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FINITE TIME BLOW–UP OF SOLUTIONS FOR SYSTEM OF κTH ORDER WAVE EQUATIONS WITH NONLINEAR AVERAGED DAMPING AND SOURCES

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Abstract. We consider basic relationships between the initial energy and the exponent of nonlinear sources in a nonlinear coupled system of κth order wave equations with nonlinear averaged damping, and we demonstrate the global nonexistence of solutions. This approach is a variant of the method of nonlinear functional analysis with a contradiction argument, which is one of the tools for proving the blow–up of solutions for nonlinear partial differential equations. A new class of nonlinear coupled wave equations with nonlinear sources is given in high-order functional spaces.

Keywords: κth order wave equations, nonlinear equations, coupled system, nonlinear averaged damping, local existence, blow–up.

Mathematics Subject Classification: 35A01; 35L20; 35B44

1. Introduction, relevance and objectives

Let $\Omega \subset \mathbb{R}^n$ $(n \geqslant 1)$ be a bounded domain with a smooth boundary Γ. We consider the initial boundary value problem for nonlinear κth order wave equations with a nonlocal damping

$$\begin{cases} u_{tt} + (-\Delta)^\kappa u + \|u_t\|^l u_t = |u|^{p-2}u|v|^p, \\ v_{tt} + (-\Delta)^\kappa v + \|v_t\|^l v_t = |v|^{p-2}v|u|^p, \end{cases} \qquad x \in \Omega, \quad t > 0, \tag{1.1}$$

with boundary conditions

$$\begin{cases} u = \dfrac{\partial u}{\partial \nu} = \ldots = \dfrac{\partial^{\kappa-1} u}{\partial \nu^{\kappa-1}} = 0, \\ v = \dfrac{\partial v}{\partial \nu} = \ldots = \dfrac{\partial^{\kappa-1} v}{\partial \nu^{\kappa-1}} = 0, \end{cases} \qquad x \in \Gamma, \tag{1.2}$$

and initial conditions

$$\begin{cases} u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \\ v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \end{cases} \qquad x \in \Omega. \tag{1.3}$$

Here ν is the outward normal to the boundary Γ, while $l > 0$, $\kappa \geqslant 1$ are integers and $p > 2$, and for $u = u(x,t)$

$$|\nabla^\kappa u|^2 = (\Delta^{\frac{\kappa}{2}} u)^2 \qquad \text{for even} \quad \kappa,$$
$$|\nabla^\kappa u|^2 = |\nabla(\Delta^{\frac{\kappa-1}{2}} u)|^2 \qquad \text{for even} \quad \kappa,$$

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and

$$|\nabla u|^2 = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2, \qquad \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

Recently, an initial boundary value problem for following wave equation with nonlinear averaged damping

$$u_{tt} - \Delta u + g\Big(\int\limits_{\Omega} |u_t|^2\,dx\Big)u_t = f(u) \tag{1.4}$$

attracted a lot of attention. Equation (1.4) with the source $f(w) = w\int\limits_{\Omega}|w|^2\,dx$ arises from a nonlinear theory of meson fields [6], [11], [14]. It can also be intrinsically connected with the study of damping phenomena in flight structures, where the $\|u_t\|^m$ reflects the effect of kinetic energy; see [2], [7], [19]. For instance, in [3], the one dimensional beam equation

$$u_{tt} - 2\varsigma\sqrt{\lambda}u_{xx} + \lambda u_{xxxx} - \gamma\Bigg(\int\limits_{L}^{-L}(\lambda|u_{xx}|^2 + |u_t|^2\,dx\Bigg)^q u_{xxt} = 0 \tag{1.5}$$

was proposed. As the first stage of the corresponding equation

$$u_{tt} + \Big(\int\limits_{\Omega} u_t^2\,dx\Big)^{\frac{\alpha}{2}} u_t - \Delta u = h(t,x),$$

the pioneering works [1] provided the above equation in a bounded domain Ω with homogeneous boundary conditions; see [1], [5], [17]. Later, the authors in [11] studied the system

$$\begin{aligned} &u_{tt} - \mu(t)\Delta u + \alpha f\Big(\int\limits_{\Omega}|u|^2\,dx\Big)u + \beta g\Big(\int\limits_{\Omega}|u_t|^2\,dx\Big)u_t = 0,\\ &u = 0 \quad \text{on} \quad \Gamma_0, \qquad \frac{\partial u}{\partial \nu} + h(\,\cdot\,,u_t) = 0 \quad \text{on} \quad \Gamma_1,\\ &u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \end{aligned} \tag{1.6}$$

under certain conditions on $\mu(t)$, $f(s)$, $g(s)$, $h(s)$. Global existence in time was shown by using the well–known Galerkin method. In addition, a decay rate of the solution was obtained by two different methods, both Lyapunov functional and Nakao's lemma. Zhang [9] extended and improved the result in [11]. In [22], [23], the long time behavior of the equations

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|^2)\Delta u + k(\|u_t\|)u_t + f(u) = h(x) \tag{1.7}$$

$$u_{tt} - \Delta u + k(\|u_t\|)u_t + f(u) = \int\limits_{\Omega} K(x,y)u_t(y)dy + h(x) \tag{1.8}$$

was considered. The global well–posedness of (1.7) (1.8) was established with the damping $\|u_t\|^m u_t$, $m \geqslant 0$, see [10], [13], [20] and references therein. In [7], the authors investigated the nonhomogeneous n–dimensional version of (1.5)

$$u_{tt} - \kappa\Delta u + \Delta^2 u - \gamma\Big(\int\limits_{\Omega}(|\Delta u|^2 + |u_t|^2)\,dx\Big)^q \Delta u_t + f(u) = 0 \quad \text{in} \quad \Omega \times R^+ \tag{1.9}$$

subject to a boundary condition and with $q \geqslant 1$. The authors proved the existence of a unique global solution and obtained the polynomial stability and a non-exponential decay prospect of the solutions provided that $q \geqslant 1$, see [4], [15], [18].

For the equation

$$u_{tt} - \Delta u + \Big(\int\limits_{\Omega} |u_t|^2\, dx\Big)^{\frac{m}{2}} u_t = \Big(\int\limits_{\Omega} |u|^p\, dx\Big)^{r} |u|^{p-2}u \tag{1.10}$$

with homogeneous boundary conditions, Hu, Li, Liu and Zhang [8] provided sufficient conditions for finite time blow–up of weak solutions, with suitable conditions on initial data, by an ordinary differential inequality for an appropriately chosen functional. When the terms $\big(\int\limits_{\Omega} |u|^p\, dx\big)^r |u|^{p-2}u$ and $-\Delta u$ in equation (1.10) are replaced by $|u|^{p-2}u$ and $\Delta^2 u$, respectively, Zennir and Miyasita [21] proved the global non-existence in time for a large class of pseudo-parabolic equations with weak viscoelasticity under suitable conditions on the variable exponents with negative initial energy.

To our knowledge, there are few results on the non-linear coupled system of κth order wave equations with nonlinear averaged damping and sources. The main goal of this paper is to study the blow–up conditions of a solution to problem (1.1)–(1.3) with the averaged damping term and the source term, which differ from the existing literature mentioned above. To the best of our knowledge, this is the first result on the nonlinear coupled system of κth order wave equations with nonlinear averaged damping and sources.

The paper is organized as follows. Section 2 is concerned with some notation and necessary materials. In Section 3, we demonstrate the global nonexistence of solutions with positive or negative initial energy.

2. Preliminaries

Let $L^2(\Omega), H^p(\Omega)$ and $H_0^2(\Omega)$ be the usual Sobolev spaces, where

$$(u,v) = \int\limits_{\Omega} uv\, dx, \qquad \|u\| = \Big(\int\limits_{\Omega} |u|^2\, dx\Big)^{\frac{1}{2}}, \qquad \|u\|_p = \Big(\int\limits_{\Omega} |u|^p\, dx\Big)^{\frac{1}{p}}.$$

Definition 2.1. *Functions*

$$(u,v) \in C\Big([0,T]; H^{\kappa}(\Omega)\Big) \cap C^1\Big([0,T]; L^2(\Omega)\Big) \times C\Big([0,T]; H^{\kappa}(\Omega)\Big) \cap C^1\Big([0,T]; L^2(\Omega)\Big),$$

with the initial data given in (1.3) *are*

1. *a strong solution to problem* (1.1)–(1.3) *on the interval* $[0,T]$ *if*
$$(u,v) \in W^{1,1}(\alpha,\beta; H_0^{\kappa}(\Omega)) \times W^{1,1}(\alpha,\beta; H_0^{\kappa}(\Omega)),$$
$$(u_t,v_t) \in W^{1,1}(\alpha,\beta; L^2(\Omega)) \times W^{1,1}(\alpha,\beta; L^2(\Omega)),$$
for all $0 < \alpha < \beta < T$ *and*
$$\Delta^{\kappa} u + \Delta^{\kappa} v + \|u_t(t)\|^l u_t + \|v_t(t)\|^l v_t \in L^2(\Omega) \quad \textit{for a.e.} \quad t \in [0,T],$$
and (1.1) *is satisfied in* $L^2(\Omega)$ *for all* $t \in [0,T]$;
2. *a generalized solution to* (1.1)-(1.3) *on the interval* $[0,T]$ *if there is a sequence of strong solutions* $(\{u_j(t)\},\{v_j(t)\})$ *to* (1.1)-(1.3) *with initial data* $(u_0^j, v_0^j), (u_1^j, v_1^j)$ *instead of* $(u_0,v_0),(u_1,v_1)$ *such that*
$$(u^j,v^j) \to (u,v) \quad \textit{in} \quad C([0,T]; H_0^{\kappa}(\Omega)) \times C([0,T]; H_0^{\kappa}(\Omega)) \quad \textit{as} \quad j \to +\infty,$$
$$(u_t^j,v_t^j) \to (u_t,v_t) \quad \textit{in} \quad C([0,T]; L^2(\Omega)) \times C([0,T]; L^2(\Omega)) \quad \textit{as} \quad j \to +\infty.$$

3. *a distributed solution to* (1.1)–(1.3) *on* $[0,T]$ *if*

$$\begin{aligned}&\int_\Omega \big(u_t(x,t)\phi(x)+v_t(x,t)\psi(x)\big)\,dx\\ &\quad+\int_0^t\left(\int_\Omega \nabla^\kappa u(x,s)\nabla^\kappa\phi(x)\,dx+\|u_t(s)\|^l\int_\Omega u_t(x,s)\phi(x)\,dx\right)ds\\ &\quad+\int_0^t\left(\int_\Omega \nabla^\kappa v(x,s)\nabla^\kappa\psi(x)\,dx+\|v_t(s)\|^l\int_\Omega v_t(x,s)\psi(x)\,dx\right)ds\\ &=\int_\Omega u_1\phi(x)\,dx+\int_0^t\int_\Omega |u|^{p-2}u(x,s)|v(x,s)|^p\phi(x)\,dxds\\ &\quad+\int_\Omega v_1\psi(x)\,dx+\int_0^t\int_\Omega |v|^{p-2}v(x,s)|u(x,s)|^p\psi(x)\,dxds\end{aligned}\tag{2.1}$$

holds for every test function $\phi,\psi\in H_0^\kappa(\Omega)\cap L^2(\Omega)$ *and for all* $t\in[0,T]$.

Lemma 2.1. [22] *For* $u_t, w_t\in L^2(\Omega)$, *we have*

$$\|u_t-w_t\|^{l+2}\leqslant\left(\|u_t\|^l u_t-\|w_t\|^l w_t, u_t-w_t\right).\tag{2.2}$$

The associated energy of (1.1)-(1.3) is defined by

$$E(t)=\frac{1}{2}\|u_t\|^2+\frac{1}{2}\|v_t\|^2+J(u,v),\quad 0\leqslant t,\tag{2.3}$$

$$J(u,v)=\frac{1}{2}\|\nabla^\kappa u\|^2+\frac{1}{2}\|\nabla^\kappa v\|^2-\frac{1}{p}\|uv\|_p^p,\tag{2.4}$$

and

$$E|_{t=0}=\frac{1}{2}\|u_1\|^2+\frac{1}{2}\|v_1\|^2+J(u_0,v_0),\tag{2.5}$$

$$\|u\|_p\leqslant C_0\|\nabla^\kappa u\|,\ C_0>0,\ \forall u\in H_0^\kappa(\Omega),$$

where

$$\begin{cases}2\leqslant p<+\infty & \text{if}\quad n=\kappa,2\kappa\\ 2\leqslant p\leqslant\dfrac{2n}{n-2\kappa} & \text{if}\quad n\geqslant 3\kappa.\end{cases}\tag{2.6}$$

3. Finite time blow up of solutions

We state the local existence in time of (1.1)–(1.3) by following the lines of [4].

Theorem 3.1. *If* $(u_0,v_0)\in(H^{2\kappa}(\Omega)\cap H_0^\kappa(\Omega))\times(H^{2\kappa}(\Omega)\cap H_0^\kappa(\Omega))$ *and* $(u_1,v_1)\in H_0^\kappa(\Omega)\times H_0^\kappa(\Omega)$ *are such that*

$$\nabla^\kappa u_0+\nabla^\kappa v_0+k\|u_1\|^l u_1+k\|v_1\|^l v_1\in L^2(\Omega).$$

Suppose that $l>0$ *and* p *satisfies*

$$\begin{cases}1\leqslant p<+\infty & \textit{if}\quad n=\kappa,2\kappa\\ 1\leqslant p\leqslant\dfrac{4\kappa-n}{n-2\kappa} & \textit{if}\quad n\geqslant 3\kappa.\end{cases}\tag{3.1}$$

Then there exists $t_{\max} \leqslant +\infty$ such that the problem (1.1)–(1.3) has a unique strong solution (u, v) on $[0, t_{\max})$.

If $(u_0, v_0) \in H_0^\kappa(\Omega) \times H_0^\kappa(\Omega)$ and $(u_1, v_1) \in L^2(\Omega) \times L^2(\Omega)$, then there exists $t_{\max} \leqslant +\infty$ such that (1.1)–(1.3) admits a unique generalized solution, which is also the distributional solution.

With positive and negative initial data, both the strong solution and the weak solution (u, v) cannot be global solutions in $[0, t_{max})$, $t_{\max} = +\infty$, so for $t_{\max} < \infty$ we have

$$\lim_{t \to t_{\max}} (\|\nabla^\kappa u\| + \|\nabla^\kappa v\| + \|u_t\| + \|v_t\|) = +\infty, \tag{3.2}$$

The energy relation is satisfied

$$E(t) + \int_0^t (\|u_t(s)\|^{l+2} + \|v_t(s)\|^{l+2}) ds = E(0), \quad 0 \leqslant t < t_{\max}. \tag{3.3}$$

3.1. Negative initial energy.

Our first main result is as follows.

Theorem 3.2. *Let $E(0) < 0$. Under the assumptions of Theorem 3.1 there exists a finite time $t_{\max} < +\infty$, at which the local solution to (1.1)–(1.3) blows up.*

Proof. Let

$$\mathcal{K}(t) = -E(t).$$

The contradiction argument will be used. By (2.4) and (3.3) we obtain

$$\begin{aligned} &\mathcal{K}'(t) = -E'(t) \geqslant 0, \\ &0 < -E(0) = \mathcal{K}(t = 0) \leqslant \mathcal{K}(t) \leqslant \frac{1}{p}\|uv\|_p^p. \end{aligned} \tag{3.4}$$

We denote

$$G(t) = \int_\Omega (\|u_t\|^l u_t u + \|v_t\|^l v_t v) \, dx,$$

then, by Hölder and ϖ-Young's inequalities for each $\varpi > 0$ we obtain

$$\begin{aligned} &\|u_t\|^2 \, \|u\|^2 \leqslant \varepsilon \|u_t\|^2 + \frac{1}{4\varepsilon}\|u\|^2, \\ &|G(t)| \leqslant \|u_t\|^l \|u_t\| \|u\| + \|v_t\|^l \|v_t\| \|v\| \\ &\qquad \leqslant \|u_t\|^l \left(\varpi \|u_t\|^2 + \frac{1}{4\varpi}\|u\|^2\right) + \|v_t\|^l \left(\varpi \|v_t\|^2 + \frac{1}{4\varpi}\|v\|^2\right) \\ &\qquad \leqslant \varpi [\|u_t\|^{l+2} + \|v_t\|^{l+2}] + \frac{1}{4\varpi}[\|u_t\|^l \|u\|^2 + \|v_t\|^l \|v\|^2] \end{aligned}$$

Again, using Young's inequality with

$$p = \frac{l+2}{l}, \qquad q = \frac{l+2}{2}, \qquad \frac{1}{p} + \frac{1}{q} = 1,$$

for each $\varpi > 0$ we obtain,

$$\|u_t\|^l \|u\|^2 \leqslant \varpi \|u_t\|^{lp} + \varpi^{-\frac{2}{l}} \|u\|^{2q} = \varpi \|u_t\|^{l+2} + \varpi^{-\frac{2}{l}} \|u\|^{l+2},$$

and similarly,

$$\|v_t\|^l \|v\|^2 \leqslant \varpi \|v_t\|^{l+2} + \varpi^{-\frac{2}{l}} \|v\|^{l+2}.$$

Hence,

$$\|u_t\|^l \|u\|^2 + \|v_t\|^l \|v\|^2 \leqslant \varpi(\|u_t\|^{l+2} + \|v_t\|^{l+2}) + \varpi^{-\frac{2}{l}}(\|u\|^{l+2} + \|v\|^{l+2}),$$

which leads by (3.3) to

$$\begin{aligned}|G(t)| \leqslant&(\varpi+\frac{1}{4})[\|u_t\|^{l+2}+\|v_t\|^{l+2}]+\frac{\varpi^{-\frac{2+l}{l}}}{4}(\|u\|^{l+2}+\|v\|^{l+2})\\ \leqslant&(\varpi+\frac{1}{4})\mathcal{K}'(t)+\frac{\varpi^{-\frac{2+l}{l}}}{4}(\|u\|^{l+2}+\|v\|^{l+2}).\end{aligned} \tag{3.5}$$

We define the functional

$$z(t)=\mathcal{K}^{1-\nu}(t)+\epsilon\int_\Omega u_t u\,dx+v_t v\,dx, \qquad \epsilon>0, \qquad 0<\nu<1. \tag{3.6}$$

Taking the derivative of (3.6) in t, by (1.1), and (3.5), we have

$$\begin{aligned}z'(t)=&(1-\nu)\mathcal{K}^{-\nu}(t)\mathcal{K}'(t)+\epsilon\int_\Omega(u_{tt}u+v_{tt}v)\,dx+\epsilon(\|u_t\|^2+\|v_t\|^2)\\ =&(1-\nu)\mathcal{K}^{-\nu}(t)\mathcal{K}'(t)+\epsilon\Big(\|u_t\|^2+\|v_t\|^2-\|\nabla^\kappa u\|^2-\|\nabla^\kappa v\|^2+\|uv\|_p^p-G(t)\Big)\\ \geqslant&\Big((1-\nu)\mathcal{K}^{-\nu}(t)-\frac{1}{4}\epsilon\varpi\Big)\mathcal{K}'(t)+\epsilon\big(\|u_t\|^2+\|v_t\|^2-\|\nabla^\kappa u\|^2-\|\nabla^\kappa v\|^2+\|uv\|_p^p\big)\\ &-\frac{\varpi^{-\frac{2+l}{l}}}{4}\big(\|u\|^{l+2}+\|v\|^{l+2}\big).\end{aligned} \tag{3.7}$$

We can choose the Young parameter ϖ as a time–dependent weight of $\mathcal{K}(t)$, namely

$$\varpi=k\,\mathcal{K}^{-\nu}(t), \qquad k>0.$$

Moreover,

$$\varpi^{-\frac{l+2}{l}}=k^{-\frac{l+2}{l}}\,\mathcal{K}^{\frac{\nu(l+2)}{l}}(t),$$

where the constant k is redefined if necessary.

Substituting these estimates into (3.7), we obtain

$$\begin{aligned}z'(t)\geqslant&\Big(1-\nu-\epsilon\frac{k}{4}\Big)\mathcal{K}^{-\nu}(t)\mathcal{K}'(t)\\ &+\epsilon\Big(\|u_t\|^2+\|v_t\|^2-\|\nabla^\kappa u\|^2-\|\nabla^\kappa v\|^2+\|uv\|_p^p-\frac{1}{4k^{\frac{l+2}{l}}}\mathcal{K}^{\frac{\nu(l+2)}{l}}(t)\big(\|u\|^{l+2}+\|v\|^{l+2}\big)\Big),\end{aligned}$$

We take

$$0\leqslant\alpha<\frac{p-2}{p}. \tag{3.8}$$

Then,

$$\begin{aligned}z'(t)\geqslant&\Big(1-\nu-\epsilon\frac{k}{4}\Big)\mathcal{K}^{-\nu}(t)\mathcal{K}'(t)+\epsilon p(1-\alpha)\mathcal{K}(t)+\epsilon\Big(\frac{p(1-\alpha)}{2}+1\Big)(\|u_t\|^2+\|v_t\|^2)\\ &+\epsilon\left(\frac{p(1-\alpha)}{2}-1\right)\left(\|\nabla^\kappa u\|^2+\|\nabla^\kappa v\|^2\right)+\epsilon p\|uv\|_p^p\\ &-\epsilon\frac{1}{4k^{\frac{l+2}{l}}}\mathcal{K}^{\frac{\nu(l+2)}{l}}(t)\big(\|u\|^{l+2}+\|v\|^{l+2}\big).\end{aligned} \tag{3.9}$$

As $l>0$, choosing ν so that

$$0<\nu<\frac{l}{l+2},$$

and using the upper bound in (3.4), we obtain

$$\mathcal{K}^{\frac{\nu(l+2)}{l}}(t)\leqslant\mathcal{K}(t)\leqslant\frac{1}{p}\|uv\|_p^p,$$

and hence

$$\epsilon\frac{1}{4k^{\frac{l+2}{l}}}\mathcal{K}^{\frac{\nu(l+2)}{l}}(t)\big(\|u\|^{l+2}+\|v\|^{l+2}\big)\leqslant\epsilon\frac{1}{4k^{\frac{l+2}{l}}p}\|uv\|_p^p\big(\|u\|^{l+2}+\|v\|^{l+2}\big).$$

Since

$$\|u\|^{l+2}+\|v\|^{l+2}\geqslant 0,$$

it follows immediately that

$$\epsilon\frac{1}{4k^{\frac{l+2}{l}}}\mathcal{K}^{\frac{\nu(l+2)}{l}}(t)\big(\|u\|^{l+2}+\|v\|^{l+2}\big)\ \leqslant\ \epsilon\frac{1}{4k^{\frac{l+2}{l}}p}\|uv\|_p^p.$$

Then

$$\begin{aligned} z'(t)\geqslant&\left(1-\nu-\epsilon\frac{k}{4}\right)\mathcal{K}^{-\nu}(t)\mathcal{K}'(t)+\epsilon p(1-\alpha)\mathcal{K}(t)+\epsilon\left(\frac{p(1-\alpha)}{2}+1\right)(\|u_t\|^2+\|v_t\|^2)\\ &+\epsilon\left(\frac{p(1-\alpha)}{2}-1\right)(\|\nabla^\kappa u\|^2+\|\nabla^\kappa v\|^2)+\epsilon\left(p-\epsilon\frac{1}{4k^{\frac{l+2}{l}}p}\right)\|uv\|_p^p. \end{aligned}\tag{3.10}$$

Choosing α and k such that

$$p(1-\alpha)>2,$$
$$k>\left(\frac{\epsilon}{4p^2}\right)^{\frac{l}{l+2}},\tag{3.11}$$

and then ϵ such that

$$1-\nu-\epsilon\frac{k}{4}\geqslant 0,$$

and we have

$$z(t)=\mathcal{K}^{1-\nu}(0)+\epsilon\int_\Omega[u_1u_0+v_1v_0]\,dx>0.$$

We can pick

$$\nu\in\left(0,\frac{l}{l+2}\right),\quad \alpha\in\left[0,\frac{p-2}{p}\right),\quad k\in\left(\left(\tfrac{\epsilon}{4p^2}\right)^{\frac{l}{l+2}},\frac{4(1-\nu)}{\epsilon}\right]$$

so that we can choose parameters satisfying all conditions simultaneously. Then (3.10) becomes

$$z'(t)\geqslant\epsilon K\left(\mathcal{K}(t)+\|u_t\|^2+\|v_t\|^2+\|uv\|_p^p\right)\geqslant 0,\qquad K>0.\tag{3.12}$$

The functional $z(t)$ is increasing on $(0,t_{\max})$ with

$$z(t)\geqslant z(0)>0\quad\text{for all}\quad t\geqslant 0.$$

We estimate $z^{\frac{1}{1-\nu}}(t)$ by using the elementary inequality

$$|\alpha+\beta|^s\leqslant 2^{s-1}(|\alpha|^s+|\beta|^s),\quad s\geqslant 1,$$

and the Young's inequality

$$\begin{aligned} z^{\frac{1}{1-\nu}}(t)&\leqslant 2^{\frac{1}{1-\nu}}\left(\mathcal{K}(t)+\epsilon\|u\|^{\frac{1}{1-\nu}}\|u_t\|^{\frac{1}{1-\nu}}+\epsilon\|v\|^{\frac{1}{1-\nu}}\|v_t\|^{\frac{1}{1-\nu}}\right)\\ &\leqslant K\left(\mathcal{K}(t)+\|u_t\|^2+\|v_t\|^2+\|u\|^{\frac{2}{1-2\nu}}+\|v\|^{\frac{2}{1-2\nu}}\right). \end{aligned}\tag{3.13}$$

Since $\nu<\frac{1}{2}-\frac{1}{p}$, using the inequality

$$\xi^\tau\leqslant\left(1+\frac{1}{s}\right)(s+\xi),\quad \xi\geqslant 0,\quad 0\leqslant\tau\leqslant 1,\quad s>0,$$

and taking

$$\xi=\|u\|_2^p,\qquad \tau=\frac{2}{(1-2\nu)p}<1,$$

and $s=\mathcal{K}(0)$. We get, for some constants $K_1, K_2>0$,

$$\begin{aligned}\|u\|^{\frac{2}{1-2\nu}}+\|v\|^{\frac{2}{1-2\nu}} &\leqslant \left(1+\frac{1}{\mathcal{K}(t=0)}\right)(\mathcal{K}(0)+\|u\|_2^p)+\left(1+\frac{1}{\mathcal{K}(t=0)}\right)(\mathcal{K}(0)+\|v\|_2^p)\\ &\leqslant K_1\left(\mathcal{K}(t)+\|u\|_p^p+\|v\|_p^p\right)\leqslant K_2\left(\mathcal{K}(t)+\sigma\|uv\|_p^p+C(\sigma)\right),\quad \sigma>0.\end{aligned} \tag{3.14}$$

Substituting (3.14) into (3.13), we find

$$z^{\frac{1}{1-\nu}}(t)\leqslant K_3\left(\mathcal{K}(t)+\|u_t\|^2+\|v_t\|^2+\|uv\|_p^p\right),\quad K_3>0. \tag{3.15}$$

By (3.12), (3.15), we get

$$z'(t)\geqslant K_4 z^{\frac{1}{1-\nu}}(t), K_4>0. \tag{3.16}$$

A simple integration of (3.16) over $(0,t)$ then yields

$$z(t)\geqslant\left(z^{\frac{-\nu}{1-\nu}}(t=0)-K_4\frac{\nu t}{1-\nu}\right)^{\frac{\nu-1}{\nu}}.$$

Thus, there exists t_2 such that

$$\begin{aligned}&t_2\leqslant t_*=\frac{z^{\frac{-\nu}{1-\nu}}(t=0)(1-\nu)}{K_4\nu},\\ &\lim_{t\to t_2^-} z(t)=+\infty.\end{aligned} \tag{3.17}$$

The proof is complete. □

3.2. Positive initial energy.

Let

$$\begin{aligned}&0\leqslant E(t=0)\leqslant E_1,\\ &E_1=\left(\frac{1}{2}-\frac{1}{p}\right)C_0^{-\frac{2p}{p-2}},\end{aligned} \tag{3.18}$$

and C_0 is the best Poincaré constant.

Lemma 3.1. [16] *If*

$$0\leqslant E(t=0)<E_1,\qquad \|\nabla^\kappa u_0\|+\|\nabla^\kappa v_0\|>\lambda_1,$$

then there exists $\lambda_2>\lambda_1$ such that

$$\|\nabla^\kappa u\|+\|\nabla^\kappa v\|\geqslant\lambda_2,\qquad \lambda_1=C_0^{-\frac{p}{p-2}}.$$

Theorem 3.3. *Under the assumptions of Theorem 3.1, for*

$$0\leqslant E(0)<E_1,\qquad \|\nabla^\kappa u_0\|+\|\nabla^\kappa v_0\|>\lambda_1,$$

the solution to problem (1.1)–(1.3) blows up, that is, $t_{\max}<+\infty$.

Proof. We set

$$\mathcal{K}(t)=E_2-E(t),\qquad E_2=\frac{E(t=0)+E_1}{2}.$$

Proceeding as in (3.9) with $\alpha=0$, we obtain

$$\begin{aligned}z'(t)\geqslant&\left(1-\nu-\epsilon\frac{k}{4}\right)\mathcal{K}^{-\nu}(t)\mathcal{K}'(t)+\epsilon p\mathcal{K}(t)+\epsilon\left(\frac{p}{2}+1\right)(\|u_t\|^2+\|v_t\|^2)\\ &+\epsilon\left(\frac{p}{2}-1\right)(\|\nabla^\kappa u\|^2+\|\nabla^\kappa v\|^2)+\epsilon p\|uv\|_p^p\\ &-\frac{\epsilon}{4k^{\frac{l+2}{l}}}\mathcal{K}^{\frac{\nu(l+2)}{l}}(t)(\|u\|^{l+2}+\|v\|^{l+2})-\epsilon pE_2.\end{aligned} \tag{3.19}$$

Moreover,

$$\begin{aligned}\epsilon\left(\frac{p}{2}-1\right)(\|\nabla^\kappa u\|^2+\|\nabla^\kappa v\|^2)-\epsilon pE_2 =&\epsilon\left(\left(\frac{p}{2}-1\right)(\|\nabla^\kappa u\|^2+\|\nabla^\kappa v\|^2)-pE_2\right)\\ =&\epsilon\Big(\left(\frac{p}{2}-1\right)\frac{\lambda_2^2-\lambda_1^2}{\lambda_2^2}(\|\nabla^\kappa u\|^2+\|\nabla^\kappa v\|^2)\\ &+\left(\frac{p}{2}-1\right)\lambda_1^2\frac{\|\nabla^\kappa u\|^2+\|\nabla^\kappa v\|^2}{\lambda_2^2}-pE_2\Big)\\ \geqslant&\epsilon\left(K_5(\|\nabla^\kappa u\|^2+\|\nabla^\kappa v\|^2)+K_6\right),\end{aligned}\tag{3.20}$$

where λ_2 is given in Lemma 3.1,

$$K_5=\left(\frac{p}{2}-1\right)\frac{\lambda_2^2-\lambda_1^2}{\lambda_2^2}>0,\qquad K_6=\left(\frac{p}{2}-1\right)\lambda_1^2-pE_2.$$

By (3.18) and Lemma 3.1, we observe that

$$K_6=\left(\frac{p}{2}-1\right)\lambda_1^2-\frac{p\,(E_1+E(t=0))}{2}=\frac{p\,(E_1-E(t=0))}{2}>0.\tag{3.21}$$

Thus, by (3.20), (3.21) and (3.19), we arrive at

$$\begin{aligned}z'(t)\geqslant&\left(1-\nu-\epsilon\frac{k}{4}\right)\mathcal{K}^{-\nu}(t)\mathcal{K}'(t)+\epsilon p\mathcal{K}(t)+\epsilon\left(\frac{p}{2}+1\right)(\|u_t\|^2+\|v_t\|^2)\\ &+\epsilon K_5(\|\nabla^\kappa u\|^2+\|\nabla^\kappa v\|^2)+\epsilon p\|uv\|_p^p-\epsilon\frac{1}{4k^{\frac{l+2}{l}}}\mathcal{K}^{\frac{\nu(l+2)}{l}}(t)(\|u\|^{l+2}+\|v\|^{l+2}).\end{aligned}$$

Similarly to (3.14) we have

$$\begin{aligned}z'(t)\geqslant&\left(1-\nu-\epsilon\frac{k}{4}\right)\mathcal{K}^{-\nu}(t)\mathcal{K}'(t)+\epsilon p\mathcal{K}(t)+\epsilon\left(\frac{p}{2}+1\right)(\|u_t\|^2+\|v_t\|^2)\\ &+\epsilon K_5(\|\nabla^\kappa u\|^2+\|\nabla^\kappa v\|^2)+\epsilon\left(p-\epsilon\frac{\sigma}{4k^{\frac{l+2}{l}}p}\right)\|uv\|_p^p+C(\sigma).\end{aligned}$$

Choosing k as in (3.11), we can complete the proof as in Theorem 3.2. □

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