ISSN 2304-0122 Ufa Mathematical Journal. Vol. 18. No 2 (2026). P. 12-20.

doi:10.13108/2026-18-2-12

IDENTIFICATION OF FORMAL POWER–LOGARITHMIC EXPANSIONS FOR SOLUTIONS TO q–DIFFERENCE EQUATIONS

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Abstract. We consider an algebraic q–difference equation. We propose a sufficient condition for the existence of a formal power–logarithmic expansion in the vicinity of zero of the solution to such an equation. We apply this sufficient condition to construct the formal expansion of a solution to a certain q–difference analogue of the fifth Painlevé equation for particular values of the parameters in the equations. We consider two different values of q, which lead to qualitatively different formal asymptotic expansions for the solutions.

Keywords: asymptotic expansions, q–difference equation, Newton polygon, power–logarithmic expansion.

Mathematics Subject Classification: 39B32, 34E05

1. Introduction

Nowadays the theory of q–difference equations and systems is being actively developed [8], [3]. Methods for finding solutions in the form of formal power expansions are well–mastered [6], there are proofs of their convergence [7]. However, even linear q–difference equations can have solutions with logarithmic terms [5]. In our previous paper [4] we obtained a sufficient condition for the existence of a solution with a power–logarithmic expansion in non–negative integer powers of the independent variable (Dulac series). Developing these results, in the present paper we extend the methods and results of power geometry [1], [2] to the case of q–difference equations, formulate sufficient conditions for the existence of formal solutions to an algebraic q–difference equation. We formulate sufficient conditions ensuring the existence of formal solutions with power–logarithmic series of a more general form than in the previous paper, and we present a method for constructing these solutions. We also provide an example of applying the obtained theorem to construction of a formal expansion for a solution to a certain q–difference analogue of the fifth Painlevé equation for particular values of the parameters. We consider two different values of the number q, which lead to qualitatively different formal asymptotic expansions for the solutions to the analogue of the fifth Painlevé equation.

N.V. Gaianov, A.V. Parusnikova, Identification of formal power–logarithmic expansions for solutions to q–difference equations.

The paper was prepared in the framework of the Basic Research Program at HSE University (HSE-BR-2025-079).

Submitted March 5, 2025.

2. Main definitions

In this work we consider the case $x \to 0$.

Suppose that y is a univalent complex–valued function of a complex variable x, and $q \neq 0$. We define the *q–difference differentiation operator* σ by the formula

$$(\sigma y)(x) = y(qx),$$

the function σy is called *q–difference derivative of function y*. We also suppose that $q^k \neq 1$ for all $k \in \mathbb{N}$. An *algebraic q–difference equation of the n-th order* is the equation

$$f(x, y, \sigma y, \dots, \sigma^n y) = 0, \tag{2.1}$$

where f is a polynomial of $n+2$ variables.

We borrow the known definitions in the power geometry for differential equations from [2] to construct a similar theory for q–difference equations. We denote $X = (x, y)$. A *q–difference monomial* $b(x, y)$ is the product of a monomial $cx^{r_1}y^{r_2}$, where $c = \text{const}$, $r_1, r_2 \in \mathbb{R}$, and of finitely many q–difference derivatives $\sigma^l y$, $l \in \mathbb{N}$. A *q–difference sum* is the sum of q–difference monomials

$$f(X) = \sum a_i(X). \tag{2.2}$$

To each q–difference monomial $b(X)$, we assign a vector exponent $Q(b)$ as follows:

$$Q(cx^{r_1}y^{r_2}) = (r_1, r_2); \qquad Q(\sigma^l y) = (0, 1); \qquad Q(b_1(X)b_2(X)) = Q(b_1(X)) + Q(b_2(X)).$$

We denote by $S(f)$ the set of vector exponents q of the difference sum $f(X)$, and by $f_Q(X)$ we denote the sum of all monomials b_i for which $Q(b_i) = Q$. Then

$$f(X) = \sum_{Q \in S(f)} f_Q(X).$$

The set $S(f)$ is called the *support* of sum $f(X)$ and of equation (2.1). We observe that the support of the algebraic q–difference equation (2.1) lies in $\mathbb{Z}^2_+$.

The *Newton polygon* of equation (2.1) (as well as of q–difference sum (2.2)) is the convex hull of the set $S(f)$, which is denoted by $\Gamma(f)$. The boundary $\partial\Gamma(f)$ of the polygon $\Gamma(f)$ consists of the vertices $\Gamma^{(0)}_j$ and edges $\Gamma^{(1)}_j$, which are (generalized) faces $\Gamma^{(d)}_j$, where the superscript d indicates the dimension of the face. Each generalized face has a corresponding boundary subset

$$S^{(d)}_j = S(f) \cap \Gamma^{(d)}_j,$$

the *truncated sum*

$$\hat{f}^{(d)}_j(X) = \sum_{Q \in S^{(d)}_j} f_Q(X)$$

and the *truncated equation*

$$f^{(d)}_j(X) = 0. \tag{2.3}$$

3. Relations between solutions to original and truncated equations

If equation (2.1) possesses a solution of the form

$$y = cx^r + O(x^{r+\varepsilon}), \quad r \in \mathbb{R}, \quad x \to 0, \quad \varepsilon > 0, \quad c \in \mathbb{C} \setminus \{0\}, \tag{3.1}$$

then its *truncated solution* is

$$y = cx^r, \tag{3.2}$$

and the *normal cone of solution* (3.1) is the ray $\lambda(-1, -r)$, where $\lambda > 0$.

The *normal cone* of the face $\Gamma_j^{(d)}$ is the set

$$U_j^{(d)} = \left\{P:\ \langle P,Q\rangle = \langle P,Q'\rangle,\ Q,Q' \in S_j^{(d)},\ \langle P,Q\rangle > \langle P,Q''\rangle,\ Q'' \in S(f)\setminus S_j^{(d)}\right\},$$

where $\langle P,Q\rangle := p_1q_1 + p_2q_2$ is the scalar product of vectors $P=(p_1,p_2)$, $Q=(q_1,q_2)$.

Theorem 3.1. *If equation* (2.1) *has a solution* (3.1) *and the normal cone U of solution* (3.1) *is such that $U \subset U_j^{(d)}$, then truncated solution* (3.2) *is a solution to corresponding truncated equation* (2.3).

Proof. We substitute solution (3.1) into equation (2.1)

$$f(X) = \sum_{Q\in S(f)} f_Q(x, cx^r + O(x^{r+\varepsilon})) = \sum_{Q\in S_j^{(d)}} f_Q(x, cx^r) + O(x^{r'}) + O(x^{r''}) = 0,$$

where

$$r' = \min_{Q\in S_j^{(d)}} \langle Q, (1, r+\varepsilon)\rangle, \qquad r'' = \min_{Q\in S(f)\setminus S_j^{(d)}} \langle Q, (1,r)\rangle.$$

Since $U \subset U_j^{(d)}$, we find $r'' > \langle Q,(1,r)\rangle$ for all $Q \in S_j^{(d)}$. We obtain

$$f(X) = \sum_{Q\in S_j^{(d)}} f_Q(x, cx^r)(1+o(1)), \quad x \to 0,$$

and to ensure the identity, we need

$$\sum_{Q\in S_j^{(d)}} f_Q(x, cx^r) = 0,$$

what is exactly truncated equation (2.3). The proof is complete. □

4. Solutions to truncated equations

We consider the truncated equation $\hat f^{(0)} = 0$ corresponding to the vertex $\Gamma^{(0)} = (q_1, q_2)$ of the Newton polygon. Substituting here $y = cx^r$ and canceling the powers of x and c, we obtain the equation

$$c^{-q_2}x^{-q_1-q_2r}\hat f^{(0)}(x, cx^r) = \chi(r) = 0,$$

which depends only on r and, generally speaking, on q. The polynomial $\chi(r)$ is called the *characteristic polynomial of the q–difference sum* $\hat f^{(0)}(X)$. Among its roots, we need to select ones, for which the vector $(-1,-r)$ lies in the normal cone $U^{(0)}$.

Consider the truncated equation corresponding to the edge $\Gamma^{(1)}$ lying on the line $q_1 + rq_2 + c = 0$. In ensure that the solution $y = cx^r$ solves the truncated equation $\hat f^{(1)}(x,y) = 0$, it is necessary to have $(-1,-r) \in U^{(1)}$, which uniquely determines the value of r. The value of c_r is found by the *determining equation*

$$x^c \hat f_j^{(1)}(x, c_r x^r) = 0.$$

Thus, each truncated equation has one or several appropriate solutions with $U \subset U^{(d)}$.

5. Critical values of truncated solution

If truncated solution (3.2) is found, the change $y = cx^r + z$ reduces equation (2.1) to

$$\tilde{f}(x, z) = f(x, cx^r + z) = 0. \tag{5.1}$$

In many cases, up to a possible reduction by some power of x, equation (5.1) reads

$$\tilde{f}(x, z) := \mathcal{L}(\sigma)z + h(x, z) = 0, \tag{5.2}$$

where $\mathcal{L}(\sigma)$ is a linear q–difference operator with constant coefficients

$$\mathcal{L}(\sigma) = a_m\sigma^m + \ldots + a_1\sigma + a_0, \tag{5.3}$$

the point $Q\left(\mathcal{L}(\sigma)z\right) = (0, 1)$ is located in the support of equation (5.2) and is a vertex of the polygon $\Gamma(\tilde{f})$, and the support of $S(h)$ does not contain the point $(0, 1)$.

We define the *characteristic polynomial* of q–difference sum $\mathcal{L}(\sigma)z$ by the formula

$$\nu(k) = x^{-k}\mathcal{L}(\sigma)[x^k].$$

If $\nu(k) \not\equiv 0$, then the roots $k_1, \ldots, k_s$ of the polynomial $\nu(k)$ are called *eigenvalues* of truncated solution (3.2). Real eigenvalues $k \in \{k_1, \ldots, k_s\}$, for which $k > r$, are called *critical values*.

We introduce some additional notation. We translate the support of $S(\tilde{f})$ by $(0, -1)$ and denote the new set by $S'(\tilde{f}) = S(\tilde{f}) - (0, 1)$. Let r be a number such that for each point $Q' \in S'(\tilde{f})$ we have $\langle R, Q'\rangle \geqslant 0$, where $R = (1, r)$. By $S'_+(\tilde{f})$ we denote the set of finite sums of points $Q' \in S'(\tilde{f})$ and vectors $(k_1, -1), \ldots, (k_s, -1)$, where $k_1, \ldots, k_s$ are the critical values of the truncated solution $y = cx^r$. Let $K(k_1, \ldots, k_s)$ be the set of $q_1 \in \mathbb{R}$ such that $(q_1, -1) \in S'_+(\tilde{f})$.

6. Power–logarithmic expansions of solutions

As it was shown in [4], not each algebraic q–difference equation has power solutions, i.e., solutions in the space

$$\mathbb{C}[[x]] = \left\{ \sum_{k=0}^{\infty} c_k x^k, c_k = \text{const} \in \mathbb{C} \right\}.$$

In [4], we restricted ourselves to formal solutions represented by the series

$$\sum_{k=0}^{\infty} p_k(\log_q x)x^k,$$

where p_k are polynomials with complex coefficients.

In this paper, we consider formal series of more general form (6.1), which are *power–logarithmic asymptotic expansions of solutions*.

In what follows we prove a sufficient condition for the existence of a formal power–logarithmic asymptotic expansion of a solution to a q–difference algebraic equation, which is similar to that for a differential equation [2].

Theorem 6.1. *We consider equation (2.1) and its truncated solution (3.2). Suppose that after the change $y = cx^r + z$ and transformations we obtain equation (5.2) and let the following conditions be satisfied for the support of equation (5.2):*

(1) *the point $(0, 1)$ is the vertex of $\Gamma(\tilde{f})$;*

(2) *in the equation $\tilde{f}(x, z) = 0$, the vertex $(0, 1)$ corresponds only to the term $\mathcal{L}(\sigma)z$, where $\mathcal{L}(\sigma)$ is a linear q–difference operator defined by formula (5.3) ($\mathcal{L}(\sigma)$ is a polynomial in σ with constant coefficients).*

Then equation (5.2) has a formal solution of the form

$$z = \sum_k \beta_k(\log_q x)x^k, \quad k \in K(k_1, \dots, k_s), \quad k > r, \tag{6.1}$$

where β_k are polynomials in the variable $\log_q x$, and $k_1, \dots, k_s$ are critical values of truncated solution (3.2).

Proof. We rewrite equation (5.2) as

$$\mathcal{L}(\sigma)z = -h(x, z).$$

We substitute formal solution (6.1) into equation (5.2). After the substitution, the left hand side contains terms with powers of x^k, where $k \in K(k_1, \dots, k_s)$, $k > r$. The right hand side involves terms with powers of the form $x^{\langle Q,(1,k)\rangle}$, where $Q = (q_1, q_2) \in S(\tilde{f})$.

But $\langle Q,(1,k)\rangle \in K(k_1, \dots, k_s)$ since $\langle Q,(1,k)\rangle = q_1 + kq_2$. The set $K(k_1, \dots, k_s)$ contains the abscissa of the vertex $Q' + q_2(k, -1) = (q_1, q_2 - 1) + q_2(k, -1) = (q_1 + kq_2, -1)$; here we use the fact that $q_2 \in \mathbb{Z}_+$, i.e. we add the point $(k, -1)$ to itself finitely many times.

Therefore, equating the coefficients at the like powers of x^k, we obtain a family of difference equations on β_k; hereinafter $t = \log_q x$;

$$\mathcal{L}(q^k T)\beta_k(t) + \theta_k(t) = 0, \tag{6.2}$$

where θ_k is a polynomial in the functions β_ℓ, where $\ell \in K(k_1, \dots, k_s)$, $\ell < k$; the translation operator T is defined by the formula $(Tf)(t) = f(t + 1)$. The proof is complete. □

Remark 6.1. *Equations (6.2) obtained in Theorem 6.1 are similar to ones in [4], but now the indices k are not only non–negative integers.*

A critical values k is said to satisfy the *compatibility condition* if $\theta_k(t) \equiv 0$ in equation (6.2).

Remark 6.2. *Under the assumptions of Theorem 6.1, let the compatibility condition be satisfied for all critical values k, and let all critical values k be non–multiple. Then formal solution (6.1) to equation (5.1) contains no logarithms.*

Proof. The functions β_k appear as solutions of difference equations (6.2).

If k is a non–multiple critical value and the compatibility condition is satisfied for it, then the equation for β_k reads

$$\mathcal{L}(q^k T)\beta_k(t) = 0,$$

then its solution is $\beta_k(t) = C_k$, $C_k \in \mathbb{C}$ is an arbitrary constant.

Let k be non–critical. The function $\theta_k(t) = A_k$ in equation (6.2) is constant, the equation for β_k reads

$$\mathcal{L}(q^k T)\beta_k(t) = -A_k = \mathrm{const},$$

its solution $\beta_k(t) = -A_k/\mathcal{L}(q^k)$ is a uniquely determined constant. The proof is complete. □

7. Powers of logarithms in expansion

Let us consider how the powers of logarithms grow in expansion (6.1). We denote by $\mu(j)$ the multiplicity of q^j as a root of $\mathcal{L}(s)$.

Theorem 7.1. *Let the assumptions of Theorem 6.1 be satisfied. Then the degrees of the polynomials β_k in solution (6.1) satisfy the estimates*

$$\deg \beta_k \leqslant C(k - r) \sum_{r<j\leqslant k} \mu(j) \tag{7.1}$$

for $C = 1 + \max\limits_{\ell \in K(k_1,\dots,k_s)\cap\{\ell>r\}} \frac{1}{\ell - r}$.

Proof. We stress that the degree of the zero polynomial is supposed to be zero.

We prove by induction. We consider $k^{(0)} = \min\left(K(k_1,\dots,k_s)\cap\{k>r\}\right)$, which is the minimal degree in expansion (6.1). Then $\deg\beta_{k^{(0)}} \leqslant \mu(k^{(0)})$, and since $C > \frac{1}{k^{(0)}-r}$, inequality (7.1) is satisfied.

Suppose that inequality (7.1) is true for all β_j for $j<k$. For β_k we have equation (6.2), in which θ_k is a linear combination of monomials of the form

$$\beta_{i_{0,0}}^{\alpha_{0,0}}(t)\dots\beta_{i_{0,N_0}}^{\alpha_{0,N_0}}(t)\,\beta_{i_{1,0}}^{\alpha_{1,0}}(t+1)\dots\beta_{i_{1,N_1}}^{\alpha_{1,N_1}}(t+1)\dots\beta_{i_{n,0}}^{\alpha_{n,0}}(t+n)\dots\beta_{i_{n,N_n}}^{\alpha_{n,N_n}}(t+n), \tag{7.2}$$

where $\alpha_{0,0}i_{0,0}+\dots+\alpha_{n,N_n}i_{n,N_n}\leqslant k$. This inequality also implies that

$$\alpha_{0,0}(i_{0,0}-r)+\dots+\alpha_{n,N_n}(i_{n,N_n}-r)\leqslant k-r.$$

By the induction assumption, the degree of each monomial (7.2) does not exceed

$$\begin{aligned}&\alpha_{0,0}\deg\beta_{i_{0,0}}+\dots+\alpha_{n,N_n}\deg\beta_{i_{n,N_n}}\\ &\qquad\leqslant C\left(\alpha_{0,0}(i_{0,0}-r)\sum_{r<j\leqslant i_{0,0}}\mu(j)+\dots+\alpha_{n,N_n}(i_{n,N_n}-r)\sum_{r<j\leqslant i_{n,Nn}}\mu(j)\right)\\ &\qquad\leqslant C(k-r)\sum_{r<j<k}\mu(j).\end{aligned}$$

Then

$$\deg\beta_k\leqslant\deg\theta_k+\mu(k)\leqslant C(k-r)\sum_{r<j<k}\mu(j)+\mu(k)\leqslant C(k-r)\sum_{r<j\leqslant k}\mu(j).$$

The latter inequality is true since $C(k-r)>1$. The proof is complete. □

8. Example of power–logarithmic expansion

Let us consider the fifth Painlevé equation

$$\frac{d^2y}{dx^2}=\left(\frac{1}{2y}+\frac{1}{y-1}\right)\left(\frac{dy}{dx}\right)^2-\frac{1}{x}\frac{dy}{dt}+\frac{(y-1)^2}{x^2}\left(a_1y+\frac{a_2}{y}\right)+a_3\frac{y}{x}+a_4\frac{y(y+1)}{y-1},$$

where a_1, a_2, a_3, a_4 are complex parameters. Letting $a_1=a_2=0$, a_3, $a_4\neq0$ and passing to the operator $\delta=x\frac{d}{dx}$, we obtain the equation

$$\frac{\delta^2y-\delta y}{x^2}=\left(\frac{1}{2y}+\frac{1}{y-1}\right)\frac{(\delta y)^2}{x^2}-\frac{1}{x^2}\delta y+a_3\frac{y}{x}+a_4\frac{y(y+1)}{y-1}.$$

We formally replace the operator δ in this equation by σ and obtain some q–difference equation

$$\frac{\sigma^2y-\sigma y}{x^2}=\left(\frac{1}{2y}+\frac{1}{y-1}\right)\frac{(\sigma y)^2}{x^2}-\frac{1}{x^2}\sigma y+a_3\frac{y}{x}+a_4\frac{y(y+1)}{y-1}.$$

We simplify it by multiplying by $x^2y(y-1)$ and we get the equation

$$-a_3xy^3+a_3xy^2-a_4x^2y^3-a_4x^2y^2+y^2\sigma^2y-\frac{3(\sigma y)^2y}{2}-y\sigma^2y+\frac{(\sigma y)^2}{2}=0. \tag{8.1}$$

The Newton polygon Γ of this equation is shown on Fig. 1.

We are interested in the left vertical edge of the polygon Γ. Its determining equation is

$$\frac{c^2}{2}-\frac{3c^3}{2}-c^2+c^3=0,$$

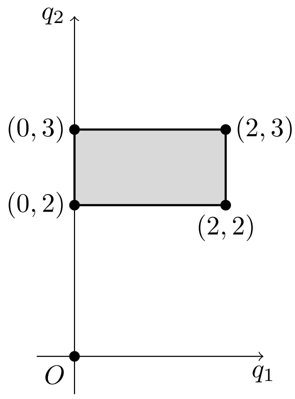

Figure 1. Newton polygon of equation (8.1).

and the only non–zero solution of this equation is $c = -1$. We make the substitution $y = z - 1$ in equation (8.1) and pass to the equation

$$-a_3xz^3 + 4a_3xz^2 - 5a_3xz + 2a_3x - a_4x^2z^3 + 2a_4x^2z^2 - a_4x^2z + z^2\sigma^2z - z^2 - \frac{3}{2}(\sigma z)^2z + 3z\sigma z - 3z\sigma^2z + \frac{3z}{2} + 2(\sigma z)^2 - 4\sigma z + 2\sigma^2z = 0, \tag{8.2}$$

the Newton polygon of which is shown on Fig. 2.

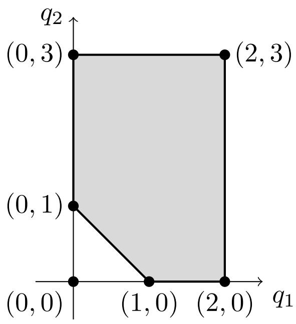

Figure 2. Newton polygon of equation (8.2).

The linear part of equation (8.2) reads $\mathcal{L}(\sigma)z = 2\sigma^2z - 4\sigma z + \frac{3z}{2}$, and hence the characteristic equation is

$$\nu(k) = 2q^{2k} - 4q^k + \frac{3}{2} = 0,$$

the roots k of which are determined by the equation $q^k = \frac{1}{2}$ or $q^k = \frac{3}{2}$.

We continue the truncated solution $y = -1$ to a formal expansion of a solution to equation (8.1). We consider several values of the parameter q.

Case 1. First let $q = \frac{1}{2}$, then the eigenvalues are $k_1 = 1, k_2 = 1 - \log_2 3$. Only k_1 is a critical value ($k_2 < 0$).

Since the coordinates of all points of the support of equation (8.2) are non–negative integers, the critical value is a natural number, and the support does not contain the point $(0, 0)$, we have $K(k_1) \subset \mathbb{N}$. On the other hand, the set $S'_+(\tilde{f})$ contains the points $(0, 1)$ and $(1, -1)$, their finite sums cover all numbers of the form $(n, -1), n \in \mathbb{N}$, and hence $K(k_1) = \mathbb{N}$. Thus, equation (8.2) has a power–logarithmic solution with support lying in $\mathbb{N}$.

We proceed to finding a solution. The second term in the expansion satisfies the equation

$$2\sigma^2 z - 4\sigma z + \frac{3}{2}z + 2a_3 x = 0.$$

Substituting the solution in the form $z = \beta_1(\log_q x)x$, $\beta_1 \in \mathbb{C}[\log_q x]$ into the above equation and passing to the variable $t = \log_q x$, we obtain the difference equation for β_1:

$$\frac{1}{2}\beta_1(t+2) - 2\beta_1(t+1) + \frac{3}{2}\beta_1(t) + 2a_3 = 0,$$

all polynomial solutions of which are written as $\beta_1(t) = 2a_3 t + C$, $C \in \mathbb{C}$.

Thus, we obtain the initial sum of the formal expansion of solution to equation (8.1) for $q = \dfrac{1}{2}$:

$$y(x) = -1 + (C - 2a_3 \log_2 x)\, x + \ldots, \quad C \in \mathbb{C}.$$

Case 2. Now let $q = \dfrac{1}{4}$, then the eigenvalues are $k_1 = \dfrac{1}{2}$, $k_2 = -\log_4\left(\dfrac{3}{2}\right)$, and only k_1 is a critical value.

Since the coordinates of all points of the support of the equation (8.2) are non–negative integers, the critical value is a positive half–integer, and the support does not contain the point $(0,0)$, we have $K(k_1) \subset \mathbb{N}/2$. On the other hand, the set $S'_+(\tilde{f})$ contains the points $(0,1)$ and $\left(\dfrac{1}{2}, -1\right)$, their finite sums cover all numbers of the form $\left(\dfrac{n}{2}, -1\right)$, $n \in \mathbb{N}$, and hence $K(k_1) = \mathbb{N}/2$.

Let's find the first term in the expansion of solution. The equation for $\beta_{\frac{1}{2}}(t)$ reads

$$\frac{1}{2}\beta_{\frac{1}{2}}(t+2) - 2\beta_{\frac{1}{2}}(t+1) + \frac{3}{2}\beta_{\frac{1}{2}}(t) = 0.$$

This equation satisfies the compatibility condition. All polynomial solutions of such a difference equation are constants: $\beta_{\frac{1}{2}}(t) = C$, $C \in \mathbb{C}$ is an arbitrary constant. We find the next term in the expansion, namely, the difference equation for β_1 has the form

$$\frac{1}{8}\beta_1(t+2) - \beta_1(t+1) + \frac{3}{2}\beta_1(t) = -2a_3 - \frac{C^2}{4}.$$

The only polynomial solution of this difference equation is $\beta_1(t) = -\dfrac{2}{5}(8a_3 + C^2)$. This gives the initial sum in the expansion of a solution to equation (8.1)

$$y(x) = -1 + C\sqrt{x} + -\frac{2}{5}(8a_3 + C^2)x + \ldots.$$

Thus, we have illustrated that, depending on the value of the parameter q of the equation, the power truncated solution of algebraic q–difference equation (8.1) can be continued as a power–logarithmic expansion of form (6.1) for $q = \dfrac{1}{2}$ or a power expansion, which is an expansion in a formal Puiseux series, for $q = \dfrac{1}{4}$.

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