STABILITY ESTIMATES IN INVERSE PROBLEM FOR POTENTIALS IN SYSTEM OF TWO SCHRÖDINGER EQUATIONS WITH DIRICHLET AND NEUMANN CONDITIONS

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Abstract. We study the inverse problem on identifying two unknown potential coefficients in a system of coupled Schrödinger equations in a bounded domain in $\mathbb{R}^n$ subject to non–homogeneous Dirichlet and Neumann boundary conditions by using Neumann and Dirichlet boundary measurements. Under specific convexity assumptions on the geometry of domain and minimal regularity conditions on the data, we establish the Lipschitz stability of this inverse problem by employing uniqueness results as well as the observability inequality.

Keywords: inverse problems, uniqueness, stability, coupled Schrödinger equations, observability inequality.

Mathematics Subject Classification: 35R30, 35L10, 35Q40, 49K20

1. Introduction

Inverse problems for partial differential equations (PDEs) play a central role in many scientific and engineering applications. Unlike direct problems, where the solution of a PDE is determined from known inputs such as boundary conditions, coefficients, and source terms, inverse problems aim to deduce these unknown inputs from observed data. These problems are inherently more complex, as they are often ill-posed in the sense of Hadamard, meaning that solutions may not exist, may not be unique, or may not depend continuously on the data. Solving inverse problems typically involves developing robust mathematical models, regularization techniques to deal with ill-posedness, and computational algorithms for implementation. Analytical approaches such as uniqueness theorems and stability estimates are also crucial for understanding the theoretical aspects of inverse problems. Historically, the Carleman estimation method was introduced in the field of inverse problems by Bukhgeim and Klibanov, who first presented it in [5]. This method is referred to as the Bukhgeim — Klibanov method. The applications of such inequalities are numerous. The major use of these inequalities is in demonstrating uniqueness results and stability inequalities in inverse problems. In [9] and [10], local Carleman inequalities and their roles in uniqueness results and Hölder- type stability estimates were discussed. They were also used in [11] for the numerical resolution of a class of inverse problems on determining the coefficients of certain partial differential equations from boundary measurements of the solution or a function of the solution. Our work follows the same approach as that in [5].

There is a large literature dealing with inverse problems for partial differential equations, see for example [4], [6], [8], [11], [17] and the references therein. Inverse problems for the Schrödinger equation were studied, for instance, in [15], [1], [14], [3], [16], [7], [21]. In [15], the authors

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Submitted January 27, 2025.

employed Carleman inequalities, which are powerful tools in the analysis of partial differential equations, with specific focus on degenerate weights.

The main objective of the study is to establish stability estimates for the inverse problem of recovering potential coefficients in the Schrödinger equation from partial boundary measurements. In [1], the authors studied the inverse problem related to the Schrödinger equation aimed on determining unknown potential functions in the Schrödinger equation based on observed measurements. The authors explored the conditions, under which this inverse problem has a unique solution, as well as the stability of these solutions under data perturbations. The paper also provides theoretical results concerning the well-posedness of the problem, offering insights into both uniqueness and stability. In [3], an inverse problem for the magnetic Schrödinger equation was treated. The authors focused on estimating the stability of the problem, specifically addressing how the magnetic potential can be determined from the Dirichlet–to–Neumann map. The Dirichlet–to–Neumann map is a mathematical object that connects the values of a solution to a partial differential equation on the boundary of a domain with the values of its normal derivative on the boundary. The main contribution of the paper is a stability estimate, which quantifies the extent to which the solution to the inverse problem can be influenced by small errors or perturbations in the data. The results in the paper have significant implications for understanding the uniqueness and stability of inverse problems in quantum mechanics, especially in the context of magnetic fields. Nakamura, Sun, and Uhlmann [16] studied on the global identifiability of an inverse problem for the Schrödinger equation in the presence of a magnetic field. The inverse problem refers to determining unknown parameters (such as the magnetic potential) of a system by observing its behavior. In this work, the authors explored the conditions, under which these parameters can be uniquely identified from the solution of the Schrödinger equation. They provided mathematical results related to the global identifiability of these parameters, contributing to the understanding of inverse problems in quantum mechanics. The study is significant for applications in areas like quantum physics and medical imaging. Triggiani and Zhang [21] investigated the inverse problem on determining the electric potential coefficient in Schrödinger equations on Riemannian manifolds. The focused on the global uniqueness and stability of the solution to this inverse problem. The authors provided mathematical proofs and established conditions, under which the potential coefficient can be uniquely determined from boundary measurements. Dou and Yamamoto [7] addressed an inverse problem related to coupled Schrödinger equations. The focus was on the stability of the inverse problem, specifically logarithmic stability. This means the authors investigated how small changes in the data can affect the accuracy of the solution to the inverse problem, which is crucial for practical applications, where data may not be perfectly accurate. The study provides mathematical analysis and results that help to understand the behavior and limitations of solutions to this class of inverse problems. In [19], the authors employed Carleman estimates for the Schrödinger equations as established by Lasiecka et al. [13], along with the associated observability inequality, to demonstrate uniqueness and Lipschitz stability for the inverse problem of recovering the unknown potential coefficient of the Schrödinger equation in a bounded domain of $\mathbb{R}^n$. In [14] the inverse problem of recovering unknown potential parameters for a more general coupled system of Schrödinger equations with magnetic potential terms was studied with Neumann boundary conditions. A uniqueness result was established by using Carleman inequalities.

The system we study in this work is a particular case of the general model in [14], but with mixed Dirichlet and Neumann boundary conditions. To the best of our knowledge, the stability of this inverse problem has not been studied in the literature. The aim of the present paper is to fill this gap. More specifically, we prove a stability inequality for the simultaneous determination of two unknown potential coefficients in a coupled system of two Schrödinger equations with non–homogeneous Dirichlet and Neumann boundary data from observations of the Neumann and Dirichlet traces.

2. Problem statement

Let $T > 0$ and let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ be an open bounded domain with boundary of class C^2. Throughout this paper, we use the notation

$$\Gamma = \partial\Omega = \overline{\Gamma_0 \cup \Gamma_1}, \quad \Gamma_0 \cap \Gamma_1 = \emptyset, \qquad \nabla v = \left(\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_n}\right), \qquad \Delta v = \sum_{i=1}^{n} \frac{\partial^2 v}{\partial x_i^2},$$

by $nu = (\nu_1, \dots, \nu_n)$ we denote the unit outward normal vector to $\Gamma = \partial\Omega$, and

$$\frac{\partial v}{\partial \nu} = \nabla v \cdot \nu$$

is the normal derivative. Following [12], [13], [20], [14], we make the following assumptions.

Assumption 2.1. *There exists a non–negative function $d : \overline{\Omega} \to \mathbb{R}_+$ of class C^3 such that*

(i) *The normal derivative of d is non–negative on Γ_0,*

$$\nu(x)\nabla d(x) \geqslant 0 \quad \text{on} \quad \Gamma_0. \tag{2.1}$$

(ii) *The function d is strictly convex in $\overline{\Omega}$, namely, there exists $\rho > 0$ such that for all $x \in \overline{\Omega}$ and all $\xi \in \mathbb{R}^n$,*

$$\mathcal{H}_d(x)\xi \cdot \xi \geqslant \rho|\xi|^2, \tag{2.2}$$

where $\mathcal{H}_d(x)$ denotes the Hessian matrix of $d(x)$.

(iii) *The function d has no critical point on $\overline{\Omega}$*

$$\inf_{x \in \overline{\Omega}} \|\nabla d(x)\| = s > 0. \tag{2.3}$$

Choosing the strictly convex potential function $d(x)$ satisfying Assumption 2.1 and $d(x) \geqslant d_0 > 0$, we next introduce the pseudo–convex function $\varphi(x,t)$

$$\varphi(x,t) = d(x) - c\left(t - \frac{T}{2}\right)^2, \qquad x \in \Omega, \quad t \in [0,T], \tag{2.4}$$

where $T > 0$ is arbitrary.

We study the system of two Schrödinger equations for the unknowns $w = w(x,t)$, $z = z(x,t)$:

$$iw_t + \Delta w = n(x)w + q(x)z \qquad \text{in} \quad Q, \tag{2.5}$$

$$iz_t + \Delta z = m(x)z + p(x)w \qquad \text{in} \quad Q, \tag{2.6}$$

$$w\left(\cdot, \tfrac{T}{2}\right) = w_0(x) \qquad \text{in} \quad \Omega, \tag{2.7}$$

$$z\left(\cdot, \tfrac{T}{2}\right) = z_0(x) \qquad \text{in} \quad \Omega, \tag{2.8}$$

$$w = g_1(x,t), \qquad \frac{\partial z}{\partial \nu} = g_2(x,t) \quad \text{on} \quad \Sigma, \tag{2.9}$$

where $\Sigma = \Gamma \times [0,T]$, $Q = \Omega \times [0,T]$. Here $w_0(x)$, $z_0(x)$ are the given initial conditions and $g_1(x,t)$, $g_2(x,t)$ are given Dirichlet and Neumann boundary conditions. Instead, the potentials $q(x)$, $p(x)$ are time–independent unknown coefficients.

This paper treats two kinds of inverse problems. We first note that the map $\{q,p\}|_\Omega \to \{\frac{\partial w}{\partial \nu}, z\}|_{\Gamma_1 \times [0,T]}$ is nonlinear and hence we consider the following nonlinear inverse problem.

1. Nonlinear inverse problem. Let $\{w = w(q,p),\ z = z(q,p)\}$ be a solution to system (2.5)–(2.9). We are interesting in following two problems.

Uniqueness in the nonlinear inverse problem. Under Assumption 2.1, does the implication holds?

$$\left\{ \begin{array}{l} \dfrac{\partial w(q_1,p_1)}{\partial \nu}\Big|_{\Gamma_1\times[0,T]} = \dfrac{\partial w(q_2,p_2)}{\partial \nu}\Big|_{\Gamma_1\times[0,T]} \\ z(q_1,p_1)\big|_{\Gamma_1\times[0,T]} = z(q_2,p_2)\big|_{\Gamma_1\times[0,T]} \end{array} \right\} \Longrightarrow \left\{ \begin{array}{l} q_1(x) = q_2(x) \\ p_1(x) = p_2(x) \end{array} \right\} \quad \text{on} \quad \Omega? \tag{2.10}$$

Stability in the nonlinear inverse problem. Under Assumption 2.1, is it possible to estimate $q_1 - q_2|_\Omega$ and $p_1 - p_2|_\Omega$ by

$$\Big(\frac{\partial w(q_1,p_1)}{\partial \nu} - \frac{\partial w(q_2,p_2)}{\partial \nu}\Big)\Big|_{\Gamma_1\times[0,T]} \quad \text{and} \quad \big(z(q_1,p_1) - z(q_2,p_2)\big)\big|_{\Gamma_1\times[0,T]}$$

in suitable norms?

As usual, the nonlinear inverse problem is converted into a linear inverse problem for an auxiliary, corresponding problem. Let

$$\begin{aligned} &f(x) = q_1(x) - q_2(x), \qquad && g(x) = p_1(x) - p_2(x), \\ &R_1(x,t) = z(q_2,p_2)(x,t), && R_2(x,t) = w(q_2,p_2)(x,t), \\ &u(x,t) = w(q_1,p_1)(x,t) - w(q_2,p_2)(x,t), && \\ &v(x,t) = z(q_1,p_1)(x,t) - z(q_2,p_2)(x,t). && \end{aligned} \tag{2.11}$$

Subtracting system (2.5)–(2.9) with coefficients (q_1,p_1) from the same system with coefficients (q_2,p_2), we obtain

$$iu_t + \Delta u = n(x)u + q(x)v + f(x)R_1(x,t) \quad \text{in} \quad Q, \tag{2.12}$$

$$iv_t + \Delta v = m(x)v + p(x)u + g(x)R_2(x,t) \quad \text{in} \quad Q, \tag{2.13}$$

$$u\big(\cdot, \tfrac{T}{2}\big) = 0 \quad \text{in} \quad \Omega, \tag{2.14}$$

$$v\big(\cdot, \tfrac{T}{2}\big) = 0 \quad \text{in} \quad \Omega, \tag{2.15}$$

$$u = 0, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \Sigma. \tag{2.16}$$

Here q in (2.12) is actually q_1 in the notation of (2.11), while p in (2.13) is actually p_1 in (2.11). The terms f, g are unknown time–independent coefficients. The $\{u,v\}$-problem has the advantage over the original $\{w,z\}$-problem in (2.5)–(2.9) that the map $\{f,g\}|_\Omega \to \{\frac{\partial u}{\partial \nu}, v\}|_{\Gamma_1\times[0,T]}$ is linear.

2. *Linear inverse problem.* We are interesting in following two problems.

Uniqueness in the linear inverse problem Let $\{u = u(f,g),\ v = v(f,g)\}$ be the solution to system (2.12)–(2.16). Under Assumption 2.1, does the implication holds

$$\left\{ \begin{array}{l} \dfrac{\partial u(f,g)}{\partial \nu}\Big|_{\Gamma_1\times[0,T]} = 0 \\ v(f,g)\big|_{\Gamma_1\times[0,T]} = 0 \end{array} \right\} \Longrightarrow \left\{ \begin{array}{l} f(x) = 0 \\ g(x) = 0 \end{array} \right\} \quad \text{on} \quad \Omega? \tag{2.17}$$

Stability in the linear inverse problem. Under Assumption 2.1, is it possible to estimate $f|_\Omega$ and $g|_\Omega$ by $\frac{\partial u(f,g)}{\partial \nu}\big|_{\Gamma_1\times[0,T]}$ and $v(f,g)\big|_{\Gamma_1\times[0,T]}$ in suitable norms?

The goal of the present paper is to give an affirmative and quantitative answer to the above stability questions for the linear and nonlinear inverse problems.

3. Important previous results

In this section we present the most important theorems that play a key role in proving our main results.

3.1. Uniqueness in linear case.

Theorem 3.1. *Under Assumption 2.1 let*

$$T > 0 \tag{3.1}$$

and in $\{u, \upsilon\}$-system (2.12)–(2.16) we suppose

$$\begin{aligned}
&n,\, m \in W^{1,\infty}(\Omega), \quad q,\, p \in W^{1,\infty}(\Omega), \quad f,\, g \in W^{1,\infty}(\Omega), \qquad f|_\Gamma = 0,\\
&R_i,\, R_{it},\, R_{itt} \in C\big([0,T], H^1(\Omega)\big), \qquad\qquad (3.2)\\
&R_{ix_j}(x, \tfrac{T}{2}),\, R_{ix_jx_k}(x, \tfrac{T}{2}),\, R_{ix_jx_kx_l}(x, \tfrac{T}{2}) \in L^\infty(\Omega), \qquad i = 1, 2, \quad 1 \leqslant j, k, l \leqslant n,\\
&fR_1,\, fR_{1t},\, fR_{1tt} \in C\big([0,T], H_0^1(\Omega)\big), \qquad gR_2,\, gR_{2t},\, gR_{2tt} \in C\big([0,T], H^1(\Omega)\big), \qquad (3.3)\\
&|R_1(x, \tfrac{T}{2})| \geqslant r_1 > 0, \qquad |R_2(x, \tfrac{T}{2})| \geqslant r_2 > 0, \qquad x \in \overline{\Omega}, \qquad (3.4)
\end{aligned}$$

with some positive constants r_1, r_2. Assume that the solution $\{u = u(f, g), \upsilon = \upsilon(f, g)\}$ to problem (2.12)–(2.16) possesses the smoothness

$$u, u_t, u_{tt} \in C\big([0,T], H_0^1(\Omega)\big), \quad \upsilon, \upsilon_t, \upsilon_{tt} \in C\big([0,T], H^1(\Omega)\big). \tag{3.5}$$

If

$$\frac{\partial u(f, g)}{\partial \nu}(x, t) = 0, \qquad \upsilon(f, g)(x, t) = 0, \qquad (x, t) \in \Gamma_1 \times [0, T], \tag{3.6}$$

then

$$f(x) = 0, \qquad g(x) = 0, \qquad x \in \Omega. \tag{3.7}$$

Доказательство. The methodology for proving the uniqueness result (3.6), (3.7) is analogous to that employed in [14, Sect. 4], where a more general coupled Schrödinger system

$$\begin{aligned}
iu_t + \Delta u &= a(x) \cdot \nabla u + n(x)u + \beta(x) \cdot \nabla \upsilon + q(x)\upsilon + f(x)R_1,\\
i\upsilon_t + \Delta \upsilon &= b(x) \cdot \nabla \upsilon + m(x)\upsilon + \gamma(x) \cdot \nabla u + p(x)u + g(x)R_2,
\end{aligned}$$

was considered with Neumann boundary conditions for both equations. Despite the presence of additional first–order terms and the difference in boundary conditions (our system (2.12)–(2.16) involves a Dirichlet condition for u and a Neumann condition for υ on Σ), the same essential steps in [14, Sect. 4] can be adapted to our setting. Consequently, the uniqueness result follows exactly as in [14]. The proof is complete. □

3.2. Uniqueness in the nonlinear case.

Theorem 3.2. *Let Assumption 2.1 holds. Consider problem (2.5)–(2.9) on $[0, T]$, with T as in (3.1) and with potential coefficients $q_1, p_1 \in W^{1,\infty}(\Omega)$, and potential coefficients $q_2, p_2 \in W^{1,\infty}(\Omega)$, and suppose that*

$$\begin{aligned}
&n, m \in W^{1,\infty}(\Omega), \quad q_1, p_1, q_2, p_2 \in W^{1,\infty}(\Omega), \quad \{q_1, q_2\}|_\Gamma = 0, \qquad (3.8)\\
&w, w_t, w_{tt} \in C\big([0,T], H_0^1(\Omega)\big), \quad z, z_t, z_{tt} \in C\big([0,T], H^1(\Omega)\big), \qquad (3.9)\\
&w_{0x_j},\, w_{0x_jx_k},\, w_{0x_jx_kx_l},\, z_{0x_j},\, z_{0x_jx_k},\, z_{0x_jx_kx_l} \in L^\infty(\Omega), \quad 1 \leqslant j, k, l \leqslant n. \qquad (3.10)\\
&|w_0(x)| \geqslant w_0 > 0, \qquad |z_0(x)| \geqslant z_0 > 0, \quad x \in \overline{\Omega}. \qquad (3.11)
\end{aligned}$$

Then

$$\frac{\partial w(q_1, p_1)}{\partial \nu} = \frac{\partial w(q_2, p_2)}{\partial \nu}, \qquad z(q_1, p_1) = z(q_2, p_2) \quad \text{on} \quad \Gamma_1 \times [0, T], \tag{3.12}$$

implies

$$q_1(x) = q_2(x), \qquad p_1(x) = p_2(x), \quad x \in \Omega. \tag{3.13}$$

Доказательство. The result is a direct consequence of Theorem 3.1. Define f, g, R_1, R_2, u, v as in (2.11). By (3.8)–(3.10) we obtain that n, m, q, p (with $f = q_1 - q_2$, $g = p_1 - p_2$) belong to $W^{1,\infty}(\Omega)$, that $f, g \in W_0^{1,\infty}(\Omega)$, and that R_1, R_2 satisfy required regularity (3.2), (3.3). Condition (3.11) implies (3.4) with $r_1 = z_0$, $r_2 = w_0$. The regularity assumptions on w, z given in (3.9) guarantee (3.5) for u, v. Moreover, condition (3.12) becomes exactly (3.6). Theorem 3.1 yields $f \equiv 0$ and $g \equiv 0$ in Ω, that is, $q_1 \equiv q_2$ and $p_1 \equiv p_2$. The proof is complete. □

3.3. Observability inequality. Our keys for the proofs of Theorem 4.1 are an observability inequality and a compactness–uniqueness argument. In this section, we state an observability inequality for a system of Schrödinger equations with homogeneous Dirichlet and Neumann boundary conditions

$$iu_t(x,t) + \Delta u(x,t) = n(x)u(x,t) + q(x)v(x,t) \quad \text{in} \quad Q, \tag{3.14}$$

$$iv_t(x,t) + \Delta v(x,t) = m(x)v(x,t) + p(x)u(x,t) \quad \text{in} \quad Q, \tag{3.15}$$

$$u\left(\cdot, \tfrac{T}{2}\right) = u_0(x) \quad \text{in} \quad \Omega, \tag{3.16}$$

$$v\left(\cdot, \tfrac{T}{2}\right) = v_0(x) \quad \text{in} \quad \Omega, \tag{3.17}$$

$$u = 0, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \Sigma, \tag{3.18}$$

where

$$\{u_0, v_0\} \in H_0^1(\Omega) \times H^1(\Omega), \qquad n, m \in W^{1,\infty}(\Omega), \qquad q, p \in W^{1,\infty}(\Omega). \tag{3.19}$$

Then

$$\{u, v\} \in C\big([0,T], H_0^1(\Omega) \times H^1(\Omega)\big). \tag{3.20}$$

The next theorem states the observability inequality.

Theorem 3.3. *Let Assumption 2.1 hold as well as (3.19), and $T > 0$. Let $\{u, v\}$ be solutions of problem (3.14)–(3.18). Then there exists a constant $C = C(\Omega, T, q, p) > 0$ such that*

$$C_T E(0) \leqslant \int_0^T \int_{\Gamma_1} \left(\left|\frac{\partial u}{\partial \nu}\right|^2 + |v|^2 + |v_t|^2 \right) d\Gamma_1 dt, \tag{3.21}$$

where

$$E(t) = E_u(t) + E_v(t), \tag{3.22}$$

and

$$E_w(t) = \int_\Omega \big(|\nabla w(x,t)|^2 + |w(x,t)|^2\big) d\Omega.$$

Доказательство. The observability result (3.21) is the same as the observability result in [20, Thm. 1.4, Case 3], where a Carleman estimate different from the Carleman estimate in [14, Thm. 3.1] was employed. However, the approach from [20, Sect. 3] still works and allows one to demonstrate estimate (3.21) by using Carleman estimate [14, Thm. 3.1]. The proof is compplete. □

4. Main Results

Our main results are the following theorems.

Theorem 4.1 (Stability of the Linear Inverse Problem). *Consider problem (2.12)–(2.16) and we suppose that Assumption 2.1, (3.1), (3.2), (3.3), and (3.4) hold. Then there exists a constant*

$$C = C(\Omega, T, \Gamma_1, \varphi, q, p, R_1, R_2) > 0,$$

depending on the data of problem (2.12)–(2.16), but not on the unknown coefficients f and g, such that

$$\|f\|_{H_0^1(\Omega)} + \|g\|_{H^1(\Omega)} \leqslant C\left(\left\|\frac{\partial u(f,g)}{\partial \nu}\right\|_{H^1(0,T;L^2(\Gamma_1))} + \left\|v_t(f,g)\right\|_{H^1(0,T;L^2(\Gamma_1))}\right) \tag{4.1}$$

for all $f, g \in W^{1,\infty}(\Omega)$.

Theorem 4.2 (Stability of Nonlinear Inverse Problem). *Suppose Assumption 2.1, (3.8), (3.9), (3.10), and (3.11) hold. Let $\{w(q_1,p_1), z(q_1,p_1)\}$, $\{w(q_2,p_2), z(q_2,p_2)\}$ be respectively solutions of problem (2.5)–(2.9) on $[0,T]$, with T as in (3.1) and with potential coefficients $q_1, p_1 \in W^{1,\infty}(\Omega)$ and $q_2, p_2 \in W^{1,\infty}(\Omega)$. Then there exists a constant*

$$C = C(\Omega, T, \Gamma_1, \varphi, M, w_0, w_1, z_0, z_1, g_1, g_2) > 0,$$

such that

$$\begin{aligned}\|q_1 - q_2\|_{H_0^1(\Omega)} + \|p_1 - p_2\|_{H^1(\Omega)} \leqslant C\bigg\{&\left\|\frac{\partial w(q_1,p_1)}{\partial \nu} - \frac{\partial w(q_2,p_2)}{\partial \nu}\right\|_{H^1(0,T;L^2(\Gamma_1))} \\ &+ \left\|z_t(q_1,p_1) - z_t(q_2,p_2)\right\|_{H^1(0,T;L^2(\Gamma_1))}\bigg\}.\end{aligned} \tag{4.2}$$

5. Stability in the linear case

In this section we prove Theorem 4.1. Before providing the proof, we establish an auxiliary statement.

Lemma 5.1. *Consider the system*

$$i(\tilde{u}_t)_t + \Delta(\tilde{u}_t) = n(x)\tilde{u}_t + q(x)\tilde{v}_t + f(x)R_{1t}(x,t) \quad \text{in} \quad Q, \tag{5.1}$$

$$i(\tilde{v}_t)_t + \Delta(\tilde{v}_t) = m(x)\tilde{v}_t + p(x)\tilde{u}_t + g(x)R_{2t}(x,t) \quad \text{in} \quad Q, \tag{5.2}$$

$$\tilde{u}_t\big(\cdot, \tfrac{T}{2}\big) = 0 \quad \text{in} \quad \Omega, \tag{5.3}$$

$$\tilde{v}_t\big(\cdot, \tfrac{T}{2}\big) = 0 \quad \text{in} \quad \Omega, \tag{5.4}$$

$$\tilde{u}_t = 0, \quad \frac{\partial \tilde{v}_t}{\partial \nu} = 0 \quad \text{on} \quad \Sigma. \tag{5.5}$$

with data

$$\begin{aligned}&n,\, m \in W^{1,\infty}(\Omega), \qquad q,\, p \in W^{1,\infty}(\Omega), \qquad f,\, g \in W^{1,\infty}(\Omega), \qquad f|_\Gamma = 0, \\ &R_{it},\, R_{itt} \in C\big([0,T], H^1(\Omega)\big), \qquad i = 1, 2.\end{aligned} \tag{5.6}$$

Then the operators

$$\begin{aligned}&\big(K\{f,g\}\big)(x,t) = \frac{\partial \tilde{u}_t}{\partial \nu} \,:\, W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) \longrightarrow L^2(\Gamma_1 \times [0,T]), \\ &\big(L\{f,g\}\big)(x,t) = \tilde{v}_t(x,t) \,:\, W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) \longrightarrow L^2(\Gamma_1 \times [0,T]), \\ &\big(L_1\{f,g\}\big)(x,t) = \tilde{v}_{tt}(x,t) \,:\, W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) \longrightarrow L^2(\Gamma_1 \times [0,T])\end{aligned} \tag{5.7}$$

are compact.

Доказательство. Step (i). By assumptions (5.6) we have the following regularity for solution to system (5.1)–(5.5):

$$\tilde{u}_t \in C\big([0,T], H_0^1(\Omega)\big), \qquad \tilde{v}_t \in C\big([0,T], H^1(\Omega)\big). \tag{5.8}$$

By differentiating in t we get

$$i(\tilde{u}_{tt})_t + \Delta(\tilde{u}_{tt}) = n(x)\tilde{u}_{tt} + q(x)\tilde{v}_{tt} + f(x)R_{1tt}(x,t) \quad \text{in} \quad Q, \tag{5.9}$$
$$i(\tilde{v}_{tt})_t + \Delta(\tilde{v}_{tt}) = m(x)\tilde{v}_{tt} + p(x)\tilde{u}_{tt} + g(x)R_{2tt}(x,t) \quad \text{in} \quad Q, \tag{5.10}$$
$$\tilde{u}_{tt}\big(\cdot,\tfrac{T}{2}\big) = -if(x)R_{1t}\big(x,\tfrac{T}{2}\big) \in H_0^1(\Omega) \quad \text{in} \quad \Omega, \tag{5.11}$$
$$\tilde{v}_{tt}\big(\cdot,\tfrac{T}{2}\big) = -ig(x)R_{2t}\big(x,\tfrac{T}{2}\big) \in H^1(\Omega) \quad \text{in} \quad \Omega, \tag{5.12}$$
$$\tilde{u}_{tt} = 0, \quad \frac{\partial \tilde{v}_{tt}}{\partial \nu} = 0 \quad \text{on} \quad \Sigma. \tag{5.13}$$

Under the assumptions (5.6), we obtain also the following regularity for solution to system (5.9)–(5.13)

$$\tilde{u}_{tt} \in C\big([0,T], H_0^1(\Omega)\big), \qquad \tilde{v}_{tt} \in C\big([0,T], H^1(\Omega)\big). \tag{5.14}$$

Step (ii). Under conditions (5.6), (5.8) problem (5.1)–(5.5), for $\{\tilde{u}_t, \tilde{v}_t\}$ yields that the map

$$W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) \longrightarrow C\big([0,T], H^1(\Omega)\big) \qquad \{f,g\} \longmapsto \{\tilde{v}_t\},$$

is continuous. Hence, by the trace theory, the map

$$W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) \longrightarrow C\big([0,T], H^{1/2}(\Gamma)\big) \qquad \Upsilon_1\{f,g\} \longmapsto \Upsilon\{f,g\} = \big\{\tilde{v}_t|_\Sigma\big\} \tag{5.15}$$

is continuous. Similarly, under conditions (5.6), (5.14), problem yields that the map

$$W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) \longrightarrow C\big([0,T], H^1(\Omega)\big) \qquad \{f,g\} \longmapsto \{\tilde{v}_{tt}\},$$

continuous and by the trace theory, the map

$$W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) \longrightarrow C\big([0,T], H^{1/2}(\Gamma)\big) \qquad \Upsilon_2\{f,g\} \longmapsto \Upsilon\{f,g\} = \big\{\tilde{v}_{tt}|_\Sigma\big\} \tag{5.16}$$

is continuous. Thus, by (5.6) for R_{it}, R_{itt}, $i = 1,2$ and (3.3), it follows from (5.15) and (5.16) that the mappings

$$f,g \in W^{1,\infty}(\Omega) \ \longmapsto \ \tilde{v}_t(f,g)|_\Sigma \in C\big([0,T], H^{1/2}(\Gamma)\big), \quad \tilde{v}_{tt}(f,g)|_\Sigma \in C\big([0,T], H^{1/2}(\Gamma)\big)$$

are continuously, i.e.,

$$\big\|\tilde{v}_t(f,g)\big\|_{C([0,T],H^{1/2}(\Gamma))} \leqslant C_{R_{1t},R_{2t}}\big(\|f\|_{H^1(\Omega)} + \|g\|_{H^1(\Omega)}\big), \tag{5.17}$$
$$\big\|\tilde{v}_{tt}(f,g)\big\|_{C([0,T],H^{1/2}(\Gamma))} \leqslant C_{R_{1t},R_{2t},R_{1tt},R_{2tt}}\big(\|f\|_{H^1(\Omega)} + \|g\|_{H^1(\Omega)}\big). \tag{5.18}$$

Compactness of L, L_1. By (5.17), (5.18) we see that

$$\big(L\{f,g\}\big)(x,t) = \tilde{v}_t(x,t) : W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) \longrightarrow L^2(\Gamma_1 \times [0,T]),$$
$$\big(L_1\{f,g\}\big)(x,t) = \tilde{v}_{tt}(x,t) : W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) \longrightarrow L^2(\Gamma_1 \times [0,T])$$

are compact operators because the embedding $H^{\frac{1}{2}}(\Gamma) \hookrightarrow L^2(\Gamma)$ is compact.

Step (iii). We apply the regularity property [2, Lm. 2] to (5.1)–(5.5) and (5.9)–(5.13) and we obtain

$$\big\|\tilde{u}_t\big\|_{C(0,T;H_0^1(\Omega))} \leqslant C\big(\|f\|_{H_0^1(\Omega)} + \|g\|_{H^1(\Omega)}\big), \tag{5.19}$$
$$\big\|\tilde{v}_t\big\|_{C(0,T;H^1(\Omega))} \leqslant C\big(\|f\|_{H_0^1(\Omega)} + \|g\|_{H^1(\Omega)}\big), \tag{5.20}$$

and

$$\left\|\tilde{u}_{tt}\right\|_{C(0,T;H_0^1(\Omega))} \leqslant C\big(\|f\|_{H_0^1(\Omega)} + \|g\|_{H^1(\Omega)}\big), \tag{5.21}$$

$$\left\|\tilde{v}_{tt}\right\|_{C(0,T;H^1(\Omega))} \leqslant C\big(\|f\|_{H_0^1(\Omega)} + \|g\|_{H^1(\Omega)}\big). \tag{5.22}$$

By (5.1), (5.2) we find

$$\begin{cases} \Delta\tilde{u}_t = -i(\tilde{u}_t)_t + n(x)\tilde{u}_t + q(x)\tilde{v}_t + f(x)R_{1t}(x,t), \\ \Delta\tilde{v}_t = -i(\tilde{v}_t)_t + m(x)\tilde{v}_t + p(x)\tilde{u}_t + g(x)R_{2t}(x,t), \end{cases} \tag{5.23}$$

so that, by (5.19), (5.20), (5.21),

$$\|\Delta\tilde{u}_t\|_{C(0,T;H^1(\Omega))} \leqslant C\big(\|f\|_{H_0^1(\Omega)} + \|g\|_{H^1(\Omega)}\big), \tag{5.24}$$

then

$$\|\Delta\tilde{u}_t\|_{C(0,T;L^2(\Omega))} \leqslant C\big(\|f\|_{H_0^1(\Omega)} + \|g\|_{H^1(\Omega)}\big). \tag{5.25}$$

On the other hand, we apply the regularity property [2, Lm. 1] to (5.1)–(5.5)

$$\|\tilde{u}_t\|_{C(0,T;L^2(\Omega))} \leqslant C\big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}\big), \tag{5.26}$$

hence

$$\|\tilde{u}_t\|_{C(0,T;L^2(\Omega))} \leqslant C\big(\|f\|_{H_0^1(\Omega)} + \|g\|_{H^1(\Omega)}\big). \tag{5.27}$$

Combining (5.25) with (5.27), by [18, Ch. 5, Cor. 4] we obtain

$$\|\tilde{u}_t\|_{C(0,T;H^2(\Omega))} \leqslant C\big(\|f\|_{H_0^1(\Omega)} + \|g\|_{H^1(\Omega)}\big). \tag{5.28}$$

Therefore, by the trace theorem,

$$\left\|\frac{\partial\tilde{u}_t}{\partial\nu}\right\|_{C(0,T;H^{1/2}(\Gamma))} \leqslant C\big(\|f\|_{H_0^1(\Omega)} + \|g\|_{H^1(\Omega)}\big). \tag{5.29}$$

Compactness of K. Similarly, by (5.29) we see that

$$\big(K\{f,g\}\big)(x,t) = \frac{\partial\tilde{u}_t}{\partial\nu} : W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) \longrightarrow L^2(\Gamma_1 \times [0,T])$$

is a compact operator, because the embedding $H^{\frac{1}{2}}(\Gamma) \hookrightarrow L^2(\Gamma)$ is compact. The proof is complete. □

Proof of Theorem 4.1. Step 1. We differentiate system (2.12)–(2.16) in t

$$i(u_t)_t + \Delta(u_t) = n(x)u_t + q(x)v_t + f(x)R_{1t}(x,t) \quad \text{in} \quad Q, \tag{5.30}$$

$$i(v_t)_t + \Delta(v_t) = m(x)v_t + p(x)u_t + g(x)R_{2t}(x,t) \quad \text{in} \quad Q, \tag{5.31}$$

$$u_t\big(\cdot\,,\tfrac{T}{2}\big) = -if(x)R_1\big(\cdot\,,\tfrac{T}{2}\big) \in H_0^1(\Omega) \quad \text{in} \quad \Omega, \tag{5.32}$$

$$v_t\big(\cdot\,,\tfrac{T}{2}\big) = -ig(x)R_2\big(\cdot\,,\tfrac{T}{2}\big) \in H^1(\Omega) \quad \text{in} \quad \Omega, \tag{5.33}$$

$$u_t = 0, \quad \frac{\partial v_t}{\partial\nu} = 0 \quad \text{on} \quad \Sigma. \tag{5.34}$$

We let

$$u_t = \bar{u}_t + \tilde{u}_t, \qquad v_t = \bar{v}_t + \tilde{v}_t. \tag{5.35}$$

Then, by the linearity of (5.30)–(5.34), we have

$$i(\bar{u}_t)_t + \Delta(\bar{u}_t) = n(x)\bar{u}_t + q(x)\bar{v}_t \quad \text{in} \quad Q, \tag{5.36}$$

$$i(\bar{v}_t)_t + \Delta(\bar{v}_t) = m(x)\bar{v}_t + p(x)\bar{u}_t \quad \text{in} \quad Q, \tag{5.37}$$

$$\bar{u}_t\left(\cdot, \tfrac{T}{2}\right) = -if(x)R_1\left(\cdot, \tfrac{T}{2}\right) \in H_0^1(\Omega) \quad \text{in} \quad \Omega, \tag{5.38}$$

$$\bar{v}_t\left(\cdot, \tfrac{T}{2}\right) = -ig(x)R_2\left(\cdot, \tfrac{T}{2}\right) \in H^1(\Omega) \quad \text{in} \quad \Omega, \tag{5.39}$$

$$\bar{u}_t = 0, \qquad \frac{\partial \bar{v}_t}{\partial \nu} = 0 \quad \text{on} \quad \Sigma. \tag{5.40}$$

and system (5.1)–(5.5).

Step 2. In view of condition (3.19) of Theorem 3.3, we apply the continuous observability inequality (3.21) to $\{\bar{u}_t, \bar{v}_t\}$-problem (5.36)–(5.40)

$$\begin{aligned} &\left\|-if(x)R_1\left(\cdot, \tfrac{T}{2}\right)\right\|^2_{H_0^1(\Omega)} + \left\|-ig(x)R_2\left(\cdot, \tfrac{T}{2}\right)\right\|^2_{H^1(\Omega)} \\ &\qquad \leqslant C^2_{T,q,p} \int_0^T \int_{\Gamma_1} \left(\left|\frac{\partial \bar{u}_t}{\partial \nu}\right|^2 + |\bar{v}_t|^2 + |\bar{v}_{tt}|^2 \right) d\Gamma_1 dt, \end{aligned} \tag{5.41}$$

Using (3.4) and (5.35), we obtain

$$\begin{aligned} \|f\|_{H_0^1(\Omega)} + \|g\|_{H^1(\Omega)} \leqslant& C\left(\left\|\frac{\partial \bar{u}_t}{\partial \nu}\right\|_{L^2(\Gamma_1\times[0,T])} + \|\bar{v}_t\|_{L^2(\Gamma_1\times[0,T])} + \|\bar{v}_{tt}\|_{L^2(\Gamma_1\times[0,T])} \right) \\ =& C\Bigg(\left\|\frac{\partial (u_t - \tilde{u}_t)}{\partial \nu}\right\|_{L^2(\Gamma_1\times[0,T])} + \|v_t - \tilde{v}_t\|_{L^2(\Gamma_1\times[0,T])} \\ &\quad + \|v_{tt} - \tilde{v}_{tt}\|_{L^2(\Gamma_1\times[0,T])} \Bigg) \\ \leqslant& C\left(\left\|\frac{\partial u}{\partial \nu}\right\|_{H^1(0,T;L^2(\Gamma_1))} + \|v_t\|_{H^1(0,T;L^2(\Gamma_1))} \right) \\ &+ C\left(\left\|\frac{\partial \tilde{u}_t}{\partial \nu}\right\|_{L^2(\Gamma_1\times[0,T])} + \|\tilde{v}_t\|_{L^2(\Gamma_1\times[0,T])} + \|\tilde{v}_{tt}\|_{L^2(\Gamma_1\times[0,T])} \right). \end{aligned} \tag{5.42}$$

Once we eliminate the term

$$\left\|\frac{\partial \tilde{u}_t}{\partial \nu}\right\|_{L^2(\Gamma_1\times[0,T])} + \|\tilde{v}_t\|_{L^2(\Gamma_1\times[0,T])} + \|\tilde{v}_{tt}\|_{L^2(\Gamma_1\times[0,T])}$$

in (5.42), we can complete the proof of Theorem 4.1.

Step 3. We drop the term

$$\left\|\frac{\partial \tilde{u}_t}{\partial \nu}\right\|_{L^2(\Gamma_1\times[0,T])} + \|\tilde{v}_t\|_{L^2(\Gamma_1\times[0,T])} + \|\tilde{v}_{tt}\|_{L^2(\Gamma_1\times[0,T])},$$

using a compactness–uniqueness argument.

Step 4. We use Lemma 5.1 to delete the terms

$$\|K\{f,g\}\|_{L^2(\Sigma_1)} = \left\|\frac{\partial \tilde{u}_t}{\partial \nu}\right\|_{L^2(\Sigma_1)}, \quad \|L\{f,g\}\|_{L^2(\Sigma_1)} = \|\tilde{v}_t\|_{L^2(\Sigma_1)}, \quad \|L_1\{f,g\}\|_{L^2(\Sigma_1)} = \|\tilde{v}_{tt}\|_{L^2(\Sigma_1)},$$

in estimate (5.42), via the compactness–uniqueness argument.

Assume that inequality in (4.1) fails. Then there exist sequences $f_n \in H_0^1(\Omega)$, $g_n \in H^1(\Omega)$, $n \geqslant 1$, such that

$$\|f_n\|_{H_0^1(\Omega)} = \|g_n\|_{H^1(\Omega)} = 1, \qquad n \geqslant 1, \tag{5.43}$$

$$\lim_{n\to+\infty}\Big(\Big\|\frac{\partial u(f_n,g_n)}{\partial \nu}\Big\|_{H^1(0,T;L^2(\Gamma_1))} + \|\upsilon_t(f_n,g_n)\|_{H^1(0,T;L^2(\Gamma_1))}\Big) = 0. \tag{5.44}$$

By (5.43), there exist subsequences, still denoted by f_n and g_n, such that

$$\{f_n\} \text{ converges weakly in } H_0^1(\Omega) \text{ to some } f_0 \in H_0^1(\Omega), \tag{5.45}$$

$$\{g_n\} \text{ converges weakly in } H^1(\Omega) \text{ to some } g_0 \in H^1(\Omega). \tag{5.46}$$

Then Lemma 5.1 yields

$$\lim_{m,n\to+\infty}\big\|K\{f_n,g_n\} - K\{f_m,g_m\}\big\|_{L^2(\Sigma_1)} = 0, \tag{5.47}$$

$$\begin{aligned}\lim_{m,n\to+\infty}\big\|L\{f_n,g_n\} - L\{f_m,g_m\}\big\|_{L^2(\Sigma_1)} &= \lim_{m,n\to+\infty}\big\|L_1\{f_n,g_n\} - L_1\{f_m,g_m\}\big\|_{L^2(\Sigma_1)} \\ &= 0.\end{aligned} \tag{5.48}$$

It follows from (5.42) and (5.7) that

$$\begin{aligned}
&\|f_n - f_m\|_{H_0^1(\Omega)} + \|g_n - g_m\|_{H^1(\Omega)} \\
&\leqslant C\Big(\Big\|\frac{\partial u(f_n,g_n)}{\partial \nu} - \frac{\partial u(f_m,g_m)}{\partial \nu}\Big\|_{H^1(0,T;L^2(\Gamma_1))} \\
&\qquad + \|\upsilon_t(f_n,g_n) - \upsilon_t(f_m,g_m)\|_{H^1(0,T;L^2(\Gamma_1))}\Big) \\
&\quad + C\Big(\|K\{f_n,g_n\} - K\{f_m,g_m\}\|_{L^2(\Sigma_1)} + \|L\{f_n,g_n\} - L\{f_m,g_m\}\|_{L^2(\Sigma_1)} \\
&\qquad + \|L_1\{f_n,g_n\} - L_1\{f_m,g_m\}\|_{L^2(\Sigma_1)}\Big) \\
&\leqslant C\Big(\Big\|\frac{\partial u(f_n,g_n)}{\partial \nu}\Big\|_{H^1(0,T;L^2(\Gamma_1))} + \Big\|\frac{\partial u(f_m,g_m)}{\partial \nu}\Big\|_{H^1(0,T;L^2(\Gamma_1))} \\
&\qquad + \|\upsilon_t(f_n,g_n)\|_{H^1(0,T;L^2(\Gamma_1))} + \|\upsilon_t(f_m,g_m)\|_{H^1(0,T;L^2(\Gamma_1))}\Big) \\
&\quad + C\Big(\|K\{f_n,g_n\} - K\{f_m,g_m\}\|_{L^2(\Sigma_1)} + \|L\{f_n,g_n\} - L\{f_m,g_m\}\|_{L^2(\Sigma_1)} \\
&\qquad + \|L_1\{f_n,g_n\} - L_1\{f_m,g_m\}\|_{L^2(\Sigma_1)}\Big).
\end{aligned} \tag{5.49}$$

Then, by (5.44) and (5.47), (5.48), we find

$$\lim_{n,m\to+\infty}\|f_n - f_m\|_{H_0^1(\Omega)} = 0, \qquad \lim_{n,m\to+\infty}\|g_n - g_m\|_{H^1(\Omega)} = 0. \tag{5.50}$$

By uniqueness of the limit and (5.45), (5.46) we obtain

$$\lim_{n\to+\infty}\|f_n - f_0\|_{H_0^1(\Omega)} = 0, \qquad \lim_{n\to+\infty}\|g_n - g_0\|_{H^1(\Omega)} = 0. \tag{5.51}$$

Thus, in view of (5.43), it follows from (5.51) that

$$\|f_0\|_{H_0^1(\Omega)} = \|g_0\|_{H^1(\Omega)} = 1. \tag{5.52}$$

For problem (2.12)–(2.16) under assumptions (3.2) we see that the map

$$W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) \longrightarrow C\big([0,T], H^1(\Omega)\big), \qquad \{f,g\} \longmapsto \{\upsilon(f,g)\} \tag{5.53}$$

is continuous; by the trace theory, the map

$$\Upsilon : W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) \longrightarrow C\big([0,T], H^{1/2}(\Gamma)\big), \qquad \{f,g\} \longmapsto \Upsilon\{f,g\} = \{\upsilon(f,g)|_\Sigma\} \tag{5.54}$$

is continuous. The map

$$\{f,g\} \longmapsto \Big\{\frac{\partial u(f,g)}{\partial \nu}|_{\Sigma}\Big\} \tag{5.55}$$

is also continuous, see [1, Lm. 7].

Since for the $\{u,\upsilon\}$-problem (2.12)–(2.16) the map $\{f,g\} \mapsto \{u(f,g)|_{\Sigma}, \upsilon(f,g)|_{\Sigma}\}$ is linear, it follows from (5.54) and (5.55) that

$$\begin{aligned}
&\Big\|\frac{\partial u(f_n,g_n)}{\partial \nu} - \frac{\partial u(f_0,g_0)}{\partial \nu}\Big\|_{C([0,T];L^2(\Gamma_1))} \leqslant C_{R_1,R_2}\big(\|f_n - f_0\|_{H_0^1(\Omega)} + \|g_n - g_0\|_{H^1(\Omega)}\big),\\
&\|\upsilon(f_n,g_n) - \upsilon(f_0,g_0)\|_{C([0,T];L^2(\Gamma_1))} \leqslant C_{R_1,R_2}\big(\|f_n - f_0\|_{H_0^1(\Omega)} + \|g_n - g_0\|_{H^1(\Omega)}\big).
\end{aligned} \tag{5.56}$$

Replacing (5.51) on the rightвЂ‘hand side of (5.56), we find

$$\lim_{n\to+\infty}\Big\|\frac{\partial u(f_n,g_n)}{\partial \nu} - \frac{\partial u(f_0,g_0)}{\partial \nu}\Big\|_{C([0,T];L^2(\Gamma_1))} = 0, \tag{5.57}$$

$$\lim_{n\to+\infty}\big\|\upsilon(f_n,g_n)|_{\Sigma_1} - \upsilon(f_0,g_0)|_{\Sigma_1}\big\|_{C([0,T];L^2(\Gamma_1))} = 0. \tag{5.58}$$

Then, by virtue of (5.44) together with (5.57), for $t \in [0,T]$ we obtain

$$\frac{\partial u(f_0,g_0)}{\partial \nu}(x,t) = 0, \qquad (x,t) \in \Sigma_1. \tag{5.59}$$

Similarly, for $\{u_t,\upsilon_t\}$-problem (5.30)–(5.34),

$$\lim_{n\to+\infty}\big\|\upsilon_t(f_n,g_n)|_{\Sigma_1} - \upsilon_t(f_0,g_0)|_{\Sigma_1}\big\|_{C([0,T];L^2(\Gamma_1))} = 0, \tag{5.60}$$

hence, by (5.44) and (5.60),

$$\upsilon_t(f_0,g_0)|_{\Sigma_1} = 0. \tag{5.61}$$

Returning to problem (2.12)–(2.16) with $f = f_n \in W^{1,\infty}(\Omega)$, $g = g_n \in W^{1,\infty}(\Omega)$, the initial conditions give

$$\upsilon(f_n,g_n)\big(\cdot,\tfrac{T}{2}\big) = 0, \quad x \in \overline{\Omega},$$

therefore

$$\upsilon(f_n,g_n)\big(\cdot,\tfrac{T}{2}\big) = 0, \quad x \in \Gamma_1, \tag{5.62}$$

in the sense of trace in $H^{\frac{1}{2}}(\Gamma_1)$. Thus, in view of (5.62), it follows from (5.58) that

$$\upsilon(f_0,g_0)\big(\cdot,\tfrac{T}{2}\big) = 0, \quad x \in \Gamma_1,$$

and next, by (5.61),

$$\upsilon(f_0,g_0) = 0, \qquad x \in \Gamma_1, \quad 0 < t < T. \tag{5.63}$$

Finally, by virtue of (5.59) and (5.63), we apply Theorem 3.1

$$f_0(x) = g_0(x) = 0, \qquad x \in \Omega. \tag{5.64}$$

Then (5.64) contradicts (5.52). This completes the proof. □

6. Stability in nonlinear case

Here we prove Theorem 4.2, which is in fact a direct consequence of Theorem 4.1. By (2.11) we have

$$\begin{aligned}
&f(x) = q_1(x) - q_2(x), \qquad g(x) = p_1(x) - p_2(x),\\
&R_1(x,t) = z(q_2,p_2)(x,t), \qquad R_2(x,t) = w(q_2,p_2)(x,t),\\
&u(x,t) = w(q_1,p_1)(x,t) - w(q_2,p_2)(x,t),\\
&\upsilon(x,t) = z(q_1,p_1)(x,t) - z(q_2,p_2)(x,t).
\end{aligned} \tag{6.1}$$

Since

$$q_1(x), q_2(x), p_1(x), p_2(x) \in W^{1,\infty}(\Omega), \qquad \{q_1, q_2\}|_\Gamma = 0,$$
$$w(q_2, p_2)(x,t) \in C([0,T], H_0^1(\Omega)), \qquad z(q_2, p_2)(x,t) \in C([0,T], H^1(\Omega)),$$

it follows from (6.1) that

$$f(\cdot) \in W^{1,\infty}(\Omega), \qquad f|_\Gamma = 0, \qquad g(\cdot) \in W^{1,\infty}(\Omega), \qquad R_1(\cdot,\cdot), R_2(\cdot,\cdot) \in C([0,T], H^1(\Omega))$$

as assumed in Theorem 4.1. Thus, we can apply Theorem 4.1 to the variables $\{u = w(q_1,p_1) - w(q_2,p_2),\ v = z(q_1,p_1) - z(q_2,p_2)\}$, solve problem (2.12)–(2.16), and we then obtain the desired stability estimate (4.2), i.e.

$$\|q_1 - q_2\|_{H_0^1(\Omega)} + \|p_1 - p_2\|_{H^1(\Omega)} \leqslant C\biggl(\biggl\|\frac{\partial w(q_1,p_1)}{\partial\nu} - \frac{\partial w(q_2,p_2)}{\partial\nu}\biggr\|_{H^1(0,T;L^2(\Gamma_1))} + \Bigl\|z_t(q_1,p_1) - z_t(q_2,p_2)\Bigr\|_{H^1(0,T;L^2(\Gamma_1))}\biggr).$$

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