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ON DISCRETIZATION METHOD FOR THREE-DIMENSIONAL SEMI-DISCRETE TODA LATTICE TYPE EQUATIONS

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Abstract. In this paper we develop the classification method of three-dimensional integrable discrete lattices based on Darboux integrable finite-field reductions of equations with one continuous and two discrete independent variables. As an example we consider a semi-discrete Toda lattice. In the framework of the present study we obtain the Toda lattice.

Keywords: three-dimensional equations, characteristic integrals, Darboux integrability, finite-field reductions, semi-discrete Toda lattice.

Mathematics Subject Classification: 35Q51, 39A14

1. INTRODUCTION

In this paper we discuss a method for classifying discrete equations with three independent variables of Hirota — Miwa type equation

$$u_{n,m+1}^{j+1} = h(u_{n,m+1}^j, u_{n,m}^{j+1}, u_{n+1,m+1}^j, u_{n,m}^j, u_{n-1,m}^{j+1}), \quad (1.1)$$

where the function $u = u_{n,m}^j$ depends on three discrete variables n , j , and m . The approach is based on the discretization of three-dimensional equations of the semi-discrete Toda lattice type

$$u_{n,x}^{j+1} = g(u_{n,x}^j, u_n^{j+1}, u_{n+1}^j, u_n^j, u_{n-1}^{j+1}), \quad (1.2)$$

where the function $u = u_n^j(x)$ depends on the continuous variable x and the discrete variables n and j . In other words, to obtain a list of integrable equations of form (1.1), it is necessary to study the list of all known equations of form (1.2), which includes the lattices

$$(L_1) \quad u_{n,x}^{j+1} = u_{n,x}^j + e^{u_n^j - u_{n-1}^{j+1}} - e^{u_{n+1}^j - u_n^{j+1}};$$

$$(L_2) \quad u_{n,x}^{j+1} = u_{n,x}^j - e^{u_{n-1}^{j+1} - u_n^j} + e^{u_n^{j+1} - u_{n+1}^j} - e^{u_{n-1}^{j+1} - u_n^{j+1}} + e^{u_n^j - u_{n+1}^j};$$

$$(L_3) \quad u_{n,x}^{j+1} = u_{n,x}^j \frac{(u_n^{j+1})^2}{u_{n+1}^j u_{n-1}^{j+1}};$$

$$(L_4) \quad u_{n,x}^{j+1} = u_{n,x}^j \frac{u_{n+1}^j - u_n^{j+1}}{u_n^j - u_{n-1}^{j+1}};$$

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$$(L_5) \quad u_{n,x}^{j+1} = u_{n,x}^j \frac{u_n^{j+1} (u_n^{j+1} - u_{n+1}^j)}{u_{n+1}^j (u_{n-1}^{j+1} - u_n^j)};$$

$$(L_6) \quad u_{n,x}^{j+1} = u_{n,x}^j \frac{(u_n^{j+1} - u_{n-1}^{j+1}) (u_n^{j+1} - u_{n+1}^j)}{(u_{n+1}^j - u_n^j) (u_{n-1}^{j+1} - u_n^j)};$$

$$(L_7) \quad u_{n,x}^{j+1} = u_{n,x}^j + e^{u_{n-1}^{j+1} - u_n^{j+1} - u_n^j + u_{n+1}^j};$$

$$(L_8) \quad u_{n,x}^{j+1} = u_{n,x}^j + e^{u_{n+1}^j} + e^{u_{n-1}^{j+1}} - e^{u_n^{j+1}} - e^{u_n^j};$$

$$(L_9) \quad u_{n,x}^{j+1} = u_{n,x}^j + (u_n^{j+1} - u_n^j) (u_n^{j+1} + u_n^j - u_{n-1}^{j+1} - u_{n+1}^j).$$

Integrable lattices (L_1) – (L_9) were found in the framework of various studies in the following works. Equations (L_1) and (L_8) were obtained as iterations of the Bäcklund transformation for nonlinear differential–difference models by Date, Jimba, and Miwa [8]. Equation (L_7) was found as an equation for the Laplace invariants of linear differential–difference equations with variable coefficients by Startsev and Adler [1]. Lattices (L_2) – (L_6) were obtained as a result of solving the problem on classifying integrable equations of form (1.2), under some additional assumption, by the dispersion deformations method by Ferapontov, Novikova and Rustemoglou [9]. Equation (L_9) was derived while studying the properties of tangential mapping by Adler in [7].

Remark 1.1. *In our recent joint works with I.T. Habibullin [12], [6] we obtained a complete list of equations (L_1) – (L_9) by discretizing all known three–dimensional equations of the two–dimensional Toda lattice type*

$$u_{n,xy} = f(u_{n,x}, u_{n,y}, u_{n+1}, u_n, u_{n-1}), \quad (1.3)$$

which consists of six equations

$$(E_1) \quad u_{n,xy} = e^{u_{n+1} - 2u_n + u_{n-1}},$$

$$(E_2) \quad u_{n,xy} = e^{u_{n+1}} - 2e^{u_n} + e^{u_{n-1}},$$

$$(E_3) \quad u_{n,xy} = e^{u_{n+1} - u_n} - e^{u_n - u_{n-1}},$$

$$(E_4) \quad u_{n,xy} = (u_{n+1} - 2u_n + u_{n-1}) u_{n,x},$$

$$(E_5) \quad u_{n,xy} = (e^{u_{n+1} - u_n} - e^{u_n - u_{n-1}}) u_{n,x},$$

$$(E_6) \quad u_{n,xy} = \alpha_n u_{n,x} u_{n,y}, \quad \alpha_n = \frac{1}{u_n - u_{n-1}} - \frac{1}{u_{n+1} - u_n}.$$

Equations (E_1) – (E_3) are known from classical works, see, for example, review [2]; equations (E_4) , (E_5) were obtained within the framework of the classification of equations of form (1.3) in [14]; lattice (E_6) was found independently in [14] and [4]. Moreover, in [3], [13], [5] the problem on a complete description of integrable cases of equation (1.3) with a linear dependence on the derivatives $u_{n,x}$, $u_{n,y}$ was solved by the method of Darboux integrable reductions. To the best of our knowledge, there are no examples in the literature of integrable equations of form (1.3) that are not linear in the derivatives. Therefore, it is likely that the list (E_1) – (E_6) is a complete one of integrable equations of form (1.3). In this regard, it can be conjectured that the list (L_1) – (L_9) is also complete.

Let us discuss some notion we shall need later. We consider a system of differential–difference equations

$$\begin{cases} u_{1,x}^{j+1} = g_1(u_{1,x}^j, u_1^{j+1}, u_2^j, u_n^j), \\ u_{n,x}^{j+1} = g(u_{n,x}^j, u_n^{j+1}, u_{n+1}^j, u_n^j, u_{n-1}^{j+1}), & 2 \leq n \leq N-1, \\ u_{N,x}^{j+1} = g_N(u_{N,x}^j, u_N^{j+1}, u_{N+1}^j, u_N^j, u_{N-1}^{j+1}). \end{cases} \quad (1.4)$$

Definition 1.1. System of equations (1.4) is called Darboux integrable if it admits complete sets of integrals in both characteristic directions x and j .

Definition 1.2. The function

$$J = J(x, j, u_1^j, u_2^j, \dots, u_N^j, u_1^{j+1}, u_2^{j+1}, \dots, u_N^{j+1}, \dots, u_1^{j+k}, u_2^{j+k}, \dots, u_N^{j+k}), \quad (1.5)$$

where $k = 1, 2, \dots$, is called an x -integral of system (1.4) of order k if, for at least one pair of numbers $l, s = 1, 2, \dots, N$, the product

$$\frac{\partial J}{\partial u_l^j} \cdot \frac{\partial J}{\partial u_s^{j+k}}$$

does not vanish identically, and, in addition, the relation

$$D_x J = 0,$$

holds where D_x denotes the operator of total differentiation with respect to x . The derivative of the variables $u_1^{j+k}, u_2^{j+k}, \dots, u_N^{j+k}$, $k = 1, 2, \dots$, with respect to x is calculated by means of system (1.4).

Suppose we are given a set of x -integrals of minimal orders k_l (the definition of minimality can be found in [15])

$$J_l = J_l(x, j, u_1^j, u_2^j, \dots, u_N^j, u_1^{j+1}, u_2^{j+1}, \dots, u_N^{j+1}, \dots, u_1^{j+k_l}, u_2^{j+k_l}, \dots, u_N^{j+k_l}), \quad (1.6)$$

where $l = 1, 2, \dots, N$. The set (1.6) is a complete set of x -integrals for a hyperbolic system (1.4) if and only if

$$\det \left(\frac{\partial J_l}{\partial u_n^{j+k_l}} \right) := \begin{vmatrix} \frac{\partial J_1}{\partial u_1^{j+k_1}} & \frac{\partial J_1}{\partial u_2^{j+k_1}} & \cdots & \frac{\partial J_1}{\partial u_N^{j+k_1}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial J_N}{\partial u_1^{j+k_N}} & \frac{\partial J_N}{\partial u_2^{j+k_N}} & \cdots & \frac{\partial J_N}{\partial u_N^{j+k_N}} \end{vmatrix} \neq 0. \quad (1.7)$$

Definition 1.3. A function

$$I = I(x, j, u_1^j, u_2^j, \dots, u_N^j, u_{1,x}^j, u_{2,x}^j, \dots, u_{N,x}^j, \dots, u_{1,[r]}^j, u_{2,[r]}^j, \dots, u_{N,[r]}^j), \quad (1.8)$$

where $u_{[r]} = \frac{\partial^r u}{\partial x^r}$, $r = 1, 2, \dots$, is called j -integral of order r of system (1.4) if the relation

$$D_j I = I$$

holds, where D_j is the shift operator, which acts by the rule $D_j(f(j)) = f(j+1)$. The shift of j variables $u_{1,[r]}^j, u_{2,[r]}^j, \dots, u_{N,[r]}^j$, $r = 1, 2, \dots$, is calculated by means of system of equations (1.4).

Suppose we are given a set of j -integrals of minimal orders r_s

$$I_s = I_s(x, j, u_1^j, u_2^j, \dots, u_N^j, u_{1,x}^j, u_{2,x}^j, \dots, u_{N,x}^j, \dots, u_{1,[r_s]}^j, u_{2,[r_s]}^j, \dots, u_{N,[r_s]}^j), \quad (1.9)$$

where $s = 1, 2, \dots, N$. Set (1.9) is a complete set of j -integrals of system (1.4) if and only if [10]

$$\det \left(\frac{\partial I_s}{\partial u_{n,[r_s]}^j} \right) := \begin{vmatrix} \frac{\partial I_1}{\partial u_{1,[r_1]}^j} & \frac{\partial I_1}{\partial u_{2,[r_1]}^j} & \cdots & \frac{\partial I_1}{\partial u_{N,[r_1]}^j} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial I_N}{\partial u_{1,[r_N]}^j} & \frac{\partial I_N}{\partial u_{2,[r_N]}^j} & \cdots & \frac{\partial I_N}{\partial u_{N,[r_N]}^j} \end{vmatrix} \neq 0. \quad (1.10)$$

2. ALGORITHM FOR DISCRETIZATION OF SEMI-DISCRETE TODA TYPE LATTICE

Let us discuss a method for discretizing integrable three-dimensional lattices of the form (1.2) based on its finite-field reduction

$$\begin{cases} u_{1,x}^{j+1} = g_1(u_{1,x}^j, u_1^{j+1}, u_2^j, u_n^j), \\ u_{n,x}^{j+1} = g(u_{n,x}^j, u_n^{j+1}, u_{n+1}^j, u_n^j, u_{n-1}^{j+1}), \\ u_{N,x}^{j+1} = g_N(u_{N,x}^j, u_N^{j+1}, u_{N+1}^j, u_N^j, u_{N-1}^{j+1}). \end{cases} \quad 2 \leq n \leq N-1, \quad (2.1)$$

Here the functions g_1 and g_N are chosen such that system (2.1) is Darboux integrable. Boundary conditions for the lattices (L_1) – (L_9) ensuring Darboux integrable reductions were found in the works [11], [12], [6].

Suppose that the functions

$$J_l = J_l(x, j, u^j, u^{j+1}, \dots, u^{j+k_l}), \quad l = 1, 2, \dots, N, \quad (2.2)$$

where $u^j = (u_1^j, u_2^j, \dots, u_N^j)$, form a complete set of characteristic integrals of minimal orders in the direction x of system (2.1). In other words, the functions J_l satisfy the relation $D_x J_l = 0$, where D_x denotes the operator of the total derivative with respect to the variable x . This means that the derivative D_x of the function J_l is calculated by means of original system (2.1).

In what follows we suppose that the dynamic variables $u_1^j, u_2^j, \dots, u_N^j$ depend not only on x and j , but also on an additional discrete variable m , so we introduce the subscript $u_{1,m}^j, u_{2,m}^j, \dots, u_{N,m}^j$. The dependence on the variable m will be determined by the following difference system

$$\begin{cases} u_{1,m+1}^{j+1} = h_1(u_{1,m+1}^j, u_{1,m}^{j+1}, u_{2,m+1}^j, u_{1,m}^j), \\ u_{n,m+1}^{j+1} = h(u_{n,m+1}^j, u_{n,m}^{j+1}, u_{n+1,m+1}^j, u_{n,m}^j, u_{n-1,m}^{j+1}), \\ u_{N,m+1}^{j+1} = h_N(u_{N,m+1}^j, u_{N,m}^{j+1}, u_{N,m}^j, u_{N-1,m}^{j+1}) \end{cases} \quad 2 \leq n \leq N-1, \quad (2.3)$$

with unknown right hand sides. To find the functions h_1, h_2 , and h , we shall employ the assumption that the functions J_l , which are x -integrals of the (2.1) system, after introducing the dependence on the discrete variable m , are m -integrals of sought system (2.3). We denote these functions by \tilde{J}_l :

$$\tilde{J}_l = \tilde{J}_l(x, j, m, u_m^j, u_m^{j+1}, \dots, u_m^{j+k_l}), \quad l = 1, 2, \dots, N, \quad (2.4)$$

where $u_m^j = (u_{1,m}^j, u_{2,m}^j, \dots, u_{N,m}^j)$. In other words, the functions \tilde{J}_l should satisfy the condition

$$D_m \tilde{J}_l = \tilde{J}_l,$$

where D_m is the shift operator of the variable m , the shift along the the variable m is made by means of system (2.3).

At the same time, to exclude a trivial result, the functions $\tilde{J}_1, \tilde{J}_2, \dots, \tilde{J}_l$ should form a complete set of independent m -integrals of minimal orders for (2.3).

Suppose that system (2.3) is found and it is not trivial, then we can easily determine the type of discrete three-dimensional lattice

$$u_{n,m+1}^{j+1} = h(u_{n,m+1}^j, u_{n,m}^{j+1}, u_{n+1,m+1}^j, u_{n,m}^j, u_{n-1,m}^{j+1}). \quad (2.5)$$

We note that to solve the discretization problem it is sufficient to restrict ourselves to considering the case $N = 3$.

3. EXAMPLE

As an illustration of the discretization algorithm, we consider the semi-discrete Toda lattice (L_1)

$$u_{n,x}^{j+1} = u_{n,x}^j + e^{u_n^j - u_{n-1}^{j+1}} - e^{u_{n+1}^j - u_n^{j+1}}. \quad (3.1)$$

The finite–field reduction of equation (3.1) is obtained from this lattice by imposing the boundary conditions $u_0 = \infty$, $u_4 = -\infty$, and it reads

$$\begin{cases} \bar{u}_x^{j+1} = \bar{u}_x^j - e^{\bar{v}^j - \bar{u}^{j+1}}, \\ \bar{v}_x^{j+1} = \bar{v}_x^j + e^{\bar{v}^j - \bar{u}^{j+1}} - e^{\bar{w}^j - \bar{v}^{j+1}}, \\ \bar{w}_x^{j+1} = \bar{w}_x^j + e^{\bar{w}^j - \bar{v}^{j+1}}, \end{cases} \quad (3.2)$$

where $\bar{u}^j = u_1^j$, $\bar{v}^j = u_2^j$, $\bar{w}^j = u_3^j$. We shall employ x –integrals of system (3.2), which read

$$\begin{aligned} \bar{J}_1 &= e^{\bar{u}^{j+1} + \bar{v}^{j+1} + \bar{w}^{j+1} - \bar{u}^j - \bar{v}^j - \bar{w}^j}, \\ \bar{J}_2 &= e^{\bar{v}^j + \bar{w}^j - \bar{v}^{j-1} - \bar{w}^{j-1}} + e^{\bar{u}^{j+1} + \bar{w}^j - \bar{u}^j - \bar{w}^{j-1}} + e^{\bar{u}^{j+1} + \bar{v}^{j+1} - \bar{u}^j - \bar{v}^j}, \\ \bar{J}_3 &= e^{\bar{u}^{j+2} - \bar{u}^{j+1}} + e^{\bar{v}^{j+1} - \bar{v}^j} + e^{\bar{w}^j - \bar{w}^{j-1}}. \end{aligned}$$

We are going to show that the integrals \bar{J}_1 – \bar{J}_3 form a complete set of independent x –integrals. Let us calculate determinant (1.7). We obtain

$$\begin{vmatrix} \frac{\partial \bar{J}_1}{\partial \bar{u}^{j+1}} & \frac{\partial \bar{J}_1}{\partial \bar{v}^{j+1}} & \frac{\partial \bar{J}_1}{\partial \bar{w}^{j+1}} \\ \frac{\partial D_j(\bar{J}_2)}{\partial \bar{u}^{j+2}} & \frac{\partial D_j(\bar{J}_2)}{\partial \bar{v}^{j+2}} & \frac{\partial D_j(\bar{J}_2)}{\partial \bar{w}^{j+2}} \\ \frac{\partial D_j(\bar{J}_3)}{\partial \bar{u}^{j+3}} & \frac{\partial D_j(\bar{J}_3)}{\partial \bar{v}^{j+3}} & \frac{\partial D_j(\bar{J}_3)}{\partial \bar{w}^{j+3}} \end{vmatrix} = e^{\bar{u}^{j+3} + \bar{v}^{j+2} + \bar{w}^{j+1} - \bar{u}^j - \bar{v}^j - \bar{w}^j} \neq 0. \quad (3.3)$$

Therefore, the integrals \bar{J}_1 – \bar{J}_3 form a complete set of independent x –integrals.

Now we introduce the dependence on m in the variables $\bar{u}^j = \bar{u}_m^j$, $\bar{v}^j = \bar{v}_m^j$, $\bar{w}^j = \bar{w}_m^j$ and make the substitution $e^{\bar{u}_m^j} = u_m^j$, $e^{\bar{v}_m^j} = v_m^j$, $e^{\bar{w}_m^j} = w_m^j$, then the x –integrals \bar{J}_1 – \bar{J}_3 can be rewritten as

$$\begin{aligned} J_1 &= \frac{u_m^{j+1} v_m^{j+1} w_m^{j+1}}{u_m^j v_m^j w_m^j}, \\ J_2 &= \frac{v_m^j w_m^j}{v_m^{j-1} w_m^{j-1}} + \frac{u_m^{j+1} w_m^j}{u_m^j w_m^{j-1}} + \frac{u_m^{j+1} v_m^{j+1}}{u_m^j v_m^j}, \\ J_3 &= \frac{u_m^{j+2}}{u_m^{j+1}} + \frac{v_m^{j+1}}{v_m^j} + \frac{w_m^j}{w_m^{j-1}}. \end{aligned}$$

In what follows we suppose that the functions J_1 – J_3 are m –integrals of the system

$$\begin{cases} u_{m+1}^{j+1} = h_1(u_m^j, u_{m+1}^j, u_m^{j+1}, v_{m+1}^j), \\ v_{m+1}^{j+1} = h_2(v_m^j, v_{m+1}^j, v_m^{j+1}, w_{m+1}^j, u_m^{j+1}), \\ w_{m+1}^{j+1} = h_3(w_m^j, w_{m+1}^j, w_m^{j+1}, v_m^{j+1}), \end{cases} \quad (3.4)$$

which is obtained from the lattice

$$u_{n,m+1}^{j+1} = h_2(u_{n,m}^j, u_{n,m+1}^j, u_{n,m}^{j+1}, u_{n+1,m+1}^j, u_{n-1,m}^{j+1}). \quad (3.5)$$

By conditions $(D_m - 1)J_i = 0$ we obtain the following three equations

$$\begin{aligned}
(D_m - 1)J_1 &= \frac{u_{m+1}^{j+1}v_{m+1}^{j+1}w_{m+1}^{j+1}}{u_{m+1}^jv_{m+1}^jw_{m+1}^j} - \frac{u_m^{j+1}v_m^{j+1}w_m^{j+1}}{u_m^jv_m^jw_m^j} = 0, \\
(D_m - 1)J_2 &= \frac{v_{m+1}^jw_{m+1}^j}{v_{m+1}^{j-1}w_{m+1}^{j-1}} + \frac{u_{m+1}^{j+1}w_{m+1}^j}{u_{m+1}^jw_{m+1}^{j-1}} + \frac{u_{m+1}^{j+1}v_{m+1}^{j+1}}{u_{m+1}^jv_{m+1}^j} \\
&\quad - \frac{v_m^jw_m^j}{v_m^{j-1}w_m^{j-1}} - \frac{u_m^{j+1}w_m^j}{u_m^jw_m^{j-1}} - \frac{u_m^{j+1}v_m^{j+1}}{u_m^jv_m^j} = 0, \\
(D_m - 1)J_3 &= \frac{u_{m+1}^{j+2}}{u_{m+1}^{j+1}} + \frac{v_{m+1}^{j+1}}{v_{m+1}^j} + \frac{w_{m+1}^j}{w_{m+1}^{j-1}} - \frac{u_m^{j+2}}{u_m^{j+1}} - \frac{v_m^{j+1}}{v_m^j} - \frac{w_m^j}{w_m^{j-1}} = 0.
\end{aligned} \tag{3.6}$$

We focus on the first equation. We rewrite it bearing in mind system (3.4)

$$\begin{aligned}
&\frac{h_1(u_m^j, u_{m+1}^j, u_m^{j+1}, v_{m+1}^j)h_2(v_m^j, v_{m+1}^j, v_m^{j+1}, w_{m+1}^j, u_m^{j+1})h_3(w_m^j, w_{m+1}^j, w_m^{j+1}, v_{m+1}^{j+1})}{u_{m+1}^jv_{m+1}^jw_{m+1}^j} \\
&\quad - \frac{u_m^{j+1}v_m^{j+1}w_m^{j+1}}{u_m^jv_m^jw_m^j} = 0.
\end{aligned} \tag{3.7}$$

We differentiate equation (3.7) in u_{m+1}^j and obtain that the function $h_1(u_m^j, u_{m+1}^j, u_m^{j+1}, v_{m+1}^j)$ satisfies the equation

$$u_{m+1}^j \frac{\partial}{\partial u_{m+1}^j} h_1(u_m^j, u_{m+1}^j, u_m^{j+1}, v_{m+1}^j) - h_1(u_m^j, u_{m+1}^j, u_m^{j+1}, v_{m+1}^j) = 0. \tag{3.8}$$

Solving the obtained equation, we find

$$h_1(u_m^j, u_{m+1}^j, u_m^{j+1}, v_{m+1}^j) = h_{11}(u_m^j, u_m^{j+1}, v_{m+1}^j)u_{m+1}^j. \tag{3.9}$$

Then equation (3.7) can be rewritten as

$$\begin{aligned}
&\frac{h_{11}(u_m^j, u_m^{j+1}, v_{m+1}^j)h_2(v_m^j, v_{m+1}^j, v_m^{j+1}, w_{m+1}^j, u_m^{j+1})h_3(w_m^j, w_{m+1}^j, w_m^{j+1}, v_{m+1}^{j+1})}{v_{m+1}^jw_{m+1}^j} \\
&\quad - \frac{u_m^{j+1}v_m^{j+1}w_m^{j+1}}{u_m^jv_m^jw_m^j} = 0.
\end{aligned} \tag{3.10}$$

We differentiate this equation twice in w_m^{j+1} , analyze the result and find that the function $h_3(w_m^j, w_{m+1}^j, w_m^{j+1}, v_{m+1}^{j+1})$ reads

$$h_3(w_m^j, w_{m+1}^j, w_m^{j+1}, v_{m+1}^{j+1}) = h_{31}(w_m^j, w_{m+1}^j, v_{m+1}^{j+1})w_m^{j+1}. \tag{3.11}$$

Then equation (3.10) can be rewritten as

$$\frac{h_{11}(u_m^j, u_m^{j+1}, v_{m+1}^j)h_2(v_m^j, v_{m+1}^j, v_m^{j+1}, w_{m+1}^j, u_m^{j+1})h_{31}(w_m^j, w_{m+1}^j, v_{m+1}^{j+1})}{v_{m+1}^jw_{m+1}^j} - \frac{u_m^{j+1}v_m^{j+1}}{u_m^jv_m^jw_m^j} = 0. \tag{3.12}$$

Since the functions $h_{11}(u_m^j, u_m^{j+1}, v_{m+1}^j)$, $h_2(v_m^j, v_{m+1}^j, v_m^{j+1}, w_{m+1}^j, u_m^{j+1})$, $h_{31}(w_m^j, w_{m+1}^j, v_{m+1}^{j+1})$ are non-zero, equation (3.12) can be presented by one of the following versions

$$\begin{aligned}
&\frac{h_{11}(u_m^j, u_m^{j+1}, v_{m+1}^j)u_m^j}{v_{m+1}^jw_{m+1}^j} - \frac{u_m^{j+1}v_m^{j+1}}{v_m^jw_m^jh_2(v_m^j, v_{m+1}^j, v_m^{j+1}, w_{m+1}^j, u_m^{j+1})h_{31}(w_m^j, w_{m+1}^j, v_{m+1}^{j+1})} = 0; \\
&\frac{h_2(v_m^j, v_{m+1}^j, v_m^{j+1}, w_{m+1}^j, u_m^{j+1})v_m^j}{v_{m+1}^jw_{m+1}^j} - \frac{u_m^{j+1}v_m^{j+1}}{u_m^jw_m^jh_{11}(u_m^j, u_m^{j+1}, v_{m+1}^j)h_{31}(w_m^j, w_{m+1}^j, v_{m+1}^{j+1})} = 0; \\
&\frac{h_{31}(w_m^j, w_{m+1}^j, v_{m+1}^{j+1})w_m^j}{v_{m+1}^jw_{m+1}^j} - \frac{u_m^{j+1}v_m^{j+1}}{u_m^jv_m^jh_{11}(u_m^j, u_m^{j+1}, v_{m+1}^j)h_2(v_m^j, v_{m+1}^j, v_m^{j+1}, w_{m+1}^j, u_m^{j+1})} = 0.
\end{aligned} \tag{3.13}$$

Differentiating these equations respectively in u_m^j , v_m^j and w_m^j , we determine the dependence of sought functions on the variables

$$\begin{aligned} h_{11}(u_m^j, u_{m+1}^{j+1}, v_{m+1}^j) &= \frac{h_{12}(u_m^{j+1}, v_{m+1}^j)}{u_m^j}; \\ h_2(v_m^j, v_{m+1}^j, v_{m+1}^{j+1}, w_{m+1}^j, u_m^{j+1}) &= \frac{h_{22}(v_{m+1}^j, v_{m+1}^{j+1}, w_{m+1}^j, u_m^{j+1})}{v_m^j}; \\ h_{31}(w_m^j, w_{m+1}^j, v_{m+1}^{j+1}) &= \frac{h_{32}(w_{m+1}^j, v_{m+1}^{j+1})}{w_m^j}. \end{aligned} \quad (3.14)$$

Then equation (3.12) is rewritten as

$$\frac{h_{12}(u_m^{j+1}, v_{m+1}^j)h_{22}(v_{m+1}^j, v_{m+1}^{j+1}, w_{m+1}^j, u_m^{j+1})h_{32}(w_{m+1}^j, v_{m+1}^{j+1})}{v_{m+1}^j w_{m+1}^j} - u_m^{j+1} v_m^{j+1} = 0. \quad (3.15)$$

We express the function $h_{22}(v_{m+1}^j, v_{m+1}^{j+1}, w_{m+1}^j, u_m^{j+1})$ by this relation and we find

$$h_{22}(v_{m+1}^j, v_{m+1}^{j+1}, w_{m+1}^j, u_m^{j+1}) = \frac{u_m^{j+1} v_m^{j+1} v_{m+1}^j w_{m+1}^j}{h_{12}(u_m^{j+1}, v_{m+1}^j) h_{32}(w_{m+1}^j, v_{m+1}^{j+1})}. \quad (3.16)$$

As a result, by the first equation in (3.6) we find preliminary form of sought functions $h_1(u_m^j, u_{m+1}^j, u_m^{j+1}, v_{m+1}^j)$, $h_2(v_m^j, v_{m+1}^j, v_{m+1}^{j+1}, w_{m+1}^j, u_m^{j+1})$ and $h_3(w_m^j, w_{m+1}^j, w_m^{j+1}, v_m^{j+1})$:

$$\begin{aligned} h_1(u_m^j, u_{m+1}^j, u_m^{j+1}, v_{m+1}^j) &= \frac{h_{12}(u_m^{j+1}, v_{m+1}^j) u_{m+1}^j}{u_m^j}; \\ h_2(v_m^j, v_{m+1}^j, v_{m+1}^{j+1}, w_{m+1}^j, u_m^{j+1}) &= \frac{u_m^{j+1} v_m^{j+1} v_{m+1}^j w_{m+1}^j}{v_m^j h_{12}(u_m^{j+1}, v_{m+1}^j) h_{32}(w_{m+1}^j, v_{m+1}^{j+1})}; \\ h_3(w_m^j, w_{m+1}^j, w_m^{j+1}, v_m^{j+1}) &= \frac{h_{32}(w_{m+1}^j, v_{m+1}^{j+1}) w_m^{j+1}}{w_m^j}. \end{aligned} \quad (3.17)$$

We proceed to studying the second equation in (3.6), which in view of found relations (3.17) becomes

$$\begin{aligned} \frac{h_{32}(w_{m+1}^j, v_{m+1}^{j+1}) h_{12}(u_m^{j+2}, v_{m+1}^{j+1})}{u_m^{j+1} w_{m+1}^j} + \frac{u_m^{j+2} v_m^{j+2} h_{32}(w_{m+1}^j, v_{m+1}^{j+1})}{u_m^{j+1} v_m^{j+1} h_{32}(w_{m+1}^{j+1}, v_m^{j+2})} + \frac{u_m^{j+1} v_m^{j+1}}{v_m^j h_{12}(u_m^{j+1}, v_{m+1}^j)} \\ - \frac{v_m^{j+1}}{v_m^j} - \frac{u_m^{j+2}}{u_m^{j+1}} - \frac{w_m^j u_m^{j+2} v_m^{j+2}}{w_{m+1}^{j+1} u_m^{j+1} v_m^{j+1}} = 0, \end{aligned} \quad (3.18)$$

where

$$\begin{aligned} h_{12}(u_m^{j+2}, v_{m+1}^{j+1}) &= D_j(h_{12}(u_m^{j+1}, v_{m+1}^j)) = h_{12}\left(u_m^{j+2}, \frac{u_m^{j+1} v_m^{j+1} v_{m+1}^j w_{m+1}^j}{v_m^j h_{12}(u_m^{j+1}, v_{m+1}^j) h_{32}(w_{m+1}^j, v_{m+1}^{j+1})}\right), \\ h_{32}(w_{m+1}^{j+1}, v_m^{j+2}) &= D_j(h_{32}(w_{m+1}^j, v_{m+1}^{j+1})) = h_{32}\left(\frac{h_{32}(w_{m+1}^j, v_{m+1}^{j+1}) w_m^{j+1}}{w_m^j}, v_m^{j+2}\right). \end{aligned}$$

We note that in equation (3.18) only the function $h_{12}(u_m^{j+2}, v_{m+1}^{j+1})$ contains the variable u_m^{j+2} , similarly only $h_{32}(w_{m+1}^{j+1}, v_m^{j+2})$ contains v_m^{j+2} . Therefore, differentiating equation (3.18) with respect to u_m^{j+2} twice and, accordingly, with respect to v_m^{j+2} twice, we obtain the following two equations for the sought functions

$$\frac{\partial^2}{\partial (u_m^{j+2})^2} h_{12}(u_m^{j+2}, v_{m+1}^{j+1}) = 0,$$

$$\frac{\partial^2}{\partial (v_m^{j+2})^2} h_{32}(w_{m+1}^{j+1}, v_m^{j+2}) - \frac{2 \left(\frac{\partial}{\partial v_m^{j+2}} h_{32}(w_{m+1}^{j+1}, v_m^{j+2}) \right)^2}{h_{32}(w_{m+1}^{j+1}, v_m^{j+2})} + \frac{2}{v_m^{j+2}} \frac{\partial}{\partial v_m^{j+2}} h_{32}(w_{m+1}^{j+1}, v_m^{j+2}) = 0.$$

We apply the operator D_j^{-1} to two latter equations. Then we obtain equations for the unknown functions $h_{12}(w_m^{j+1}, v_{m+1}^j)$ and $h_{32}(w_{m+1}^j, v_m^{j+1})$. This gives

$$\begin{aligned} h_{12}(w_m^{j+1}, v_{m+1}^j) &= h_{13}(v_{m+1}^j) w_m^{j+1} + h_{14}(v_{m+1}^j), \\ h_{32}(w_{m+1}^j, v_m^{j+1}) &= -\frac{v_m^{j+1}}{h_{33}(w_{m+1}^j) v_m^{j+1} - h_{34}(w_{m+1}^j)}. \end{aligned}$$

Thus, the sought functions become

$$\begin{aligned} h_1(w_m^j, u_{m+1}^j, u_m^{j+1}, v_{m+1}^j) &= \frac{(h_{13}(v_{m+1}^j) w_m^{j+1} + h_{14}(v_{m+1}^j)) w_{m+1}^j}{u_m^j}; \\ h_2(v_m^j, v_{m+1}^j, v_m^{j+1}, w_{m+1}^j, u_m^{j+1}) &= -\frac{u_m^{j+1} v_{m+1}^j w_{m+1}^j (h_{33}(w_{m+1}^j) v_m^{j+1} - h_{34}(w_{m+1}^j))}{v_m^j (h_{13}(v_{m+1}^j) u_m^{j+1} + h_{14}(v_{m+1}^j))}; \\ h_3(w_m^j, w_{m+1}^j, w_m^{j+1}, v_m^{j+1}) &= -\frac{w_m^{j+1} v_m^{j+1}}{w_m^j (h_{33}(w_{m+1}^j) v_m^{j+1} - h_{34}(w_{m+1}^j))}. \end{aligned} \quad (3.19)$$

We rewrite equation (3.18) bearing in mind the found functions and equating the coefficients at the independent variables u_m^{j+2} and v_m^{j+2} . We then obtain the following three equations

$$\begin{aligned} 1) \quad & h_{33} \left(-\frac{w_m^{j+1} v_m^{j+1}}{w_m^j (h_{33}(w_{m+1}^j) v_m^{j+1} - h_{34}(w_{m+1}^j))} \right) - \frac{w_m^j (h_{33}(w_{m+1}^j) v_m^{j+1} - h_{34}(w_{m+1}^j))}{v_m^{j+1} w_m^{j+1}} = 0, \\ 2) \quad & h_{13} \left(-\frac{u_m^{j+1} v_m^{j+1} v_{m+1}^j w_{m+1}^j (h_{33}(w_{m+1}^j) v_m^{j+1} - h_{34}(w_{m+1}^j))}{v_m^j (h_{13}(v_{m+1}^j) u_m^{j+1} + h_{14}(v_{m+1}^j)) v_m^{j+1}} \right) v_m^{j+1} \\ & + h_{34} \left(-\frac{w_m^{j+1} v_m^{j+1}}{w_m^j (h_{33}(w_{m+1}^j) v_m^{j+1} - h_{34}(w_{m+1}^j))} \right) w_{m+1}^j \\ & - (h_{33}(w_{m+1}^j) v_m^{j+1} - h_{34}(w_{m+1}^j)) w_{m+1}^j = 0, \\ 3) \quad & h_{14} \left(-\frac{u_m^{j+1} v_m^{j+1} v_{m+1}^j w_{m+1}^j (h_{33}(w_{m+1}^j) v_m^{j+1} - h_{34}(w_{m+1}^j))}{v_m^j (h_{13}(v_{m+1}^j) u_m^{j+1} + h_{14}(v_{m+1}^j)) v_m^{j+1}} \right) \\ & + \frac{u_m^{j+1} w_{m+1}^j (h_{33}(w_{m+1}^j) v_m^{j+1} - h_{34}(w_{m+1}^j))}{v_m^j} \\ & - \frac{w_{m+1}^j (u_m^{j+1})^2 (h_{33}(w_{m+1}^j) v_m^{j+1} - h_{34}(w_{m+1}^j))}{v_m^j (h_{13}(v_{m+1}^j) u_m^{j+1} + h_{14}(v_{m+1}^j))} = 0. \end{aligned}$$

We focus on the second equation. We note that only the function h_{13} depends on the variable v_m^j and only h_{34} does on w_m^j . We therefore find

$$h_{13} = C_1, \quad h_{34} = C_2.$$

By the same equation we find explicitly the function h_{33}

$$h_{33} = -\frac{C_1}{w_{m+1}^j}.$$

Analyzing the first equation in the triple, we conclude that $C_1 = 1$. It remains to consider the third equation, which reads

$$h_{14} \left(\frac{u_m^{j+1} v_{m+1}^j (C_2 w_{m+1}^j + v_m^{j+1})}{v_m^j (u_m^{j+1} + h_{14} (v_{m+1}^j))} \right) - \frac{u_m^{j+1} h_{14} (v_{m+1}^j) (C_2 w_{m+1}^j + v_m^{j+1})}{v_m^j (u_m^{j+1} + h_{14} (v_{m+1}^j))} = 0.$$

Differentiating this identity twice in w_{m+1}^j and applying the operator D_j^{-1} to the resulting equation, we find

$$h_{14} (v_{m+1}^j) = C_3 v_{m+1}^j.$$

As a result we obtain that the sought functions read

$$\begin{aligned} h_1(u_m^j, u_{m+1}^j, u_m^{j+1}, v_{m+1}^j) &= \frac{u_{m+1}^j (u_m^{j+1} + C_3 v_{m+1}^j)}{u_m^j}, \\ h_2(v_m^j, v_{m+1}^j, v_m^{j+1}, w_{m+1}^j, u_m^{j+1}) &= \frac{u_m^{j+1} v_{m+1}^j (v_m^{j+1} + C_2 w_{m+1}^j)}{v_m^j (u_m^{j+1} + C_3 v_{m+1}^j)}, \\ h_3(w_m^j, w_{m+1}^j, w_m^{j+1}, v_{m+1}^j) &= \frac{w_m^{j+1} v_m^{j+1} w_{m+1}^j}{w_m^j (v_m^{j+1} + C_2 w_{m+1}^j)}. \end{aligned} \quad (3.20)$$

It remains to verify the third equation in system (3.6), i.e., to verify that the function J_3 is the m -integral of found system (3.20). By direct calculation we find that system (3.20) satisfies the third equation in system (3.6).

Thus, we finally obtain a system of equations

$$\begin{cases} u_{m+1}^{j+1} = \frac{u_{m+1}^j (u_m^{j+1} + C_3 v_{m+1}^j)}{u_m^j}, \\ v_{m+1}^{j+1} = \frac{u_m^{j+1} v_{m+1}^j (v_m^{j+1} + C_2 w_{m+1}^j)}{v_m^j (u_m^{j+1} + C_3 v_{m+1}^j)}, \\ w_{m+1}^{j+1} = \frac{w_m^{j+1} v_m^{j+1} w_{m+1}^j}{w_m^j (v_m^{j+1} + C_2 w_{m+1}^j)}. \end{cases} \quad (3.21)$$

Since the found system is assumed to be obtained from lattice (3.5), we have

$$u_{n,m+1}^{j+1} = \frac{u_{n-1,m}^{j+1} u_{n,m+1}^j (C u_{n+1,m+1}^j + u_{n,m}^{j+1})}{u_{n,m}^j (C u_{n,m+1}^j + u_{n-1,m}^{j+1})}, \quad (3.22)$$

We let $C = -1$ since it can be removed by the change $u_{n,m}^j = (-C)^{-m} v_{n,m}^j$. This leads to known lattice Toda lattice

$$u_{n,m+1}^{j+1} = \frac{u_{n-1,m}^{j+1} u_{n,m+1}^j (u_{n+1,m+1}^j - u_{n,m}^{j+1})}{u_{n,m}^j (u_{n,m+1}^j - u_{n-1,m}^{j+1})}. \quad (3.23)$$

Its Lax pair reads

$$\begin{cases} \psi_{n,m}^{j+1} = \frac{u_{n,m}^{j+1}}{u_{n,m}^j} \psi_{n,m}^j - \psi_{n+1,m}^j, \\ \psi_{n,m+1}^j = \psi_{n,m}^j - \frac{u_{n,m+1}^j}{u_{n-1,m}^j} \psi_{n-1,m}^j. \end{cases} \quad (3.24)$$

BIBLIOGRAPHY

1. V.E. Adler, S.Ya. Startsev. *Discrete analogues of the Liouville equation* // Theor. Math. Phys. **121**:2, 1484–1495 (1999). <https://doi.org/10.1007/BF02557219>
2. E.I. Ganzha, S.P. Tsarev. *Classical methods of integration of second order hyperbolic systems and equations*. Siberian Federal University, Krasnoyarsk (2007). (in Russian).
3. M.N. Poptsova, I.T. Khabibullin. *Algebraic properties of quasilinear two-dimensional lattices connected with integrability* // Ufa Math. J. **10**:3, 86–105 (2018). <https://doi.org/10.13108/2018-10-3-86>
4. E.V. Ferapontov. *Laplace transformations of hydrodynamic-type systems in Riemann invariants* // Theor. Math. Phys. **110**:1, 68–77 (1997). <https://doi.org/10.1007/BF02630370>
5. I.T. Habibullin, M.N. Kuznetsova. *A classification algorithm for integrable two-dimensional lattices via Lie – Rinehart algebras* // Theor. Math. Phys. **203**:1, 569–581 (2020). <https://doi.org/10.1134/S0040577920040121>
6. I.T. Habibullin, A.R. Khakimova. *Integrable discrete systems* // Sb. Math. **217**:6 (2026), to appear.
7. V.E. Adler. *The tangential map and associated integrable equations* // J. Phys. A Math. Theor. **42**:33, 332004 (2009). <https://doi.org/10.1088/1751-8113/42/33/332004>
8. E. Date, M. Jimbo, T. Miwa. *Method for generating discrete soliton equation. II* // J. Phys. Soc. Japan, **51**, 4125–4131 (1982). <https://doi.org/10.1143/JPSJ.51.4116>
9. E.V. Ferapontov, V.S. Novikov, I. Roustemoglou. *On the classification of discrete Hirota-type equations in 3D* // Int. Math. Res. Not. **2015**:13, 4933–4974 (2015). <https://doi.org/10.1093/imrn/rnu086>
10. I.T. Habibullin, M.N. Kuznetsova. *An algebraic criterion of the Darboux integrability of differential-difference equations and systems* // J. Phys. A Math. Theor. **54**:50, 505201 (2021). <https://doi.org/10.1088/1751-8121/ac37e8>
11. I.T. Habibullin, A.R. Khakimova. *Characteristic Lie algebras of integrable differential-difference equations in 3D* // J. Phys. A Math. Theor. **54**:29, 295202 (2021). <https://doi.org/10.1088/1751-8121/ac070c>
12. I.T. Habibullin, A.R. Khakimova. *Nonlinear integrable lattices with three independent variables* // Ufa Math. J. **17**:2, 105–119 (2025). <https://doi.org/10.13108/2025-17-2-105>
13. M.N. Kuznetsova. *Classification of a subclass of quasilinear two-dimensional lattices by means of characteristic algebras* // Ufa Math. J. **11**:3, 109–131 (2019).
14. A.B. Shabat, R.I. Yamilov. *To a transformation theory of two-dimensional integrable systems* // Phys. Lett., A **227**:1-2, 15–23 (1997). [https://doi.org/10.1016/S0375-9601\(96\)00922-X](https://doi.org/10.1016/S0375-9601(96)00922-X)
15. A.V. Zhiber, R.D. Murtazina, I.T. Habibullin, A.B. Shabat. *Characteristic Lie rings and nonlinear integrable equations*. Inst. Comp. Studies, Moscow — Izhevsk (2012). (in Russian).

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