also show that each entire function of exponential type 0 in $\mathbb{C}^N$ can be expanded into the corresponding multidimensional Newton series. This allows us to prove a certain multidimensional version of the Wigert — Leau theorem in terms of the defining compact set of an analytic functional in Section 4. In Section 5 we establish that each operator of Hadamard type in $H(\mathbb{C}^N)$ is an Euler operator of form (1.1) (and (1.2)).

2. Conditions for coefficients

2.1. Applicability of $\mathcal{E}_a$. We first study the action of the operator $\mathcal{E}_a$ in $H(\Omega)$. For $L \in \mathbb{N}$, $c \in \mathbb{C}^L$, $c \neq 0$, $d \in \mathbb{C}$, we introduce the hyperplane in $\mathbb{C}^L$

$$T(c,d) := \{t \in \mathbb{C}^L : \langle c, t\rangle = d\}.$$

Here $\langle z, w\rangle := \sum_{k=1}^{L} z_k w_k$, $z, w \in \mathbb{C}^L$. The proof of the necessary condition for the applicability of the Euler operator involves a certain local condition of linear convexity. It is formulated in terms of the existence of a complex hyperplane passing through the boundary point of the set and not intersecting it. Further, $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$. For a set Q in $\mathbb{C}^L$, we denote by ∂Q the boundary of Q in $\mathbb{C}^L$;

$$t^\beta := t_1^{\beta_1}\cdots t_L^{\beta_L}, \qquad |\beta| := \beta_1 + \cdots + \beta_L, \qquad \beta! := \beta_1!\cdots\beta_L!, \qquad t \in \mathbb{C}^L, \qquad \beta \in \mathbb{N}_0^L.$$

Theorem 2.1. (i) *If*

$$\lim_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} = 0,$$

then for each nonempty open set Ω in $\mathbb{C}^N$ and each function $f \in H(\Omega)$ series (1.1) converges absolutely in $H(\Omega)$.

(ii) *Let*

$$\Omega = \Omega_1 \times \cdots \times \Omega_M, \qquad 1 \leqslant M \leqslant N, \qquad N_m \in \mathbb{N}, \qquad \sum_{m=1}^{M} N_m = N,$$

and $\Omega_m \neq \mathbb{C}^{N_m}$ be a a non–empty open set in $\mathbb{C}^{N_m}$ with following property in $\mathbb{C}^{N_m}$: there exist $c^{(m)} \in (\mathbb{C}^*)^{N_m}$, $d_m \in \mathbb{C}$, points

$$w^{(m)} \in T(c^{(m)}, d_m) \cap (\mathbb{C}^*)^{N_m} \cap \partial\Omega_m$$

such that

$$T(c^{(m)}, d_m) \cap \Omega_m = \emptyset.$$

If the operator $\mathcal{E}_a$ is applicable to $H(\Omega)$, then

$$\lim_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} = 0.$$

Proof. Statement (i) can proved in the standard way by means of estimates obtained by using the Cauchy formula.

(ii): We write the point $t \in \mathbb{C}^N$ as $t = (t^{(1)}, \ldots, t^{(M)})$, where $t^{(m)} \in \mathbb{C}^{N_m}$, $1 \leqslant m \leqslant M$. There exists a sequence $t(n) \in \Omega \cap (\mathbb{C}^*)^N$, $n \in \mathbb{N}$, converging to the point $w = (w^{(1)}, \ldots, w^{(M)})$ in $\mathbb{C}^N$. Since the function

$$f(t) = \prod_{m=1}^{M} \frac{1}{d_m - \langle c^{(m)}, t^{(m)}\rangle}$$

is holomorphic in Ω and

$$f^{(\beta)}(t) = \prod_{m=1}^{M} \frac{|\beta^{(m)}|!(c^{(m)})^{\beta^{(m)}}}{(d_m - \langle c^{(m)}, t^{(m)}\rangle)^{|\beta^{(m)}|+1}}, \qquad \beta \in \mathbb{N}_0^N,$$

for each $n \in \mathbb{N}$ there exists a constant $A_n > 0$ obeying

$$|a_\beta| \left(\prod_{m=1}^{M} |\beta^{(m)}|! \right) |(t(n))^\beta||c^\beta| \leqslant A_n \prod_{m=1}^{M} |d_m - \langle c^{(m)}, (t(n))^{(m)} \rangle|^{|\beta^{(m)}|+1}, \qquad \beta \in \mathbb{N}_0^N. \tag{2.1}$$

There exits $\delta > 0$ such $|(t(n))^\beta||c^\beta| \geqslant \delta^{|\beta|}$ for all $n \in \mathbb{N}$, $\beta \in \mathbb{N}_0^N$. By (2.1) and the estimate $\beta! \leqslant \prod\limits_{m=1}^{M} |\beta^{(m)}|!$ we find

$$\beta! |a_\beta| \leqslant \frac{A_n}{\delta^{|\beta|}} \prod_{m=1}^{M} |d_m - \langle c^{(m)}, (t(n))^{(m)} \rangle|^{|\beta^{(m)}|+1}, \qquad n \in \mathbb{N},$$

and

$$\begin{aligned}
(\beta! |a_\beta|)^{\frac{1}{|\beta|}} \leqslant & \frac{(A_n)^{\frac{1}{|\beta|}}}{\delta} \prod_{m=1}^{M} |d_m - \langle c^{(m)}, (t(n))^{(m)} \rangle|^{\frac{|\beta^{(m)}|+1}{|\beta|}} \\
\leqslant & \frac{(A_n)^{\frac{1}{|\beta|}}}{\delta} \left(\prod_{m=1}^{M} |d_m - \langle c^{(m)}, (t(n))^{(m)} \rangle| \right)^{\frac{1}{|\beta|}} \\
& \cdot \sum_{m=1}^{M} |d_m - \langle c^{(m)}, (t(n))^{(m)} \rangle|, \quad n \in \mathbb{N}, \beta \in \mathbb{N}_0^N.
\end{aligned}$$

This yields that for all $n \in \mathbb{N}$

$$\limsup_{|\beta| \to \infty} (\beta! |a_\beta|)^{\frac{1}{|\beta|}} \leqslant \frac{1}{\delta} \sum_{m=1}^{M} |d_m - \langle c^{(m)}, (t(n))^{(m)} \rangle|.$$

This is why the limit $\lim\limits_{|\beta| \to \infty} (\beta! |a_\beta|)^{\frac{1}{|\beta|}}$ exists and it is zero. The proof is complete. □

Remark 2.1. *The proven theorem is known for $N = 1$ [9]. In this case, the additional condition in (ii) is satisfied by any nonempty open set Ω in $\mathbb{C}$ distinct from $\mathbb{C}$ and $\mathbb{C}^*$.*

For $N \geqslant 2$, the set $\Omega \subset \mathbb{C}^N$ possesses the property assumed in statement (ii) of Theorem 2.1, for example, in the following situations: if $\Omega = \Omega_1 \times \cdots \times \Omega_N$, where Ω_j, $1 \leqslant j \leqslant N$, is a nonempty open set in $\mathbb{C}$ distinct from $\mathbb{C}$ and $\mathbb{C}^*$; if Ω is a bounded convex domain in $\mathbb{C}^N$, each point of its boundary is smooth, i.e., through each point in $\partial\Omega$ there passes a unique (real) support hyperplane to Ω [12, Def. 10.7].

Let us justify the last example. Let $H_\Omega(t) := \sup\limits_{z \in \Omega} \operatorname{Re}\langle z, t \rangle$, $t \in \mathbb{C}^N$, be the support function of Ω, $S := \left\{ z \in \mathbb{C}^N : \sum\limits_{j=1}^{N} |z_j|^2 = 1 \right\}$ be the unit sphere in $\mathbb{C}^N$, $\omega : \partial\Omega \to S$ be the mapping that assigns $z \in \partial\Omega$ to a point $\omega(z) \in S$ such that the real hyperplane

$$W(z) = \{t \in \mathbb{C}^N : \operatorname{Re}\langle \omega(z), t \rangle = H_\Omega(\omega(z))\}$$

is a support hyperplane to Ω at z. By [12, Thm. 10.8], *the mapping $\omega : \partial\Omega \to S$ is continuous (when $\partial\Omega$ and S are equipped with the topologies induced by the Euclidean topology in $\mathbb{C}^N$). Since ω is bijective and $\partial\Omega$ is compact, ω is a homeomorphism of $\partial\Omega$ onto S. Fix $z \in (\partial\Omega) \cap (\mathbb{C}^*)^N$. There exists a neighborhood U of z in $\mathbb{C}^N$ such that $(\partial\Omega) \cap U \subset (\mathbb{C}^*)^N$. The set $V = \omega((\partial\Omega) \cap U)$ is a neighborhood of $\omega(z)$ in S, and there exists a point $c \in V \cap (\mathbb{C}^*)^N$. The complex hyperplane $T(w(c), \langle w(c), c \rangle)$ passes through c, is contained in $W(c)$, and therefore does not intersect Ω.*

We proceed to the case $\Omega = \mathbb{C}^N$. We denote by $\mathbf{1}$ a point in $\mathbb{C}^N$ (and a multi–index in $\mathbb{N}_0^N$), all coordinates of which are equal to 1.

Theorem 2.2. *If the operator $\mathcal{E}_a$ is applicable to $H(\mathbb{C}^N)$, then*

$$\limsup_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} < +\infty.$$

If the last condition is satisfied, then series (1.1) converges absolutely in $H(\mathbb{C}^N)$ for each function $f \in H(\mathbb{C}^N)$.

Proof. Let us prove that the condition

$$\limsup_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} < +\infty$$

is necessary for the applicability of $\mathcal{E}_a$. We introduce the space of sequences

$$\Lambda_\infty = \left\{c \in \mathbb{C}^{\mathbb{N}_0^N} : \forall m \in \mathbb{N} \sum_{\beta\in\mathbb{N}_0^N} |c_\beta|\frac{m^{|\beta|}}{\beta!} < +\infty\right\}.$$

As it is known, for $d \in \mathbb{C}^{\mathbb{N}_0^N}$, the series $\sum\limits_{\beta\in\mathbb{N}_0^N} |d_\beta c_\beta|$ converges for all $c \in \Lambda_\infty$ if and only if there exists $m \in \mathbb{N}$, for which

$$\sup_{\beta\in\mathbb{N}_0^N} \frac{|d_\beta|\beta!}{m^{|\beta|}} < +\infty,$$

see, for example, [10, Sect. 2].

For each $c \in \Lambda_\infty$, the function

$$f_c(t) = \sum_{\beta\in\mathbb{N}_0} \frac{c_\beta}{\beta!}(t - \mathbf{1})^\beta$$

is entire in $\mathbb{C}^N$. Therefore, for each $c \in \Lambda_\infty$, the series $\sum\limits_{\beta\in\mathbb{N}_0^N} |a_\beta||f_c^{(\beta)}(\mathbf{1})|$ converges, i.e. $\sum\limits_{\beta\in\mathbb{N}_0^N} |a_\beta||c_\beta| < +\infty$. Therefore, there exists $m \in \mathbb{N}$ for which

$$\sup_{\beta\in\mathbb{N}_0^N} \frac{\beta!|a_\beta|}{m^{|\beta|}} < +\infty,$$

and

$$\limsup_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} < +\infty.$$

And in this case, the fact that the condition

$$\limsup_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} < +\infty$$

ensures the absolute convergence of series (1.1) in $H(\mathbb{C}^N)$ is shown in the standard way by using the Cauchy formula. The proof is complete. □

Remark 2.2. *The applicability (in another sense) to $H(\Omega)$ of infinite order differential operators*

$$\mathcal{L}(f)(t) = \sum_{\beta\in\mathbb{N}_0^N} \varphi_\beta(t) f^{(\beta)}(t),$$

where φ_β, $\beta \in \mathbb{N}_0^N$ are functions defined on a compact subset of the holomorphy domain Ω in $\mathbb{C}^N$, was studied in [1].

2.2. Applicability $\mathcal{E}_{\theta,a}$. Let us study the applicability of the operator $\mathcal{E}_{\theta,a}$. We let $f_\alpha(t) := t^\alpha := t_1^{\alpha_1}\cdots t_N^{\alpha_N}$, $\alpha \in \mathbb{Z}^N$. The following statement on the action of the operator θ^β on the functions f_α is obvious.

Lemma 2.1. *For all $\alpha \in \mathbb{Z}^N$, $\beta \in \mathbb{N}_0^N$ in the domain of f_α the identity holds $\theta^\beta(f_\alpha) = \alpha^\beta f_\alpha$.*

We introduce the dilation operator $M_v(f)(t) := f(vt)$, $vt := (v_j t_j)_{j=1}^N$, $v, t \in \mathbb{C}^N$.

Theorem 2.3. (i) *If $\lim\limits_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} = 0$, then for each non–empty open set Ω in $\mathbb{C}^N$ and each function $f \in H(\Omega)$ series (1.2) converges absolutely in $H(\Omega)$.*

(ii) *Let $\Omega = \Omega_1 \times \cdots \times \Omega_N$, where Ω_j, $1 \leqslant j \leqslant N$, is an open set in $\mathbb{C}$, distinct from $\mathbb{C}$ and $\mathbb{C}^*$. If the operator $\mathcal{E}_{\theta,a}$ is applicable to $H(\Omega)$, then*

$$\lim_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} = 0.$$

(iii) *If the operator $\mathcal{E}_{\theta,a}$ is applicable to $H(\mathbb{C}^N)$, then*

$$\limsup_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} < +\infty.$$

If the latter condition is satisfied, then for each function $f \in H(\mathbb{C}^N)$ series (1.2) converges absolutely in $H(\mathbb{C}^N)$.

Proof. Statement (i) was essentially proved in [26]; see the proof of implication (c)⇒(d) in Theorem 2.2 in [26], which goes back to [28, Lm. 11.2] and uses the Cauchy integral formula and upper bounds for the modulus of function $\theta^\beta(g_w)$, where

$$g_w(t) := \frac{1}{(w_1 - t_1)\cdots(w_N - t_N)}.$$

The proof of (ii) reduces to the single variable case considered in [31]. There exist $v_j \in \mathbb{C}^* \cap (\partial\Omega_j)$, $1 \leqslant j \leqslant N$. Since $\theta^\beta M_v(f) = M_v\theta^\beta(f)$ for all $\beta \in \mathbb{N}_0^N$, $f \in H(\Omega)$, the operator $\mathcal{E}_{\theta,a}$ is applicable to $H\left(\frac{1}{v}\Omega\right)$, where where

$$\frac{1}{v}\Omega := \left\{\left(\frac{t_1}{v_1}, \ldots, \frac{t_N}{v_N}\right) : t \in \Omega\right\}.$$

The function

$$g(t) = \frac{1}{(1-t_1)\cdots(1-t_N)}$$

is holomorphic in $\frac{1}{v}\Omega$ and

$$\theta^\beta(g)(t) = \frac{A_{\beta_1}(t_1)}{(1-t_1)^{\beta_1+1}}\cdots\frac{A_{\beta_N}(t_N)}{(1-t_N)^{\beta_N+1}}, \qquad \beta \in \mathbb{N}_0^N,$$

where A_n, $n \in \mathbb{N}_0$, is an Euler polynomial of degree n. By the proof of implication (3)⇒(5) in Theorem 6.3 in [31] and due to the fact that the polynomials A_n have non–positive real roots we conclude that there exists a sequence $w^{(k)} \in \frac{1}{v}\Omega$, $k \in \mathbb{N}$, such that $w^{(k)} \to \mathbf{1}$ and the inequalities

$$|\theta^\beta(g)(w^{(k)})| \geqslant k^{|\beta|+N}2^N\beta!, \quad \beta \in \mathbb{N}_0^N, \quad k \in \mathbb{N},$$

hold. This implies

$$\lim_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} = 0.$$

(iii): The condition

$$\limsup_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} < +\infty$$

ensures the absolute convergence of series (1.2) in $H(\mathbb{C}^N)$ for each function $f \in H(\mathbb{C}^N)$ due to Lemma 2.1 and estimates implied by the Cauchy integral formula.

Suppose that series (1.2) converges absolutely for all $f \in H(\mathbb{C}^N)$, $t \in \mathbb{C}^N$. For each sequence $c \in \Lambda_\infty$, the function

$$g_c(t) := \sum_{\alpha\in\mathbb{N}_0^N} \frac{|c_\alpha|}{\alpha!} t^\alpha$$

is entire in $\mathbb{C}^N$. This is why

$$+\infty > \sum_{\beta\in\mathbb{N}_0^N} |a_\beta||\theta^\beta(g_c)(\mathbf{1})| \geqslant \sum_{\beta\in\mathbb{N}_0^N} |a_\beta| \frac{|c_\beta|}{\beta!}\theta^\beta(f_\beta)(\mathbf{1}) = \sum_{\beta\in\mathbb{N}_0^N} |a_\beta| \frac{|c_\beta|}{\beta!}\beta^\beta.$$

This yields

$$\limsup_{|\beta|\to\infty}(\beta!|a_\beta|)^{\frac{1}{|\beta|}} < +\infty.$$

The proof is complete. □

Remark 2.3. (i) *The condition*

$$\limsup_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} < +\infty$$

is equivalent to the fact that the function

$$a(t) = \sum_{\beta\in\mathbb{N}_0^N} a_\beta t^\beta$$

is an entire function of exponential type in $\mathbb{C}^N$, *and the condition*

$$\lim_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} = 0$$

is equivalent to the fact that a *is an entire function of exponential type* 0 *in* $\mathbb{C}^N$ [5], [18, Ch. 3, Sect. 1], [26, Lm. 2.1].

(ii) *If* $\limsup_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} < +\infty$, *then the identities hold*

$$\mathcal{E}_a(f_\beta) = \lambda_\beta f_\beta, \quad \textit{where} \quad \lambda_\beta = \beta! \sum_{0\leqslant\gamma\leqslant\beta} \frac{a_\gamma}{(\beta-\gamma)!}, \quad \beta \in \mathbb{N}_0^N,$$

and

$$\mathcal{E}_{\theta,a}(f_\beta) = a(\beta) f_\beta, \quad \beta \in \mathbb{N}_0^N.$$

In what follows, $\mathrm{Exp}(\mathbb{C}^N)$ (respectively, $\mathrm{Exp}(\{0\})$) is the space of all entire functions of exponential type (respectively, of exponential type 0) in $\mathbb{C}^N$.

2.3. Case $\Omega = (\mathbb{C}^*)^N$. We consider the case $\Omega = (\mathbb{C}^*)^N$. The set Ω does not satisfy the additional assumptions made in the proof of the necessary applicability conditions in Theorems 2.1, 2.3. We will use multidimensional decreasing factorials

$$(t)_\beta := (t_1)_{\beta_1} \cdots (t_N)_{\beta_N}, \quad t \in \mathbb{C}^N, \quad \beta \in \mathbb{N}_0^N,$$

where

$$(z)_n := z(z-1)\cdots(z-n+1), \quad n \in \mathbb{N}, \quad (z)_0 := 1, \quad z \in \mathbb{C}.$$

Lemma 2.2. *For eachg compact set Q in $(\mathbb{C}^*)^N$, each function $f \in H((\mathbb{C}^*)^N)$, each $R > 1$, there exists a constant $C > 0$ such that*

$$|t^\beta||f^{(\beta)}(t)| \leqslant CR^{|\beta|}, \quad t \in Q, \quad \beta \in \mathbb{N}_0^N.$$

Proof. For $R > 1$, we let $\varepsilon := 1 - \dfrac{1}{R}$. The function $f \in H((\mathbb{C}^*)^N)$ can be expanded into the Laurent series

$$f(t) = \sum_{\nu \in \mathbb{Z}^N} d_\nu t^\nu, \quad t \in (\mathbb{C}^*)^N.$$

For all $\beta \in \mathbb{N}_0^N$, $t \in (\mathbb{C}^*)^N$

$$t^\beta f^{(\beta)}(t) = \sum_{\nu \in \mathbb{Z}^N} d_\nu (\nu)_\beta t^\nu,$$

and hence

$$|t^\beta||f^{(\beta)}(t)| \leqslant \sum_{\nu \in \mathbb{Z}^N} |d_\nu||(\nu)_\beta||t^\nu|.$$

We employ the Cauchy inequalities for the coefficients of the Laurent series

$$|d_\nu| \leqslant \frac{M(f; r_1, \ldots, r_N)}{(r_1)^{\nu_1} \cdots (r_N)^{\nu_N}}, \quad \nu \in \mathbb{Z}^N,$$

for all $r_j > 0$, $1 \leqslant j \leqslant N$, where $M(f; r_1, \ldots, r_N) := \max\{|f(z)| : |z_j| = r_j, 1 \leqslant j \leqslant N\}$. There exist $\tau, T > 0$ such that $\tau \leqslant |z_j| \leqslant T$, $1 \leqslant j \leqslant N$, for each $z \in Q$. For $\nu \in \mathbb{Z}^N$, we define the numbers r_j, $1 \leqslant j \leqslant N$. If $\nu_j \leqslant -1$, then $r_j := \varepsilon\tau$; if $\nu_j \geqslant 0$, then we let $r_j := \dfrac{T}{\varepsilon}$. Then, for all $t \in Q$, $\nu \in \mathbb{Z}^N$, $1 \leqslant j \leqslant N$, the inequality $\left(\dfrac{|t_j|}{r_j}\right)^{\nu_j} \leqslant \varepsilon^{|\nu_j|}$ holds. Therefore, for all $t \in Q$, $\beta \in \mathbb{N}_0^N$, for constants

$$B := \sup\{M(f; \rho_1, \ldots, \rho_N) : \varepsilon\tau \leqslant \rho_j \leqslant T/\varepsilon, 1 \leqslant j \leqslant N\}, \qquad C := 2^{N+1} B R^N,$$

we have

$$|t^\beta||f^{(\beta)}(t)| \leqslant B2^{N+1} \sum_{\mu \in \mathbb{N}_0^N} \varepsilon^{|\mu|} C^\mu_{\mu+\beta} = \frac{B2^{N+1}}{(1-\varepsilon)^N} \frac{\beta!}{(1-\varepsilon)^{|\beta|}} = C\beta! R^{|\beta|}.$$

The proof is complete. $\square$

We introduce the space of sequences

$$\Lambda_1 := \left\{ d \in \mathbb{C}^{\mathbb{N}_0^N} : \forall n \in \mathbb{N} \sup_{\beta \in \mathbb{N}_0^N} \frac{|d_\beta|}{e^{|\beta|/n}} < +\infty \right\}.$$

It coincides with the set of all sequences $d \in \mathbb{C}^{\mathbb{N}_0^N}$ obeying

$$\limsup_{|\beta| \to \infty} |d_\beta|^{\frac{1}{|\beta|}} \leqslant 1.$$

We extend Vostretsov lemma [3] to the multi–dimensional case.

Lemma 2.3. *For each $d \in \Lambda_1$ there exists a function $g \in \operatorname{Exp}(\{0\})$ such that $g(\beta) \geqslant |d_\beta|$ for all $\beta \in \mathbb{N}_0^N$.*

Proof. Let $c_\beta := \max\{1; |d_\beta|\}$, $\beta \in \mathbb{N}_0^N$. Then $\lim\limits_{|\beta| \to \infty} (c_\beta)^{\frac{1}{|\beta|}} = 1$. We define the majorant sequences

$$x_{j,n} := \max\{c_\gamma : 0 \leqslant \gamma_k \leqslant n, 1 \leqslant k \leqslant N, k \neq j; \gamma_j = n\}, \quad n \in \mathbb{N}_0, \quad 1 \leqslant j \leqslant N.$$

The identities $x_{j,n} \geqslant 1$, $1 \leqslant j \leqslant N$, $n \in \mathbb{N}_0$, and $c_\beta \leqslant x_{1,\beta_1} \cdots x_{N,\beta_N}$, $\beta \in \mathbb{N}_0^N$, hold. Indeed, for $\beta_k = \max\{\beta_j : 1 \leqslant j \leqslant N\}$ we have

$$c_\beta \leqslant x_{k,\beta_k} \leqslant x_{1,\beta_1} \cdots x_{N,\beta_N}.$$

The relations

$$\begin{aligned}(x_{j,n})^{\frac{1}{n}} &= \max\left\{(c_\gamma)^{\frac{1}{n}} : 0 \leqslant \gamma_k \leqslant n,\, 1 \leqslant k \leqslant N, k \neq j; \gamma_j = n\right\}\\ &\leqslant \max\left\{(c_\gamma)^{\frac{N}{|\gamma|}} : 0 \leqslant \gamma_k \leqslant n,\, 1 \leqslant k \leqslant N, k \neq j; \gamma_j = n\right\},\end{aligned}$$

$1 \leqslant j \leqslant N$, $n \in \mathbb{N}$, yield

$$\lim_{n\to\infty} (x_{j,n})^{\frac{1}{n}} = 1, \qquad 1 \leqslant j \leqslant N.$$

By the lemma in [3], there exist entire functions h_j, $1 \leqslant j \leqslant N$, in $\mathbb{C}$ of zero exponential type with non–negative Taylor coefficients such that

$$h_j(n) \geqslant x_{j,n}, \quad n \in \mathbb{N}.$$

For entire functions $g_j(z) = x_{j,0} + zh_j(z)$, $1 \leqslant j \leqslant N$, in $\mathbb{C}$ of exponential type zero the inequalities $h_j(n) \geqslant x_{j,n}$ hold for all $n \in \mathbb{N}_0$. The function $g(t) := g_1(t_1) \cdots g_N(t_N)$, $t \in \mathbb{C}^N$, is an entire function of exponential type zero in $\mathbb{C}^N$, and $g(\beta) \geqslant c_\beta \geqslant |d_\beta|$ for each $\beta \in \mathbb{N}_0^N$. The proof is complete. □

Theorem 2.4. (i) *If*

$$\limsup_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} < 1,$$

then series (1.1) *converges absolutely in* $H((\mathbb{C}^*)^N)$. *If the operator* $\mathcal{E}_a$ *is applicable to* $H((\mathbb{C}^*)^N)$, *then*

$$\limsup_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} < 1.$$

(ii) *The condition*

$$\limsup_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} < +\infty$$

implies the absolute convergence of series (1.2) *in* $H((\mathbb{C}^*)^N)$. *If the operator* $\mathcal{E}_{\theta,a}$ *is applicable to* $H((\mathbb{C}^*)^N)$, *then*

$$\limsup_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} < +\infty.$$

Proof. (i): Suppose that

$$\limsup_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} < 1.$$

By Lemma 2.2, for each compact set Q in $\mathbb{C}^N$, the series

$$\sum_{\beta\in\mathbb{N}_0^N} |a_\beta| \sup_{t\in Q}(|t^\beta||f^{(\beta)}(t)|)$$

converges.

Let the operator $\mathcal{E}_a$ be applicable to $H((\mathbb{C}^*)^N)$. Fix $d \in \Lambda_1$. By Lemma 2.3, there exists a function

$$g \in \operatorname{Exp}(\{0\}), \qquad g(t) = \sum_{\gamma\in\mathbb{N}_0^N} \frac{c_\gamma}{\gamma!} t^\gamma,$$

for which $g(\beta) \geqslant |d_\beta|$ for all $\beta \in \mathbb{N}_0^N$ and $c_\gamma \geqslant 0$, $\gamma \in \mathbb{N}_0^N$. The function

$$f(t) = \sum_{\gamma\in\mathbb{N}_0^N} \frac{c_\gamma}{t^{\gamma+\mathbf{1}}}$$

is holomorphic in $(\mathbb{C}^*)^N$, and for each $t \in (\mathbb{C}^*)^N$ the series converges

$$\sum_{\beta\in\mathbb{N}_0^N} |a_\beta t^\beta| \left|f^{(\beta)}(t)\right| = \sum_{\beta\in\mathbb{N}_0^N} |a_\beta||t^\beta| \left|\sum_{\gamma\in\mathbb{N}_0^N} \frac{(-1)^{|\beta|}c_\gamma(\gamma+\beta)!}{\gamma! t^{\gamma+\beta+\mathbf{1}}}\right|$$
$$= \sum_{\beta\in\mathbb{N}_0^N} |a_\beta| \left|\sum_{\gamma\in\mathbb{N}_0^N} \frac{c_\gamma(\gamma+\beta)!}{\gamma! t^{\gamma+\mathbf{1}}}\right|.$$

Hence, the latter series converges for $t = \mathbf{1}$, that is, the series converges

$$\sum_{\beta\in\mathbb{N}_0^N} |a_\beta|\beta! \sum_{\gamma\in\mathbb{N}^N} \frac{c_\gamma(\gamma+\beta)!}{\gamma!\beta!} \geqslant \sum_{\beta\in\mathbb{N}_0^N} |a_\beta|\beta! \sum_{\gamma\in\mathbb{N}_0^N} \frac{c_\gamma}{\gamma!}\beta^\gamma \geqslant \sum_{\beta\in\mathbb{N}_0^N} |a_\beta|\beta!|d_\beta|.$$

Thus,

$$\sum_{\beta\in\mathbb{N}_0^N} |a_\beta|\beta!|d_\beta| < +\infty$$

for each sequence $d \in \Lambda_1$. This implies

$$\limsup_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} < 1.$$

(ii): Suppose that

$$\limsup_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} < +\infty.$$

We let

$$|a|(t) := \sum_{\beta\in\mathbb{N}_0^N} |a_\beta| t^\beta, \qquad t \in \mathbb{C}^N.$$

We take a function $f \in H((\mathbb{C}^*)^N)$ and expand it into the Laurent series in $(\mathbb{C}^*)^N$

$$f(t) = \sum_{\nu\in\mathbb{Z}^N} d_\nu t^\nu.$$

We fix a compact set Q in $(\mathbb{C}^*)^N$. There exists $\rho > 0$ such that $|t^\nu| \leqslant \rho^{|\nu_1|+\cdots+|\nu_N|}$ for all $t \in Q$, $\nu \in \mathbb{Z}^N$. Since $a \in \mathrm{Exp}(\mathbb{C}^N)$, we also have $|a| \in \mathrm{Exp}(\mathbb{C}^N)$, and there exist constants $C, R > 0$ such that

$$|a|(|t_1|, \ldots, |t_N|) \leqslant Ce^{R(|t_1|+\cdots+|t_N|)}, \quad t \in \mathbb{C}^N.$$

Hence,

$$\sum_{\beta\in\mathbb{N}_0^N} |a_\beta| \sup_{t\in Q}(|\theta^\beta(f)(t)|) \leqslant \sum_{\beta\in\mathbb{N}_0^N} |a_\beta| \sum_{\nu\in\mathbb{Z}^N} |d_\nu||\nu^\beta|\rho^{|\nu_1|+\cdots+|\nu_N|}$$
$$= \sum_{\nu\in\mathbb{Z}^N} |d_\nu|\rho^{|\nu_1|+\cdots+|\nu_N|}|a|(|\nu_1|, \ldots, |\nu_N|)$$
$$\leqslant C \sum_{\nu\in\mathbb{Z}^N} |d_\nu| \left(\rho e^R\right)^{|\nu_1|+\cdots+|\nu_N|} < +\infty.$$

Now suppose that the operator $\mathcal{E}_{\theta,a}$ is applicable to $H((\mathbb{C}^*)^N)$. Then, for each entire function f in $\mathbb{C}^N$, the series

$$\sum_{\beta\in\mathbb{N}_0^N} |a_\beta||\theta^\beta(f)(\mathbf{1})|$$

converges. By the proof of Assertion (iii) in Theorem 2.3, this implies

$$\limsup_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} < +\infty.$$

The proof is complete. □

Theorems 2.1–2.4 and Banach — Steinhaus theorem imply

Corollary 2.1. (i) *If*

$$\lim_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} = 0,$$

then the operators $\mathcal{E}_a$, $\mathcal{E}_{\theta,a}$ are linear and continuous in $H(\Omega)$ for each nonempty open set Ω in $\mathbb{C}^N$.

(ii) *If*

$$\limsup_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} < +\infty,$$

then the operators $\mathcal{E}_a$, $\mathcal{E}_{\theta,a}$ are linear and continuous in $H(\mathbb{C}^N)$, while $\mathcal{E}_{\theta,a}$ also is linear and continuous in $H((\mathbb{C}^*)^N)$.

(iii) *If*

$$\limsup_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} < 1,$$

then the operator $\mathcal{E}_a$ is linear and continuous in $H((\mathbb{C}^*)^N)$.

3. Relations between representations for Euler operators

For $\beta, \gamma \in \mathbb{N}_0^N$, $\gamma \leqslant \beta$, we let

$$s(\beta,\gamma) := \prod_{j=1}^{N} s(\beta_j,\gamma_j), \qquad S(\beta,\gamma) := \prod_{j=1}^{N} S(\beta_j,\gamma_j),$$

where $s(n,k)$, respectively, $S(n,k)$, for $k, n \in \mathbb{N}_0$, $k \leqslant n$, is the Stirling number of the first, respectively, second kind. The notation $\gamma \leqslant \beta$ means that $\gamma_j \leqslant \beta_j$, $1 \leqslant j \leqslant N$. These numbers are defined in various equivalent ways. One definition uses decreasing factorials; Stirling numbers are defined by the identities [17, Ch. 2, Sect. 7]

$$(z)_n = \sum_{k=0}^{n} s(n,k) z^k, \qquad z^n = \sum_{k=0}^{n} S(n,k)(z)_k, \quad n \in \mathbb{N}_0, \quad z \in \mathbb{C}.$$

For all $t \in \mathbb{C}^N$, $\beta \in \mathbb{N}_0^N$

$$(t)_\beta = \sum_{0\leqslant\gamma\leqslant\beta} s(\beta,\gamma) t^\gamma, \qquad t^\beta = \sum_{0\leqslant\gamma\leqslant\beta} S(\beta,\gamma)(t)_\gamma.$$

The relations in the following statement are implied by known similar equalities for one variable [17, Ch. 2, Probl. 18].

Lemma 3.1. *For each non–empty open set Ω in $\mathbb{C}^N$, all $\beta \in \mathbb{N}_0^N$, $f \in H(\Omega)$, $t \in \Omega$*

$$\theta^\beta(f)(t) = \sum_{0\leqslant\gamma\leqslant\beta} S(\beta,\gamma) t^\gamma f^{(\gamma)}(t), \qquad t^\beta f^{(\beta)}(t) = \sum_{0\leqslant\gamma\leqslant\beta} s(\beta,\gamma)\theta^\gamma(f)(t).$$

Let

$$a \in \operatorname{Exp}(\mathbb{C}^N), \qquad a(t) = \sum_{\beta\in\mathbb{N}_0^N} a_\beta t^\beta, \qquad t \in \mathbb{C}^N.$$

To analyze the relationship between infinite order Euler operators, we need series

$$\widetilde{a}(t) := \sum_{\beta\in\mathbb{N}_0^N} a_\beta (t)_\beta, \tag{3.1}$$

$$\widehat{a}(t) := \sum_{\gamma\in\mathbb{N}_0^N} \left(\sum_{\beta\geqslant\gamma} S(\beta,\gamma) a_\beta \right) t^\gamma, \quad t\in\mathbb{C}^N.$$

We are going to show that the series

$$\sum_{\beta\geqslant\gamma} S(\beta,\gamma) a_\beta, \qquad \gamma\in\mathbb{N}_0^N,$$

are absolutely convergent; we let

$$\widehat{a}_\gamma := \sum_{\beta\geqslant\gamma} S(\beta,\gamma) a_\beta.$$

Series (3.1) (formal in the general case) is a multidimensional version of Newton interpolation series; for holomorphic functions of one variable, it was studied, for example, in [7], [4], [14, Sect. 1.10].

We shall use some properties of Stirling numbers, in particular, the estimates for $S(\beta,\gamma)$ and $|s(\beta,\gamma)|$,

Lemma 3.2. (i) *For all $\beta,\gamma\in\mathbb{N}_0^N$, $\gamma\leqslant\beta$,*

$$\sum_{\gamma\leqslant\nu\leqslant\beta} s(\beta,\nu) S(\nu,\gamma) = \sum_{\gamma\leqslant\nu\leqslant\beta} S(\beta,\nu) s(\nu,\gamma) = \delta_{\gamma,\beta},$$

$\delta_{\gamma,\beta}$ is the Kronecker delta.

(ii) $S(\beta,\gamma) \leqslant C_\beta^\gamma \gamma^{\beta-\gamma}, \quad \beta,\gamma\in\mathbb{N}_0^N, \quad \gamma\leqslant\beta.$

(iii) $|s(\beta,\gamma)| \leqslant C_\beta^\gamma \dfrac{\beta!}{\gamma!}, \quad \beta,\gamma\in\mathbb{N}_0^N, \quad \gamma\leqslant\beta.$

(iv) *For all $t\in\mathbb{C}^N$, $\gamma\in\mathbb{N}_0^N$ the identity holds*

$$\sum_{\beta\geqslant\gamma} S(\beta,\gamma)\frac{t^\beta}{\beta!} = \frac{(e^{t_1}-1)^{\gamma_1}\cdots(e^{t_N}-1)^{\gamma_N}}{\gamma!}.$$

Proof. The identities in (i) are implied by the same identities for $N=1$ [17, Ch. 2, Sect. 7, (39)].

(ii): By [21, Probl. 7] the identities hold

$$S(\beta,\gamma) \leqslant C_{\beta-\mathbf{1}}^{\gamma-\mathbf{1}} \gamma^{\beta-\gamma}, \quad \beta,\gamma\in\mathbb{N}^N, \gamma\leqslant\beta.$$

This implies inequalities in (ii) for all $\beta,\gamma\in\mathbb{N}_0^N$, $\gamma\leqslant\beta$.

(iii): Let $n\in\mathbb{N}$. Since

$$z(z+1)\cdots(z+n-1) = \sum_{k=1}^n |s(n,k)| z^k, \quad z\in\mathbb{C},$$

we have

$$|s(n,k)| \leqslant C_{n-1}^{k-1}\frac{(n-1)!}{(k-1)!} \leqslant C_n^k \frac{n!}{k!}, \quad 1\leqslant k\leqslant n.$$

The above can be also obtained from the identity

$$\sum_{j=k}^n |s(n,j)| S(j,k) = C_{n-1}^{k-1}\frac{n!}{k!}$$

[17, Ch. 2, Probl. 16 (d)] and the identity $S(k,k)=1$. The above inequalities imply the estimate in (iii) for all $\beta,\gamma\in\mathbb{N}_0^N$, $\gamma\leqslant\beta$.

(iv): For $N=1$ this relation was given, for instance, in [17, Ch. 2, Probl. 14]. We note that the inequalities in (ii) ensure the absolute convergence of the series in (iv) for each $t\in\mathbb{C}^N$. The proof is complete. □

Lemma 3.3. (i) *If $a\in\mathrm{Exp}(\mathbb{C}^N)$, then $\widehat{a}\in\mathrm{Exp}(\mathbb{C}^N)$.*

(ii) *If*

$$\limsup_{|\beta|\to\infty}(\beta!|a_\beta|)^{\frac{1}{|\beta|}}<1,$$

then $\widetilde{a}\in\mathrm{Exp}(\mathbb{C}^N)$ and $a=\widehat{\widetilde{a}}$.

(iii) *If*

$$\limsup_{|\beta|\to\infty}(\beta!|a_\beta|)^{\frac{1}{|\beta|}}<\log 2,$$

then

$$\limsup_{|\beta|\to\infty}(\beta!|\widehat{a}_\beta|)^{\frac{1}{|\beta|}}<1$$

and $a=\widehat{\widetilde{a}}=\widetilde{\widehat{a}}$.

(iv) *If $a\in\mathrm{Exp}(\{0\})$, then $\widetilde{a},\widehat{a}\in\mathrm{Exp}(\{0\})$ and $a=\widehat{\widetilde{a}}=\widetilde{\widehat{a}}$.*

Proof. (i): Let

$$a\in\mathrm{Exp}(\mathbb{C}^N),\qquad a(t)=\sum_{\beta\in\mathbb{N}_0^N}a_\beta t^\beta,\qquad \sigma:=\limsup_{|\beta|\to\infty}(\beta!|a_\beta|)^{\frac{1}{|\beta|}}<+\infty.$$

We fix $\varepsilon>0$. There exists $A>0$ such that

$$|a_\beta|\leqslant A\frac{(\sigma+\varepsilon)^{|\beta|}}{\beta!},\qquad \beta\in\mathbb{N}_0^N.$$

By Lemma 3.2 for $\gamma\in\mathbb{N}_0^N$

$$\sum_{\beta\geqslant\gamma}S(\beta,\gamma)|a_\beta|\leqslant A\sum_{\beta\geqslant\gamma}S(\beta,\gamma)\frac{(\sigma+\varepsilon)^{|\beta|}}{\beta!}=A\frac{(e^{\sigma+\varepsilon}-1)^{|\gamma|}}{\gamma!}.$$

Hence,

$$\limsup_{|\gamma|\to\infty}(\gamma!|\widehat{a}_\gamma|)^{\frac{1}{|\gamma|}}\leqslant e^\sigma-1,\qquad \widehat{a}_\gamma=\sum_{\beta\geqslant\gamma}S(\beta,\gamma)a_\beta,$$

and an entire function

$$\widehat{a}(t)=\sum_{\gamma\in\mathbb{N}_0^N}\widehat{a}_\gamma t^\gamma,\qquad t\in\mathbb{C}^N,$$

of exponential type is well–defined in $\mathbb{C}^N$.

(ii): We choose $\delta<1$ such that

$$\limsup_{|\beta|\to\infty}(\beta!|a_\beta|)^{\frac{1}{|\beta|}}<\delta.$$

There exists a constant $B > 0$ such that $|a_\beta| \leqslant B\dfrac{\delta^{|\beta|}}{\beta!}$ for each $\beta \in \mathbb{N}_0^N$. In view of Lemma 3.2 we obtain

$$\sum_{\beta \geqslant \gamma} |s(\beta,\gamma)||a_\beta| \leqslant B \sum_{\beta \geqslant \gamma} C_\beta^\gamma \frac{\beta!}{\gamma!}\frac{\delta^{|\beta|}}{\beta!} = B \sum_{\beta \in \mathbb{N}_0^N} C_{\beta+\gamma}^\gamma \frac{\delta^{|\beta|+|\gamma|}}{\gamma!}$$
$$\leqslant B\frac{\delta^{|\gamma|}}{\gamma!} \sum_{\beta \in \mathbb{N}_0^N} \frac{(\beta+\gamma)!}{\beta!\gamma!}\delta^{|\beta|} = B\frac{\delta^{|\gamma|}}{\gamma!(1-\delta)^{|\gamma|+N}}.$$

Therefore,

$$\sum_{\beta \in \mathbb{N}_0^N} |a_\beta| \sum_{0 \leqslant \gamma \leqslant \beta} |s(\beta,\gamma)||t^\gamma| < +\infty$$

for each $t \in \mathbb{C}^N$ and once

$$\widetilde{a}(t) = \sum_{\gamma \in \mathbb{N}_0^N} \widetilde{a}_\gamma t^\gamma, \quad t \in \mathbb{C}^N,$$

we have

$$\limsup_{|\gamma| \to \infty} (\gamma!|\widetilde{a}_\gamma|)^{\frac{1}{|\gamma|}} \leqslant \frac{\delta}{1-\delta}.$$

At the same time,

$$\widetilde{a}_\gamma = \sum_{\beta \in \mathbb{N}_0^N} s(\beta,\gamma)a_\beta, \qquad \gamma \in \mathbb{N}_0^N.$$

Thus, $\widetilde{a} \in \operatorname{Exp}(\mathbb{C}^N)$. By (i), an entire function $\widehat{\widetilde{a}}$ of exponential type is well–defined. For $\beta \in \mathbb{N}_0^N$, by Lemma 3.2,

$$\sum_{\nu \geqslant \beta} S(\nu,\beta)\widetilde{a}_\nu = \sum_{\nu \geqslant \beta} S(\nu,\beta) \sum_{\gamma \geqslant \nu} s(\gamma,\nu)a_\gamma$$
$$= \sum_{\gamma \geqslant \beta} a_\gamma \sum_{\beta \leqslant \nu \leqslant \gamma} s(\gamma,\nu)S(\nu,\beta) = a_\beta.$$

Hence, $a = \widehat{\widetilde{a}}$.

Statement (iii) follows from (i) and (ii).

(iv): The belonging $\widetilde{a}, \widehat{a} \in \operatorname{Exp}(\{0\})$ is implied by the estimates obtained in the proof of (i) and (ii). The identities hold due to (iii). The proof is complete. □

We interpret statement (iii) in Lemma 3.3 in terms of the representation of an entire function by its Newton series. For an entire function a in $\mathbb{C}^N$ we introduce the numbers

$$\Delta^\gamma(a) := \sum_{0 \leqslant \nu \leqslant \gamma} (-1)^{\gamma-\nu} C_\gamma^{\gamma-\nu} a(\nu), \quad \gamma \in \mathbb{N}_0^N.$$

For $N = 1$, they coincide with the finite difference of order γ of the function a at the point 0. Let

$$\limsup_{|\beta| \to \infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} < \log 2.$$

By Lemma 3.3, the function $\widehat{a}$ is well–defined. We are going to show that

$$\frac{1}{\gamma!}\Delta^\gamma(a) = \widehat{a}_\gamma, \qquad \gamma \in \mathbb{N}_0^N,$$

where $\widehat{a}_\gamma$ are the Taylor coefficients of $\widehat{a}$. To prove this, we use the "direct" identities for $S(\beta,\gamma)$, $0\leqslant\gamma\leqslant\beta$,

$$S(\beta,\gamma)=\frac{1}{\gamma!}\sum_{0\leqslant\nu\leqslant\gamma}(-1)^{\gamma-\nu}C_\gamma^{\gamma-\nu}\nu^\beta,$$

which are implied by the corresponding relations for $N=1$ [17, Ch. 2, Sect. 7, Ident. (38)]). For $\gamma\in\mathbb{N}_0^N$

$$\frac{\Delta^\gamma(a)}{\gamma!}=\frac{1}{\gamma!}\sum_{0\leqslant\nu\leqslant\gamma}(-1)^{\gamma-\nu}C_\gamma^{\gamma-\nu}\sum_{\beta\in\mathbb{N}_0^N}a_\beta\nu^\beta$$
$$=\sum_{\beta\in\mathbb{N}_0^N}a_\beta\frac{1}{\gamma!}\sum_{0\leqslant\nu\leqslant\gamma}(-1)^{\gamma-\nu}C_\gamma^{\gamma-\nu}\nu^\beta=\sum_{\beta\geqslant\gamma}a_\beta S(\beta,\gamma).$$

The latter identity holds since

$$\sum_{0\leqslant\nu\leqslant\gamma}(-1)^{\gamma-\nu}C_\gamma^{\gamma-\nu}\nu^\beta=0$$

for $\gamma\nleqslant\beta$. This is, for instance, since $g^{(m)}(0)=0$ for $m,n\in\mathbb{N}_0$, $m<n$, where

$$g(z)=(e^z-1)^n=\sum_{k=0}^{n}(-1)^{n-k}C_n^{n-k}e^{kz},\qquad z\in\mathbb{C}.$$

Thus, the identity $a=\widetilde{\widehat{a}}$ in Lemma 3.3 (iii) means that each entire function a obeying

$$\limsup_{|\beta|\to\infty}(\beta!|a_\beta|)^{\frac{1}{|\beta|}}<\log 2$$

can be expanded into the Newton series. For $N=1$, this is known, see, for example, [14, Thm. 5.11]. In [14], the proof was made by using integral representations related with the function associated with a in Borel sense.

Remark 3.1. (i) *In statement (i) of Lemma 3.3, in the condition*

$$\limsup_{|\beta|\to\infty}(\beta!|a_\beta|)^{\frac{1}{|\beta|}}<1,$$

the number 1 cannot be replaced by a larger number. Indeed, for $a_\beta:=\dfrac{1}{\beta!}$, $\beta\in\mathbb{N}_0^N$ (and then $a(t)=e^{\langle 1,t\rangle}$), for $t_j=-1$, $1\leqslant j\leqslant N$, the series in the definition of function $\widetilde{a}(t)$ is not convergent for any permutation of indices from $\mathbb{N}_0^N$.

In the condition

$$\limsup_{|\beta|\to\infty}(\beta!|a_\beta|)^{\frac{1}{|\beta|}}<\log 2$$

of statement (iii) in Lemma 3.3, the number $\log 2$ *also cannot be replaced by a larger one. For example, if $a(t)=2^{\langle\mathbf{1},t\rangle}$, then*

$$\limsup_{|\beta|\to\infty}(\beta!|a_\beta|)^{\frac{1}{|\beta|}}=\log 2,\qquad \widehat{a}(t)=e^{\langle\mathbf{1},t\rangle},$$

and the function $\widetilde{\widehat{a}}$ is ill–defined.

(ii) *The quantity*

$$\sigma=\limsup_{|\beta|\to\infty}(\beta!|a_\beta|)^{\frac{1}{|\beta|}}$$

can be characterized in terms of the set

$$L_1 := \left\{ x \in [0, +\infty)^N : \sum_{j=1}^{N} x_j \leqslant 1 \right\}.$$

For $a \in \operatorname{Exp}(\mathbb{C}^N)$ the number σ coincides with the L_1–type of function a [18, Ch. 3, Sect. 1].

The following statement is implied the solvability of a more general interpolation problem (see, for example, [20, Thm. 1]; for $N = 1$, its solvability was proved in [13, Thm. 2]). We present an independent, simple proof that employs the specificity of the node set in this situation. It is made by the method applied in the proof of its one–dimensional analog in [2, Thm. 1.2.1].

Lemma 3.4. *For each sequence $g_\beta \in \mathbb{C}$, $\beta \in \mathbb{N}_0^N$ such that*

$$\limsup_{|\beta|\to\infty} (|g_\beta|)^{\frac{1}{|\beta|}} < +\infty,$$

there exists a function $f \in \operatorname{Exp}(\mathbb{C}^N)$ such that $f(\beta) = g_\beta$ for each $\beta \in \mathbb{N}_0^N$.

Proof. The function $g(t) = \sum_{\beta\in\mathbb{N}_0^N} g_\beta t^\beta$ is holomorphic on the closed polydisk $\{t \in \mathbb{C}^N \mid |t_j| \leqslant \varepsilon, 1 \leqslant j \leqslant N\}$ for some $\varepsilon > 0$. By the Cauchy integral formula we have

$$g_\beta = \frac{1}{(2\pi i)^N} \int\limits_{|t_1|=\varepsilon} \cdots \int\limits_{|t_N|=\varepsilon} \frac{g(t)}{t^{\beta+\mathbf{1}}} dt_N \cdots dt_1, \quad \beta \in \mathbb{N}_0^N.$$

We make the change $\tau_j = \log t_j$, $1 \leqslant j \leqslant N$ ($\log z$ denotes any continuous univalent branch of the logarithm). We obtain

$$g_\beta = \frac{1}{(2\pi i)^N} \int\limits_{\Gamma_1} \cdots \int\limits_{\Gamma_N} g\left(e^{\tau_1}, \dots, e^{\tau_N}\right) e^{-\langle \beta,\tau\rangle} d\tau_N \cdots d\tau_1,$$

where Γ_j is the vertical segment in $\mathbb{C}$. The function

$$f(z) := \frac{1}{(2\pi i)^N} \int\limits_{\Gamma_1} \cdots \int\limits_{\Gamma_N} g\left(e^{\tau_1}, \dots, e^{\tau_N}\right) e^{-\langle z,\tau\rangle} d\tau_N \cdots d\tau_1$$

is an entire function of exponential type in $\mathbb{C}^N$ and $f(\beta) = g_\beta$ for all $\beta \in \mathbb{N}_0^N$. The proof is complete. □

Theorem 3.1. (i) *For each function $a \in \operatorname{Exp}(\{0\})$, each non–empty open set Ω in $\mathbb{C}^N$ in $H(\Omega)$, the identities $\mathcal{E}_a = \mathcal{E}_{\theta,\widetilde{a}}$ and $\mathcal{E}_{\theta,a} = \mathcal{E}_{\widehat{a}}$ hold.*

(ii) *For each function $a \in \operatorname{Exp}(\mathbb{C}^N)$ in $H(\mathbb{C}^N)$ the identity $\mathcal{E}_{\theta,a} = \mathcal{E}_{\widehat{a}}$ holds.*

(iii) *For each function $a \in \operatorname{Exp}(\mathbb{C}^N)$ there exists $c \in \operatorname{Exp}(\mathbb{C}^N)$ such that $\mathcal{E}_a = \mathcal{E}_{\theta,c}$ in $H(\mathbb{C}^N)$.*

(iv) *If*

$$\limsup_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} < 1,$$

then $\mathcal{E}_a = \mathcal{E}_{\theta,\widetilde{a}}$ in $H(\mathbb{C}^N)$.

(v) *If*

$$\limsup_{|\beta|\to\infty} (\beta!|a_\beta|)^{\frac{1}{|\beta|}} < 1,$$

then $\mathcal{E}_a = \mathcal{E}_{\theta,\widetilde{a}}$ *in* $H((\mathbb{C}^*)^N)$. *If*

$$\limsup_{|\beta|\to\infty}(\beta!|a_\beta|)^{\frac{1}{|\beta|}} < \log 2,$$

then $\mathcal{E}_{\theta,a} = \mathcal{E}_{\widehat{a}}$ *in* $H((\mathbb{C}^*)^N)$.

Proof. (i): By the proofs of Lemma 3.3 and Theorems 2.1 and 2.3 for all $f \in H(\Omega)$, $t \in \Omega$ we have

$$\sum_{\gamma\in\mathbb{N}_0^N} |\theta^\gamma(f)(t)| \sum_{\beta\geqslant\gamma} |s(\beta,\gamma)||a_\beta| < +\infty, \tag{3.2}$$

$$\sum_{\gamma\in\mathbb{N}_0^N} |t^\gamma||f^{(\gamma)}(t)| \sum_{\gamma\geqslant\gamma} S(\beta,\gamma)|a_\beta| < +\infty. \tag{3.3}$$

Due to (3.2) and Lemma 3.1 for all $f \in H(\Omega)$, $t \in \Omega$,

$$\mathcal{E}_{\theta,\widetilde{a}}(f)(t) = \sum_{\gamma\in\mathbb{N}_0^N} \theta^\gamma(f)(t) \sum_{\beta\geqslant\gamma} s(\beta,\gamma)a_\beta = \sum_{\beta\in\mathbb{N}_0^N} a_\beta \sum_{0\leqslant\gamma\leqslant\beta} s(\beta,\gamma)\theta^\gamma(f)(t) = \mathcal{E}_a(f)(t).$$

By (3.3) and Lemma 3.1, for all $f \in H(\Omega)$, $t \in \Omega$,

$$\mathcal{E}_{\widehat{a}}(f)(t) = \sum_{\gamma\in\mathbb{N}_0^N} t^\gamma f^{(\gamma)}(t) \sum_{\beta\geqslant\gamma} S(\beta,\gamma)a_\beta = \sum_{\beta\in\mathbb{N}_0^N} a_\beta \sum_{0\leqslant\gamma\leqslant\beta} S(\beta,\gamma)\theta^\gamma(f)(t) = \mathcal{E}_{\theta,a}(f)(t).$$

(ii): In this case, for all $f \in H(\mathbb{C}^N)$, $t \in \mathbb{C}^N$ the relation (3.3) is valid, and hence, $\mathcal{E}_{\widehat{a}}(f)(t) = \mathcal{E}_{\theta,a}(f)(t)$.

(iii): The identities hold

$$\mathcal{E}_a(f_\beta) = \lambda_\beta f_\beta, \qquad \lambda_\beta = \beta! \sum_{0\leqslant\gamma\leqslant\beta} \frac{a_\gamma}{(\beta-\gamma)!}, \qquad \beta \in \mathbb{N}_0^N.$$

Let us estimate $|\lambda_\beta|$ from above. Since

$$|\lambda_\beta| \leqslant \beta! \sum_{0\leqslant\gamma\leqslant\beta} \frac{|a_\gamma|}{(\beta-\gamma)!}$$

and there exist $C, \sigma > 0$ such that $|a_\nu| \leqslant C\dfrac{\sigma^{|\nu|}}{\nu!}$ for all $\nu \in \mathbb{N}_0^N$, we have

$$|\lambda_\beta| \leqslant C \sum_{0\leqslant\gamma\leqslant\beta} \frac{\beta!\sigma^{|\gamma|}}{\gamma!(\beta-\gamma)!} = C(1+\sigma)^{|\beta|}, \quad \beta \in \mathbb{N}_0^N.$$

By Lemma 3.4, there exists a function $c \in \mathrm{Exp}(\mathbb{C}^N)$ such that $c(\beta) = \lambda_\beta$ for all $\beta \in \mathbb{N}_0^N$. Since $\mathcal{E}_{\theta,c}(f_\beta) = c(\beta)f_\beta = \mathcal{E}_a(f_\beta)$ for each $\beta \in \mathbb{N}_0^N$, the set of all polynomials is dense in $H(\mathbb{C}^N)$ and the operators $\mathcal{E}_{\theta,c}$ and $\mathcal{E}_a$ are linear and continuous in $H(\mathbb{C}^N)$, then $\mathcal{E}_{\theta,c} = \mathcal{E}_a$ on the entire space $H(\mathbb{C}^N)$.

(iv): By (ii) and Lemma 3.3 $\mathcal{E}_{\theta,\widetilde{a}} = \mathcal{E}_{\widehat{\widetilde{a}}} = \mathcal{E}_a$ in $H(\mathbb{C}^N)$.

(v): If $\limsup\limits_{|\beta|\to\infty}(\beta!|a_\beta|)^{\frac{1}{|\beta|}} < 1$, by Theorem 2.4 and the proof of Lemma 3.3 (ii) for all $f \in H((\mathbb{C}^*)^N)$, $t \in (\mathbb{C}^*)^N$ relation (3.2) holds. This is why $\mathcal{E}_a = \mathcal{E}_{\theta,\widetilde{a}}$ in $H((\mathbb{C}^*)^N)$.

If

$$\limsup_{|\beta|\to\infty}(\beta!|a_\beta|)^{\frac{1}{|\beta|}} < \log 2,$$

then by Theorem 2.4 and the proof of Assertion (i) in Lemma 3.3, for all $f \in H((\mathbb{C}^*)^N)$, $t \in (\mathbb{C}^*)^N$ relation (3.3) holds. This is why $\mathcal{E}_{\theta,a} = \mathcal{E}_{\widehat{a}}$ in $H((\mathbb{C}^*)^N)$. The proof is complete. □

4. Multi–dimensional verions of Wigert — Leau theorem

We present one result that can be considered a multi–dimensional version of the Wigert — Leau theorem [2, Thm. 1.3.2], the proof of which uses Lemma 3.3. To justify this terminology, we note that a fact for functions of one variable t essentially contained in [11, Sect. 2]. The function $f(t)$ is holomorphic in a neighborhood of 0 and holomorphically extends to $\overline{\mathbb{C}}\backslash\{1\}$ if and only if the function $\frac{1}{t}f\left(\frac{1}{t}\right)$ is holomorphic in a neighborhood of infinity and holomorphically extends to $\overline{\mathbb{C}}\backslash\{1\}$. Thus, if

$$g(t)=\sum_{n=0}^{\infty}\frac{g^{(n)}(0)}{n!}t^n$$

is a nonzero entire function of exponential type in $\mathbb{C}$, then the function

$$g_0(t)=\sum_{n=0}^{\infty}g^{(n)}(0)t^n$$

is entire in $\frac{1}{1-t}$ if and only if the conjugate diagram of g coincides with $\{1\}$. Therefore, the Wigert — Leau theorem can be reformulated as follows: the conjugate diagram of an entire function g of exponential type coincides with $\{1\}$ if and only if there exists an entire function a of exponential type 0 such that $a(n)=g^{(n)}(0)$ for all $n\in\mathbb{N}_0$. Therefore, the theorem below can be considered a multidimensional analogue of the Wigert — Leau theorem. We will use information on analytic functionals from [19, Ch. 4, Sect. 4.5]. Below, $H(\mathbb{C}^N)'$ is the topological dual space of $H(\mathbb{C}^N)$; $\mathcal{F}:H(\mathbb{C}^N)'\to \mathrm{Exp}(\mathbb{C}^N)$ is the Laplace transform

$$\mathcal{F}(\varphi)(t):=\varphi_z(e^{\langle z,t\rangle}),\quad \varphi\in H(\mathbb{C}^N)',\quad t\in\mathbb{C}^N.$$

Theorem 4.1. *For each non-zero functional $\varphi\in H(\mathbb{C}^N)'$ the following conditions are equivalent:*

(i) *The defining compact set of φ coincides with $\{\mathbf{1}\}$.*

(ii) *There exists a function $a\in \mathrm{Exp}(\{0\})$ such that $a(\beta)=(\mathcal{F}(\varphi))^{(\beta)}(0)$ for each $\beta\in\mathbb{N}_0^N$.*

Proof. (i)⇒(ii): We introduce the function $a(t)=\mathcal{F}(\varphi)(t)e^{-\langle 1,t\rangle}$, $t\in\mathbb{C}^N$. Since $a\in\mathrm{Exp}(\{0\})$, then by Lemma 3.3 the function $\widetilde{a}\in\mathrm{Exp}(\{0\})$ is well–defined, and

$$\widetilde{a}(\beta)=\beta!\sum_{0\leqslant\gamma\leqslant\beta}\frac{a_\gamma}{(\beta-\gamma)!}$$

for all $\beta\in\mathbb{N}_0^N$. Since $\mathcal{F}(\varphi)(t)=a(t)e^{\langle 1,t\rangle}$, $t\in\mathbb{C}^N$, we obtain $(\mathcal{F}(\varphi))^{(\beta)}(0)=\widetilde{a}(\beta)$, $\beta\in\mathbb{N}_0^N$.

(ii)⇒(i): Let $a=\mathcal{F}(\psi)$ for a functional $\psi\in H(\mathbb{C}^N)'$, the defining compact set of which coincides with $\{0\}$. For each $t\in\mathbb{C}^N$

$$\mathcal{F}(\varphi)(t)=\sum_{\beta\in\mathbb{N}_0^N}\frac{(\mathcal{F}(\varphi))^{(\beta)}(0)}{\beta!}t^\beta=\sum_{\beta\in\mathbb{N}_0^N}\frac{a(\beta)}{\beta!}t^\beta=\sum_{\beta\in\mathbb{N}_0^N}\frac{\psi_u(e^{\langle\beta,u\rangle})}{\beta!}t^\beta$$
$$=\psi_u\left(\sum_{\beta\in\mathbb{N}_0^N}\frac{e^{\langle\beta,u\rangle}}{\beta!}t^\beta\right)=\psi_u\left(\sum_{\beta\in\mathbb{N}_0^N}\frac{(e^{u_1})^{\beta_1}\cdots(e^{u_N})^{\beta_N}t^\beta}{\beta!}\right)=\psi_u\left(e^{e^{u_1}t_1+\cdots e^{u_N}t_N}\right).$$

We fix $\varepsilon>0$ and choose $\delta>0$ such that $|e^w-1|<\varepsilon$ for all $w\in\mathbb{C}^N$ satisfying the inequality $|w|<\delta$. Since the defining compact set ψ coincides with $\{0\}$, there exists $C>0$ such that for each $t \in \mathbb{C}^N$

$$
\begin{aligned}
|\mathcal{F}(\psi)(t)| &\leqslant C \sup_{|u_j| \leqslant \delta, 1 \leqslant j \leqslant N} |e^{e^{u_1} t_1 + \dots + e^{u_N} t_N}| \\
&\leqslant C \sup_{|u_j| \leqslant \delta, 1 \leqslant j \leqslant N} e^{\mathrm{Re}(e^{u_1} t_1 + \dots e^{u_N} t_N)} \\
&= C \left(\sup_{|u_j| \leqslant \delta, 1 \leqslant j \leqslant N} e^{\mathrm{Re}((e^{u_1}-1)t_1) + \dots + \mathrm{Re}((e^{u_N}-1)t_N)} \right) e^{\mathrm{Re}\, t_1 + \dots + \mathrm{Re}\, t_N} \\
&\leqslant C e^{\varepsilon |t_1| + \dots + \varepsilon |t_N|} e^{\mathrm{Re}\langle \mathbf{1}, t\rangle}.
\end{aligned}
$$

Since $\mathrm{Re}\, t_1 + \dots + \mathrm{Re}\, t_N = \mathrm{Re}\langle \mathbf{1}, t\rangle$, this implies that defining compact set of φ coincides with $\{\mathbf{1}\}$. The proof is complete. □

5. Each Hadamard operator on space of all entire functions is Euler operator

Below, $\mathcal{L}_h(H(\mathbb{C}^N))$ is the space of all Hadamard type (Hadamard) operators in $H(\mathbb{C}^N)$, i.e., of continuous linear operators A in $H(\mathbb{C}^N)$, for which each monomial f_α, $\alpha \in \mathbb{N}_0^N$, is an eigenvector. By [8, Thm. 2], the mapping $\varphi \mapsto A_\varphi$, where

$$
A_\varphi(f)(t) := \varphi_z(f(tz)), \qquad t \in \mathbb{C}^N,
$$

is bijective from $H(\mathbb{C}^N)'$ into $\mathcal{L}_h(H(\mathbb{C}^N))$. Let $\mathcal{L}_{\mathcal{E}}(H(\mathbb{C}^N))$ be the set of all operators $\mathcal{E}_a$, $a \in \mathrm{Exp}(\mathbb{C}^N)$. Each operator $\mathcal{E}_a$ is Hadamard; it coincides with A_{φ_a}, where

$$
\varphi_a = \sum_{\beta \in \mathbb{N}_0^N} a_\beta \delta_{1,\beta}, \qquad \delta_{1,\beta}(f) = f^{(\beta)}(\mathbf{1}).
$$

Let us show that each Hadamard operator in $H(\mathbb{C}^N)$ is an Euler operator of form (1.1), and hence also (1.2).

Theorem 5.1. *The identity holds $\mathcal{L}_h(H(\mathbb{C}^N)) = \mathcal{L}_{\mathcal{E}}(H(\mathbb{C}^N))$.*

Proof. For all $\varphi \in H(\mathbb{C}^N)'$, $f \in H(\mathbb{C}^N)$

$$
\varphi(f) = \varphi_z \left(\sum_{\beta \in \mathbb{N}_0^N} \frac{f^{(\beta)}(\mathbf{1})}{\beta!} (z - \mathbf{1})^\beta \right) = \sum_{\beta \in \mathbb{N}_0^N} f^{(\beta)}(\mathbf{1}) \frac{1}{\beta!} \varphi_z((z - \mathbf{1})^\beta).
$$

There exist $C, R > 0$ such that for each $\beta \in \mathbb{N}_0^N$

$$
|\varphi_z((z - \mathbf{1})^\beta)| \leqslant C \sup_{|z_j - 1| \leqslant R, 1 \leqslant j \leqslant N} |(z - \mathbf{1})^\beta| = C R^{|\beta|}.
$$

Hence, if $c_\beta := \varphi_z((z - \mathbf{1})^\beta)$, then

$$
\limsup_{|\beta| \to \infty} |c_\beta|^{\frac{1}{|\beta|}} \leqslant R
$$

and

$$
c(t) = \sum_{\beta \in \mathbb{N}_0^N} \frac{c_\beta}{\beta!} t^\beta, \qquad t \in \mathbb{C}^N,
$$

is an entire function of exponential type in $\mathbb{C}^N$. Therefore, $\varphi = \varphi_c$ and $A_\varphi = \mathcal{E}_c$. Thus, $\mathcal{L}_h(H(\mathbb{C}^N)) = \mathcal{L}_{\mathcal{E}}(H(\mathbb{C}^N))$. The proof is complete. □

Remark 5.1. *Let*

$$\Omega := \Big\{ z \in \mathbb{C}^N : \sum_{j=1}^{N} |z_j|^2 < R^2 \Big\}, \qquad R \in (0, +\infty), \qquad \lambda \in \mathbb{C}^N,$$
$$|\lambda_j| \leqslant 1, \qquad 1 \leqslant j \leqslant N, \qquad \lambda \neq \mathbf{1}.$$

The dilation operator $M_\lambda(f)(t) = f(\lambda_1 t_1, \dots, \lambda_N t_N)$ is Hadamard in $H(\Omega)$, but is not an Euler operator of form (1.1) and (1.2) for $a \in \mathrm{Exp}(\{0\})$. Indeed, suppose that there exists a function $a \in \mathrm{Exp}(\{0\})$ such that $M_\lambda = \mathcal{E}_a$ in $H(\Omega)$. Then, for $f(t) = e^{\langle \mathbf{1}, t\rangle}$, for all $t \in \mathbb{C}^N$, the identities $a(t)f(t) = \mathcal{E}_a(f)(t) = f(\lambda t)$ hold, and hence, $a(t) = e^{\langle \lambda - \mathbf{1}), t\rangle}$. We obtain a contradiction since the function $e^{\langle \lambda - \mathbf{1}, t\rangle}$ does not belong to $\mathrm{Exp}(\{0\})$.

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