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CONVOLUTION KERNEL IDENTIFICATION PROBLEM FOR MULTI–DIMENSIONAL TIME FRACTIONAL DIFFUSION–WAVE EQUATION IN BOUNDED DOMAIN

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Abstract. We study the inverse problem on determining the convolution kernel of integral term in an initial boundary value problem for a multi–dimensional time fractional diffusion–wave equation with the uniformly elliptic operator in a divergent form. Moreover, as an overdetermination condition, a single observation at the point $x_0 \in \Omega$ of the diffusion–wave process serves, where $\Omega \subset \mathbb{R}^n$ is a bounded domain. By the Fourier spectral method and fractional integro–differentiating technics the inverse problem is reduced to a convolution nonlinear Volterra integral equation of the second kind. The fixed point argument proves the local existence and global uniqueness results. Also the stability estimate for solution to inverse problem is obtained.

Keywords: integro–differential equation, inverse problem, kernel, spectral problem, fixed point theorem, Grönwall inequality, existence, uniqueness

Mathematics Subject Classification: 34A08, 34K37, 34M50, 35R11

1. Introduction and statement of problem

Nowadays, equations of mathematical physics with fractional derivatives have received considerable attention in various areas of applied mathematics. These equations involving nonlocal integro–differential operators are used to formulate many real world problems, for example heat and mass transfer processes. Number of interesting features of time fractional diffusion–wave equations (TFDWEs) are presented in the book [16] and they indicate a certain similarity of these equations with second order partial differential equations. It also presents the studies of a wide class of well–posed initial (Cauchy), initial boundary value and boundary value problems for TFDWEs.

Recently, the attention to inverse zero order coefficient (potential) problems for TFDWEs increased extensively. In [27] and [14], the uniqueness of determining the potential by finitely many measurements on the boundary and the flux measurements in one dimension, respectively was proved. In [29] the authors considered an inverse potential term problem from the overdetermined final time data and gave an efficient regularized iterative algorithm based on mollification in one dimensional case. The paper [15] studied an inverse potential problem from overposed final time data, and recovered numerically the unknown coefficient by an iterative Newton type method. In [13], a locally Lipschitz conditional stability for an inverse potential coefficient from the terminal datawas obtained. In [23], [24], [25], the authors investigated the

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uniqueness in determining the fractional order(s) and the potential simultaneously for the single term and multi term TFDEs, respectively, and gave a valid numerical method. In [18], an inverse potential coefficient from the integral overdetermination data was studied by using the well–known Rothe method. In [1]-[3] the inverse time depending potential problems by single observations at $x = 0$ on corresponding direct problems solutions was studied on the base of the integral equations method.

In this paper we consider the following integro–differential equation with fractional derivative

$$\partial_t^\alpha u - Lu = \int_0^t k(t-\tau)u(x,\tau)\,d\tau + g(x,t), \qquad 1<\alpha\leqslant 2, \quad (x,t)\in Q_T, \tag{1.1}$$

with initial

$$u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x), \qquad x\in\overline{\Omega}, \tag{1.2}$$

and boundary conditions

$$u(x,t) = 0, \qquad (x,t)\in\partial Q_T, \tag{1.3}$$

where ∂_t^α is the regularized fractional derivative (Caputo derivative) in the variable t [16, Ch. II, Sect. 4, Thm 2.1]:

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(2-\alpha)}\int_0^t \frac{u_{\tau\tau}(x,\tau)}{(t-\tau)^{\alpha-1}}d\tau, \qquad \partial_t^2 u(x,t) = \frac{\partial^2}{\partial t^2}u(x,t),$$

where

$$L = \sum_{i,j=1}^n \frac{\partial}{\partial x_i}a_{ij}(x)\frac{\partial}{\partial x_j} - c(x)$$

is the uniformly elliptic operator ($n\geqslant 1$), the coefficients satisfy the conditions

$$a_{ij}(x) = a_{ji}(x), \qquad c(x)\geq 0, \quad x\in\bar{\Omega},$$

and

$$Q_T := \Omega\times(0,T], \qquad \Omega\subset\mathbb{R}^n$$

is a bounded domain with a smooth boundary $\partial\Omega$, and $\partial Q_T := \partial\Omega\times[0,T]$, while $\varphi(x)$, $\psi(x)$ and $g(x,t)$ are given functions.

In linear viscoelasticity a large variety of regular kernels were classically employed depending on the mechanical properties of the materials to be modeled. In (1.1), the main feature is that the memory kernel k depends on time, allowing for instance to describe the dynamics of aging composite materials.

The direct problem is to find the function $u(x,t)$ from (1.1), (1.2), (1.3), with known $k(t)$.

Many artificial and natural materials exhibit viscoelastic properties. Most often, the need to take into consideration such properties arises in relation to polymers and composites with a polymer matrix. The governing equations for viscoelastic bodies contain integral operators, which significantly complicates the mathematical description of the deformation behavior of such bodies. The equation (1.1) for $\alpha = 2$ is a viscoelasticity equation written for one of the components of the displacement vector. The kernel determination problems in these equations are a relatively new direction in the theory of inverse problems for differential equations. The unique solvability of such inverse problems were studied, for example, in [21], [22], [4], [5], [6], [7], [8], [12], [17]. In recent monographs [19], [10], a wide range of inverse problems of determining the convolution kernel in equations describing phenomena with memory were studied.

Inverse problems studied in [28], [9] are close to the problem considered here. In these problem, the kernel of the convolution of the integral, as in (1.1), was determined in the time fractional diffusion equation ($0<\alpha<1$) with the Caputo operator from the integral overdetermination condition in the first and from the simple observation of the direct problem solution at the point $x = 0$ in the second one. Theorems for unique solvability were obtained and estimates for the stability of the solution were established.

By $C^{0,1}\left(\overline{Q}_T\right)$ we denote the class of continuous and continuously differentiable in t in $\overline{Q}_T$ functions. Let $C^{m,\alpha}\left(Q_T\right)$ be the class of functions, which are m times continuously differentiable with respect to x in Q_T and for which a continuous derivative ∂_t^α exists.

Definition 1.1. *A classical solution of initial and boundary value problem (1.1), (1.2), (1.3) is a function $u(x,t) \in C^{m,\alpha}\left(Q_T\right) \cap C^{0,1}\left(\overline{Q}_T\right)$, which satisfies all the equation in (1.1), (1.2), (1.3) in the usual sense.*

The inverse problem consists in determining the unknown coefficient $k(t)$, $t > 0$, from the available additional data on the solution to the direct problem at some point $x_0 \in \Omega$

$$u(x_0, t) = h(t), \qquad 0 \leqslant t \leqslant T, \tag{1.4}$$

where $h(t)$ is the given function.

Definition 1.2. *A solution to inverse problem (1.1), (1.2), (1.3), (1.4) is a pair of functions $u(x,t)$ and $k(t)$ from the classes $C^{2,\alpha}\left(Q_T\right) \cap C^{0,1}\left(\overline{Q}_T\right)$ and $C[0,T]$, respectively, which satisfy (1.1), (1.2), (1.3), (1.4).*

In this paper, we investigate the local existence, uniqueness and the conditional stability estimate for inverse problem. We begin with studying the direct problem.

2. Direct problem

We begin with considering the numbers λ such that the Dirichlet problem

$$Lv + \lambda v = 0, \quad x \in \Omega, \tag{2.1}$$

$$v\big|_{\partial\Omega} = 0, \tag{2.2}$$

has a nontrivial solution. It is known (see, [11, Sect. 1, Eq. 8]) that spectral problem (2.1), (2.2) in $L_2(\Omega)$ has a complete set of orthonormal eigenfunctions $v_m(x)$, $m \geqslant 1$, and the corresponding eigenvalues λ_m are positive, form a countable set and can be arranged counting their multiplicities as $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots$, $\lim\limits_{k\to\infty} \lambda_k = \infty$.

A solution to problem (1.1), (1.2), (1.3) is sought in the form of Fourier series

$$u(x,t) = \sum_{m=1}^{\infty} u_m(t) v_m(x), \tag{2.3}$$

where $v_m(x)$ are the eigenfunctions of problem (2.1), (2.2), and $u_m(t)$ are the Fourier coefficients defined by the formula

$$u_m(t) = \int_\Omega u(x,t) v_m(x)\, dx. \tag{2.4}$$

Substituting (2.3) into equations (1.1), (1.2), we obtain the problem for the Fourier coefficients $u_m(t)$

$$\partial_t^\alpha u_m(t) + \lambda_m u_m(t) - \int_0^t k(t-\tau) u_m(\tau)\, d\tau = g_m(t), \tag{2.5}$$

$$u_m(0) = \varphi_m, \quad u_m'(0) = \psi_m, \tag{2.6}$$

where φ_m, ψ_m, $g_m(t)$ are the Fourier coefficients of functions $\varphi(x)$, $\psi(x)$, $g(x,t)$:

$$\varphi(x) = \sum_{m=1}^{\infty} \varphi_m v_m(x), \qquad \psi(x) = \sum_{m=1}^{\infty} \psi_m v_m(x), \qquad g(x,t) = \sum_{m=1}^{\infty} g_m(t) v_m(x). \tag{2.7}$$

The problem (2.5) and (2.6) is equivalent to the integral equation [16, Ch. III, Sect. 1, Eq. 3.1.11]:

$$
\begin{aligned}
u_m(t) =&\varphi_m E_{\alpha,1}\left(-\lambda_m t^\alpha\right) + \psi_m t E_{\alpha,2}\left(-\lambda_m t^\alpha\right) \\
&+ \int\limits_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_m (t-s)^\alpha\right) G_m(s;k,u_m)\, ds,
\end{aligned} \tag{2.8}
$$

where

$$
G_m(t;k,u_m) := g_m(t) + \int\limits_0^t k(t-\tau) u_m(\tau)\, d\tau, \tag{2.9}
$$

and $E_{\alpha,\beta}(z)$ is the Mittag — Leffler function defined by the series [16, Ch. I, Sect. 8, Eq. (1.8.1)]:

$$
E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)},
$$

with $\alpha, z, \beta \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$ ($\mathrm{Re}(\alpha)$ stands for real part of α). Here we list some important properties of the Mittag — Leffler function, which will be employed in what follows.

Proposition 2.1. [20, Ch. I, Sect. 2, Thm 1.6]. *Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$ be arbitrary. We suppose that κ is such that $\pi\alpha/2 < \kappa < \min\{\pi, \pi\alpha\}$. Then there exists a constant $C = C(\alpha, \beta, \kappa) > 0$ such that*

$$
|E_{\alpha,\beta}(z)| \leqslant \frac{C}{1+|z|}, \quad \kappa \leqslant |\arg(z)| \leqslant \pi.
$$

Proposition 2.2. [16, Ch. I, Sec. 8]. *For $\lambda > 0$, $a > 0$, and $t > 0$ we have*

$$
\frac{d}{dt} E_{\alpha,1}\left(-\lambda t^\alpha\right) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda t^\alpha\right), \qquad \frac{d}{dt} t^{\beta-1} E_{\alpha,\beta}\left(-\lambda t^\alpha\right) = t^{\beta-2} E_{\alpha,\beta-1}\left(-\lambda t^\alpha\right).
$$

Equation (2.8) is an integral Volterra equation of second kind for the function $u_m(t)$. It follows from the theory of integral equations that this equation is uniquely solvable and the solution can be obtained by the method of successive approximations. Then next two lemmas will used in the proof of our main result.

Lemma 2.1. *The estimates hold*

$$
|u_m(t)| \leqslant C\left(|\varphi_m| + T|\psi_m| + \frac{T^\alpha}{\alpha}\|g_m\|\right) e^{\frac{C\|k\|T^{1+\alpha}}{\alpha(1+\alpha)}}, \tag{2.10}
$$

$$
\begin{aligned}
|\partial_t^\alpha u_m(t)| \leqslant& C\left(|\varphi_m| + T|\psi_m|\right)\left(\lambda_m + T\|k\|\left(1+\lambda_m\frac{T^\alpha}{\alpha}\right) e^{\frac{C\|k\|T^{1+\alpha}}{\alpha(1+\alpha)}}\right) \\
&+ C\|g_m\|\left(1 + \frac{T^{1+\alpha}}{\alpha}\|k\|\left(1+\lambda_m\frac{T^\alpha}{\alpha}\right) e^{\frac{C\|k\|T^{1+\alpha}}{\alpha(1+\alpha)}}\right),
\end{aligned} \tag{2.11}
$$

where

$$
\|g_m\| = \max_{0\leqslant t\leqslant T} |g_m(t)|, \qquad \|k\| = \max_{0\leqslant t\leqslant T} |k(t)|.
$$

Proof. By Proposition 2.1, it follows from formula (2.8) that

$$
\begin{aligned}
|u_m(t)| \leqslant& C\left(|\varphi_m| + |\psi_m| t\right) \\
&+ \left| \int\limits_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_m (t-s)^\alpha\right) \left(g_m(s) + \int\limits_0^s k(s-\tau) u_m(\tau)\, d\tau\right)(s)\, ds \right|
\end{aligned}
$$

$$\leqslant C\Big(|\varphi_m| + T|\psi_m| + \frac{T^\alpha}{\alpha}\|g_m\|\Big) + \frac{C\|k\|}{\alpha}\int\limits_0^t (t-\tau)^\alpha |u_m(\tau)|\,d\tau.$$

By Grönwall integral inequality we now arrive at (2.10).

It follows from (2.5) that

$$\partial_t^\alpha \int\limits_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_m (t-s)^\alpha\right) G_m(s;k,u_m)\,ds$$
$$= G_m(t;k,u_m) - \lambda_m \int\limits_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_m (t-s)^\alpha\right) G_m(s;k,u_m)\,ds.$$

Taking this identity into consideration, we apply the operator ∂_t^α to both sides of (2.8) and on the base of the relations [16, Ch. II, Sec. 4]

$$\partial_t^\alpha E_{\alpha,1}\left(-\lambda_m t^\alpha\right) = -\lambda_m E_{\alpha,1}\left(-\lambda_m t^\alpha\right), \qquad \partial_t^\alpha \left(t E_{\alpha,2}\left(-\lambda_m t^\alpha\right)\right) = -\lambda_m t E_{\alpha,2}\left(-\lambda_m t^\alpha\right),$$

we obtain

$$\begin{aligned}\partial_t^\alpha u_m(t) = &-\lambda_m \varphi_m E_{\alpha,1}\left(-\lambda_m t^\alpha\right) - \lambda_m \psi_m t E_{\alpha,2}\left(-\lambda_m t^\alpha\right) + G_m(t;k,u_m)\\ &- \lambda_m \int\limits_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_m (t-s)^\alpha\right) G_m(s;k,u_m) ds.\end{aligned} \tag{2.12}$$

Estimating the right hand side of this equation and using (2.10), we obtain (2.11). The proof is complete. □

The main result of this section is as follows.

Theorem 2.1. *Let $k(t) \in C[0,T]$ and*

1. $a_{ij} \in C^{[\frac{N}{2}]+2}(\bar\Omega)$, $c \in C^{[\frac{N}{2}]+1}(\bar\Omega)$;
2. $\{\varphi\,\psi\} \in H^{[\frac{N}{2}]+3}(\Omega)$, $g \in C\left(H^{[\frac{N}{2}]+3};[0,T]\right)$;
3. $\{\varphi,\,\psi,\,g(\,\cdot\,,t)\}\,, L\{\varphi,\,\psi,\,g(\,\cdot\,,t)\}\,,\ldots, L^{[\frac{N}{4}+1]}\{\varphi,\,\psi,\,g(\,\cdot\,,t)\} \in H_0^1(\Omega)$.

Then there exists a unique classical solution to problem (1.1), (1.2), (1.3).

Proof. Applying (formally) the operators L and ∂_t^α to the series (2.3), we get

$$\partial_t^\alpha u(x,t) = \sum_{m=1}^\infty \partial_t^\alpha u_m(t) v_m(x), \tag{2.13}$$

$$Lu(x,t) = \sum_{m=1}^\infty u_m(t) L v_m(x) = -\sum_{m=1}^\infty \lambda_m u_m(t) v_m(x). \tag{2.14}$$

Let us prove the convergence of series (2.3), (2.13), (2.14).

It follows from (2.3) and (2.10) that

$$\begin{aligned}|u(x,t)| \leqslant & C\sum_{m=1}^\infty |v_m(x)|\left(|\varphi_m| + T|\psi_m| + \frac{T^\alpha}{\alpha}\|g_m\|\right) e^{\frac{C\|k\|T^{1+\alpha}}{\alpha(1+\alpha)}}\\ \leqslant & C e^{\frac{C\|k\|T^{1+\alpha}}{\alpha(1+\alpha)}}\left(\sum_{m=1}^\infty |\varphi_m||v_m| + T\sum_{m=1}^\infty |\psi_m||v_m| + \frac{T^\alpha}{\alpha}\sum_{m=1}^\infty |g_m(t)||v_m|\right).\end{aligned} \tag{2.15}$$

Proceeding similarly for series (2.13) and (2.14), we have

$$
\begin{aligned}
|\partial_t^\alpha u(x,t)| \leqslant C\Bigg(& e^{\frac{C\|k\|T^{1+\alpha}}{\alpha(1+\alpha)}}\Bigg(\left(1+\|k\|\frac{T^{1+\alpha}}{\alpha}\right)\Bigg(\sum_{m=1}^{\infty}\lambda_m|\varphi_m||v_m(x)| \\
& \qquad + T\sum_{m=1}^{\infty}\lambda_m|\psi_m||v_m(x)|\Bigg) \\
& + T\|k\|\sum_{m=1}^{\infty}\left(|\varphi_m||v_m(x)| + T\sum_{m=1}^{\infty}|\psi_m||v_m(x)|\right)\Bigg) \\
& + \left(1+\frac{T^{1+\alpha}}{\alpha}\right)\sum_{m=1}^{\infty}\lambda_m|g_m(t)||v_m(x)| \\
& + e^{\frac{C\|k\|T^{1+\alpha}}{\alpha(1+\alpha)}}\frac{T^\alpha}{\alpha}\sum_{m=1}^{\infty}|g_m(t)||v_m(x)|\Bigg),
\end{aligned} \tag{2.16}
$$

and

$$
\begin{aligned}
|Lu(x,t)| \leqslant Ce^{\frac{C\|k\|T^{1+\alpha}}{\alpha(1+\alpha)}}\Bigg(& \sum_{m=1}^{\infty}\lambda_m|\varphi_m||v_m(x)| \\
& + T\sum_{m=1}^{\infty}\lambda_m|\psi_m||v_m(x)| + \frac{T^\alpha}{\alpha}\sum_{m=1}^{\infty}\lambda_m|g_m(t)||v_m(x)|\Bigg).
\end{aligned} \tag{2.17}
$$

If we ensure the convergence of the series $\sum\limits_{m=1}^{\infty}\lambda_m\omega_m|v_m(x)|$ with $\omega_m = \{|\varphi_m|, |\psi_m|, \|g_m\|\}$, then the series (2.3), (2.13), (2.14) converge. We shall only prove the convergence of $\sum\limits_{m=1}^{\infty}\lambda_m|\varphi_m||v_m(x)|$, and for the remaining series this can be proved in a similar way. By the Cauchy —Schwarz — Bunyakovsky inequality we have

$$
\sum_{m=1}^{\infty}\lambda_m|\varphi_m||v_m(x)| = \sum_{m=1}^{\infty}\frac{|v_m(x)|}{\lambda_m^{\frac{[\frac{N}{2}]+1}{2}}}|\varphi_m|\lambda_m^{\frac{[\frac{N}{2}]+3}{2}} \leqslant \left(\sum_{m=1}^{\infty}\frac{|v_m(x)|^2}{\lambda_m^{[\frac{N}{2}]+1}}\right)^{\frac{1}{2}}\left(\sum_{m=1}^{\infty}|\varphi_m|^2\lambda_m^{[\frac{N}{2}]+3}\right)^{\frac{1}{2}}.
$$

The first series in the right side of this relation converges by [11, Lm. 1], and the second series converges by [11, Lm. 5]. Generally, if the assumptions of the theorem are satisfied, the series (2.3), (2.13), (2.14) converge uniformly and absolutely in $\overline{Q_T}$. Thus, the function $u(x,t)$ defined by series (2.3) is a solution to problem (1.1), (1.2), (1.3) in $\overline{Q_T}$.

Let us now prove the uniqueness of this solution. For $\varphi(x)\equiv 0$, $\psi(x)\equiv 0$ and $g(x,t)\equiv 0$ we obtain $\varphi_m\equiv 0$, $\psi_m\equiv 0$, $g_m(t)\equiv 0$. Then formula (2.8) implies that $u_m\equiv 0$ since u_m is a solution to the homogeneous equation

$$
u_m(t) = \int\limits_0^t ds\,(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_m(t-s)^\alpha\right)\int\limits_0^s k(s-\tau)u_m(\tau)\,d\tau.
$$

Substituting $u_m\equiv 0$ into equation (2.4), we obtain

$$
\int\limits_\Omega u(x,t)v_m(x)\,dx = 0.
$$

Since the system v_m is complete in space $L_2(\Omega)$, function $u(x,t)=0$ almost everywhere in Ω and for any $t\in[0,T]$. Since the function $u(x,t)\in C^{0,1}\left(\overline{Q}_T\right)$ we conclude that $u(x,t)\equiv 0$ in $\overline{Q}_T$. The proof is complete. □

3. Theorem on solvability of inverse problem

Denoting $r(t) = \int\limits_0^t k(s)ds$, we rewrite (2.8) as

$$\begin{aligned} u_m(t) =& \varphi_m E_{\alpha,1}\left(-\lambda_m t^\alpha\right) + \psi_m t E_{\alpha,2}\left(-\lambda_m t^\alpha\right) \\ &+ \int\limits_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_m (t-s)^\alpha\right) \mathcal{G}_m(s; r, u_m')ds, \end{aligned} \tag{3.1}$$

where

$$\mathcal{G}_m(t; r, u_m') := r(t)\varphi_m + g_m(t) + \int\limits_0^t r(t-\tau)u_m'(\tau)\, d\tau. \tag{3.2}$$

Basing on Proposition 2.2 and differentiating (3.1) in t, we get the integral equation for $u_m'(t)$

$$\begin{aligned} u_m'(t) =& -\lambda_m \varphi_m t^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_m t^\alpha\right) \\ &+ \psi_m E_{\alpha,1}\left(-\lambda_m t^\alpha\right) + \int\limits_0^t (t-s)^{\alpha-2} E_{\alpha,\alpha-1}\left(-\lambda_m (t-s)^\alpha\right) \mathcal{G}_m(s; r, u_m')ds, \end{aligned} \tag{3.3}$$

Lemma 3.1. *Let $u_m'(t)$, $\left(u_m'\right)^1(t)$, $\left(u_m'\right)^2(t)$ be solutions of (3.3), corresponding to functions $r(t)$, $r^1(t)$, $r^2(t)$, respectively. For $t \in [0,T]$ and fixed $m \in \mathbb{N}$ the estimates hold*

$$|u_m'(t)| \leqslant C\left(\left(\lambda_m T^{\alpha-1} + \|r\|\frac{T^{\alpha-1}}{\alpha-1}\right)|\varphi_m| + |\psi_m| + \|g_m\|\frac{T^{\alpha-1}}{\alpha-1}\right) e^{\frac{C\|r\|T^\alpha}{(\alpha-1)(\alpha)}}, \tag{3.4}$$

$$\left|\left(u_m'\right)^1(t) - \left(u_m'\right)^2(t)\right| \leqslant C\left\|r^1 - r^2\right\|\left(|\varphi_m|\frac{T^{\alpha-1}}{\alpha-1} + \left\|(u_m')^1\right\|\frac{T^\alpha}{(\alpha-1)\alpha}\right) e^{\frac{C\|r^2\|T^\alpha}{(\alpha-1)\alpha}}, \tag{3.5}$$

where

$$\|g_m\| = \max_{0\leqslant t\leqslant T}|g_m(t)|, \qquad \|r\| = \max_{0\leqslant t\leqslant T}|r(t)|,$$

and $\|(u_m')^1\|$ satisfies (3.4) with r^1.

Proof. The validity of (3.4) is proved similarly to the evaluation of (2.10). To prove the estimate (3.5) for $t \in [0,T]$, we use Proposition 1 and we get

$$\begin{aligned} \left|\left(u_m'\right)^1(t) - \left(u_m'\right)^2(t)\right| \leqslant& C\int\limits_0^t ds(t-s)^{\alpha-2}\Bigg(\left|r^1(t) - r^2(t)\right||\varphi_m| + \\ &\int\limits_0^t \Big(\left|r^1(t-\tau) - r^2(t-\tau)\right|\left|(u_m')^1(\tau)\right| \\ &+ \left|r^2(t-\tau)\right|\left|(u_m')^1(\tau) - (u_m')^2(\tau)\right|\Big)\, d\tau\Bigg). \end{aligned}$$

Applying the Grönwall lemma to this inequality, in view of (3.4) with r^1 we arrive at (3.5).

Since $G_m(s; k, u_m) = \mathcal{G}_m(t; r, u_m')$, we multiply (2.12) by $v_m(x_0)$ and sum over m. Then, taking into consideration (1.4), we get the integral equation for $r(t)$

$$r(t) = A(r)(t), \qquad t \in [0,T], \tag{3.6}$$

where

$$
\begin{aligned}
A(r)(t) =& r_0(t) + (\varphi(x_0))^{-1}\Bigg(\int_0^t r(\tau)\sum_{m=1}^{\infty} u'_m(t-\tau;\, r)v_m(x_0)d\tau \\
&+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_m(t-s)^{\alpha}\right) \\
&\cdot\Bigg(r(s)\sum_{m=1}^{\infty}\lambda_m\varphi_m v_m(x_0) + \int_0^s r(\tau)\sum_{m=1}^{\infty}\lambda_m u'_m(s-\tau;\, r)v_m(x_0)\, d\tau\Bigg)\, ds\Bigg),
\end{aligned}
$$

$u'_m(t;\, r)$ means that the solution of (3.3) depends on $r(t)$, and

$$
\begin{aligned}
r_0(t) = (\varphi(x_0))^{-1}\Bigg(&\partial_t^{\alpha}h(t) - g(x_0,t) + \sum_{m=1}^{\infty}\lambda_m\varphi_m v_m(x_0)E_{\alpha,1}\left(-\lambda_m t^{\alpha}\right) \\
&+ t\sum_{m=1}^{\infty}\lambda_m\psi_m v_m(x_0)E_{\alpha,2}\left(-\lambda_m t^{\alpha}\right) \\
&+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_m(t-s)^{\alpha}\right)\sum_{m=1}^{\infty}\lambda_m g_m(s)v_m(x_0)ds\Bigg).
\end{aligned}
$$

The proof is complete. □

Fix a number $\rho > 0$ and consider the ball in the class of continuous function

$$
R_T\left(r_0,\rho\right) := \big\{r(t) :\ r(t)\in C\left[0,T\right],\ \|r-r_0\|\leqslant\rho\big\}.
$$

The main result of this work is as follows.

Theorem 3.1. *Let $h\in C^2[0,T]$, $\varphi(x_0)\neq 0$ and*

1. $a_{ij}\in C^{\left[\frac{N}{2}\right]+4}(\bar{\Omega}),\quad c\in C^{\left[\frac{N}{2}\right]+3}(\bar{\Omega})$;
2. $\varphi(x)\in H^{\left[\frac{N}{2}\right]+5}(\Omega),\quad \psi)\in H^{\left[\frac{N}{2}\right]+3}(\Omega),\quad g\in C\left(H^{\left[\frac{N}{2}\right]+3};[0,T]\right)$;
3. $\{\varphi,\,\psi,\,g(\,\cdot\,,t)\}\,, L\{\varphi,\,\psi,\,g(\,\cdot\,,t)\}\,,\ldots, L^{\left[\frac{N}{4}+1\right]}\{\varphi,\,\psi,\,g(\,\cdot\,,t)\}\in H_0^1(\Omega),\ L^{\left[\frac{N}{4}+2\right]}\varphi\in H_0^1(\Omega)$.

Then there is a unique solution to the operator equation (3.6).

Proof. It can be shown that for a sufficiently small $T > 0$ the operator A is a contracting mapping of the ball $R_T\left(r_0,\rho\right)$ onto itself; i.e., the condition $r\in R_T\left(r_0,\rho\right)$ implies that $A(r)\in R_T\left(r_0,\rho\right)$ and A shrinks the distance between any elements $\{r^1(t),\, r^2(t)\}\in R_T\left(r_0,\rho\right)$.

Indeed, by (3.4), (3.5), (3.6), after some calculations we have

$$\begin{aligned}|A(r)(t)-r_0(t)| \leqslant & CT^{\alpha-1}|(\varphi(x_0))^{-1}|\Bigg(\frac{T}{\alpha}\sum_{m=1}^{\infty}\lambda_m|\varphi_m|\,|\vartheta_m(x_0)| \\ & + C\sum_{m=1}^{\infty}\Bigg(\left(\lambda_m T^{\alpha-1}+\|r\|\frac{T^{\alpha-1}}{\alpha-1}\right)|\varphi_m|+|\psi_m|+\|g_m\|\frac{T^{\alpha-1}}{\alpha-1}\Bigg) \\ & \cdot\exp\left(\frac{C\|r\|T^{\alpha}}{\alpha(\alpha-1)}\right)\|r\|\,|\vartheta_m(x_0)| \\ & + \frac{CT^2}{\alpha(\alpha+1)}\sum_{m=1}^{\infty}\Bigg(\left(\lambda_m^2T^{\alpha-1}+\|r\|\frac{T^{\alpha-1}}{\alpha-1}\lambda_m\right)|\varphi_m|+|\psi_m|\lambda_m \\ & + \frac{T^{\alpha-1}}{\alpha-1}\lambda_m\|g_m\|\Bigg)\exp\left(\frac{C\|r\|T^{\alpha}}{\alpha(\alpha-1)}\right)|v_m(x_0)|\,\|r\|,\end{aligned} \tag{3.7}$$

$$\begin{aligned}\left|A(r^1)(t)-A(r^2)(t)\right| \leqslant & CT^{\alpha-1}\left|(\varphi(x_0))^{-1}\right|\Bigg(\sum_{m=1}^{\infty}\frac{\lambda_m|\varphi_m|\,|v_m(x_0)|}{\alpha} \\ & + \sum_{m=1}^{\infty}\Bigg(\left(\lambda_m T^{\alpha-1}+\|r^1\|\frac{T^{\alpha-1}}{\alpha-1}\right)|\varphi_m|+|\psi_m| \\ & + \|g_m\|\frac{T^{\alpha-1}}{\alpha-1}\Bigg)\exp\left(\frac{C\|r^1\|T^{\alpha}}{\alpha(\alpha-1)}\right)|v_m(x_0)| \\ & + \frac{T^2}{\alpha(\alpha+1)}\sum_{m=1}^{\infty}\Bigg(\left(\lambda_m^2T^{\alpha-1}+\|r^1\|\frac{\lambda_m^2T^{\alpha-1}}{\alpha-1}\right)|\psi_m| \\ & + \lambda_m|\psi_m|+\lambda_m\|g_m\|\frac{T^{\alpha-1}}{\alpha-1}\Bigg)\exp\left(\frac{C\|r^1\|T^{\alpha}}{\alpha(\alpha-1)}\right)|v_m(x_0)| \\ & + \|r^2\|\sum_{m=1}^{\infty}\Bigg(|\varphi_m|\frac{T^{\alpha-1}}{\alpha-1}+C\Bigg(\left(\lambda_m T^{\alpha-1}+\|r^2\|\frac{T^{\alpha-1}}{\alpha-1}\right)|\varphi_m| \\ & + |\psi_m|+\|g_m\|\frac{T^{\alpha-1}}{\alpha-1}\Bigg)\exp\left(\frac{C\|r^2\|T^{\alpha}}{\alpha(\alpha-1)}\right)\frac{T^{\alpha}}{\alpha(\alpha+1)}\Bigg) \\ & \cdot|v_m(x_0)|\exp\left(\frac{C\|r^2\|T^{\alpha}}{\alpha(\alpha-1)}\right) \\ & + \frac{\|r^2\|T^{\alpha}}{\alpha(\alpha+1)}\sum_{m=1}^{\infty}\Bigg(\lambda_m|\varphi_m|\frac{T^{\alpha-1}}{\alpha-1} \\ & + C\Bigg(\left(\lambda_m^2T^{\alpha-1}+\lambda_m\|r\|\frac{T^{\alpha-1}}{\alpha-1}\right)|\varphi_m| \\ & + \lambda_m|\psi_m|+\lambda_m\|g_m\|\frac{T^{\alpha-1}}{\alpha-1}\Bigg)\exp\left(\frac{C\|r^2\|T^{\alpha}}{\alpha(\alpha-1)}\right)\frac{T^{\alpha}}{\alpha(\alpha+1)}\Bigg) \\ & \cdot\exp\left(\frac{C\|r^2\|T^{\alpha}}{\alpha(\alpha-1)}\right)|v_m(x_0)|\Bigg)\|r^1-r^2\|.\end{aligned} \tag{3.8}$$

Under the assumptions of the theorem, all series in the above formulas converge.

Note that the expressions on the right sides of these inequalities are monotonically increasing functions of T, $\|r\|$, $\|r^1\|$, $\|r^1\|$, and since $r,\, r^1,\, r^2\in R_T\left(r_0,\rho\right)$, it yields the inequality

$$\left\|(r,\, r^1, r^2)\right\| \leqslant \rho+\|r_0\|$$

In view of these facts, for $(x,t) \in Q_T$ we obtain

$$\|A(r)(t) - r_0(t)\| \leqslant m_1(T),$$
$$\left\|A[r^1](t) - A[r^2(t)]\right\| = m_2(T)\left\|r^1 - r^2\right\|,$$

where $m_1(T)$ denotes the right hand side of inequality (3.7) and $m_2(T)$ is the factor at $\|r^1 - r^2\|$ in the right hand side of (3.8) with the only difference that r, r^1, r^2 are replaced by $\rho + \|r_0\|$ in both cases.

Note that $m_i(T)$, $i = 1, 2$, are positive monotonically increasing functions of T, and $m_i(0) = 0$. Hence, the equations $m_1(T) = \rho$ and $m_2(T) = 1$ have unique positive roots. We denote these roots by T_1 and T_2, respectively. Then, it is clear that if we choose $T^* < \min(T_1, T_2)$, then the operator A is a contraction on the ball $R_T(r_0, \rho)$. By the Banach theorem, the operator A has a unique fixed point in the ball $R_T(r_0, \rho)$; i.e., there exists a unique solution of the equation (3.6). The proof is complete. □

Since we have found that the function $r \in C[0,T^*]$, provided $r \in C^1[0,T^*]$, we have $k(t) = r'(t)$. Let us find the conditions for the given functions, under which $r \in C^1[0,T^*]$. Differentiating both sides of equation (3.6), we obtain the linear integral equation for k

$$k(t) = \bar{A}(k)(t), \qquad t \in [0,T], \tag{3.9}$$

where the integral operator $\bar{A}(k)$ involves $r_0'(t)$ as a free term and the integral terms have similar structures as in $A(r)$; they may also include the already known function r.

On the base of Proposition 2.2, by direct calculations we show

$$r_0'(t) = (\varphi(x_0))^{-1}\Bigg(\frac{d}{dt}\left(\partial_t^\alpha h(t)\right) - g_t(x_0,t) + t^{\alpha-1}\sum_{m=1}^{\infty}\lambda_m^2\varphi_m v_m(x_0)E_{\alpha,\alpha}\left(-\lambda_m t^\alpha\right)$$
$$+ \sum_{m=1}^{\infty}\lambda_m\psi_m v_m(x_0)E_{\alpha,1}\left(-\lambda_m t^\alpha\right)$$
$$+ \int_0^t (t-s)^{\alpha-2}E_{\alpha,\alpha-1}\left(-\lambda_m(t-s)^\alpha\right)\sum_{m=1}^{\infty}\lambda_m g_m(s)v_m(x_0)\,ds\Bigg).$$

Theorem 3.2. *Let $h(t) \in C^3[0,T]$, $\varphi(x_0) \neq 0$ and*

1. $a_{ij} \in C^{[\frac{N}{2}]+4}(\bar{\Omega}), |\ c \in C^{[\frac{N}{2}]+3}(\bar{\Omega})$;
2. $\varphi \in H^{[\frac{N}{2}]+5}(\Omega)$, $\psi \in H^{[\frac{N}{2}]+3}(\Omega)$, $g \in C^1\left(H^{[\frac{N}{2}]+3}(\Omega); [0,T]\right)$;
3. $\{\varphi,\ \psi,\ g(\cdot,t)\}, L\{\varphi,\ \psi,\ g(\cdot,t)\}, \cdots, L^{[\frac{N}{4}+1]}\{\varphi,\ \psi,\ g(\cdot,t)\} \in H_0^1(\Omega)$, $L^{[\frac{N}{4}+2]}\varphi \in H_0^1(\Omega)$.

Then there exists a unique solution to the inverse problem (1.1), (1.2), (1.3), (1.4).

To prove this theorem, we note that under the assumptions of this theorem the function r_0' is continuous. Then equation (3.9) is a linear Volterra integral equation of the second kind with continuous free term and kernels. This equation has a unique solution $k \in C[0,T^*]$. Then the solution to direct problem (1.1), (1.2), (1.3) is found by formula (2.3), where $u_m(t)$ is the solution to integral equation (2.8).

Let T be a positive fixed number. Consider the set $\Psi(\omega_0)$ ($\omega_0 > 0$ is some fixed number) of the given functions $(g,\ \varphi,\ \psi,\ h)$, for which all assumptions of Theorem 3.1 are satisfied and

$$\max\left\{\|g\|_{C^1\left(H^{[\frac{N}{2}]+3}(\Omega);[0,T]\right)}, \|\varphi\|_{H^{[\frac{N}{2}]+5}(\Omega)}, \|\psi\|_{H^{[\frac{N}{2}]+3}(\Omega)}, \|h\|_{C^3[0,T]}\right\} \leqslant \omega_0.$$

By $K(\omega_1)$ we denote the class of functions $k \in C\,[0,\,T]$ satisfying the inequality $\|k\| \leqslant \omega_1$ with some fixed positive number ω_1.

The following theorems characterize the conditional stability and global uniqueness the solution to the inverse problem. They can be proved quite analogously to the corresponding theorems in [1].

Theorem 3.3. *Let $(g,\ \varphi,\ \psi,\ h) \in \Psi(\omega_0)$, $\left(\tilde g,\ \tilde\varphi,\ \tilde\psi,\ \tilde h\right) \in \Psi(\omega_0)$ and $\left(k,\ \tilde k\right) \in K(\omega_1)$. Then the solution of inverse problem (1.1), (1.2), (1.3), (1.4) satisfies the estimate*

$$\|k - \tilde k\| \leqslant d\kappa,$$

where

$$\kappa := \|g - \tilde g\|_{C^1\left(H^{\left[\frac{N}{2}\right]+3}(\Omega);[0,T]\right)} + \|\varphi - \tilde\varphi\|_{H^{\left[\frac{N}{2}\right]+5}(\Omega)} + \|\psi - \tilde\psi\|_{H^{\left[\frac{N}{2}\right]+3}(\Omega)} + \|h - \tilde h\|_{C^3[0,T]},$$

the constant d depends only on T, α N, ω_0, ω_1 and norms of a_{ij}, c in the corresponding spaces given in Theorem 3.2 for these functions.

This theorem implies the uniqueness theorem for each $T > 0$.

Theorem 3.4. *Let the functions k, g, φ, ψ, h and $\tilde k$, $\tilde g$, $\tilde\varphi$, $\tilde\psi$, $\tilde h$ have the same meaning as in Theorem 3.3. Moreover, if $g = \tilde g$, $\varphi = \tilde\varphi$, $\psi = \tilde\psi$, $h = \tilde h$ for $t \in [0,\,T]$, then $k(t) = \tilde k(t)$ for $t \in [0,\,T]$.*

4. Particular case

Let us consider a special case of problem (1.1), (1.2), (1.3), (1.4) for $n = 1$. In this case, we assume that $a_{11}(x)$ and $c(x)$ are constant, namely, $a_{11}(x) \equiv 1$, $c(x) \equiv 0$. Then in the domain $Q_T = \{(x,t):\ 0 < x < l,\ 0 < t \leqslant T\}$ we have the problem

$$\partial_t^\alpha u - u_{xx} = \int_0^t k(t-\theta)u(x,\theta)\,d\theta + g(x,t), \qquad (x,t) \in Q_T, \tag{4.1}$$

with initial

$$u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x), \qquad 0 \leqslant x \leqslant l, \tag{4.2}$$

and boundary conditions

$$u(0,t) = 0, \quad u(l,t) = 0, \qquad 0 \leqslant t \leqslant T, \tag{4.3}$$

where $\varphi(x)$, $\psi(x)$ and $g(x,t)$ are given functions.

The inverse problem requires to find the function $k(t)$ if there is additional information regarding the solution of the direct problem (1.1, (1.2), (1.3)

$$u(x_0,t) = h(t), \qquad 0 \leqslant t \leqslant T, \tag{4.4}$$

where $h(t)$ is a given function, $x_0 \in (0,l)$ is a given number. It is known that in this case the eigenfunctions and the corresponding eigenvalues of this problem have the form [26, Ch. II, Sec. 3, Eq. 14]

$$v_m(x) = \sqrt{\frac{2}{l}}\sin\sqrt{\lambda_m}x, \qquad \lambda_m = \left(\frac{\pi m}{l}\right)^2, \qquad m \in \mathbb{N}.$$

For this case by Sobolev embedding theorem and Theorem 3.4 implies the following statement.

Theorem 4.1. *Let $h \in C^3[0,T]$, $\varphi(x_0) \neq 0$ and*

1. $\varphi \in C^4[0,l]$, $\varphi^{(V)}(x) \in L^2(0,l)$, $\psi \in C^2[0,l]$, $\psi''' \in L^2[0,l]$, $g \in C^{2,1}(Q_T)$ $g_{xxx}(\cdot,t) \in L^2(0,l)$;

2. $\varphi(0) = \varphi(l) = \varphi''(0) = \varphi''(l) = \varphi^{(IV)}(0) = \varphi^{(IV)}(l)$, $\psi(0) = \psi(l) = \psi''(0) = \psi''(l)$, $g(0,t) = g(l,t) = g_{xx}(0,t) = g_{xx}(l,t)$.

Then there exists a unique solution to the inverse problem (4.1), (4.2), (4.3), (4.4).

It should be noted that for this particular case, it is also possible to formulate theorems on conditional stability and the global unique solvability of the inverse problem (4.1), (4.2), (4.3), (4.4) similar to Theorems 3.3 and 3.4.

5. Conclusion

We have studied the inverse problem on identification the convolution kernel in the integral term of an initial boundary value problem for a multidimensional time fractional diffusion–wave equation. The spatial operator is assumed to be time independent and uniformly elliptic in divergence form. As an additional (overdetermination) condition, a single pointwise observation of the diffusion–wave process is provided at a fixed point $x_0 \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain.

By employing the Fourier spectral method in combination with techniques from the fractional calculus, the inverse problem is reduced to a nonlinear Volterra integral equation of the second kind with a convolution kernel. Using a fixed point approach, we establish the local existence and global uniqueness of the solution. In addition, we derive a stability estimate for the solution to the inverse problem. Finally, we present the one–dimensional case and prove a theorem on the unique solvability in this setting.

The following cases appear to be promising for further investigation:

• The study of the inverse problem of determining the memory kernel $k(t)$ from equation (1.1) with conditions (1.2) and (1.3), based on an additional condition of the form

$$\int\limits_{\Omega} h(x)u(t,x)\,dx = g(t), \quad 0 \leqslant t \leqslant T,$$

where $h(x)$ and $g(t)$ are given functions. A key question is how this choice of supplementary condition influences the stability of the solution.

• The study of the inverse problem (1.1), (1.2), (1.3), (1.4) in the case where the integral term in equation (1.1) reads

$$\int\limits_0^t k(t-\tau)Lu(x,\tau)\,d\tau,$$

where L is the operator defined in the problem statement.

• Extending the results of this paper to nonlinear analogs of equation (1.1), for example by considering $k(u)$ in place of $k(t)$.

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BIBLIOGRAPHY

1. D.K. Durdiev. *Inverse coefficient problem for the time–fractional diffusion equation* // Eurasian J. Math. Comput. Appl. **9**:1, 44–54 (2021). https://doi.org/10.32523/2306-6172-2021-9-1-44-54
2. D.K. Durdiev, A.A. Rahmonov, Z.R. Bozorov. *A two–dimensional diffusion coefficient determination problem for the time–fractional equation* // Math. Meth. Appl. Sci. **44**:13, 10753 –10761 (2021). https://doi.org/10.1002/mma.7442

3. U.D. Durdiev. *Problem of determining the reaction coefficient in a fractional diffusion equation* // Differ. Equ. **57**:9, 1195–1204 (2021). https://doi.org/10.1134/S0012266121090081
4. D.K. Durdiev, Z.D. Totieva. *The problem of determining the one-dimensional kernel of viscoelasticity equation with a source of explosive type* // J. Inverse Ill–Posed Probl. **28**:1, 43–52 (2020). https://doi.org/10.1515/jiip-2018-0024
5. D.K. Durdiev, Z.D. Totieva. *About global solvability of a multidimensional inverse problem for an equation with memory* // Sib. Math. J. **62**:2, 215–229 (2021). https://doi.org/10.1134/S0037446621020038
6. D.K. Durdiev, Z.S. Safarov. *Inverse problem of determining the one–dimensional kernel of the viscoelasticity equation in a bounded domain* // Math. Notes **97**:6, 867–877 (2015). https://doi.org/10.1134/S0001434615050223
7. D.K. Durdiev, Z.D. Totieva. *The problem of determining the one-dimensional matrix kernel of the system of viscoelasticity equations* Math. Methods Appl. Sci. **41**:17, 8019–8032 (2018). https://doi.org/10.1002/mma.5267
8. D. K. Durdiev, J. J. Jumaev, D. D. Atoev. *Inverse problem on determining two kernels in integro-differential equation of heat flow* // Ufa Math. J. **15**:2, 120–135 (2023). https://doi.org/10.13108/2023-15-2-119
9. D.K. Durdiev. *Convolution kernel determination problem for the time-fractional diffusion equation* // Physica D **457**:1, 133959 (2024). https://doi.org/10.1016/j.physd.2023.133959
10. D.K. Durdiev, Z.D. Totieva. *Kernel Determination Problems in Hyperbolic Integro–Differential Equations*. Springer, Singapore (2023).
11. V.A. Il'in. *The solvability of mixed problems for hyperbolic and parabolic equations* // Russ. Math. Surv. **15**:2, 85–142 (1960). https://doi.org/10.1070/rm1960v015n02ABEH004217
12. J. Janno, L. Von Wolfersdorf. *Inverse problems for identification of memory kernels in viscoelasticity* // Math. Methods Appl. Sci. **20**:4, 291–314 (1997). https://doi.org/10.1002/(SICI)1099-1476(19970310)20:4%3C291::AID-MMA860%3E3.0.CO;2-W
13. B.T. Jin, Z. Zhou. *An inverse potential problem for subdiffusion: stability and reconstruction* // Inverse Probl. **37**:1, 015006 (2020).
14. B.T. Jin, W. Rundell. *An inverse problem for a one–dimensional time–fractional diffusion problem* // Inverse Probl. **28**:7, 075010 (2012). https://doi.org/10.1088/0266-5611/28/7/075010
15. B. Kaltenbacher, W. Rundell. *On an inverse potential problem for a fractional reaction–diffusion equation* // Inverse Probl. **35**:6, 065004 (2019). https://doi.org/10.1088/1361-6420/ab109e
16. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo. *Theory and Application of Fractional Differential Equations.* Elsevier, Amsterdam (2006).
17. A. Lorenzi, E. Paparoni. *Direct and inverse problems in the theory of materials with memory* // Rend. Semin. Mat. Univ. Padova **87**, 105–138 (1992). https://eudml.org/doc/108245
18. W. Ma, L. Sun. *Inverse potential problem for a semilinear generalized fractional diffusion equation with spatio–temporal dependent coefficients* // Inverse Probl. **39**:1, 015005 (2023). https://doi.org/10.1088/1361-6420/aca49e
19. L. Pandolfi. *Systems with Persistent Memory. Controllability, Stability, Identification.* Springer, Cham (2021).
20. I. Podlubny. *Fractional differential equations.* Academic Press, San Diego, CA (1999).
21. V.G. Romanov. *Inverse problems for equations with memory* // Eurasian J. Math. Comput. Appl. **2**:4, 51–80 (2014). https://doi.org/10.32523/2306-6172-2014-2-4-51-80
22. V.G. Romanov. *On the determination of the coefficients in the viscoelasticity equations* // Sib. Math. J. **55**:3, 503–510 (2014). https://doi.org/10.1134/S0037446614030124
23. L.L. Sun, Y.S. Li, Y. Zhang. *Simultaneous inversion of the potential term and the fractional orders in a multi-term time-fractional diffusion equation* // Inverse Probl. **37**:5, 055007 (2021). https://doi.org/10.1088/1361-6420/abf162
24. L. Sun, T. Wei. *Identification of zero–order coefficient in a time–fractional diffusion equation* // Appl. Numer. Math. **111**, 160–180 (2017). https://doi.org/10.1016/j.apnum.2016.09.005
25. L. Sun, Y. Zhang, T. Wei. *Recovering the time–dependent potential function in a multi–term time–fractional diffusion equation* // Appl. Numer. Math. **135**, 228–245 (2019).

https://doi.org/10.1016/j.apnum.2018.09.001
26. A.N. Tikhonov, A.A. Samarskii. *Equations of Mathematical Physics*. Dover Publications, Inc., New York (1977).
27. V.K. Tuan. *Inverse problem for fractional diffusion equation* // Fract. Calc. Appl. Anal. **14**:1, 31–55 (2011). https://doi.org/10.2478/s13540-011-0004-x
28. H. Wang, B.Wu. *On the well–posedness of determination of two coefficients in a fractional integro–differential equation* // Chin. Ann. Math. Ser. B **35**:3, 447–468 (2014). https://doi.org/10.1007/s11401-014-0832-1
29. Z. Zhang, Z. Zhou. *Recovering the potential term in a fractional diffusion equation* // IMA J. Appl. Math. **82**:3, 579–600 (2017). https://doi.org/10.1093/imamat/hxx004

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