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COARSE–GRAINING STATES ON ULTRAPRODUCTS OF PROBABILITY ALGEBRAS

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Abstract. A probability algebra, which is a kind of analogue of a probability space, is designed to describe a physical system. If we have a prior distribution defined by a faithful state on this algebra and we can define the behavior of the system in terms of this state, then we can also make some prediction about its behavior. In practice, however, it is impossible to measure all observable quantities on the original algebra, so we only perform a partial measurement, that is, we measure the observable quantities on a certain subalgebra. This measurement then determines another state on this subalgebra. If the resulting state extends to the original algebra without introducing additional information about the behavior of system, then we call the resulting state on the original algebra coarse–graining.

In the paper we discuss the existence of coarse–graining states, referred to as $(\mathcal{B},\omega)$–coarse–graining of a state, and we also consider ultraproducts of sequences of probability algebras and states. We show that under the introduced definition, ultraproducts of sequences of probability algebras are probability algebras, and we discuss the existence of conditional mathematical expectations, coarse–grained states, and information.

Keywords: probability algebra, equivalent states, coarse–graining states, information, ultraproducts.

Mathematics Subject Classification: 81Qxx, 46M07

1. Introduction

This work continues a series of studies of ultraproducts of various quantum structures [1], [2], [6]–[9]. The author is interested in studying state relationships, such as absolute continuity and singularity of states. However, the definitions of absolute continuity of a state with respect to to another state may differ for different structures. This is due to the fact that sometimes it is necessary to consider only faithful states, which, according to the traditional definition, are trivially mutually absolutely continuous with respect to other faithful states.

It is also interesting to study questions related to the concept of absolute continuity, namely, the existence of Radon — Nikodym derivatives, conditional expectation, coarse–graining states, and information on probability algebras. The concept of probability algebra goes back to Day and Segal [4], [12].

To preserve the useful properties of the "factors" in the ultraproduct, we modify the definition of the ultraproduct subject to structure. We consider ultrafilters only in the set of natural numbers that is motivated by questions of state equivalence and mutual contiguity of sequences of states. The concept of contiguity was first introduced by Le Cam [10] in the statistical problem of distinguishing closely related hypotheses with increasing sample size.

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2. EQUIVALENCE AND ORTHOGONALITY OF STATES ON PROBABILITY ALGEBRAS

Definition 2.1 ([5]). *Suppose that we are given an* **–algebra* $\mathcal{A}$ *over the field of complex numbers with the involution* * *and a unity. A linear functional* $\omega : \mathcal{A} \to \mathbb{C}$ *is called a state if*

i) *ω is positive:* $\omega(x^*x) \geqslant 0$ *for all* $x \in \mathcal{A}$*;*

ii) *ω is normalized:* $\omega(\mathbf{1}) = 1$.

A state ω is faithful if $\omega(x^*x) = 0$ *implies* $x = \mathbf{0}$*. Let* $\omega : \mathcal{A} \to \mathbb{C}$ *be a faithful state on* $\mathcal{A}$*. Then the triple* $(\mathcal{A},\, ^*,\, \omega)$ *is called a probability algebra.*

Let $(\mathcal{A},\, ^*,\, \omega)$ be a probability algebra. We introduce the scalar product on $\mathcal{A}$ by letting

$$\langle x, y\rangle = \omega(y^*x).$$

We denote by H the Hilbert space that is the completion of $\mathcal{A}$ with respect to this scalar product. The extension of the state ω from $\mathcal{A}$ to H is denoted by the same symbol ω.

Definition 2.2 ([11]). *A representation π of an algebra $\mathcal{E}$ in a Hilbert space $\tilde{H}$ is a mapping from $\mathcal{E}$ into the set of linear operators defined in a common dense domain $D(\pi)$ (dense in $\tilde{H}$) and satisfying the conditions*

1. $\pi(\mathbf{1}) = \mathbf{1}$*;*
2. $\pi(ax + y)z = a\pi(x)z + \pi(y)z \quad x, y \in \mathcal{E}, z \in D(\pi), a \in \mathbb{C}$*;*
3. $\pi(x)D(\pi) \subset D(\pi)$ *for all* $x \in \mathcal{E}$ *and* $\pi(x)\pi(y)z = \pi(xy)z, \quad x, y \in \mathcal{E}, \quad z \in D(\pi)$.
4. $\pi(x^*) \subset \pi(x)^*$ *for all* $x \in \mathcal{A}$.

Definition 2.3 ([11]). *A representation of π^*-algebra $\mathcal{E}$ in a Hilbert space H is called a Hermitian or *-representation if* $(y, \pi(x)z) = (\pi(x^*)y, z)$ *for all* $y, z \in D(\pi)$ *and* $x \in \mathcal{E}$.

We note that the representation π is Hermitian if and only if for each Hermitian $x \in \mathcal{E}$, that is, $x = x^*$, the operator $\pi(x)$ is Hermitian.

Let $(\mathcal{A},\, ^*,\, \omega)$ be a probability algebra. For $x \in \mathcal{A}$ we define the operator $\pi(x) : \mathcal{A} \to \mathcal{A}$ by the formula

$$\pi(x)y = xy. \tag{2.1}$$

Then the operator π is a Hermitian *–representation of the probability algebra $\mathcal{A}$ in a dense invariant domain $D(\pi) = \mathcal{A} \subseteq H$.

We define the adjoint representation of π^* as

$$D(\pi^*) = \{\cap D(\pi(x)^*) : x \in \mathcal{A}\}, \qquad \pi^*(x) = [\pi(x^*)]^*|D(\pi^*).$$

In what follows we consider representations 2.1.

We denote by $L(\mathcal{A})$ the set of all linear mappings $C : \mathcal{A} \to H$.

Definition 2.4 ([5]). *The set*

$$\pi(\mathcal{A})^c = \big\{C \in L(\mathcal{A}) : \langle C\pi(x)y, z\rangle = \langle Cy, \pi(x^*)z\rangle, \quad x, y, z \in \mathcal{A}\big\}$$

is called the weak unbounded commutator subgroup of the operator π.

It can be shown that if $C \in \pi(\mathcal{A})^c$, then $C : \mathcal{A} \to D(\pi^*)$ and

$$C\pi(x) = \pi^*(x)C$$

for all $x \in \mathcal{A}$.

Example 2.1. *Let $(H, \mathcal{M}, \tau)$ be a regular probability gauge space, that is, H is a complex Hilbert space, $\mathcal{M}$ is the von Neumann algebra on H, and τ is a faithful normal trace state on $\mathcal{M}$ (see [12] for more detail). In the algebra $\mathcal{M}$ we take a set (P_α) of mutually orthogonal projections such that $\sum_\alpha P_\alpha = \mathbf{1}$. We consider the set $\mathcal{A}$ of simple operators on H, in other words, finite sums $T = \sum \lambda_i P_i$, where $\lambda_i \in \mathbb{C}$, $P_i \in (P_\alpha)$. We define the faithful state ω on $\mathcal{A}$ by letting*

$$\omega(T) = \sum \lambda_i \tau(P_i).$$

Then the triple $(\mathcal{A}, {}^*, \omega)$ is a probability algebra.

Definition 2.5. *Let $(\mathcal{A}, {}^*, \omega)$ be a probability algebra.*

1) *A state ν is called absolutely continuous with respect to a state ω (denoted $\nu \ll \omega$) if $\omega(x_k^* x_k) \to 0$ implies $\nu(y x_k) \to 0$, $(k \to \infty)$ for each $y \in \mathcal{A}$. Mutually absolutely continuous states are called equivalent.*
2) *States ω and ν are called orthogonal if there exists a sequence $(x_k) \in \mathcal{A}$ such that $\omega(x_k^* x_k) \to 0$ and $\nu(1 - y x_k) \to 0$, $k \to \infty$, for each $y \in \mathcal{A}$.*

It is known that that on probability algebras the analog of Radon — Nikodym theorem holds: a state ν is absolutely continuous with respect to a state ω if and only if there exists a non–negative element $C =: \dfrac{d\nu}{d\omega} \in \pi(\mathcal{A})^c$ such that $\omega(C\mathbf{1}) = 1$ and $\nu(x) = \omega(Cx)$ for all $x \in \mathcal{A}$ [5]. If the state ν is faithful, then $C > 0$.

Let us define the concept of contiguity for sequences of states defined on probability algebras.

Definition 2.6. *Let $(\mathcal{A}_n, {}^*, \omega_n)_{n \geqslant 1}$ be a sequence of probability algebras.*

1) *A sequence of states (ν_n) defined on a sequence of algebras $(\mathcal{A}_n, {}^*, \omega_n)_{n \geqslant 1}$ is called contiguous with respect to the sequence (ω_n) if*

$$\lim_{n\to\infty} \omega_n(x_n^* x_n) = 0 \quad \textit{implies} \quad \lim_{n\to\infty} \nu_n(y_n x_n) = 0 \quad \textit{for each} \quad y_n \in \mathcal{A}_n.$$

If the sequence (ω_n) is contiguous with respect to the sequence (ν_n), then these sequences are called mutually contiguous or, simply, contiguous;

2) *Sequences of states (ω_n) and (ν_n) are called completely asymptotically separable if there exists a subsequence $(n') \subseteq (n)$ and $(x_{n'}) \in \mathcal{A}_{n'}$ such that*

$$\lim_{n'\to\infty} \omega_{n'}(x_{n'}^* x_{n'}) = 0, \qquad \lim_{n'\to\infty} \nu_{n'}(\mathbf{1} - y_{n'} x_{n'}) = 0 \quad \textit{for each} \quad y_{n'} \in \mathcal{A}_{n'}.$$

It is clear that the contiguity of sequences of states is a generalization of the concept of equivalence of states, and the complete asymptotic separability generalizes the concept of orthogonality of states.

Let $(\mathcal{A}, {}^*, \omega)$ be a probability algebra and $\mathcal{B}$ be a * subalgebra of $\mathcal{A}$. In this case $\mathcal{B}$ is a subspace of H and its closure $\bar{\mathcal{B}}$ is a (closed) subspace of H. We denote by $\pi|\mathcal{B}$ the *-representation of the *-subalgebra $\mathcal{B}$.

Definition 2.7 ([5])**.** *For $x \in \mathcal{A}$, the conditional mathematical expectation of an element x with respect to a *–subalgebra $\mathcal{B}$ is an element $\mathbb{E}(x|\mathcal{B})$ if*

$$\mathbb{E}(x|\mathcal{B}) \in D[(\pi|\mathcal{B})^*] \subseteq \bar{\mathcal{B}}, \quad \omega(yx) = \omega[(\pi|\mathcal{B})^*(y)\mathbb{E}(x|\mathcal{B})]$$

for all $y \in \mathcal{B}$.

The existence of the conditional expectation is implied by the following fact [5]. Suppose that for all $x \in \mathcal{A}$, a state $\omega_x : \mathcal{B} \to \mathbf{C}$ is defined as follows: we denote $\omega_x(y) = \omega(\pi(x)y) = \omega(xy)$, which turns out to be absolutely continuous with respect to the state $\omega|\mathcal{B}$. We call this a transformation of the state ω with respect to the representation π. Then $\mathbb{E}(x|\mathcal{B}) = (d\omega_{x^*}/d(\omega|\mathcal{B}))\mathbf{1}$.

It is easy to see that for $y = \mathbf{1}$ the identity holds

$$\omega(x) = \omega(\mathbb{E}(x|\mathcal{B})).$$

Another property of conditional mathematical expectations is also known

$$\mathbb{E}(x|\mathcal{B}) = P_{\mathcal{B}}(x),$$

where $P_{\mathcal{B}} : H \to \bar{\mathcal{B}}$ is the projection.

Definition 2.8 ([5]). *Let $(\mathcal{A}, {}^*, \omega)$ be a probability algebra, and $\mathcal{B} \subseteq \mathcal{A}$ be a subalgebra. Let ν be a state on $\mathcal{B}$ such that $\nu \ll \omega|\mathcal{B}$. The $(\mathcal{B},\omega)$-coarse–graining ν_c of a state ν on the algebra $\mathcal{A}$ is a state, which obeys the conditions*

1) *$\nu_c|\mathcal{B} = \nu$;*
2) *$\nu_c \ll \omega$;*
3) *$\nu_c(x) = \nu_c[\mathbb{E}(x|\mathcal{B})]$ for all $x \in \mathcal{A}$.*

Let us consider one more concept that will be needed later.

Definition 2.9 ([5]). *A subalgebra of $\mathcal{B}$ is said to preserve positivity if for each $x \in \mathcal{A}$ there exists a sequence (y_k), $y_k \in \mathcal{B}$, such that*

$$y_k^* y_k \to P_{\mathcal{B}}(x^* x), \quad k \to \infty,$$

where the convergence is considered as that in the norm on the *–subalgebra $\mathcal{B}$.

We provide some known results on the existence and properties of $(\mathcal{B},\omega)$-coarse–graining.

Theorem 2.1 ([3]). *Let $(\mathcal{A}, {}^*, \omega)$ be a probability algebra and $\mathcal{B} \subseteq \mathcal{A}$ be a *–subalgebra. Suppose that *–subalgebra $\mathcal{B}$ preserves positivity, and ν is a state on $\mathcal{B}$ such that $\nu \ll \omega|\mathcal{B}$. Then $(\mathcal{B},\omega)$-coarse–graining of ν exists.*

Theorem 2.2 ([5]). *Let $(\mathcal{A}, {}^*, \omega)$ be a probability algebra and $\mathcal{B} \subseteq \mathcal{A}$ be a *-subalgebra. If a $(\mathcal{B},\omega)$-coarse–grain ν_c of a state ν exists and is unique, then ν_c is an absolutely continuous extension of the state ν to the algebra $\mathcal{A}$, and $(d\nu_c/d\omega)|\mathcal{B} = d\nu/d(\omega|\mathcal{B})$.*

Definition 2.10. *Let $(\mathcal{A}, {}^*, \omega)$ be a probability algebra and ν be a state on $\mathcal{A}$. Information about a state ν with respect to the state ω is a mapping $\nu \to I_\omega(\nu)$, which satisfies*

1) *$I_\omega(\nu) \geqslant 0, I_\omega(\omega) = 1$;*
2) *for states $\nu_1 \ll \omega, \nu_2 \ll \omega$ such that $\langle (d\nu_1/d\omega)\mathbf{1}, (d\nu_2/d\omega)\mathbf{1}\rangle = 0$,*

$$I_\omega(\nu_1 + \nu_2) = I_\omega(\nu_1) + I_\omega(\nu_2).$$

It is known [5, Thm. 4] that the information $I_\omega(\nu)$ exists and is unique, and

$$I_\omega(\nu) = \|(d\nu/d\omega)\mathbf{1}\|^2,$$

and if $(\mathcal{B},\omega)$-coarse–graining of state ν exists, then it is the unique continuous extension of ν to the algebra $\mathcal{A}$ with minimal information with respect to the state ω.

3. ULTRAPRODUCT OF PROBABILITY ALGEBRAS

Let $(\mathcal{A}_n,\, ^*,\, \omega_n)_{n\geqslant 1}$ be a sequence of probability algebras, and $\mathscr{U}$ be a nontrivial ultrafilter on the set $\mathbb{N}$. We consider the ultraproduct $((\mathcal{A}_n)_{\mathscr{U}},\, ^*,\, \omega_{\mathscr{U}})$ of this sequence (see definition in [9]).

Theorem 3.1 ([9]). *The ultraproduct of a sequence of probability algebras is a probability algebra.*

In a natural way, on the probability algebra $((\mathcal{A}_n)_{\mathscr{U}},\, ^*,\, \omega_{\mathscr{U}})$, we introduce the *–representation $\pi_{\mathscr{U}}$ and its adjoint $\pi^*_{\mathscr{U}}$:

$$\pi_{\mathscr{U}}(x)y = (\pi_n(x_n)y_n)_{\mathscr{U}} = (x_n)_{\mathscr{U}}(y_n)_{\mathscr{U}}, \quad x = (x_n)_{\mathscr{U}}, \quad y = (y_n)_{\mathscr{U}}.$$

Theorem 3.2. *Let $(\mathcal{A}_n,\, ^*,\, \omega_n)_{n\geqslant 1}$ be a sequence of probability algebras.*

1) *If sequences of states (ω_n) and (ν_n) are mutually contiguous, then the states $\omega_{\mathscr{U}}$ and $\nu_{\mathscr{U}}$ are equivalent on the ultraproduct $(\mathcal{A}_n)_{\mathscr{U}}$ for each nontrivial ultrafilter $\mathscr{U}$ on $\mathbb{N}$.*
2) *If sequences of states (ω_n) and (ν_n) are completely asymptotically separable, then the states $\omega_{\mathscr{U}}$ and $\nu_{\mathscr{U}}$ are orthogonal on the ultraproduct $(\mathcal{A}_n)_{\mathscr{U}}$ for some nontrivial ultrafilter $\mathscr{U}$ on $\mathbb{N}$.*

Proof. Assertion 1) was proved in [6]. Here prove Assertion 2).

We suppose the contrary, that is, the states $\omega_{\mathscr{U}}$ and $\nu_{\mathscr{U}}$ on the ultraproduct $(\mathcal{A}_n)_{\mathscr{U}}$ are non–orthogonal. This means that for each ultrafilter $\mathscr{U}$, for each sequence $(x_k) \in (\mathcal{A}_n)_{\mathscr{U}}$, there exists an element $y \in (\mathcal{A}_n)_{\mathscr{U}}$ such that

$$\lim_{k\to\infty} \omega_{\mathscr{U}}(x_k^* x_k) = 0, \qquad \lim_{k\to\infty} \nu_{\mathscr{U}}(\mathbf{1} - y x_k) > \delta \quad \text{for some} \quad \delta > 0.$$

This implies that there exists k_0 such that for all $k > k_0$

$$\omega_{\mathscr{U}}(x_k^* x_k) < \varepsilon, \qquad \nu_{\mathscr{U}}(\mathbf{1} - y x_k) > \delta.$$

Hence, for all $k > k_0$ there exists $U_k \in \mathscr{U}$, and for all $n(k) \in U_k$

$$\omega_{n(k)}(x^*_{n(k)} x_{n(k)}) < \varepsilon, \qquad \nu_{n(k)}(\mathbf{1} - y_{n(k)} x_{n(k)}) > \delta.$$

The latter contradicts the fact that the sequences of states (ω_n) and (ν_n) are completely asymptotically separable. The proof is complete. ☐

We provive the conditions ensuring the existence of a conditional mathematical expectation and the information associated with it in the ultraproduct of probability algebras.

Theorem 3.3. *Let $(\mathcal{A}_n,\, ^*,\, \omega_n)_{n\geqslant 1}$ be a sequence of probability algebras, $\mathcal{B}_n$ be a *–subalgebra of $\mathcal{A}_n$, $n \in \mathbb{N}$, $\mathscr{U}$ be an ultrafilter in the set $\mathbb{N}$. Assume that*

1) *for each n there is a conditional mathematical expectation $\mathbb{E}(x_n|\mathcal{B}_n)$;*
2) *the sequence of states $((\omega_n)_{x_n^*})$ defined on the sequence of *–subalgebras $(\mathcal{B}_n)$ is contiguous with respect to the sequence $(\omega_n|\mathcal{B}_n)$, where $(\omega_n)_{x_n}(y_n) = \omega_n(x_n y_n)$, $y_n \in \mathcal{B}_n$;*
3) *the sequence of states (ν_n) is contiguous with respect to the sequence (ω_n).*

Then on the ultraproduct $((\mathcal{A}_n)_{\mathscr{U}},\, ^*,\, \omega_{\mathscr{U}})$ there exists a conditional mathematical expectation with respect to the subalgebra $(\mathcal{B}_n)_{\mathscr{U}}$ and

$$\mathbb{E}((x_n)_{\mathscr{U}}|(\mathcal{B}_n)_{\mathscr{U}}) = (\mathbb{E}(x_n|\mathcal{B}_n))_{\mathscr{U}},$$

and there exists an information about the state of $\nu_{\mathscr{U}}$ with respect to the state of $\omega_{\mathscr{U}}$ and

$$I_{\omega_{\mathscr{U}}}(\nu_{\mathscr{U}}) = \lim_{\mathscr{U}} \|(d\nu_n/d\omega_n)\mathbf{1_n}\|^2.$$

Proof. The proof of the theorem follows immediately from the definition of the ultraproduct and Theorem 3.2. Indeed, Condition 2) ensures that the state $((\omega_n)_{x_n^*})_{\mathscr{U}}$ is absolutely continuous with respect to the state $(\omega_n|\mathcal{B}_n)_{\mathscr{U}}$, and, consequently, there exists the Radon — Nikodym derivative

$$d(\omega_{x_n^*})_{\mathscr{U}}/d(\omega_n|\mathcal{B}_n)_{\mathscr{U}} = \lim_{\mathscr{U}} d(\omega_{x_n^*})/d(\omega_n|\mathcal{B}_n)$$

and $[d(\omega_{x_n^*})_{\mathscr{U}}/d(\omega_n|\mathcal{B}_n)_{\mathscr{U}}]\mathbf{1} = \mathbb{E}((x_n)_{\mathscr{U}}|(\mathcal{B}_n))_{\mathscr{U}}$.

In the same way, the existence of information and the necessary identity are derived from Condition 3). The proof is complete. □

We introduce the following definition.

Definition 3.1. *Let $(\mathcal{A}_n, {}^*, \omega_n)_{n\geqslant 1}$ be a sequence of probability algebras,*

$$\mathcal{A} = \{(x_n),\ x_n \in \mathcal{A}_n : \sup_n \|x_n\|_n < \infty\}.$$

We say that a sequence $(x^{(k)})_{k\geqslant 1} = (x_n^{(k)})_{k\geqslant 1} \in \mathcal{A}$ converges uniformly to an element $x = (x_n) \in \mathcal{A}$ as $k \to \infty$ if for each $\varepsilon > 0$ there exists k_0 such that for all $k > k_0$ and all $n \in \mathbb{N}$

$$\|x_n^{(k)} - x_n\|_n < \varepsilon.$$

Theorem 3.4. *Let $(\mathcal{A}_n, {}^*, \omega_n)_{n\geqslant 1}$ be a sequence of probability algebras, $\mathcal{B}_n \subseteq \mathcal{A}_n$ be *–subalgebras ($n \in \mathbb{N}$), and $(\mathcal{B}_n)_{\mathscr{U}}$ be the ultraproduct of sequence $(\mathcal{B}_n)$.*

If *–subalgebra $\mathcal{B}_n$ preserves positivity, $n \in \mathbb{N}$, and for each $x_n \in \mathcal{A}_n$ there exists a sequence (y_n^k), $y_n^k \in \mathcal{B}_n$, such that $y_n^{k*}y_n^k \to P_{\mathcal{B}_n}(x_n^* x_n)$ uniformly as $k \to \infty$, then *-subalgebra $(\mathcal{B}_n)_{\mathscr{U}}$ preserves positivity.

Proof. We take an arbitrary sequence $x^k \in (\mathcal{A}_n)_{\mathscr{U}}$, $x^k = (x_n^k)_{\mathscr{U}}$. The assumptions of theorem implies that for each n there exists a sequence (y_n^k) such that $\lim_{k\to\infty} \|y_n^{k*}y_n^k - P_{\mathcal{B}_n}(x_n^{k*}x_n^k)\| = 0$. Let $y_k = (y_n^k)_{\mathscr{U}}$, $\mathcal{B} = (\mathcal{B}_n)_{\mathscr{U}}$. Then

$$\begin{aligned}\lim_{k\to\infty} \|y^{k*}y^k - P_{\mathcal{B}}(x^{k*}x^k)\| &= \lim_{k\to\infty}\lim_{\mathscr{U}} \|y_n^{k*}y_n^k - P_{\mathcal{B}_n}(x_n^{k*}x_n^k)\| \\ &= \lim_{\mathscr{U}}\lim_{k\to\infty} \|y_n^{k*}y_n^k - P_{\mathcal{B}_n}(x_n^{k*}x_n^k)\| = 0.\end{aligned}$$

The interchange of limits in the double limit is ensured by the condition of uniform convergence of the corresponding sequence. The proof is complete. □

Theorem 3.5. 1) *Let $(\mathcal{A}_n, {}^*, \omega_n)_{n\geqslant 1}$ be a sequence of probability algebras, $\mathcal{B}_n \subseteq \mathcal{A}_n$ be *–subalgebras ($n \in \mathbb{N}$), ν_n be a state on $\mathcal{B}_n$ such that $\nu_n \ll \omega_n|\mathcal{B}_n$ and the sequence (ν_n) be contiguous with respect to $(\omega_n|\mathcal{B}_n)$, and $\mathscr{U}$ be a nontrivial ultrafilter on the set $\mathbb{N}$. If there exists $(\mathcal{B}_n, \omega_n)$–coarse–graining ν_{nc} of the state ν_n on the algebra $\mathcal{A}_n$ and *–subalgebra $(\mathcal{B}_n)_{\mathscr{U}}$ preserves positivity, then the state $\nu_{\mathscr{U}c}$ on the algebra $(\mathcal{A}_n)_{\mathscr{U}}$ is a coarse–graining of the state $\nu_{\mathscr{U}}$, where*

$$\nu_{\mathscr{U}c}(\cdot) = \lim_{\mathscr{U}} \nu_{nc}(\cdot);$$

2) *If (ν_{nc}) is contiguous with respect to (ω_n), then the state $\nu_{\mathscr{U}c}$ is the only absolutely continuous extension of the state $\nu_{\mathscr{U}}$ to the algebra $(\mathcal{A}_n)_{\mathscr{U}}$ with minimal information with respect to $\omega_{\mathscr{U}}$.*

Proof. The existence of coarse–graining on the ultraproduct is implied by Theorem 2.2 and 3.4. Assertion 2) follows from Theorem 3.3. □

BIBLIOGRAPHY

1. S.G. Haliullin. *Ultraproducts of quantum mechanical systems* // Ufa Math. J. **14**:2, 90–96 (2022). https://doi.org/10.13108/2022-14-2-90
2. S.G. Khaliullin. *Extreme point of completely convex state structure* // Ufa Math. J. **16**:3, 107–112 (2024). https://doi.org/10.13108/2024-16-3-107
3. S.J. Bhatt. *Sufficiency and strong commutants in quantum probability theory* // Proc. Proc. Indian Acad. Sci., Math. Sci. **95**:2, 97–107 (1986). https://doi.org/10.1007/BF02881073
4. H.A. Dye. *The Radon — Nikodym theorem for finite rings of operators* // Trans. Amer. Math. Soc. **72**, 243–280 (1952). https://doi.org/10.2307/1990754
5. S. Gudder, R. Hudson. *A noncommutative probability theory* // Trans. Arner. Math. Soc. **245**, 1–41 (1978). https://doi.org/10.2307/1998855
6. S. Haliullin. *Contiguity and entire separability of states on von Neumann algebras* // Int. J. Theor. Phys. **56**:12, 3889–3894 (2017). https://doi.org/10.1007/s10773-017-3373-z
7. S. Haliullin. *Representations of von Neumann algebras and ultraproducts* // Int. J. Theor. Phys. **59**:4, 1010–1016 (2020). https://doi.org/10.1007/s10773-019-04318-x
8. S.G. Haliullin. *Ultraproducts for state–spaces of algebra and Radon measures* // Lobachevskii J. Math. **41**:4, 655–660 (2020). https://doi.org/10.1134/S1995080220040149
9. S.G. Haliullin. *Probability algebras and ultraproducts* // Lobachevskii J. Math. **45**:1, 412–415 (2024). https://doi.org/10.1134/S1995080224010207
10. L. Le Cam. *Locally asymptotically normal families of distributions. Certain approximations to families of distributions and their use in the theory of estimation and testing hypotheses* // Univ. Cflifornia Publ. Statist. **3**, 37–98 (1960).
11. R.T. Powers. *Self–adjoint algebras of unbounded operators* // Comm. Math. Phys. **21**:2, 85–124 (1971). https://doi.org/10.1007/BF01646746
12. I.E. Segal. *A non-commutative extension of abstract integration* // Ann. Math. (2) **57**:2, 401–457 (1953). https://doi.org/10.2307/1969729

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