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KOLMOGOROV TYPE INEQUALITIES FOR FUNCTIONS ANALYTIC IN CIRCLE

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Abstract. In the paper we obtain a series of Kolmogorov type inequalities for functions analytic in a circle of an arbitrary radius R and belonging to the Hardy space $H_{q,R}$ ($1 \leqslant q \leqslant \infty$, $R > 0$). We provide some applications of these inequalities in extremal problem of best polynomial approximation.

Keywords: Kolmogorov type inequalities, best polynomial approximation, intermediate derivatives, supremum, Hardy space.

Mathematics Subject Classification: 30E05, 30E10

1. Introduction

Since the early twentieth century, many great mathematicians, such as E. Landau, J. Hadamard, G. Hardy, J. Littlewood, A.N. Kolmogorov, have been particularly interested in obtaining precise inequalities for the norms of intermediate derivatives of functions using the norm of the function itself and the norm of its highest derivative. In modern mathematics, such inequalities are commonly referred to as Kolmogorov type inequalities. The rapid development of this topic is associated with the works of V.V. Arestov, S.B. Stechkin, L.V. Taikov, V.N. Gabushin, V.M. Tikhomirov, N.P. Kuptsov, V.N. Konovalov, N.P. Korneichuk, V.F. Babenko, G.G. Magaril-Ilyaev, A.A. Ligun, S.A. Pichugov, and many others. A detailed presentation of both modern and earlier results is given in the relatively recently published monograph by Babenko, Korneichuk, Kofanov, and Pichugov [2]. For functions of two variables, Kolmogorov type inequalities were proved in recently published papers by Vakarchuk [3], Vakarchuk and Vakarchuk [4], [5] and by Shabozov and his students [9], [10].

In our opinion, it is of great interest to find Kolmogorov type inequalities for functions analytic in an arbitrary circle of radius R, where, in comparison to functions of a real variable, just few complete results were obtained, see, for example, [3]–[5], [8]–[10], [12] and the references therein.

In this paper, we prove a series of Kolmogorov type inequalities for functions analytic in an arbitrary circle and belonging to the Hardy spaces H_q, $1 \leqslant q \leqslant \infty$.

We introduce necessary notation and notions. Let $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{R}_+$, $\mathbb{R}$, and $\mathbb{C}$ be the set of natural numbers, non-negative integers, positive numbers, real numbers, and complex numbers, respectively. Let

$$U_R := \{z \in \mathbb{C} : |z| < R\}$$

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be a disk of an arbitrary radius R in the complex plane $\mathbb{C}$, and let $A(U_R)$ be the set of functions analytic in the disk U_R. For an arbitrary function $f \in A(U_R)$ with $0 < \rho \leqslant R$ we denote

$$M_q(f,\rho) := \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} |f(\rho e^{it})|^q dt\right)^{\frac{1}{q}} & \text{if} \quad 1 \leqslant q < \infty, \\ \max\{|f(\rho e^{it})| : 0 \leqslant t < 2\pi\} & \text{if} \quad q = \infty, \end{cases}$$

where the integral is understood in the Lebesgue sense.

By the symbol $H_{q,R}$, $1 \leqslant q \leqslant \infty$, $R > 0$, we denote the Hardy space, which consists of functions $f \in A(U_R)$ with a finite norm

$$\|f\|_{H_{q,R}} := \lim_{\rho \to R-0} M_q(f,\rho). \tag{1.1}$$

It is well–known [6, Ch. 4] that norm (1.1) is attained at angular boundary values $f(Re^{it})$ of a function $f \in H_{q,R}$:

$$\|f\|_{H_{q,R}} := \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} |f(Re^{it})|^q\, dt\right)^{\frac{1}{q}} & \text{if} \quad 1 \leqslant q < \infty, \\ \operatorname{ess\,sup}\{|f(Re^{it})| : 0 \leqslant t < 2\pi\} & \text{if} \quad q = \infty. \end{cases}$$

In the case $R = 1$ we let $U := U_1$, $H_q := H_{q,1}$.

We denote $H_{q,\rho}$, $0 < \rho \leqslant R$ the Hardy space of functions $f \in A(U_\rho)$, which obey

$$\|f(z)\|_{H_{q,\rho}} := \|f(\rho z)\|_{H_q} = \|f(\rho e^{i(\cdot)})\|_{L_q[0,2\pi]} = M_q(f,\rho) < \infty.$$

The rth derivative of function $f(z)$ in the argument of the complex variable $z = \rho e^{it}$ is denoted by $f_a^{(r)}(z)$. It is obvious that

$$f_a^{(1)}(z) = \frac{\partial f(\rho e^{it})}{\partial t} = f'(z)zi$$

and for $r \geqslant 2$ ($r \in \mathbb{N}$) we let

$$f_a^{(r)}(z) := \{f_a^{(r-1)}(z)\}_a^{(1)}.$$

By the symbol $H_{q,R,a}^{(r)}$ we denote the class of functions $f \in A(U_R)$ such that $f_a^{(r)} \in H_{q,R}$, that is,

$$H_{q,R,a}^{(r)} := \left\{f \in A(U_R) : \|f_a^{(r)}\|_{q,R} = \|f_a^{(r)}(R\,\cdot)\|_q < \infty\right\}, \quad r \in \mathbb{N}.$$

In the same way for the usual derivative of rth order in the variable z of a function $f \in A(U_R)$ with the Taylor series

$$f(z) = \sum_{k=0}^{\infty} c_k(f) z^k, \tag{1.2}$$

we let

$$f^{(r)}(z) := \frac{d^r f}{dz} = \sum_{k=r}^{\infty} \alpha_{k,r} c_k(f) z^{k-r},$$

where

$$\alpha_{k,r} := k(k-1)\cdots(k-r+1), \quad k \geqslant r, \quad k \in \mathbb{N}, \quad r \in \mathbb{Z}_+, \quad \alpha_{k,0} := 1, \quad \alpha_{k,1} := k,$$

$c_k(f)$ are the coefficients in the Taylor series of function f. In what follows we let

$$H_{q,R}^{(r)} := \left\{f \in A(U_R) : \|f^{(r)}\|_{q,R} = \|f^{(r)}(R\,\cdot)\|_q < \infty\right\}, \quad r \in \mathbb{N}.$$

2. Kolmogorov type inequality for classes $H^{(r)}_{q,R,a}$ and $H^{(r)}_{q,R}$ $(r\in\mathbb{N},\ 1\leqslant q\leqslant\infty)$

In this section we prove a Kolmogorov type inequality for the classes $H^{(r)}_{q,R,a}$ and $H^{(r)}_{q,R}$ for all $r\in\mathbb{N}$, $1\leqslant q\leqslant\infty$. We have noted above that the norm of functions $f\in H_{q,R}$ is attained on its angular boundary values $f(Re^{it})\in L_q$, $1\leqslant q\leqslant\infty$, which are well–defined for almost all values $t\in[0,2\pi)$. If at the same time $1\leqslant p\leqslant q$, then the inclusion $H_{q,R}\subset H_{p,R}$ holds for each $R>0$. The results of [1] imply as a corollary that $H^{(r)}_{q,R,a}\subset H_{q,R}$, $1\leqslant q\leqslant\infty$. Hence, for an arbitrary function $f\in H^{(r)}_{q,R,a}$ for $q\geqslant 2$ we have $f^{(r)}_a\in H_{q,R}\subset H_{2,R}$. Using the Taylor series (1.2) of function f in the circle U_R, we represented the rth derivative of $f^{(r)}_a(z)$ as

$$f^{(r)}_a(z)=\sum_{k=1}^{\infty}(ik)^r c_k(f)z^k.$$

Since, as the above facts imply, the rth derivative

$$f^{(r)}_a(Re^{it})=\sum_{k=1}^{\infty}(ik)^r c_k(f)(Re^{it})^k$$

of function $f\in H^{(r)}_{q,R,a}$ has a finite norm in the space $H_{2,R}$

$$\|f^{(r)}_a\|^2_{H_{2,R}}=\sum_{k=1}^{\infty}k^{2r}|c_k(f)|^2R^{2k}<\infty, \tag{2.1}$$

by the above formula, the norms of all intermediate derivatives $f^{(r-\nu)}_a(z)$, $\nu=\overline{1,r-1}$, in this space are also finite. In particular, this implies that these derivatives belong to the Hardy space $H_{p,R}$, $1\leqslant p\leqslant 2$, that is, the relations

$$H^{(r)}_{q,R,a}\subset H^{(r-\nu)}_{p,R,a}$$

hold, where $1\leqslant p\leqslant 2\leqslant q$, $\nu=\overline{1,r}$, $H^{(0)}_{p,R}\equiv H_{p,R}$. The following theorem holds.

Theorem 2.1. *Let $\nu,r\in\mathbb{N}$, $1\leqslant\nu\leqslant r$, $0<\rho\leqslant R$ and $1\leqslant p\leqslant 2\leqslant s,t\leqslant q$. Then each function $f\in H^{(r)}_{q,R,a}$ satisfies the Kolmogorov type inequality*

$$\|f^{(r-\nu)}_a\|_{H_{p,\rho}}\leqslant\frac{\rho}{R}\|f\|^{\frac{\nu}{r}}_{H_{s,R}}\|f^{(r)}_a\|^{1-\frac{\nu}{r}}_{H_{t,R}}. \tag{2.2}$$

This inequality is sharp in the sense that there exists a function $g\in H^{(r)}_{q,R,a}$, which turns it into the identity.

Proof. Since for $\nu=r$ inequality (2.2) is obvious, we suppose that $1\leqslant\nu\leqslant r-1$. Let $f\in H^{(r)}_{q,R,a}$. Then its $(r-\nu)$th derivative obeys the relation

$$f^{(r-\nu)}_a(z)=\sum_{k=1}^{\infty}(ik)^{r-\nu}c_k(f)z^k, \tag{2.3}$$

and by the Parseval identity for the norms of functions f and $f^{(r-\nu)}$, by (1.2) and (2.3) we have

$$\|f\|^2_{H_{2,R}}=\sum_{k=0}^{\infty}|c_k(f)|^2R^{2k}, \tag{2.4}$$

$$\|f^{(r-\nu)}_a\|^2_{H_{2,R}}=\sum_{k=1}^{\infty}k^{2(r-\nu)}|c_k(f)|^2R^{2k}.$$

At an arbitrary point $z \in U_R$, $|z| = \rho$, $0 < \rho \leqslant R$ for the norms of $f_a^{(r-\nu)}(z)$ we write

$$\|f_a^{(r-\nu)}\|^2_{H_{2,\rho}} = \sum_{k=1}^{\infty} k^{2(r-\nu)} |c_k(f)|^2 \rho^{2k}. \tag{2.5}$$

We rewrite this identity as

$$\begin{aligned} \|f_a^{(r-\nu)}\|^2_{H_{2,\rho}} &= \sum_{k=1}^{\infty} k^{2(r-\nu)} |c_k(f)|^2 R^{2k} \left(\frac{\rho}{R}\right)^{2k} \\ &= \sum_{k=1}^{\infty} \left(|c_k(f)|^2 R^{2k}\right)^{\frac{\nu}{r}} \left(k^{2r}|c_k(f)|^2 R^{2k}\right)^{1-\frac{\nu}{r}} \left(\frac{\rho}{R}\right)^{2k} \\ &\leqslant \max_{k\geqslant 1} \left(\frac{\rho}{R}\right)^{2k} \sum_{k=1}^{\infty} \left(|c_k(f)|^2 R^{2k}\right)^{\frac{\nu}{r}} \left(k^{2r}|c_k(f)|^2 R^{2k}\right)^{1-\frac{\nu}{r}}. \end{aligned} \tag{2.6}$$

Applying the Hölder inequality for series

$$\sum_{k=1}^{\infty} a_k b_k \leqslant \left(\sum_{k=1}^{\infty} a_k^{\alpha}\right)^{\frac{1}{\alpha}} \left(\sum_{k=1}^{\infty} b_k^{\beta}\right)^{\frac{1}{\beta}}, \tag{2.7}$$

where $a_k, b_k \geqslant 0$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, to the right hand side of (2.6) with $\alpha := \frac{r}{\nu}$, $\beta := \frac{r}{(r-\nu)}$, in view of formulas (2.4), (2.1) and the relation

$$\max_{k\geqslant 1} \left(\frac{\rho}{R}\right)^{2k} = \left(\frac{\rho}{R}\right)^2$$

we get

$$\|f_a^{(r-\nu)}\|^2_{H_{2,\rho}} \leqslant \left(\frac{\rho}{R}\right)^2 \|f\|^{2\frac{\nu}{r}}_{H_{2,R}} \|f_a^{(r)}\|^{2(1-\frac{\nu}{r})}_{H_{2,R}}$$

or, what is the same,

$$\|f_a^{(r-\nu)}\|_{H_{2,\rho}} \leqslant \frac{\rho}{R} \|f\|^{\frac{\nu}{r}}_{H_{2,R}} \|f_a^{(r)}\|^{1-\frac{\nu}{r}}_{H_{2,R}}. \tag{2.8}$$

Taking into consideration the belonging of intermediate derivative $f_a^{(r-\nu)}$ of the function $f \in H^{(r)}_{q,R,a}$, $q \geqslant 2$, to the space $H_{p,R}$, $1 \leqslant p \leqslant 2$ and the relations $f \in H_{s,R}$, $f_a^{(r)} \in H_{t,R}$, where $2 \leqslant s, t \leqslant q$, as well as the features of the definition of norm in the Hardy space, we obtain

$$\|f_a^{(r-\nu)}\|_{H_{p,\rho}} \leqslant \|f_a^{(r-\nu)}\|_{H_{2,\rho}}, \qquad 0 < \rho \leqslant R, \qquad 1 \leqslant p \leqslant 2. \tag{2.9}$$

Moreover, since $f \in H_{2,R}$, for $1 \leqslant p \leqslant 2 \leqslant s, t \leqslant q$ we have

$$H_{2,R} \subset H_{s,R}, \qquad H^{(r)}_{2,R,a} \subset H^{(r)}_{t,R,a},$$

and the inequalities hold

$$\|f\|_{H_{2,R}} \leqslant \|f\|_{H_{s,R}}, \qquad \|f_a^{(r)}\|_{H_{2,R}} \leqslant \|f_a^{(r)}\|_{H_{t,R}}. \tag{2.10}$$

In view of inequalities (2.9) and (2.10), by (2.8) we have

$$\|f_a^{(r-\nu)}\|_{H_{p,\rho}} \leqslant \frac{\rho}{R} \|f\|^{\frac{\nu}{r}}_{H_{s,R}} \|f_a^{(r)}\|^{1-\frac{\nu}{r}}_{H_{t,R}}$$

and this proves inequality (2.2).

Let us prove the sharpness of inequality (2.2), for instance, for the extremal function $g(z) = az \in H^{(r)}_{q,R,a}$. We have

$$\|g\|_{H_{s,R}} = |a|R, \qquad \|g_a^{(r-\nu)}\|_{H_{p,\rho}} = |a|\rho, \qquad \|g_a^{(r)}\|_{H_{t,R}} = |a|R.$$

Using these values of norms, we get

$$\frac{\rho}{R} \|g\|^{\frac{\nu}{r}}_{H_{s,R}} \|g_a^{(r)}\|^{1-\frac{\nu}{r}}_{H_{t,R}} = \frac{\rho}{R} (|a|R)^{\frac{\nu}{r}} (|a|R)^{1-\frac{\nu}{r}} = |a|\rho = \|g_a^{(r-\nu)}\|_{H_{p,\rho}},$$

that implies the sharpness of (2.2) and completes the proof. □

Theorem 2.1 implies the next corollary.

Corollary 2.1. *Under the assumptions of Theorem 2.1 for $\rho = R$ the inequality holds*

$$\|f_a^{(r-\nu)}\|_{H_{p,R}} \leqslant \|f\|_{H_{s,R}}^{\frac{\nu}{r}} \|f_a^{(r)}\|_{H_{t,R}}^{1-\frac{\nu}{r}}.$$

Theorem 2.2. *Let $\nu, r \in \mathbb{N}$, $1 \leqslant \nu \leqslant r$, $0 < \rho \leqslant R$ and $1 \leqslant p \leqslant 2 \leqslant s, t \leqslant q$. Then each function $f \in H_{q,R}^{(r)}$ with the Taylor coefficients $c_k(f) = 0$, $k = \overline{r-\nu, r-1}$, satisfies the inequality*

$$\|f^{(r-\nu)}\|_{H_{p,\rho}} \leqslant \left(\frac{\rho}{R}\right)^r \frac{\alpha_{r,r-\nu}}{\alpha_{r,r}^{1-\frac{\nu}{r}}} \|f\|_{H_{s,R}}^{\frac{\nu}{r}} \|f^{(r)}\|_{H_{t,R}}^{1-\frac{\nu}{r}}, \tag{2.11}$$

which becomes the identity for the function $g_1(z) = az^r$, $a \in \mathbb{C}$, $r \in \mathbb{N}$.

Proof. Using the Taylor series (1.2) of function $f \in A(U_R)$, for the rth derivative $f^{(r)}(z)$ we write

$$f^{(r)}(z) = \sum_{k=r}^{\infty} \alpha_{k,r} c_k(f) z^{k-r},$$

where $\alpha_{k,r} := k(k-1)\cdots(k-r+1)$, $k \geqslant r$, $k, r \in \mathbb{N}$. Since for an arbitrary function $f \in H_{q,R}^{(r)}$ the norm of its rth derivative

$$\|f^{(r)}\|_{H_{2,R}}^2 = \sum_{k=r}^{\infty} \alpha_{k,r}^2 |c_k(f)|^2 R^{2(k-r)} \tag{2.12}$$

is finite in the space $H_{2,R}$, by (2.12) and representations of quantities $\alpha_{k,r}$, all its intermediate derivatives $f^{(r-\nu)}(z)$, $(\nu = \overline{1, r-1})$. also have finite norms. In particular, this implies that $f^{(r-\nu)}(z) \in H_{p,R}^{(r)}$ $(1 \leqslant p \leqslant 2)$ and $H_{q,R}^{(r)} \subset H_{p,R}^{(r-\nu)}$, where $1 \leqslant p \leqslant 2 \leqslant q$, $\nu = \overline{1, r}$.

Now let a function $f \in H_{q,R}^{(r)}$ satisfies the assumptions of Theorem 2.2 on the coefficients $c_k(f)$. Then its $(r-\nu)$th derivative has the Taylor series

$$f^{(r-\nu)}(z) = \sum_{k=r}^{\infty} \alpha_{k,r-\nu} c_k(f) z^{k-r+\nu}$$

and by the closedness condition

$$\|f^{(r-\nu)}\|_{H_{2,\rho}}^2 = \sum_{k=r}^{\infty} \alpha_{k,r-\nu}^2 |c_k(f)|^2 \rho^{2(k-r+\nu)},$$

what is represented as

$$\begin{aligned} \|f^{(r-\nu)}\|_{H_{2,R}}^2 &= \sum_{k=r}^{\infty} \left(\alpha_{k,r}^2 |c_k(f)|^2 R^{2(k-r)}\right)^{1-\frac{\nu}{r}} \left(|c_k(f)|^2 R^{2k}\right)^{\frac{\nu}{r}} \left(\frac{\rho}{R}\right)^{2(k-r+\nu)} \left(\frac{\alpha_{k,r-\nu}}{\alpha_{k,r}^{1-\frac{\nu}{r}}}\right)^2 \\ &\leqslant \max_{k \geqslant r} \left(\left(\frac{\rho}{R}\right)^{2(k-r+\nu)} \left(\frac{\alpha_{k,r-\nu}}{\alpha_{k,r}^{1-\frac{\nu}{r}}}\right)^2 \right) \sum_{k=r}^{\infty} \left(|c_k(f)|^2 R^{2k}\right)^{\frac{\nu}{r}} \left(\alpha_{k,r}^2 |c_k(f)|^2 R^{2(k-r)}\right)^{1-\frac{\nu}{r}}. \end{aligned} \tag{2.13}$$

The result of [5] implies

$$\max_{k \geqslant r} \left\{ \left(\frac{\rho}{R}\right)^{2(k-r+\nu)} \left[\frac{\alpha_{k,r-\nu}}{\alpha_{k,r}^{1-\frac{\nu}{r}}}\right]^2 \right\} = \left(\frac{\rho}{R}\right)^{2\nu} \left[\frac{\alpha_{r,r-\nu}}{\alpha_{r,r}^{1-\frac{\nu}{r}}}\right]. \tag{2.14}$$

As in the proof of previous theorem, we apply Hölder inequality (2.7) to the right hand side of (2.13) and in view of (2.14) we obtain

$$\|f^{(r-\nu)}\|^2_{H_{2,\rho}} \leqslant \left(\frac{\rho}{R}\right)^{2\nu} \frac{\alpha^2_{r,r-\nu}}{\alpha_{r,r}^{2(1-\frac{\nu}{r})}} \left(\sum_{k=r}^{\infty} |c_k(f)|^2 R^{2k}\right)^{\frac{\nu}{r}} \left(\sum_{k=r}^{\infty} \alpha^2_{k,r}|c_k(f)|^2 R^{2(k-r)}\right)^{1-\frac{\nu}{r}}$$
$$= \left(\frac{\rho}{R}\right)^{2\nu} \frac{\alpha^2_{r,r-\nu}}{\alpha_{r,r}^{2(1-\frac{\nu}{r})}} \|f\|^{2\frac{\nu}{r}}_{H_{2,R}} \|f^{(r)}\|^{2(1-\frac{\nu}{r})}_{H_{2,R}}.$$

Arguing as in the proof of Theorem 2.1, we find

$$\|f^{(r-\nu)}\|_{H_{p,\rho}} \leqslant \|f^{(r-\nu)}\|_{H_{2,\rho}}, \qquad 1 \leqslant p \leqslant 2, \quad 0 < \rho \leqslant R,$$
$$\|f^{(r)}\|_{2,R} \leqslant \|f^{(r)}\|_{H_{t,R}}, \qquad \|f\|_{H_{2,R}} \leqslant \|f\|_{H_{s,R}},$$

and this yields

$$\|f^{(r-\nu)}\|_{H_{p,\rho}} \leqslant \left(\frac{\rho}{R}\right)^{\nu} \frac{\alpha_{r,r-\nu}}{\alpha_{r,r}^{1-\frac{\nu}{r}}} \|f\|^{\frac{\nu}{r}}_{H_{s,R}} \|f^{(r)}\|^{1-\frac{\nu}{r}}_{H_{t,R}},$$

that proves inequality (2.11).

For the function $g_1(z) = az^r \in H^{(r)}_{q,R}$, $a \in \mathbb{C}$, $r \in \mathbb{N}$ inequality (2.11) becomes the identity. Indeed, for g_1 we obtain

$$\|g_1\|_{H_{s,R}} = |a|R^r, \qquad \|g_1^{(r)}\|_{H_{t,R}} = |a|\alpha_{r,r}, \qquad \|g_1^{(r-\nu)}\|_{H_{p,\rho}} = |a|\rho^{\nu}\alpha_{r,r-\nu}.$$

Substituting the values of these norms into the right hand side of (2.11), we get

$$\left(\frac{\rho}{R}\right)^{\nu} \frac{\alpha_{r,r-\nu}}{\alpha_{r,r}^{1-\frac{\nu}{r}}} \|g_1\|^{\frac{\nu}{r}}_{H_{s,R}} \|g_1^{(r)}\|^{1-\frac{\nu}{r}}_{H_{t,R}} = \left(\frac{\rho}{R}\right)^{\nu} \frac{\alpha_{r,r-\nu}}{\alpha_{r,r}^{1-\frac{\nu}{r}}} \left(|a|R^r\right)^{\frac{\nu}{r}} \left(\alpha_{r,r}|a|\right)^{1-\frac{\nu}{r}}$$
$$= |a| \left(\frac{\rho}{R}\right)^{\nu} \frac{\alpha_{r,r-\nu}}{\alpha_{r,r}^{1-\frac{\nu}{r}}} \alpha_{r,r}^{1-\frac{\nu}{r}} R^{\nu} = |a|\rho^{\nu}\alpha_{r,r-\nu} = \|g_1^{(r-\nu)}\|_{H_{p,R}}.$$

This proves the sharpness of inequality (2.11) and completes the proof. □

Corollary 2.2. *Under the assumptions of Theorem 2.2 for $\rho = R$ the inequality*

$$\|f^{(r-\nu)}\|_{H_{p,R}} \leqslant \frac{\alpha_{r,r-\nu}}{\alpha_{r,r}^{1-\frac{\nu}{r}}} \|f\|^{\frac{\nu}{r}}_{H_{s,R}} \|f^{(r)}\|^{1-\frac{\nu}{r}}_{H_{t,R}} \tag{2.15}$$

holds, and it becomes the identity for the function $g_1(z) = az^r$.

Remark 2.1. *Earlier inequality (2.15) for $p = s = t \equiv 2$, $R \equiv 1$ was proved in [3], and in the case $1 \leqslant p \leqslant 2 \leqslant s, t \leqslant q$ and $R = 1$ it was proved in [5].*

3. Kolmogorov type inequalities for best polynomial approximations of analytic functions in clasess $H^{(r)}_{q,R,a}$ and $H^{(r)}_{q,R}$, $r \in \mathbb{N}$, $1 \leqslant q \leqslant \infty$

We consider some applications of the results obtained in Theorems 2.1 and 2.2 in Section 2 to the extremal problems in the theory of polynomial approximations of functions analytic in a disk and belonging to the classes $H^{(r)}_{q,R,a}$ and $H^{(r)}_{q,R}$ $r \in \mathbb{N}$, $1 \leqslant q \leqslant \infty$. We denote by $\mathcal{P}_n$, $n \in \mathbb{Z}_+$ the subspace of algebraic polynomials of a complex variable z of degree not exceeding n.

By the symbol $E_{n-1}(f)_{q,\rho}$, $n \in \mathbb{N}$, $1 \leqslant q \leqslant \infty$, $0 < \rho \leqslant R$, we denote the value of best polynomial approximation of a function $f \in H_{q,\rho}$ by the elements in the subspace $\mathcal{P}_{n-1}$ in the metric of space $H_{q,\rho}$:

$$E_{n-1}(f)_{q,\rho} := \inf\left\{\|f - p_{n-1}\|_{q,\rho} : p_{n-1} \in \mathcal{P}_{n-1}\right\}.$$

The polynomial p^*_{n-1}, for which $E_{n-1}(f)_{q,\rho} = \|f - p^*_{n-1}\|_{q,\rho}$, is called the polynomial of best approximation of function $f \in H_{q,\rho}$. In the case $q = 2$ the polynomial of best approximation p^*_{n-1} coincides with $(n-1)$th partial sum

$$T_{n-1}(f,z) := \sum_{k=0}^{n-1} c_k(f) z^k$$

of the Taylor series of function $f \in H_{2,R}$. At the same time

$$E_{n-1}(f)_{2,\rho} = \|f - T_{n-1}(f)\|_{2,\rho} = \left\{ \sum_{k=n}^{\infty} |c_k(f)|^2 \rho^{2k} \right\}. \tag{3.1}$$

Theorem 3.1. *Let $\nu, r \in \mathbb{N}$, $1 \leqslant \nu \leqslant r$, $1 \leqslant p \leqslant 2 \leqslant s,t \leqslant q$, $0 < \rho \leqslant R$. Then for each function $f \in H^{(r)}_{q,R} \cap H^{(r)}_{q,R,a}$ and each $n \in \mathbb{N}$ the inequalities*

$$E_{n-r+\nu-1}(f^{(r-\nu)})_{p,\rho} \leqslant \left(\frac{\rho}{R}\right)^{n-r+\nu} \frac{\alpha_{n,r-\nu}}{\alpha_{n,r}^{1-\frac{\nu}{r}}} \left(E_{n-1}(f)_{s,R}\right)^{\frac{\nu}{r}} \left(E_{n-r+1}(f^{(r)})_{t,R}\right)^{1-\frac{\nu}{r}}, \quad n > r, \tag{3.2}$$

$$E_{n-1}(f_a^{(r-\nu)})_{p,\rho} \leqslant \left(\frac{\rho}{R}\right)^{n} \left(E_{n-1}(f)_{s,R}\right)^{\frac{\nu}{r}} \left(E_{n-1}(f_a^{(r)})_{t,R}\right)^{1-\frac{\nu}{r}}, \tag{3.3}$$

hold, where $f^{(0)} = f_a^{(0)} \equiv f$, which are sharp in the above stated sense.

Proof. The proofs of identities (3.2) and (3.3) follow the same scheme and this is why we prove only (3.2). We consider an arbitrary function $f \in A(U_R)$ with Taylor series (1.2), which belongs to the set $H_{q,R}$, $q \geqslant 2$, and we denote

$$r_n(f,z) := f(z) - T_{n-1}(f,z) = \sum_{k=n}^{\infty} c_k(f) z^k. \tag{3.4}$$

It is clear that $r_n(f) \in H^{(r)}_{q,R}$, and since by the assumptions of theorem $f \in H_{q,R}$, by identities (3.1) and (3.4)

$$E_{n-1}(f)_{2,R} = \|r_n(f)\|_{2,R}. \tag{3.5}$$

Now let $\nu \in [0, n-1]$ be an arbitrary natural number. By direct calculation of νth derivative we verify the identity

$$T^{(\nu)}_{n-1}(f,z) = T_{n-\nu-1}(f^{(\nu)}, z), \tag{3.6}$$

where $T^{(0)}_{n-1}(f,z) \equiv T_{n-1}(f,z)$. By identities (3.4) and (3.6) for $n \geqslant r \geqslant \nu \geqslant 1$ we obtain

$$\begin{aligned} r_n^{(r-\nu)}(f,z) &= \sum_{k=n}^{\infty} \alpha_{k,r-\nu} c_k(f) z^{k-r+\nu} \\ &= f^{(r-\nu)}(z) - T_{n-r+\nu-1}(f^{(r-\nu)}, z) = r_{n-r+\nu}(f^{(r-\nu)}, z). \end{aligned} \tag{3.7}$$

In view of identity (3.5), by (3.7) we get

$$\|r_n^{(r-\nu)}(f)\|_{2,R} = E_{n-r+\nu-1}(f^{(r-\nu)})_{2,R}, \tag{3.8}$$

$$\|r_n^{(r)}(f)\|_{2,R} = E_{n-r-1}(f^{(r)})_{2,R}. \tag{3.9}$$

Applying the scheme of proof of Theorem 2.2 for $p = s = t = 2$ in view of formulas (3.5), (3.8) and (3.9), after some simple calculations we obtain

$$E_{n-r+\nu-1}(f^{(r-\nu)})_{2,\rho} \leqslant \left(\frac{\rho}{R}\right)^{n-r+\nu} \frac{\alpha_{n,r-\nu}}{\alpha_{n,r}^{1-\frac{\nu}{r}}} \left(E_{n-1}(f)_{2,R}\right)^{\frac{\nu}{r}} \left(E_{n-r+1}(f^{(r)})_{2,R}\right)^{1-\frac{\nu}{r}}.$$

The definition of best polynomial approximation of functions $f \in H_{q,R}$ for $1 \leqslant p \leqslant q$ we have

$$E_{n-1}(f)_{p,R} \leqslant E_{n-1}(f)_{q,R}.$$

In view of the relation $H^{(r)}_{q,R}\subset H^{(r-\nu)}_{p,R}$, $1\leqslant p\leqslant 2\leqslant q$ and the implied inequality

$$E_{n-1}(f)_{2,R}\leqslant E_{n-1}(f)_{s,R},\qquad s,t\leqslant q,$$

we write

$$E_{n-r+\nu-1}(f^{(r-\nu)})_{p,\rho}\leqslant E_{n-r+\nu-1}(f^{(r-\nu)})_{2,\rho},$$
$$E_{n-r-1}(f^{(r)})_{2,R}\leqslant E_{n-r-1}(f^{(r)})_{t,R},\quad t\geqslant 2,$$

and this immediately implies inequality (3.2).

For the function $g_2(z)=az^n\in H^{(r)}_{q,R}$, $a\in\mathbb{C}$, $n\in\mathbb{N}$, $n>r$, inequality (3.2) becomes the identity. We have

$$g_2^{(r-\nu)}(z)=a\alpha_{n,r-\nu}z^{n-r+\nu},$$
$$g_2^{(r)}(z)=a\alpha_{n,r}z^{n-r},$$
$$E_{n-1}(g_2)_{s,R}=|a|R^n,$$
$$E_{n-r+\nu-1}(g_2^{(r-\nu)})_{p,\rho}=|a|\alpha_{n,r-\nu}\rho^{n-r+\nu},$$
$$E_{n-r-1}(g_2^{(r)})_{t,R}=|a|\alpha_{n,r}R^{n-r}.$$

Substituting the obtained quantities of best polynomial approximations into inequality (3.2), we make sure that it becomes the identity and hence it is sharp in the above stated sense. Indeed, we have

$$\begin{aligned}&\left(\frac{\rho}{R}\right)^{n-r+\nu}\frac{\alpha_{n,r-\nu}}{\alpha_{n,r}^{1-\frac{\nu}{r}}}\left(E_{n-1}(g_2)_{s,R}\right)^{\frac{\nu}{r}}\left(E_{n-r-1}(g_2^{(r)})_{t,R}\right)^{1-\frac{\nu}{r}}\\
&=\left(\frac{\rho}{R}\right)^{n-r+\nu}\frac{\alpha_{n,r-\nu}}{\alpha_{n,r}^{1-\frac{\nu}{r}}}\left(|a|R^n\right)^{\frac{\nu}{r}}\left(|a|\alpha_{n,r}R^{n-r}\right)^{1-\frac{\nu}{r}}\\
&=\left(\frac{\rho}{R}\right)^{n-r+\nu}\frac{\alpha_{n,r-\nu}}{\alpha_{n,r}^{1-\frac{\nu}{r}}}|a|\,R^{n\frac{\nu}{r}+n-r-(n-r)\frac{\nu}{r}}\alpha_{n,r}^{1-\frac{\nu}{r}}\\
&=\rho^{n-r+\nu}|a|\alpha_{n,r-\nu}=E_{n-r+\nu-1}(g_2^{(r-\nu)})_{p,\rho},\end{aligned}$$

that completes the proof. □

Corollary 3.1. *Under the assumptions of Theorem 3.1 for $\rho=R$ the inequalities hold*

$$E_{n-r+\nu-1}(f^{(r-\nu)})_{p,R}\leqslant\frac{\alpha_{n,r-\nu}}{\alpha_{n,r}^{1-\frac{\nu}{r}}}\left(E_{n-1}(f)_{s,R}\right)^{\frac{\nu}{r}}\left(E_{n-r-1}(f^{(r)})_{t,R}\right)^{1-\frac{\nu}{r}},\tag{3.10}$$

$$E_{n-1}(f_a^{(r-\nu)})_{p,R}\leqslant\left(E_{n-1}(f)_{s,R}\right)^{\frac{\nu}{r}}\left(E_{n-1}(f_a^{(r)})_{t,R}\right)^{1-\frac{\nu}{r}}.$$

We note that inequality (3.10) was earlier proved for $R=1$ in [5]. It is obvious that proven inequalities (3.2) and (3.3) can be written in a slightly simplified form

$$E_{n-\nu-1}(f^{(\nu)})_{p,\rho}\leqslant\left(\frac{\rho}{R}\right)^{n-\nu}\frac{\alpha_{n,\nu}}{\alpha_{n,r}^{\frac{\nu}{r}}}\left(E_{n-1}(f)_{s,R}\right)^{1-\frac{\nu}{r}}\left(E_{n-r-1}(f^{(r)})_{t,R}\right)^{\frac{\nu}{r}},\tag{3.11}$$

$$E_{n-\nu-1}(f_a^{(\nu)})_{p,\rho}\leqslant\left(\frac{\rho}{R}\right)^{n}\left(E_{n-1}(f)_{s,R}\right)^{1-\frac{\nu}{r}}\left(E_{n-1}(f_a^{(r)})\right)^{\frac{\nu}{r}}.\tag{3.12}$$

Inequalities (3.11) and (3.12) allow one to solve some extremal problems of approximation theory in a simpler way.

By $W^{(r)}_{q,R}(W^{(r)}_{q,R,a})$ we denote the class of functions $f\in H^{(r)}_{q,R}$ ($f\in H^{(r)}_{q,R,a}$), for which $\|f^{(r)}\|_{q,R}\leqslant 1$, ($\|f_a^{(r)}\|_{q,R}\leqslant 1$). We consider the following extremal problem: find exact values of quantities

$$\sup\left\{E_{n-\nu-1}(f^{(\nu)})_{p,\rho}:f\in W^{(r)}_{q,R}\right\},\qquad n\geqslant r\geqslant\nu\geqslant 1,\qquad p\leqslant 2\leqslant q,\tag{3.13}$$

$$\sup\left\{E_{n-1}(f_a^{(\nu)})_{p,\rho}: f\in W_{q,R,a}^{(r)}\right\}, \qquad n\in\mathbb{N}, \qquad p\leqslant 2\leqslant q. \tag{3.14}$$

These quantities (3.13) and (3.14) define the suprema of best polynomial approximations of intermediate derivatives $f^{(\nu)}$ $(f_a^{(\nu)})$, $1\leqslant\nu\leqslant r$, on the classes $W_{q,R}^{(r)}$ and $W_{q,R,a}^{(r)}$ respectively in the metric of Hardy space $H_{p,\rho}$, where $1\leqslant p\leqslant 2\leqslant q$, $0<\rho\leqslant R$.

The next theorems concern the calculations of mentioned quantities.

Theorem 3.2. *Let $n,r,\nu\in\mathbb{N}$ satisfy the relations $n\geqslant r\geqslant\nu\geqslant 1$, $1\leqslant p\leqslant 2\leqslant q$, and $0<\rho\leqslant R$, $R\geqslant 1$. Then the identity holds*

$$\begin{aligned}&\sup\left\{E_{n-\nu-1}(f^{(\nu)})_{p,\rho}: f\in W_{q,R}^{(r)}\right\}\\ &=\left(\frac{\rho}{R}\right)^{n-\nu}\frac{\alpha_{n,\nu}}{\alpha_{n,r}}:=\left(\frac{\rho}{R}\right)^{n-\nu}\frac{1}{(n-\nu)\cdots(n-r+1)}.\end{aligned} \tag{3.15}$$

Proof. For an arbitrary function $f\in H_{q,R}^{(r)}$, $1\leqslant q\leqslant\infty$, $r\in\mathbb{Z}_+$ the inequality holds [7], [11]:

$$E_{n-1}(f)_{q,R}\leqslant\frac{1}{\alpha_{n,r}}E_{n-r-1}(f^{(r)})_{q,R}.$$

Using the Hölder inequality, we easily verify the inclusion $W_{q,R}^{(r)}\subset W_{s,R}^{(r)}$ for $2\leqslant s\leqslant q$. Therefore,

$$E_{n-1}(f)_{s,R}\leqslant E_{n-1}(f)_{q,R}\leqslant\frac{1}{\alpha_{n,r}}E_{n-r-1}(f^{(r)})_{q,R}\leqslant\frac{1}{\alpha_{n,r}}\|f^{(r)}\|_{q,R}\leqslant\frac{1}{\alpha_{n,r}}. \tag{3.16}$$

For a function $f\in W_{q,R}^{(r)}$ with $1\leqslant p\leqslant 2\leqslant t\leqslant q$ by the inequality

$$E_{n-r-1}(f^{(r)})_{p,R}\leqslant E_{n-r-1}(f^{(r)})_{q,R} \tag{3.17}$$

for $p=t$ we have

$$E_{n-r-1}(f^{(r)})_{t,R}\leqslant E_{n-r-1}(f^{(r)})_{q,R}\leqslant\|f^{(r)}\|_{q,R}\leqslant 1. \tag{3.18}$$

Then for an arbitrary function $f\in W_{q,R}^{(r)}$ by formulas (3.16), (3.18), the inclusion $H_{q,R}^{(r)}\subset H_{p,R}^{(\nu)}$, with $1\leqslant\nu\leqslant r$, $1\leqslant p\leqslant 2\leqslant q$, and by inequality (3.11) we obtain

$$\begin{aligned}E_{n-\nu-1}(f^{(\nu)})_{p,R}&\leqslant\left(\frac{\rho}{R}\right)^{n-\nu}\frac{\alpha_{n,\nu}}{\alpha_{n,r}^{\frac{\nu}{r}}}\left(\frac{1}{\alpha_{n,r}}\right)^{1-\frac{\nu}{r}}\\ &=\left(\frac{\rho}{R}\right)^{n-\nu}\frac{\alpha_{n,\nu}}{\alpha_{n,r}}=\left(\frac{\rho}{R}\right)^{n-\nu}\frac{1}{(n-\nu)\cdots(n-r+1)}.\end{aligned}$$

This implies the upper bound

$$\sup\left\{E_{n-\nu-1}(f^{(\nu)})_{p,\rho}:\ f\in W_{q,R}^{(r)}\right\}\leqslant\left(\frac{\rho}{R}\right)^{n-\nu}\frac{1}{(n-\nu)\cdots(n-r+1)}. \tag{3.19}$$

To get the lower bound, we consider the function

$$g_3(z)=\frac{1}{R^{n-\nu}}\frac{z^n}{\alpha_{n,r}}, \quad R\geqslant 1, \quad n,r,\nu\in\mathbb{N}, \quad n\geqslant r\geqslant\nu,$$

which belongs to the class $W_{q,R}^{(r)}$. Since

$$g_3^{(\nu)}(z)=\frac{1}{R^{n-\nu}}\frac{\alpha_{n,\nu}}{\alpha_{n,r}}z^{n-\nu}, \qquad E_{n-\nu-1}(g_3^{(\nu)})_{p,\rho}=\left(\frac{\rho}{R}\right)^{n-\nu}\frac{\alpha_{n,\nu}}{\alpha_{n,r}},$$

$$\|g_3^{(r)}\|_{q,R}=\frac{R^{n-r}}{R^{n-\nu}}=\frac{1}{R^{r-\nu}}\leqslant 1, \qquad r\geqslant\nu,$$

the lower bound

$$\sup\left\{E_{n-\nu-1}(f^{(\nu)})_{p,\rho}:\ f\in W^{(r)}_{q,R}\right\}\geqslant E_{n-\nu-1}(g_3^{(\nu)})_{p,\rho}$$
$$=\left(\frac{\rho}{R}\right)^{n-\nu}\frac{\alpha_{n,\nu}}{\alpha_{n,r}}=\left(\frac{\rho}{R}\right)^{n-\nu}\frac{1}{(n-\nu)\cdots(n-r+1)} \tag{3.20}$$

holds. Desired identity (3.15) is obtained by comparing inequalities (3.19) and (3.20). The proof is complete. □

In the same way we prove the next theorem.

Theorem 3.3. *Let $n,r,\nu\in\mathbb{N}$, $1\leqslant p\leqslant 2\leqslant q$, $0<\rho\leqslant R$. Then the identity holds*

$$\sup\left\{E_{n-1}(f_a^{(\nu)})_{p,\rho}:f\in W^{(r)}_{q,R,a}\right\}=\left(\frac{\rho}{R}\right)^{n}.$$

Theorem 3.4. *Let $n,r,\nu\in\mathbb{N}$ satisfy the relations $n\geqslant r\geqslant\nu\geqslant 1$, $1\leqslant p\leqslant 2\leqslant s\leqslant q$ and $0<\rho\leqslant R$. Then the identity holds*

$$\sup\left\{\frac{E_{n-\nu-1}(f^{(\nu)})_{p,\rho}}{(E_{n-1}(f)_{s,R})^{1-\frac{\nu}{r}}}:f\in W^{(r)}_{q,R}\right\}=\left(\frac{\rho}{R}\right)^{n-\nu}\frac{\alpha_{n,\nu}}{(\alpha_{n,r})^{\frac{\nu}{r}}}. \tag{3.21}$$

If $n,r,\nu\in\mathbb{N}$ are arbitrary numbers and $1\leqslant p\leqslant 2\leqslant q$, $0<\rho\leqslant R$, then the identity holds

$$\sup\left\{\frac{E_{n-1}(f_a^{(\nu)})_{p,\rho}}{(E_{n-1}(f)_{s,R})^{1-\frac{\nu}{r}}}:\ f\in W^{(r)}_{q,R,a}\right\}=\left(\frac{\rho}{R}\right)^{n}. \tag{3.22}$$

Proof. We provide the proof (3.21), which is essentially based on (3.2). The proof of identity (3.22) follows the same scheme and is based on inequality (3.3). For an arbitrary function $f\in W^{(r)}_{q,R}$ with $p=t$ on the base of inequality (3.17) we write

$$E_{n-r-1}(f^{(r)})_{t,R}\leqslant E_{n-r-1}(f^{(r)})_{q,R}\leqslant\|f^{(r)}\|_{q,R}\leqslant 1, \tag{3.23}$$

and hence, in view of (3.16) and (3.23), the inclusion $H^{(r)}_{q,R}\subset H^{(r-\nu)}_{p,R}$, with $1\leqslant p\leqslant 2\leqslant s\leqslant q$ and (3.11), we obtain

$$E_{n-\nu-1}(f^{(\nu)})_{p,\rho}\leqslant\left(\frac{\rho}{R}\right)^{n-\nu}\frac{\alpha_{n,\nu}}{\alpha_{n,r}^{\frac{\nu}{r}}}(E_{n-1}(f)_{s,R})^{1-\frac{\nu}{r}}.$$

This immediately implies the upper bound for the quantity in the left hand side of identity (3.21):

$$\sup\left\{\frac{E_{n-\nu-1}(f^{(\nu)})_{p,\rho}}{(E_{n-1}(f)_{s,R})^{1-\frac{\nu}{r}}}:f\in W^{(r)}_{q,R}\right\}\leqslant\left(\frac{\rho}{R}\right)^{n-\nu}\frac{\alpha_{n,\nu}}{\alpha_{n,r}^{\frac{\nu}{r}}}. \tag{3.24}$$

To obtain the lower bound, we introduce the function

$$g_4(z)=\frac{1}{R^{n-r}}\frac{z^n}{\alpha_{n,r}}.$$

Taking into consideration the relations

$$g_4^{(r)}(z)=\frac{z^{n-r}}{R^{n-r}},\qquad \|g_4^{(r)}\|_{q,R}=1,\qquad g_4^{(\nu)}(z)=\frac{z^{n-\nu}}{R^{n-r}}\frac{\alpha_{n,\nu}}{\alpha_{n,r}},$$
$$E_{n-\nu-1}(g_4^{(\nu)})_{p,\rho}=\frac{\rho^{n-\nu}}{R^{n-r}}\frac{\alpha_{n,\nu}}{\alpha_{n,r}},\qquad E_{n-1}(g_4)_{s,R}=\frac{R^r}{\alpha_{n,r}},$$

we write

$$\begin{aligned}\sup\left\{\frac{E_{n-\nu-1}(f^{(\nu)})_{p,\rho}}{(E_{n-1}(f)_{s,R})^{1-\frac{\nu}{r}}}:f\in W_{q,R}^{(r)}\right\}\\ \geqslant\frac{E_{n-\nu-1}(g_4^{(\nu)})_{p,\rho}}{(E_{n-1}(g_4)_{s,R})^{1-\frac{\nu}{r}}}=\frac{\rho^{n-\nu}}{R^{n-r}}\frac{\alpha_{n,\nu}}{\alpha_{n,r}}\left(\frac{\alpha_{n,r}}{R^r}\right)^{1-\frac{\nu}{r}}=\left(\frac{\rho}{R}\right)^{n-\nu}\frac{\alpha_{n,\nu}}{\alpha_{n,r}^{\frac{\nu}{r}}}.\end{aligned}\tag{3.25}$$

Comparing upper bounds (3.24) and lower bounds (3.25), we get identity (3.21), and this completes the proof. □

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