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# PROBLEM ON SMALL MOTIONS OF MULTICOMPONENT VISCOUS INCOMPRESSIBLE FLUID

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**Abstract.** In this work we study the problem on small motions and normal oscillations of a homogeneous mixture of several viscous incompressible fluids. The considered model is a generalization of the well-known Navier — Stokes equations for the dynamics of a one-component incompressible viscous medium, and it involves the incompressibility and momentum equations. We prove that the corresponding initial boundary value problem is well-posed and solvable. In terms of the Stokes operator, we construct the spectrum and system of eigenelements for the problem on normal oscillations.

**Keywords:** mixture of liquids, viscous incompressible fluid, Cauchy problem, discrete spectrum, orthonormal basis.

**Mathematics Subject Classification:** 76T30, 76M22

## 1. INTRODUCTION

We present a formulation of a nonlinear problem describing the barotropic motion of a multicomponent viscous compressible fluid. In this paper we study a system of equations linearized with respect to the state of rest for the case of incompressible mixture components.

Let a bounded domain  $\Omega \subset \mathbb{R}^3$  with a Lipschitz boundary  $\partial\Omega$  be filled with a homogeneous mixture of several viscous compressible fluids. We introduce a coordinate system  $Ox_1x_2x_3$  such that the axis  $Ox_3$  is directed against the gravity force  $-g\mathbf{e}_3$ ,  $g > 0$ , and the origin is located inside the domain  $\Omega$ . We denote by  $\mathbf{n}$  the unit vector normal to the boundary  $\partial\Omega$  and directed outside the domain  $\Omega$ . The barotropic motion of a mixture of  $n \geq 2$  viscous compressible fluids is described by the system of equations

$$\begin{aligned} R_i \frac{\partial \mathbf{u}_i}{\partial t} + R_i (\mathbf{u}_i \cdot \nabla) \mathbf{u}_i &= \operatorname{div} \mathbf{T}_i + \sum_{j=1}^n a_{ij} (\mathbf{u}_j - \mathbf{u}_i) + R_i \mathbf{F}_i, \\ \frac{\partial R_i}{\partial t} + \operatorname{div} (R_i \mathbf{u}_i) &= 0, \quad (t, x) \in [0, +\infty) \times \Omega, \quad i = 1, \dots, n, \end{aligned} \tag{1.1}$$

where  $\mathbf{u}_i = \mathbf{u}_i(t, x) = (u_{i1}(t, x); u_{i2}(t, x); u_{i3}(t, x))^T$  ( $x = (x_1; x_2; x_3) \in \Omega$ ) is velocity field of the  $i$ th component of the mixture (the symbol  $\top$  denotes the transposition),  $R_i = R_i(t, x)$  is the density,  $a_{ij} = a_{ji} \geq 0$  are coefficients describing the intensity of momentum exchange between the components of the mixture,  $\mathbf{F}_i = \mathbf{F}_i(t, x)$  are known fields of the external mass forces. The

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stress tensors  $\mathbf{T}_i$  and the viscous stress tensors  $\mathbf{S}_i$  are defined by the identities <sup>1</sup>:

$$\mathbf{T}_i := -P_i \mathbf{I}_3 + \mathbf{S}_i, \quad \mathbf{S}_i := \sum_{j=1}^n (\lambda_{ij} \operatorname{tr} e(\mathbf{u}) \mathbf{I}_3 + 2\mu_{ij} e(\mathbf{u})),$$

where  $P_i = P_i(t, x)$  is the pressure in the  $i$ th component of the mixture,  $\mathbf{I}_3$  is the identity matrix in  $\mathbb{R}^3$ ,  $\mu_{ij}$ ,  $\lambda_{ij}$  are the components of the viscosity matrices  $\mathbf{M} := \{\mu_{ij}\}_{i,j=1}^n$ ,  $\mathbf{\Lambda} := \{\lambda_{ij}\}_{i,j=1}^n$ . The viscosity matrices obey the conditions

$$\mathbf{M} > 0, \quad 2\mathbf{M} + 3\mathbf{\Lambda} > 0.$$

The pressure and density of each component of mixture are usually related by some equation of state, and system (1.1) is considered with classical no-slip boundary conditions, or with no-flow conditions and zero tangential stresses.

System (1.1) is one of many ways to describe the motion of multicomponent fluid mixtures, and it models the motion of a homogeneous mixture of viscous compressible fluids, a multi-velocity model (for detail see [9], [17], [15], [6]). In particular, this means that at each point in space, all components of the mixture are present, which are in the same phase, but each has its own local velocity of motion. The interaction between the components is realized through the exchange of momentum and viscous friction. At the same time, the intercomponent viscous friction, which is taken into account by considering the off-diagonal viscosity matrices  $\mathbf{M}$  and  $\mathbf{\Lambda}$ , essentially determines the features of the studied model of mixture dynamics. If the viscosity matrices are all diagonal and  $a_{ij} = 0$  in (1.1), then we deal with  $n$  independent systems of Navier — Stokes equations for each component.

The mathematical study of multi-velocity models of the motion of multicomponent media with off-diagonal viscosity matrices was initiated rather recently. One of the first works, in which results on solvability in the multidimensional case were obtained, was by Frehse, Goj, and Málek [10]. In this work, the solvability of the Cauchy problem for a system without convective terms was proved in the case of a general dependence of pressures on the component densities. In [11], the same authors obtained a result on the uniqueness of weak solutions to Cauchy problem under the additional assumptions that the mass forces and terms accounting for the momentum exchange between the different components are equal to zero. In work by Frehse and Weigant [12], the existence and uniqueness of the classical solution of the boundary value problem for a quasi-stationary system without convective terms with special boundary conditions were proved. Results on the existence of solutions taking into account convective terms were obtained by Mamontov and Prokudin for a multi-velocity model in [6], [7]. Spectral analysis of some linear models of compressible viscous multicomponent media was made in [18], [2].

The aim of this work is to study the problem of small motions and normal oscillations for system (1.1) linearized at the state of rest in the case of incompressible components of the mixture. The main results are presented in Theorem 2.1.

## 2. FORMULATION OF PROBLEM AND MAIN RESULTS

We suppose that the components of the mixture are incompressible homogeneous fluids with densities  $R_i(t, x) = \rho_i > 0$ . Considering the state of rest  $\mathbf{u}_i = \mathbf{0}$ ,  $\mathbf{F}_i = -g\mathbf{e}_3$  ( $i = 1, \dots, n$ ) of system (1.1), we find the steady-state pressures in the components of mixture  $P_{i0}(x_3) = -\rho_i g x_3 + p_{i0}$ , where  $p_{i0}$  is the pressure at the origin. We suppose that

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<sup>1</sup>For a vector field  $\mathbf{u} = (u_1; u_2; u_3)^\top$  we define a set of coefficients  $e_{lk}(\mathbf{u}) := \frac{1}{2} \left( \frac{\partial u_l}{\partial x_k} + \frac{\partial u_k}{\partial x_l} \right)$  of the ( $l, k = 1, 2, 3$ ) strain rate tensor  $e(\mathbf{u})$ . By  $\operatorname{tr} e(\mathbf{u}) := \sum_{s=1}^3 e_{ss}(\mathbf{u}) \equiv \operatorname{div} \mathbf{u}$  we denote the trace of matrix  $e(\mathbf{u})$ .

$P_i(t, x) = P_{i0}(x_3) + p_i(t, x)$ ,  $\mathbf{F}_i(t, x) = -g\mathbf{e}_3 + \mathbf{f}_i(t, x)$ , where  $p_i$  is the so-called dynamic pressure,  $\mathbf{f}_i$  is the small field of external mass forces imposed on the gravitational field. Assuming that  $\mathbf{u}_i$ ,  $p_i$ ,  $\mathbf{f}_i$  are small quantities of the same order, we arrive at a linearized system. This system, the no-slip boundary conditions and the initial conditions read

$$\begin{aligned} \frac{\partial \mathbf{u}_i}{\partial t} &= -\frac{1}{\rho_i} \nabla p_i + \sum_{j=1}^n \frac{1}{\rho_i} \mu_{ij} \Delta \mathbf{u}_j + \sum_{j=1}^n \frac{1}{\rho_i} a_{ij} (\mathbf{u}_j - \mathbf{u}_i) + \mathbf{f}_i, \quad (t, x) \in [0, +\infty) \times \Omega, \\ \operatorname{div} \mathbf{u}_i &= 0, \quad (t, x) \in [0, +\infty) \times \Omega, \quad \mathbf{u}_i = \mathbf{0}, \quad (t, x) \in [0, +\infty) \times \partial\Omega, \\ \mathbf{u}_i(0, x) &= \mathbf{u}_i^0(x), \quad i = 1, \dots, n. \end{aligned} \quad (2.1)$$

Here  $\mu_{ij}$  are viscosity coefficients,  $a_{ij} = a_{ji} \geq 0$  are coefficients describing the intensity of momentum exchange between the components. The matrix  $\mathbf{M} := \{\mu_{ij}\}_{i,j=1}^n$ , called the viscosity matrix, is symmetric and positive:  $\mathbf{M} > 0$ .

By means of the orthogonal projection method, initial boundary value problem (2.1) using is treated as the Cauchy problem in the Hilbert space<sup>1</sup>  $\mathcal{H} := \oplus_{i=1}^n \mathbf{J}_0(\Omega)$ :

$$\frac{d\xi}{dt} = -\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B})\xi + \mathcal{F}(t), \quad \xi(0) = \xi^0, \quad (2.2)$$

where

$$\begin{aligned} \xi &:= (\mathbf{u}_1; \dots; \mathbf{u}_n)^\top, \quad \xi^0 := (\mathbf{u}_1^0; \dots; \mathbf{u}_n^0)^\top, \quad \mathcal{F}(t) := (P_0 \mathbf{f}_1(t); \dots; P_0 \mathbf{f}_n(t))^\top, \\ \mathcal{R} &:= \{\delta_{ij} \rho_j I\}_{i,j=1}^n, \quad \mathcal{M} := \{\mu_{ij} I\}_{i,j=1}^n, \quad \mathcal{A} := \{\delta_{ij} A\}_{i,j=1}^n, \quad \mathcal{D}(\mathcal{A}) := \bigoplus_{i=1}^n \mathcal{D}(A), \\ \mathcal{B} &:= \begin{pmatrix} \sum_{j=1}^n a_{1j} I - a_{11} I & -a_{12} I & \dots & -a_{1n} I \\ -a_{21} I & \sum_{j=1}^n a_{2j} I - a_{22} I & \dots & -a_{2n} I \\ \dots & \dots & \dots & \dots \\ -a_{n1} I & -a_{n2} I & \dots & \sum_{j=1}^n a_{nj} I - a_{nn} I \end{pmatrix}. \end{aligned}$$

Here  $\top$  is the transposition operation,  $\delta_{ij}$  is the Kronecker delta,  $I$  is the identity operator in  $\mathbf{J}_0(\Omega)$ ,  $P_0$  is the orthoprojection of  $\mathbf{L}_2(\Omega)$  onto  $\mathbf{J}_0(\Omega)$ , and  $A$  is the Stokes operator.

**Definition 2.1.** *Fields  $\mathbf{u}_i$  and functions  $p_i$  ( $i = 1, \dots, n$ ) are called the strong in time solution to initial boundary value problem (2.1) if the function  $\xi$  is a solution of Cauchy problem (2.2). In its turn, the function  $\xi$  is a solution to Cauchy problem (2.2) if  $\xi \in C^1([0, +\infty); \mathcal{H})$ ,  $\xi(t) \in \mathcal{D}(\mathcal{A})$  for all  $t \geq 0$ ,  $\mathcal{A}\xi \in C([0, +\infty); \mathcal{H})$ , the equation in (2.2) holds for all  $t \geq 0$ , and the initial condition holds as well.*

In addition to the viscosity matrix  $\mathbf{M}$ , we introduce the density matrix  $\mathbf{R} := \{\delta_{ij} \rho_j\}_{i,j=1}^n$  associated with the operator  $\mathcal{R}$ , and the momentum exchange intensity matrix  $\mathbf{B}$  associated with the operator  $\mathcal{B}$ . And in general, if  $\mathbf{S} := \{s_{ij}\}_{i,j=1}^n$  is a matrix acting in  $\mathbb{C}^n$ , then we associate with it the operator  $\mathcal{S} := \{s_{ij} I\}_{i,j=1}^n$  acting in  $\mathcal{H}$ . In this case we write  $\mathbf{S} \leftrightarrow \mathcal{S}$ .

We denote by  $\lambda_k(A)$ ,  $\mathbf{u}_k(A)$  the eigenvalues of the Stokes operator  $A$  taken in the ascending order and the associated eigenelements.

The main result of the work is the following theorem.

**Theorem 2.1.** 1) *Let  $\mathbf{u}_i^0 \in \mathcal{D}(A)$  ( $i = 1, \dots, n$ ), and let the fields  $\mathbf{f}_i$  ( $i = 1, \dots, n$ ) satisfy the local Hölder condition. Then initial boundary value problem (2.1) has a unique strong in time solution.*

<sup>1</sup>All notations for spaces and operators used in this section are explained in Section 3.

- 2) The spectrum  $\sigma$  of operator  $\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B})$  is located on the positive semi-axis, is discrete and has the asymptotic distribution

$$\lambda_k(\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B})) = \left( \frac{|\Omega|}{3\pi^2} \operatorname{tr}(\mathbf{R}^{-\frac{1}{2}}\mathbf{M}\mathbf{R}^{-\frac{1}{2}})^{-\frac{3}{2}} \right)^{-\frac{2}{3}} k^{\frac{2}{3}}(1 + o(1)), \quad k \rightarrow \infty.$$

- 3) The spectrum  $\sigma$  can be represented as  $\sigma = \{\lambda_k^{(p)}\}_{k \in \mathbb{N}, p=1, \dots, n}$ , where  $\lambda_k^{(p)}$  ( $p = 1, \dots, n$ ) are the roots of the characteristic equations

$$\det(\lambda_k(A)\mathbf{M} + \mathbf{B} - \lambda\mathbf{R}) = 0.$$

The system of eigenelements  $\{\xi_k^{(p)}\}_{k \in \mathbb{N}, p=1, \dots, n}$  of the operator  $\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B})$  forms an orthonormal basis in the space  $\mathcal{H}_{\mathcal{R}}^1$  and it can be represented as

$$\left\{ \xi_k^{(p)} = \begin{pmatrix} \varphi_{k,1}^{(p)} \\ \vdots \\ \varphi_{k,n}^{(p)} \end{pmatrix} \mathbf{u}_k(A) = \begin{pmatrix} \varphi_{k,1}^{(p)} \mathbf{u}_k(A) \\ \vdots \\ \varphi_{k,n}^{(p)} \mathbf{u}_k(A) \end{pmatrix} \right\}_{k \in \mathbb{N}, p=1, \dots, n},$$

where  $\varphi_k^{(p)} := (\varphi_{k,1}^{(p)}; \dots; \varphi_{k,n}^{(p)})^\top$  ( $p = 1, \dots, n$ ) are normalized in  $\mathbb{C}_{\mathbf{R}}^n$  eigenvectors of the matrix spectral problem

$$(\lambda_k(A)\mathbf{M} + \mathbf{B})\varphi = \lambda\mathbf{R}\varphi.$$

- 4) The solution of Cauchy problem (2.2) is given by the formula

$$\xi(t) = \mathcal{U}(t)\xi^0 + \int_0^t \mathcal{U}(t-s)\mathcal{F}(s) ds,$$

$$\mathcal{U}(t)\xi := \mathcal{U}(t) \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{pmatrix} = \sum_{k \in \mathbb{N}} \sum_{p=1}^n e^{-\lambda_k^{(p)} t} \sum_{l=1}^n \rho_l \varphi_{k,l}^{(p)}(\mathbf{u}_l, \mathbf{u}_k(A))_{\mathbf{L}_2(\Omega)} \begin{pmatrix} \varphi_{k,1}^{(p)} \mathbf{u}_k(A) \\ \vdots \\ \varphi_{k,n}^{(p)} \mathbf{u}_k(A) \end{pmatrix}.$$

### 3. WELL-POSEDNESS AND SOLVABILITY OF INITIAL BOUNDARY VALUE PROBLEM (2.1)

In section we derive problem (2.2) and prove Statement 1) of Theorem 2.1.

We introduce the Hilbert space  $\mathbf{L}_2(\Omega)$  with the scalar product and norm

$$(\mathbf{u}, \mathbf{v})_{\mathbf{L}_2(\Omega)} := \int_{\Omega} \mathbf{u}(x) \cdot \overline{\mathbf{v}(x)} d\Omega, \quad \|\mathbf{u}\|_{\mathbf{L}_2(\Omega)}^2 = \int_{\Omega} |\mathbf{u}(x)|^2 d\Omega.$$

For the space  $\mathbf{L}_2(\Omega)$ , the Weyl expansion into an orthogonal sum of solenoidal fields with zero normal component on the boundary and potential fields is valid (see, for example, [5, Ch. 2, Sect. 1, Eq. (1.18)]):

$$\mathbf{L}_2(\Omega) = \mathbf{J}_0(\Omega) \oplus \mathbf{G}(\Omega),$$

where

$$\mathbf{J}_0(\Omega) := \{ \mathbf{u} \in \mathbf{L}_2(\Omega) : \operatorname{div} \mathbf{u} = 0 \ (x \in \Omega), \quad \mathbf{u} \cdot \mathbf{n} = 0 \ (x \in \partial\Omega) \},$$

$$\mathbf{G}(\Omega) := \{ \mathbf{u} \in \mathbf{L}_2(\Omega) : \mathbf{u} = \nabla\Psi \}.$$

Here the operations of divergence and the normal component on the boundary are understood in the sense of the theory of generalized functions (distributions), see [5, Ch. 2, Sect. 1.6].

<sup>1</sup> $\mathcal{H}_{\mathcal{R}}, \mathbb{C}_{\mathbf{R}}^n$  are energy spaces for the operator  $\mathcal{R}$  and the matrix  $\mathbf{R}$ , respectively.

In what follows we suppose that the fields  $\mathbf{u}_i, \nabla p_i$  ( $i = 1, \dots, n$ ) depend on the variable  $t$  and take values in  $\mathbf{L}_2(\Omega)$ . Then  $\mathbf{u}_i(t) \in \mathbf{J}_0(\Omega)$  by the continuity equations and boundary conditions from (2.1), and  $\nabla p_i(t) \in \mathbf{G}(\Omega)$ , obviously, for each  $t \geq 0$ . We introduce orthoprojections  $P_0$  and  $P_G$  of the space  $\mathbf{L}_2(\Omega)$  onto the subspaces  $\mathbf{J}_0(\Omega)$  and  $\mathbf{G}(\Omega)$ , respectively. Applying the orthoprojections  $P_0$  and  $P_G$  to the momentum equations from (2.1), we obtain the relations

$$\frac{d\mathbf{u}_i}{dt} = \sum_{j=1}^n \frac{1}{\rho_i} \mu_{ij} P_0 \Delta \mathbf{u}_j + \sum_{j=1}^n \frac{1}{\rho_i} a_{ij} (\mathbf{u}_j - \mathbf{u}_i) + P_0 \mathbf{f}_i, \quad i = 1, \dots, n, \quad (3.1)$$

$$\mathbf{0} = -\frac{1}{\rho_i} \nabla p_i + \sum_{j=1}^n \frac{1}{\rho_i} \mu_{ij} P_G \Delta \mathbf{u}_j + P_G \mathbf{f}_i, \quad i = 1, \dots, n. \quad (3.2)$$

Having the known fields  $\mathbf{u}_i$  and the given fields  $\mathbf{f}_i$  by relations (3.2) we can recover the fields  $\nabla p_i$  ( $i = 1, \dots, n$ ), and this is why in what follows we study only relations (3.1).

We introduce the Stokes operator  $A$ , which is the Friedrichs extension of the operator  $-P_0 \Delta$  on smooth fields from  $\mathbf{J}_0(\Omega)$  with the no-slip condition on the boundary  $\partial\Omega$  of the domain  $\Omega$ ; see, for example, [5, Ch. 2, Sect. 2.6]. The operator  $A$  is self-adjoint and positive definite, its spectrum is discrete and has the following asymptotic distribution (see [16]):

$$\lambda_k(A) = \left( \frac{|\Omega|}{3\pi^2} \right)^{-\frac{2}{3}} k^{\frac{2}{3}} (1 + o(1)), \quad k \rightarrow \infty. \quad (3.3)$$

By using the Stokes operator, equations (3.1) together with initial conditions in (2.1) can be rewritten as the initial value problem for a system of operator equations in  $\mathbf{J}_0(\Omega)$ :

$$\frac{d\mathbf{u}_i}{dt} = -\frac{1}{\rho_i} \left( \sum_{j=1}^n \mu_{ij} A \mathbf{u}_j + \sum_{j=1}^n a_{ij} (\mathbf{u}_j - \mathbf{u}_i) \right) + P_0 \mathbf{f}_i, \quad \mathbf{u}_i(0) = \mathbf{u}_i^0, \quad i = 1, \dots, n. \quad (3.4)$$

Now system of operator equations and initial conditions (3.4) can be rewritten as Cauchy problem (2.2).

We present the rest of the proof in the form of several lemmas.

**Lemma 3.1.** *Let  $\det \mathbf{S} \neq 0$  and  $\mathbf{S} \leftrightarrow \mathcal{S}$ . Then the operators  $\mathcal{S}\mathcal{A}$ ,  $\mathcal{A}\mathcal{S}$  are closed on  $\mathcal{D}(\mathcal{A})$  and  $\mathcal{S}\mathcal{A}\xi = \mathcal{A}\mathcal{S}\xi$  for all  $\xi \in \mathcal{D}(\mathcal{A})$ .*

*Proof.* 1) Let us show that the operator  $\mathcal{S}\mathcal{A}$  is closed on  $\mathcal{D}(\mathcal{A})$ . Due to its diagonal structure, the operator  $\mathcal{A}$  is closed on its natural domain  $\mathcal{D}(\mathcal{A})$ . It is obvious that the operator  $\mathcal{S}$  is bounded<sup>1</sup>, that is,  $\mathcal{S} \in \mathcal{L}(\mathcal{H})$ , since all coefficients of the operator matrix  $\mathcal{S}$  are proportional to the unit operators  $I$ . The inequality  $\det \mathbf{S} \neq 0$  implies that there exists  $\mathcal{S}^{-1} \in \mathcal{L}(\mathcal{H})$ . This yields, see, for example, [3, Ch. III, Sect. 5.2, Prbl. 5.7], that the operator  $\mathcal{S}\mathcal{A}$  is closed on  $\mathcal{D}(\mathcal{A})$ . Indeed, let  $\xi_n \rightarrow \xi$  ( $\xi_n \in \mathcal{D}(\mathcal{S}\mathcal{A}) = \mathcal{D}(\mathcal{A})$ ),  $\mathcal{S}\mathcal{A}\xi_n \rightarrow \zeta$ . The last relation can be equivalently rewritten as  $\mathcal{A}\xi_n \rightarrow \mathcal{S}^{-1}\zeta$ . Together with closedness of the operator  $\mathcal{A}$  this yields that  $\xi \in \mathcal{D}(\mathcal{A})$  and  $\mathcal{A}\xi = \mathcal{S}^{-1}\zeta$  or, what is the same,  $\mathcal{S}\mathcal{A}\xi = \zeta$ .

2) Let us show that the operator  $\mathcal{A}\mathcal{S}$  is closed on  $\mathcal{D}(\mathcal{A})$ . The operator  $\mathcal{A}\mathcal{S}$  is defined on the natural domain  $\mathcal{D}(\mathcal{A}\mathcal{S}) := \{\xi \in \mathcal{H} : \mathcal{S}\xi \in \mathcal{D}(\mathcal{A})\}$  and is closed on it. Indeed, let  $\xi_n \rightarrow \xi$  ( $\xi_n \in \mathcal{D}(\mathcal{A}\mathcal{S})$ ),  $\mathcal{A}\mathcal{S}\xi_n \rightarrow \zeta$ . In view of the boundedness of the operator  $\mathcal{S}$ , these conditions imply  $\mathcal{S}\xi_n \rightarrow \mathcal{S}\xi$  ( $\mathcal{S}\xi_n \in \mathcal{D}(\mathcal{A})$ ),  $\mathcal{A}(\mathcal{S}\xi_n) \rightarrow \zeta$ . Together with the closedness of the operator  $\mathcal{A}$  this implies that  $\mathcal{S}\xi \in \mathcal{D}(\mathcal{A})$  (or  $\xi \in \mathcal{D}(\mathcal{A}\mathcal{S})$ ) and  $\mathcal{A}\mathcal{S}\xi = \zeta$ .

Let us show that  $\mathcal{D}(\mathcal{A}\mathcal{S}) = \mathcal{D}(\mathcal{A})$ . Let  $\xi \in \mathcal{D}(\mathcal{A})$ . In view of the structure of operator  $\mathcal{S}$  and the fact that the domain  $\mathcal{D}(A)$  of the Stokes operator  $A$  is a linear set, we obtain that  $\mathcal{S}\xi \in \mathcal{D}(\mathcal{A})$ , and hence  $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{A}\mathcal{S})$ . Now let  $\xi \in \mathcal{D}(\mathcal{A}\mathcal{S})$ , that is,  $\mathcal{S}\xi =: \zeta \in \mathcal{D}(\mathcal{A})$ . Then, arguing as above, we obtain  $\xi = \mathcal{S}^{-1}\zeta \in \mathcal{D}(\mathcal{A})$ , and therefore,  $\mathcal{D}(\mathcal{A}\mathcal{S}) \subset \mathcal{D}(\mathcal{A})$ .

<sup>1</sup>By  $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$  we denote the algebra of linear bounded operators from  $\mathcal{H}_1$  to  $\mathcal{H}_2$  defined on the entire space  $\mathcal{H}_1$ ,  $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$ .

3) The identity  $\mathcal{S}\mathcal{A}\xi = \mathcal{A}\mathcal{S}\xi$  for all  $\xi \in \mathcal{D}(\mathcal{A})$  can be checked straightforwardly. The proof is complete.  $\square$

**Lemma 3.2.** *The operator  $\mathcal{B}$  is bounded and non-negative in  $\mathcal{H}$ .*

*Proof.* 1) Let us show that the operator  $\mathcal{B}$  is bounded. For all  $\xi \in \mathcal{H}$  we have

$$\begin{aligned} \|\mathcal{B}\xi\|_{\mathcal{H}}^2 &= \sum_{i=1}^n \left\| -\sum_{j=1}^n a_{ij}(\mathbf{u}_j - \mathbf{u}_i) \right\|_{\mathbf{L}_2(\Omega)}^2 \leq \sum_{i=1}^n \left( \sum_{j=1}^n a_{ij} \|\mathbf{u}_j - \mathbf{u}_i\|_{\mathbf{L}_2(\Omega)} \right)^2 \\ &\leq n \sum_{i,j=1}^n a_{ij}^2 \|\mathbf{u}_j - \mathbf{u}_i\|_{\mathbf{L}_2(\Omega)}^2 \leq 2n \max_{i,j=1,\dots,n} \{a_{ij}^2\} \sum_{i,j=1}^n (\|\mathbf{u}_j\|_{\mathbf{L}_2(\Omega)}^2 + \|\mathbf{u}_i\|_{\mathbf{L}_2(\Omega)}^2) \\ &= 4n^2 \max_{i,j=1,\dots,n} \{a_{ij}^2\} \sum_{j=1}^n \|\mathbf{u}_j\|_{\mathbf{L}_2(\Omega)}^2 = 4n^2 \max_{i,j=1,\dots,n} \{a_{ij}^2\} \|\xi\|_{\mathcal{H}}^2, \end{aligned}$$

that is,  $\mathcal{B} \in \mathcal{L}(\mathcal{H})$  and  $\|\mathcal{B}\| \leq 2n \max_{i,j=1,\dots,n} \{a_{ij}\}$ .

2) We recall that  $a_{ij} = a_{ji} \geq 0$  ( $i, j = 1, \dots, n$ ). For each  $\xi \in \mathcal{H}$  we have

$$\begin{aligned} (\mathcal{B}\xi, \xi)_{\mathcal{H}} &= -\sum_{i,j=1}^n a_{ij}(\mathbf{u}_j - \mathbf{u}_i, \mathbf{u}_i)_{\mathbf{L}_2(\Omega)} = -\sum_{i>j} a_{ij}(\mathbf{u}_j - \mathbf{u}_i, \mathbf{u}_i)_{\mathbf{L}_2(\Omega)} - \sum_{i<j} a_{ij}(\mathbf{u}_j - \mathbf{u}_i, \mathbf{u}_i)_{\mathbf{L}_2(\Omega)} \\ &= -\sum_{i>j} a_{ij}(\mathbf{u}_j - \mathbf{u}_i, \mathbf{u}_i)_{\mathbf{L}_2(\Omega)} - \sum_{j<i} a_{ji}(\mathbf{u}_i - \mathbf{u}_j, \mathbf{u}_j)_{\mathbf{L}_2(\Omega)} = \sum_{i>j} a_{ij} \|\mathbf{u}_j - \mathbf{u}_i\|_{\mathbf{L}_2(\Omega)}^2 \geq 0, \end{aligned}$$

that is, the operator  $\mathcal{B}$  is self-adjoint (see, for example, [13, Ch. 2, Sect. 12, Thm. 12.3]) and non-negative in  $\mathcal{H}$ . The proof is complete.  $\square$

**Lemma 3.3.** *The operator  $\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B})$  is self-adjoint and positive definite in the energy space  $\mathcal{H}_{\mathcal{R}}$  of operator  $\mathcal{R}$ .*

*Proof.* 1) Let us show that the operator  $\mathcal{M}\mathcal{A}$  is self-adjoint in  $\mathcal{H}$ . Indeed, it follows from  $\mathcal{M} = \mathcal{M}^* \in \mathcal{L}(\mathcal{H})$  [3, Ch. III, Sect. 5, Prbl. 5.26] that  $(\mathcal{M}\mathcal{A})^* = \mathcal{A}^* \mathcal{M}^* = \mathcal{A}\mathcal{M}$  on  $\mathcal{D}((\mathcal{M}\mathcal{A})^*) = \mathcal{D}(\mathcal{A}\mathcal{M})$ . The inequality  $\det \mathbf{M} \neq 0$  and Lemma 3.1 imply  $\mathcal{A}\mathcal{M} = \mathcal{M}\mathcal{A}$  on  $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{M}\mathcal{A}) = \mathcal{D}(\mathcal{A}\mathcal{M})$ .

The operator  $\mathcal{M}\mathcal{A} + \mathcal{B}$  is self-adjoint in  $\mathcal{H}$  since  $\mathcal{B}$  is self-adjoint and bounded [3, Ch. V, Sect. 4, Thm. 4.3]. Hence, the operator  $\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B})$  is self-adjoint in the energy space  $\mathcal{H}_{\mathcal{R}}$  of the operator  $\mathcal{R}$ . Indeed, given  $\mathcal{R} = \mathcal{R}^*$ ,  $\mathcal{R}^{-1} \in \mathcal{L}(\mathcal{H})$ , for each  $\xi, \zeta \in \mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B}))$  we have

$$\begin{aligned} (\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B})\xi, \zeta)_{\mathcal{H}_{\mathcal{R}}} &= ((\mathcal{M}\mathcal{A} + \mathcal{B})\xi, \zeta)_{\mathcal{H}} = (\xi, (\mathcal{M}\mathcal{A} + \mathcal{B})\zeta)_{\mathcal{H}} \\ &= (\mathcal{R}\xi, \mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B})\zeta)_{\mathcal{H}} = (\xi, \mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B})\zeta)_{\mathcal{H}_{\mathcal{R}}}. \end{aligned}$$

2) Let us show that the operator  $\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B})$  is positive definite in  $\mathcal{H}_{\mathcal{R}}$ .

We denote by  $\lambda_j(\mathbf{M})$  and  $\varphi_j(\mathbf{M})$  ( $j = 1, \dots, n$ ) the eigenvalues and corresponding eigenvectors of matrix  $\mathbf{M}$ . The coordinates of the eigenvectors can be assumed to be real, and the system  $\{\varphi_j(\mathbf{M})\}_{j=1}^n$  is an orthonormal basis in  $\mathbb{C}^n$  since the viscosity matrix  $\mathbf{M}$  is positive definite by assumption. Let  $\mathbf{M}_{\varphi} = \mathbf{M}_{\varphi}(\varphi_1, \dots, \varphi_n)$  be the matrix, the columns of which are the eigenvectors of matrix  $\mathbf{M}$ . Then the identities  $\mathbf{M}_{\varphi}^{\top} = \mathbf{M}_{\varphi}^*$ ,  $\mathbf{M}_{\varphi}^* \mathbf{M}_{\varphi} = \mathbf{M}_{\varphi} \mathbf{M}_{\varphi}^* = \mathbf{I}_n$  hold, where  $\mathbf{I}_n$  is the identity matrix in  $\mathbb{C}^n$  and  $\mathbf{M}_{\varphi}^* \mathbf{M} \mathbf{M}_{\varphi} = \{\delta_{ij} \lambda_j(\mathbf{M})\}_{i,j=1}^n$ . We let  $\mathbf{M}_{\varphi} \leftrightarrow \mathcal{M}_{\varphi}$ . Then  $\mathcal{M}_{\varphi}^* \mathcal{M}_{\varphi} = \mathcal{M}_{\varphi} \mathcal{M}_{\varphi}^* = \mathcal{I}$ , where  $\mathcal{I}$  is the identity operator in  $\mathcal{H}$ ,  $\mathcal{M}_{\varphi}^* \mathcal{M} \mathcal{M}_{\varphi} = \{\delta_{ij} \lambda_j(\mathbf{M}) I\}_{i,j=1}^n$ .

In view of Lemmas 3.1, 3.2 for each  $\xi \in \mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B}))$  we have

$$\begin{aligned}
 (\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B})\xi, \xi)_{\mathcal{H}_{\mathcal{R}}} &= ((\mathcal{M}\mathcal{A} + \mathcal{B})\xi, \xi)_{\mathcal{H}} \geq (\mathcal{M}\mathcal{A}\xi, \xi)_{\mathcal{H}} \\
 &= (\mathcal{M}\mathcal{A}\mathcal{M}_{\varphi}\mathcal{M}_{\varphi}^*\xi, \mathcal{M}_{\varphi}\mathcal{M}_{\varphi}^*\xi)_{\mathcal{H}} \\
 &= ((\mathcal{M}_{\varphi}^*\mathcal{M}\mathcal{M}_{\varphi})\mathcal{A}\mathcal{M}_{\varphi}^*\xi, \mathcal{M}_{\varphi}^*\xi)_{\mathcal{H}} \\
 &= (\{\delta_{ij}\lambda_j(\mathbf{M})A\}_{i,j=1}^n \mathcal{M}_{\varphi}^*\xi, \mathcal{M}_{\varphi}^*\xi)_{\mathcal{H}} \\
 &\geq \lambda_1(A) \min_{j=1,\dots,n} \lambda_j(\mathbf{M})(\mathcal{M}_{\varphi}^*\xi, \mathcal{M}_{\varphi}^*\xi)_{\mathcal{H}} \\
 &= \lambda_1(A) \min_{j=1,\dots,n} \lambda_j(\mathbf{M})(\xi, \xi)_{\mathcal{H}} \\
 &\geq \lambda_1(A) \min_{j=1,\dots,n} \lambda_j(\mathbf{M}) \min_{j=1,\dots,n} \rho_j^{-1}(\mathcal{R}\xi, \xi)_{\mathcal{H}} \\
 &= \lambda_1(A) \min_{j=1,\dots,n} \lambda_j(\mathbf{M}) \min_{j=1,\dots,n} \rho_j^{-1}(\xi, \xi)_{\mathcal{H}_{\mathcal{R}}},
 \end{aligned}$$

and therefore, the operator  $\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B})$  is positive definite in  $\mathcal{H}_{\mathcal{R}}$ . The proof is complete.  $\square$

**Lemma 3.4.** *Let  $\mathbf{u}_i^0 \in \mathcal{D}(A)$  ( $i = 1, \dots, n$ ), and the fields  $\mathbf{f}_i$  ( $i = 1, \dots, n$ ) satisfy the local Hölder condition. Then initial boundary value problem (2.1) has a unique strong in time solution.*

*Proof.* By Lemma 3.3, the operator  $-\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B})$  is self-adjoint and negative definite in  $\mathcal{H}_{\mathcal{R}}$ , and hence is the generator of a holomorphic contraction semigroup; see, for example, [3, Ch. IX, Sect. 6, Thm. 1.24]. Now, to apply Crandall — Pazy theorem (see [14] or [1, Ch. II, Sect. 1, Thm. 1.4]) to the Cauchy problem, we need to show that  $\xi^0 \in \mathcal{D}(\mathcal{A})$  and that  $\mathcal{F}$  satisfies the local Hölder condition.

1) The conditions on the initial data imply  $\xi^0 := (\mathbf{u}_1^0; \dots; \mathbf{u}_n^0)^{\top} \in \oplus_{i=1}^n \mathcal{D}(A) = \mathcal{D}(\mathcal{A})$ .

2) Let us verify that the function  $\mathcal{F} := (P_0\mathbf{f}_1; \dots; P_0\mathbf{f}_n)^{\top}$  is locally Hölder continuous with values in the space  $\mathcal{H}_{\mathcal{R}}$ . By the assumption, the fields  $\mathbf{f}_i$  ( $i = 1, \dots, n$ ) satisfy the local Hölder condition, that is, for each  $T \in [0, +\infty)$  there exist numbers  $K_i = K_i(T) > 0$ ,  $k_i = k_i(T) \in (0, 1]$  such that

$$\|\mathbf{f}_i(t) - \mathbf{f}_i(s)\|_{\mathbf{L}_2(\Omega)} \leq K_i |t - s|^{k_i} \quad \text{for } 0 \leq s, t \leq T.$$

Next, we have

$$\begin{aligned}
 \|\mathcal{F}(t) - \mathcal{F}(s)\|_{\mathcal{H}_{\mathcal{R}}} &\leq \max_{j=1,\dots,n} \rho_j^{\frac{1}{2}} \|\mathcal{F}(t) - \mathcal{F}(s)\|_{\mathcal{H}} = \max_{j=1,\dots,n} \rho_j^{\frac{1}{2}} \sqrt{\sum_{i=1}^n \|P_0\mathbf{f}_i(t) - P_0\mathbf{f}_i(s)\|_{\mathbf{L}_2(\Omega)}^2} \\
 &\leq \max_{j=1,\dots,n} \rho_j^{\frac{1}{2}} \|P_0\| \sqrt{\sum_{i=1}^n \|\mathbf{f}_i(t) - \mathbf{f}_i(s)\|_{\mathbf{L}_2(\Omega)}^2} \leq \max_{j=1,\dots,n} \rho_j^{\frac{1}{2}} \sqrt{\sum_{i=1}^n K_i^2 |t - s|^{2k_i}} \\
 &= \max_{j=1,\dots,n} \rho_j^{\frac{1}{2}} \sqrt{\sum_{i=1}^n K_i^2 |t - s|^{2(k_i - \min_{j=1,\dots,n} k_j)}} \cdot |t - s|^{\min_{j=1,\dots,n} k_j} \\
 &\leq \max_{j=1,\dots,n} \rho_j^{\frac{1}{2}} \sqrt{\sum_{i=1}^n K_i^2 T^{2(k_i - \min_{j=1,\dots,n} k_j)}} \cdot |t - s|^{\min_{j=1,\dots,n} k_j} \quad \forall 0 \leq s, t \leq T.
 \end{aligned}$$

By Crandall — Pazy theorem, Cauchy problem (2.2) has a unique solution in the sense of Definition 2.1, and therefore, initial boundary value problem (2.1) has a unique strong in time solution, see Definition 2.1. The proof is complete.  $\square$

4. SPECTRAL PROPERTIES OF OPERATOR  $\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B})$ 

In this section we prove Statements 2), 3), 4) of Theorem 2.1.

For the sake of completeness of the presentation, we prove the following auxiliary proposition.

**Proposition 4.1.** *Let the operator  $T$  be self-adjoint, positive definite in  $H$ , and have a discrete spectrum. Let the distribution function of its eigenvalues  $\mathcal{N}(r, T) := \sum_{\lambda_k(T) \leq r} 1$  have a power law asymptotic distribution*

$$\mathcal{N}(r, T) = ar^\alpha(1 + o(1)), \quad r \rightarrow +\infty. \quad (4.1)$$

Then the eigenvalues of operator  $T$  satisfy the asymptotic formula

$$\lambda_k(T) = a^{-\frac{1}{\alpha}} k^{\frac{1}{\alpha}}(1 + o(1)), \quad k \rightarrow \infty. \quad (4.2)$$

The inverse is also true.

*Proof.* 1) Let the leading term of the eigenvalue distribution function  $\mathcal{N}(r, T)$  of  $T$  have a power law asymptotic behavior. Let us show that while dealing with this leading term, we can suppose that all eigenvalues of  $T$  are simple. Indeed, we renumber the eigenvalues of  $T$  in the ascending order as follows:  $\{\lambda_{l,1}(T) = \dots = \lambda_{l,n_l}(T)\}_{l \in \mathbb{N}}$ , where  $n_l \in \mathbb{N}$  is the multiplicity of the eigenvalue. We denote by  $\{u_{l,s}(T)\}_{l \in \mathbb{N}, s=1, \dots, n_l}$  the system of corresponding eigenelements, which is an orthonormal basis in the Hilbert space  $H$ .

We define the operator

$$Su := \sum_{l \in \mathbb{N}} \sum_{s=1}^{n_l} \frac{(s-1) \min\{1, \lambda_{l+1,1}(T) - \lambda_{l,1}(T)\}}{n_l} (u, u_{l,s}(T))_H u_{l,s}(T), \quad u \in H.$$

The operator  $S$  is bounded ( $S \in \mathcal{L}(H)$ ) due to the estimates

$$\begin{aligned} \|Su\|_H^2 &= \sum_{l \in \mathbb{N}} \sum_{s=1}^{n_l} \frac{(s-1)^2 \min^2\{1, \lambda_{l+1,1}(T) - \lambda_{l,1}(T)\}}{n_l^2} |(u, u_{l,s}(T))_H|^2 \\ &\leq \sum_{l \in \mathbb{N}} \sum_{s=1}^{n_l} |(u, u_{l,s}(T))_H|^2 = \|u\|_H^2 \quad \forall u \in H. \end{aligned}$$

It is easy to see that all eigenvalues of the operator  $T + S$  are simple. Let us show that the leading terms of the distribution functions  $\mathcal{N}(r, T)$  and  $\mathcal{N}(r, T + S)$  coincide. Indeed, the eigenvalues of the operators  $T$  and  $T + S$  coincide with the characteristic values of the operator pencils  $l_0(\lambda) := I - \lambda T^{-1}$  and  $l(\lambda) := I - \lambda T^{-1} + T^{-1}S$ , respectively. The desired assertion now follows from the inclusion<sup>1</sup>  $T^{-1}S \in \mathfrak{S}_\infty(H)$  and Keldysh theorem [4] on the comparison of spectra of operator pencils, see also Markus — Matsaev theorem [8].

2) Let the distribution function of the eigenvalues of the operator  $T$  have power law asymptotic distribution (4.1). By the above proven facts, we can suppose that all eigenvalues of operator  $T$  are simple. By (4.1) we have

$$\lim_{r \rightarrow +\infty} \frac{\mathcal{N}(r, T)}{ar^\alpha} = 1.$$

We use Heine definition of the limit of a function, choosing  $\{r_k = \lambda_k(T)\}_{k \in \mathbb{N}}$  as the sequence of points. Then  $\mathcal{N}(\lambda_k(T), T) = k$  and the last relation due to the convergence  $\lambda_k(T) \rightarrow +\infty$  as  $k \rightarrow \infty$  can be rewritten in the following equivalent forms:

$$\lim_{\lambda_k(T) \rightarrow +\infty} \frac{k}{a(\lambda_k(T))^\alpha} = 1 \quad \iff \quad \lim_{k \rightarrow \infty} \frac{a(\lambda_k(T))^\alpha}{k} = 1.$$

<sup>1</sup>By  $\mathfrak{S}_\infty(\mathcal{H}_1, \mathcal{H}_2)$  we denote the set of completely continuous (compact) operators from  $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ ,  $\mathfrak{S}_\infty(\mathcal{H}) := \mathfrak{S}_\infty(\mathcal{H}, \mathcal{H})$ .

The latter relations implies (4.2), and all calculations can be inverted. The proof is complete.  $\square$

**Lemma 4.1.** *The spectrum  $\sigma$  of the operator  $\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B})$  is located on the positive semi-axis, is discrete and has the asymptotic distribution*

$$\lambda_k(\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B})) = \left( \frac{|\Omega|}{3\pi^2} \operatorname{tr}(\mathbf{R}^{-\frac{1}{2}}\mathbf{M}\mathbf{R}^{-\frac{1}{2}})^{-\frac{3}{2}} \right)^{-\frac{2}{3}} k^{\frac{2}{3}}(1 + o(1)), \quad k \rightarrow \infty. \quad (4.3)$$

*Proof.* 1) Due to the compactness of the inverse operator for the Stokes operator, we have  $\mathcal{A}^{-1} = \{\delta_{ij}A^{-1}\}_{i,j=1}^n \in \mathfrak{S}_\infty(\mathcal{H})$ . Taking into account Lemma 3.1, we find that

$$(\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B}))^{-1} = \mathcal{A}^{-\frac{1}{2}}(\mathcal{M} + \mathcal{A}^{-\frac{1}{2}}\mathcal{B}\mathcal{A}^{-\frac{1}{2}})^{-1}\mathcal{A}^{-\frac{1}{2}}\mathcal{R} \in \mathfrak{S}_\infty(\mathcal{H}),$$

and therefore, the spectrum of the operator  $\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B})$  is discrete (and lies on the positive semiaxis by Lemma 3.3).

2) In the spectral problem

$$\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B})\xi = \lambda\xi,$$

we make the change  $\mathcal{R}^{\frac{1}{2}}\xi =: \zeta$  and obtain the spectral problem

$$\mathcal{R}^{-\frac{1}{2}}(\mathcal{M}\mathcal{A} + \mathcal{B})\mathcal{R}^{-\frac{1}{2}}\zeta = \lambda\zeta.$$

The spectrum of this problem coincides with the characteristic values of the operator pencil

$$\mathcal{L}(\lambda) := \mathcal{I} - \lambda\mathcal{R}^{\frac{1}{2}}(\mathcal{M}\mathcal{A})^{-1}\mathcal{R}^{\frac{1}{2}} + (\mathcal{R}^{\frac{1}{2}}(\mathcal{M}\mathcal{A})^{-1}\mathcal{R}^{\frac{1}{2}})\mathcal{R}^{-\frac{1}{2}}\mathcal{B}\mathcal{R}^{-\frac{1}{2}}.$$

By the aforementioned Keldysh theorem [4], [8], the leading term of the distribution function of the characteristic values of operator pencil  $\mathcal{L}(\lambda)$  coincides with the leading term of distribution function of characteristic values of the truncated operator pencil  $\mathcal{L}_0(\lambda) := \mathcal{I} - \lambda\mathcal{R}^{\frac{1}{2}}(\mathcal{M}\mathcal{A})^{-1}\mathcal{R}^{\frac{1}{2}}$ , if this distribution function for  $\mathcal{L}_0(\lambda)$  is, for example, power law. The problem on the spectrum of operator-function  $\mathcal{L}_0(\lambda)$  is equivalent to the spectral problem

$$(\mathcal{R}^{-\frac{1}{2}}\mathcal{M}\mathcal{R}^{-\frac{1}{2}})\mathcal{A}\zeta = \lambda\zeta.$$

Thus, if the leading term of the distribution function  $\mathcal{N}(r, (\mathcal{R}^{-\frac{1}{2}}\mathcal{M}\mathcal{R}^{-\frac{1}{2}})\mathcal{A})$  is power law, then the distribution function  $\mathcal{N}(r, \mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B}))$  also has the same leading term.

3) We let  $\mathbf{P} := \mathbf{R}^{-\frac{1}{2}}\mathbf{M}\mathbf{R}^{-\frac{1}{2}}$  and as in Lemma 3.3, we denote  $\lambda_j(\mathbf{P})$ ,  $\varphi_j(\mathbf{P})$  ( $j = 1, \dots, n$ ) the eigenvalues and the corresponding eigenvectors of matrix  $\mathbf{P}$ . The coordinates of the eigenvectors can be assumed to be real, and the system  $\{\varphi_j(\mathbf{P})\}_{j=1}^n$  is an orthonormal basis in  $\mathbb{C}^n$ . Let  $\mathbf{P}_\varphi = \mathbf{P}_\varphi(\varphi_1, \dots, \varphi_n)$  be the matrix, the columns of which are the eigenvectors of matrix  $\mathbf{P}$ . Then the identities  $\mathbf{P}_\varphi^\top = \mathbf{P}_\varphi^*$ ,  $\mathbf{P}_\varphi^*\mathbf{P}_\varphi = \mathbf{P}_\varphi\mathbf{P}_\varphi^* = \mathbf{I}_n$  hold, where  $\mathbf{I}_n$  is the identity matrix in  $\mathbb{C}^n$ ,  $\mathbf{P}_\varphi^*\mathbf{P}\mathbf{P}_\varphi = \{\delta_{ij}\lambda_j(\mathbf{P})\}_{i,j=1}^n$ . We let  $\mathbf{P}_\varphi \leftrightarrow \mathcal{P}_\varphi$ . Then  $\mathcal{P}_\varphi^*\mathcal{P}_\varphi = \mathcal{P}_\varphi\mathcal{P}_\varphi^* = \mathcal{I}$ , where  $\mathcal{I}$  is the identity operator in  $\mathcal{H}$ ,  $\mathcal{P}_\varphi^*(\mathcal{R}^{-\frac{1}{2}}\mathcal{M}\mathcal{R}^{-\frac{1}{2}})\mathcal{P}_\varphi = \{\delta_{ij}\lambda_j(\mathbf{P})I\}_{i,j=1}^n$ . We make the change  $\mathcal{P}_\varphi^*\zeta =: \eta$  in the problem

$$(\mathcal{R}^{-\frac{1}{2}}\mathcal{M}\mathcal{R}^{-\frac{1}{2}})\mathcal{A}\zeta = \lambda\zeta,$$

and in view of Lemma 3.1 we obtain the split spectral problem

$$\mathcal{P}_\varphi^*(\mathcal{R}^{-\frac{1}{2}}\mathcal{M}\mathcal{R}^{-\frac{1}{2}})\mathcal{P}_\varphi\mathcal{A}\eta \equiv \{\delta_{ij}\lambda_j(\mathbf{P})A\}_{i,j=1}^n\eta = \lambda\eta, \quad \eta \in \mathcal{D}(\mathcal{A}) = \bigoplus_{l=1}^n \mathcal{D}(A).$$

In view of Proposition 4.1 and (3.3) this yields

$$\begin{aligned} \mathcal{N}(r, (\mathcal{R}^{-\frac{1}{2}}\mathcal{M}\mathcal{R}^{-\frac{1}{2}})\mathcal{A}) &= \sum_{j=1}^n \mathcal{N}(r, \lambda_j(\mathbf{P})\mathcal{A}) = \sum_{j=1}^n \frac{|\Omega|\lambda_j^{-\frac{3}{2}}(\mathbf{P})}{3\pi^2} r^{\frac{3}{2}}(1+o(1)) \\ &= \frac{|\Omega|}{3\pi^2} \sum_{j=1}^n \lambda_j^{-\frac{3}{2}}(\mathbf{R}^{-\frac{1}{2}}\mathbf{M}\mathbf{R}^{-\frac{1}{2}}) r^{\frac{3}{2}}(1+o(1)) \\ &= \frac{|\Omega|}{3\pi^2} \operatorname{tr}(\mathbf{R}^{-\frac{1}{2}}\mathbf{M}\mathbf{R}^{-\frac{1}{2}})^{-\frac{3}{2}} r^{\frac{3}{2}}(1+o(1)), \quad r \rightarrow +\infty. \end{aligned}$$

By Proposition 4.1 and the above proven facts this implies (4.3). The proof is complete.  $\square$

**Lemma 4.2.** *The spectrum  $\sigma$  can be represented as  $\sigma = \{\lambda_k^{(p)}\}_{k \in \mathbb{N}, p=1, \dots, n}$ , where  $\lambda_k^{(p)}$  ( $p = 1, \dots, n$ ) are the roots of the characteristic equations*

$$\det(\lambda_k(A)\mathbf{M} + \mathbf{B} - \lambda\mathbf{R}) = 0.$$

*The system of eigenelements  $\{\xi_k^{(p)}\}_{k \in \mathbb{N}, p=1, \dots, n}$  of operator  $\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B})$  forms an orthonormal basis in the space  $\mathcal{H}_{\mathcal{R}}$  and it can be represented as*

$$\left\{ \xi_k^{(p)} = \begin{pmatrix} \varphi_{k,1}^{(p)} \\ \vdots \\ \varphi_{k,n}^{(p)} \end{pmatrix} \mathbf{u}_k(A) = \begin{pmatrix} \varphi_{k,1}^{(p)} \mathbf{u}_k(A) \\ \vdots \\ \varphi_{k,n}^{(p)} \mathbf{u}_k(A) \end{pmatrix} \right\}_{k \in \mathbb{N}, p=1, \dots, n},$$

where  $\varphi_k^{(p)} := (\varphi_{k,1}^{(p)}; \dots; \varphi_{k,n}^{(p)})^\top$  ( $p = 1, \dots, n$ ) are the normalized in  $\mathbb{C}_{\mathbf{R}}^n$  eigenelements of the matrix spectral problem

$$(\lambda_k(A)\mathbf{M} + \mathbf{B})\varphi = \lambda\mathbf{R}\varphi.$$

*Proof.* 1) We seek the eigenelements of the spectral problem

$$(\mathcal{M}\mathcal{A} + \mathcal{B})\xi = \lambda\mathcal{R}\xi$$

as  $\xi := \varphi \mathbf{u}_k(A)$  ( $k \in \mathbb{N}$ ), where  $\varphi := (\varphi_1; \dots; \varphi_n)^\top \in \mathbb{C}^n$ . Then, in view of the identity  $\mathcal{A}\xi = \lambda_k(A)\xi$ , we obtain the relation

$$(\mathcal{M}\mathcal{A} + \mathcal{B} - \lambda\mathcal{R})\xi = (\lambda_k(A)\mathcal{M} + \mathcal{B} - \lambda\mathcal{R})\xi = (\lambda_k(A)\mathbf{M} + \mathbf{B} - \lambda\mathbf{R})\varphi \mathbf{u}_k(A) = 0.$$

Hence, since  $\mathbf{u}_k(A)$  is an eigenelement of the Stokes operator, and therefore is nontrivial, the vector  $\varphi$  should be an eigenvector of the matrix spectral problem

$$(\lambda_k(A)\mathbf{M} + \mathbf{B} - \lambda\mathbf{R})\varphi = 0.$$

Thus, let  $\lambda_k^{(p)}$ ,  $\varphi_k^{(p)} := (\varphi_{k,1}^{(p)}; \dots; \varphi_{k,n}^{(p)})^\top$ ,  $p = 1, \dots, n$ , be the eigenvalues and corresponding eigenvectors of the matrix spectral problem

$$(\lambda_k(A)\mathbf{M} + \mathbf{B} - \lambda\mathbf{R})\varphi = 0$$

for each  $k \in \mathbb{N}$ . We can suppose that the system  $\{\varphi_k^{(p)}\}_{p=1}^n$  is an orthonormal basis in  $\mathbb{C}_{\mathbf{R}}^n$  for each  $k \in \mathbb{N}$ . Then  $\lambda_k^{(p)}$ ,  $\xi_k^{(p)} := \varphi_k^{(p)} \mathbf{u}_k(A)$  ( $k \in \mathbb{N}$ ,  $p = 1, \dots, n$ ) are the eigenvalues and corresponding eigenelements of the operator  $\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B})$ . At the same time, the system  $\{\xi_k^{(p)}\}_{k \in \mathbb{N}, p=1, \dots, n}$  is orthonormal in  $\mathcal{H}_{\mathcal{R}}$ .

2) Let us show that the system  $\{\xi_k^{(p)}\}_{k \in \mathbb{N}, p=1, \dots, n}$  is complete in  $\mathcal{H}_{\mathcal{R}}$ . Since the system of eigenelements of the operator  $\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B})$  is orthonormal, this will imply that the operator  $\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B})$  has no eigenelements different from found ones. In this case, the roots of the characteristic equations

$$\det(\lambda_k(A)\mathbf{M} + \mathbf{B} - \lambda\mathbf{R}) = 0$$

exhaust all the eigenvalues of the operator  $\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B})$ .

Suppose that the system  $\{\xi_k^{(p)}\}_{k \in \mathbb{N}, p=1, \dots, n}$  is incomplete in  $\mathcal{H}_{\mathcal{R}}$ . Then there exists an element  $\zeta = (\mathbf{v}_1; \dots; \mathbf{v}_n)^\top \in \mathcal{H}$ ,  $\zeta \neq 0$ , orthogonal to all elements of this system, that is,

$$\begin{aligned} (\xi_k^{(p)}, \zeta)_{\mathcal{H}_{\mathcal{R}}} &= \left( \mathcal{R} \begin{pmatrix} \varphi_{k,1}^{(p)} \mathbf{u}_k(A) \\ \vdots \\ \varphi_{k,n}^{(p)} \mathbf{u}_k(A) \end{pmatrix}, \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{pmatrix} \right)_{\mathcal{H}} = \left( \begin{pmatrix} \rho_1 \varphi_{k,1}^{(p)} \mathbf{u}_k(A) \\ \vdots \\ \rho_n \varphi_{k,n}^{(p)} \mathbf{u}_k(A) \end{pmatrix}, \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{pmatrix} \right)_{\mathcal{H}} \\ &= \sum_{j=1}^n \rho_j \varphi_{k,j}^{(p)} (\mathbf{u}_k(A), \mathbf{v}_j)_{\mathbf{L}_2(\Omega)} = 0 \quad \forall k \in \mathbb{N}, \quad p = 1, \dots, n. \end{aligned}$$

Then for a fixed  $k \in \mathbb{N}$  we have  $n$  relations

$$\sum_{j=1}^n \rho_j \varphi_{k,j}^{(p)} (\mathbf{u}_k(A), \mathbf{v}_j)_{\mathbf{L}_2(\Omega)} = 0, \quad p = 1, \dots, n.$$

These relations and the fact that  $\{\varphi_k^{(p)} = (\varphi_{k,1}^{(p)}; \dots; \varphi_{k,n}^{(p)})^\top\}_{p=1}^n$  is a linearly independent system, since it is an orthonormal basis in  $\mathbb{C}_{\mathbf{R}}^n$ , can be rewritten as

$$\begin{pmatrix} \varphi_{k,1}^{(1)} & \dots & \varphi_{k,n}^{(1)} \\ \vdots & \ddots & \vdots \\ \varphi_{k,1}^{(n)} & \dots & \varphi_{k,n}^{(n)} \end{pmatrix} \begin{pmatrix} \rho_1 (\mathbf{u}_k(A), \mathbf{v}_1)_{\mathbf{L}_2(\Omega)} \\ \vdots \\ \rho_n (\mathbf{u}_k(A), \mathbf{v}_n)_{\mathbf{L}_2(\Omega)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \det \begin{pmatrix} \varphi_{k,1}^{(1)} & \dots & \varphi_{k,n}^{(1)} \\ \vdots & \ddots & \vdots \\ \varphi_{k,1}^{(n)} & \dots & \varphi_{k,n}^{(n)} \end{pmatrix} \neq 0.$$

This yields that  $(\mathbf{u}_k(A), \mathbf{v}_j)_{\mathbf{L}_2(\Omega)} = 0$  for all  $k \in \mathbb{N}$ ,  $j = 1, \dots, n$ . Hence,  $\mathbf{v}_j = 0$ ,  $j = 1, \dots, n$ , since  $\{\mathbf{u}_k(A)\}_{k \in \mathbb{N}}$  is an orthonormal basis in  $\mathbf{J}_0(\Omega)$  being the system of eigenelements of Stokes operator. Thus,  $\zeta = 0$ . The obtained contradiction completes the proof of the lemma.  $\square$

**Lemma 4.3.** *The solution to Cauchy problem (2.2) is given by the formula*

$$\xi(t) = \mathcal{U}(t)\xi^0 + \int_0^t \mathcal{U}(t-s)\mathcal{F}(s) ds, \quad (4.4)$$

$$\mathcal{U}(t)\xi := \mathcal{U}(t) \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{pmatrix} = \sum_{k \in \mathbb{N}} \sum_{p=1}^n e^{-\lambda_k^{(p)} t} \sum_{l=1}^n \rho_l \varphi_{k,l}^{(p)} (\mathbf{u}_l, \mathbf{u}_k(A))_{\mathbf{L}_2(\Omega)} \begin{pmatrix} \varphi_{k,1}^{(p)} \mathbf{u}_k(A) \\ \vdots \\ \varphi_{k,n}^{(p)} \mathbf{u}_k(A) \end{pmatrix}. \quad (4.5)$$

*Proof.* If  $\mathcal{U}(t)$  is a holomorphic semigroup with the generator  $-\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B})$ , then formula (4.4) gives a solution to the Cauchy problem (2.2) by Crandall – Pazy.

Formula (4.5) is implied by the representation of the semigroup  $\mathcal{U}(t)$  via the spectral family generated by the generator  $-\mathcal{R}^{-1}(\mathcal{M}\mathcal{A} + \mathcal{B})$ . In view of Lemma 4.2 we have

$$\begin{aligned} \mathcal{U}(t)\xi &= \sum_{k \in \mathbb{N}} \sum_{p=1}^n e^{-\lambda_k^{(p)} t} (\xi, \xi_k^{(p)})_{\mathcal{H}_{\mathcal{R}}} \xi_k^{(p)} \\ &= \sum_{k \in \mathbb{N}} \sum_{p=1}^n e^{-\lambda_k^{(p)} t} \left( \begin{pmatrix} \rho_1 \mathbf{u}_1 \\ \vdots \\ \rho_n \mathbf{u}_n \end{pmatrix}, \begin{pmatrix} \varphi_{k,1}^{(p)} \mathbf{u}_k(A) \\ \vdots \\ \varphi_{k,n}^{(p)} \mathbf{u}_k(A) \end{pmatrix} \right)_{\mathcal{H}} \begin{pmatrix} \varphi_{k,1}^{(p)} \mathbf{u}_k(A) \\ \vdots \\ \varphi_{k,n}^{(p)} \mathbf{u}_k(A) \end{pmatrix} \\ &= \sum_{k \in \mathbb{N}} \sum_{p=1}^n e^{-\lambda_k^{(p)} t} \sum_{l=1}^n \rho_l \varphi_{k,l}^{(p)} (\mathbf{u}_l, \mathbf{u}_k(A))_{\mathbf{L}_2(\Omega)} \begin{pmatrix} \varphi_{k,1}^{(p)} \mathbf{u}_k(A) \\ \vdots \\ \varphi_{k,n}^{(p)} \mathbf{u}_k(A) \end{pmatrix}. \end{aligned}$$

The proof is complete.  $\square$

## 5. CONCLUSION

We studied the unique solvability and spectral properties for the initial boundary value problem on small motions of a multicomponent viscous incompressible fluid. The considered equations are certain generalizations of the well-known system of Navier — Stokes equations and they involve higher order derivatives (second order derivatives) of the velocity fields of all components. This is due to the fact that unlike the Navier — Stokes equations, in which the viscosity coefficient is a scalar, in the multicomponent case, due to the composite structure of the viscous stress tensors, the viscosity coefficients form a viscosity matrix, the elements of which are responsible for viscous friction. Diagonal elements are responsible for viscous friction within each component, and the off-diagonal elements are responsible for friction between the components. This does not allow one to extend immediately the known results for the Navier — Stokes equations to the multicomponent case. In the case of a diagonal viscosity matrix, the equations are likely related only via lower order terms. In this paper we considered a more complicated case of a non-diagonal viscosity matrix. The existence and uniqueness of a strong in time solution to the initial boundary value problem were proven with no simplifying assumptions about the structure of the viscosity matrix other than the standard physical requirements of symmetry and positive definiteness. We also proved the discreteness of spectrum in the problem on normal oscillation in the considered system, along with the asymptotic formula for the eigenvalues. The spectrum and system of eigenelements of the problem on normal oscillation were expressed in terms of the Stokes operator.

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