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# FOURIER SERIES AND DELTA–SUBHARMONIC FUNCTIONS ON OPEN SEMI–ANNULUS

**K.G. MALYUTIN[1], N.V. QUYNH, A.A. NAUMOVA[2]**

**Abstract.** We consider a class $SK(R)$ of subharmonic functions on the unbounded open semi–annulus

$$D_+(R) = \{z : |z| > R, \ \operatorname{Im} z > 0\},$$

which on each semi–annulus

$$D_+(R_1, R_2) = \{z : R < R_1 < |z| < R_2 < \infty, \ \operatorname{Im} z > 0\}$$

possesses a positive harmonic majorant. We introduce a class $JS(R)$ of subharmonic functions on $D_+(R)$, whose boundary values on the real boundary $D_+(R)$ are non–positive. We obtain some properties of functions in the classes $SK(R)$ and $JS(R)$. The class $\delta S(R)$ of delta–subharmonic functions on $D_+(R)$ is defined as the difference of classes $SK(R)$ or $JS(R)$:

$$\delta S(R) = SK(R) - SK(R) = JS(R) - JS(R).$$

For functions $v \in \delta S(R)$ we introduce a growth characteristic $T_R(r, v)$, which differs from the characteristics used for the functions defined on the upper half–plane. It determines the growth of function in the vicinity of semi–circumference $L_R = \{Re^{i\theta} : 0 \leqslant \theta \leqslant \pi\}$. For an arbitrary function of growth $\gamma$ (unbounded non–decreasing positive function defined on the real semi–axis $R_+ = \{r : r > 0\}$) we define the class $\delta S_{L_R}(R, \gamma) \subset \delta S(R)$ of delta–subharmonic functions $v$ of finite $\gamma$–type on $D_+(R)$ in the vicinity of circumference $L_R$ as

$$T_R(r, v) \leqslant A\gamma\left(\frac{B}{r - R}\right)$$

for all $R < r < 2R$ and some positive $A$ and $B$ depending on $v$, but independent of $r$. We obtain the criterions of belonging of function $v$ to the class $\delta S_{L_R}(R, \gamma)$ in terms of its Fourier coefficients.

**Keywords:** unbounded open semi–annulus, harmonic majorant, delta–subharmonic function, growth function, Fourier coefficient.

**Mathematics Subject Classification:** 30D35 31A05

## 1. Introduction

We employ the following notation and terminology. By $\mathbb{N} = \{1, 2, \dots\}$ we denote the set of integer positive (natural) numbers, $\mathbb{C} = \{z = x + iy\}$ is the complex plane with the real axis $\mathbb{R}$,

$$R_+ = \{r : r > 0\}, \qquad \operatorname{Im} z = y, \qquad \operatorname{Re} z = x,$$

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$\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is the upper half–plane without the boundary, $\infty$ denote both the infinite point in the complex plane and $+\infty$ on the real axis. The open circle of a radius $r$ centered at a point $a$ is denoted by $C(a, r)$, while $B(a, r) = \overline{C(a, r)}$ stands for the closed circle, $\overline{G}$ is the closure of a set $G$, and $G_+$ denotes the intersection of the set $G$ with the half–plane $\mathbb{C}_+$, that is, $G_+ = G \cap \mathbb{C}_+$. Subject to the context, the symbol $\partial$ can stand both for the partial derivative of a function and the boundary of a set. By $x_+ = \max\{x; 0\}$ we denote the non–negative part of a real–number $x$. In particular, $\ln_+ 0 = 0$. We denote by

$$D_+(R_1, R_2) = \{z : 0 < R_1 < |z| < R_2 < +\infty,\ \operatorname{Im} z > 0\}$$

an open semi–annulus in the upper half–plane. By

$$D_+(R) = \{z : |z| \geqslant R,\ \operatorname{Im} z > 0\}$$

we denote an unbounded open semi–annulus in the upper half–plane. By $A$, $A_1$, ..., we denote positive constants, which can vary in the proofs. The reader shall meet the following arguing: «let $f(r) \leqslant A\gamma(Br)$, then $f(2r) \leqslant A\gamma(Br)$», which should not cause misunderstanding. By $B \times G$ we denote the Cartesian product of sets $B$ and $G$.

The exposition of theory of subharmonic functions can be found in books by Privalov [12], Hayman and Kennedy [14], Tsuji [20], as well as in monograph [8]. In the paper we obtain the criterion of belonging a delta–subharmonic function on an unbounded semi–annulus to a given class, which is determined by an arbitrary growth function. The base, on which the proof of this result relies, is a generalized Carleman formula. This formula differs from formulas by Ito [16], Grishin [5], Govorov [2]. We note that these authors considered these functions on an open half–plane in contrast to Carleman, who considered meromorphic functions on a closed semi–annulus. Recently there appeared interesting works by Khabibullin [13] and its pupils [11], in which they proposed a new approach to a generalization of the Carleman formula based on the inversion of special functions with respect to a circumference.

## 2. Selection and estimates for exceptional sets

Let $\mu$ be a positive measure on the semi–annulus $D_+(R)$, and $\varphi(\alpha)$ be a continuous strictly increasing function one the segment $[0, 1]$, while $A(r)$ be a continuous strictly positive non–decreasing function on the semi–axis $(0, \infty)$.

**Definition 2.1.** *The exceptional set $F$ for a measure $\mu$ constructed by means of functions $A(r)$ and $\varphi(\alpha)$ is the set of $z \neq 0$, for which there exists $\alpha \in [0, 1]$ such that*

$$\frac{\mu(B(z, \alpha r))}{A(r)} \geqslant \varphi(\alpha), \quad r = |z|. \tag{2.1}$$

**Lemma 2.1.** *For each $z \in F$ there exists the greatest of numbers $\alpha = \alpha_z$, for which inequality (2.1) holds. In the case $\alpha_z < 1$ the identity holds*

$$\frac{\mu(B(z, \alpha r))}{A(r)} = \varphi(\alpha_z).$$

Definition 2.1 and Lemma 2.1 for the half–plane belong to Grishin [4]. For an unbounded semi–annulus the proof of Lemma 2.1 is similar to the arguing in the case of half–plane and we omit it.

We introduce the following notation

$$L(z, \zeta) = \frac{1}{\operatorname{Im} \zeta} \ln \left| \frac{z - \zeta}{z - \overline{\zeta}} \right|, \quad l(z, \zeta) = |\zeta| L(z, \zeta).$$

The written formulas define the kernels as continuous functions for $\operatorname{Im} z \geqslant 0$, $\zeta \in \mathbb{C}_+$, $z \neq \zeta$. However, these functions are continued as continuous to the set

$$\overline{\mathbb{C}}_+ \times \overline{\mathbb{C}}_+ \setminus \operatorname{diag}\left(\overline{\mathbb{C}}_+ \times \overline{\mathbb{C}}_+\right),$$

where

$$\operatorname{diag}\left(\overline{\mathbb{C}}_+ \times \overline{\mathbb{C}}_+\right) = \left\{(z,z), z \in \overline{\mathbb{C}}_+\right\}.$$

In what follows, as usually, we consider the function $L(z,\zeta)$ on the set $\mathbb{C}_+ \times \mathbb{C}$ assuming that $L(z,z) = -\infty$ for $z \in \mathbb{C}_+$ and $L(z,\zeta) = 0$ for $\operatorname{Im}\zeta < 0$. If $\zeta = t$, then

$$\begin{aligned} L(z,t) = \lim_{\eta \to +0} \frac{1}{\eta} \ln \left| \frac{z-t-i\eta}{z-t+i\eta} \right| &= - \lim_{\eta \to +0} \frac{1}{\eta} \operatorname{Re} \ln \left(1 - \frac{2i\eta}{z-t+i\eta}\right) \\ &= -\operatorname{Re} \frac{2i}{z-t} = -\frac{2y}{(x-t)^2+y^2}, \end{aligned}$$

where $z = x + iy$.

Thus, $L(z,\zeta)$ differs from the Poisson kernel for the upper half–plane just by a factor. The function $L(z,\zeta)$ is almost constant in the variable $\zeta$ on the part of circumference $|\zeta - z| = \alpha r$, $r = |z|$, located in the closed upper half–plane. We reflect this in the next lemma due to Grishin [4].

**Lemma 2.2.** *For $|\zeta - z| = \alpha r$, $\operatorname{Im}\zeta \geqslant 0$, the function $L(z,\zeta)$ satisfies the estimate*

$$\frac{1}{r} f(\alpha,\theta) \leqslant L(re^{i\theta},\zeta) \leqslant \frac{1}{r(\sin\theta+\alpha)} \ln \frac{\alpha}{2\sin\theta+\alpha},$$

$$f(\alpha,\theta) = \begin{cases} \dfrac{1}{\sin\theta-\alpha} \ln \dfrac{\alpha}{2\sin\theta-\alpha}, & \alpha < \sin\theta, \\ -\dfrac{2\sin\theta}{\alpha^2}, & \alpha \geqslant \sin\theta, \end{cases}$$

$$\frac{4}{\ln 3} \frac{1}{r(\sin\theta+\alpha)} \ln \frac{\alpha}{2\sin\theta+\alpha} \leqslant L(re^{i\theta},\zeta) \leqslant \frac{1}{r(\sin\theta+\alpha)} \ln \frac{\alpha}{2\sin\theta+\alpha}.$$

**Remark 2.1.** *We have expressed the presence of latter inequality by words that the function $L(z,\zeta)$ is almost constant on the circumference $|\zeta - z| = \alpha r$.*

**Theorem 2.1.** *Let $\mu$ be a positive finite measure on the semi–annulus $\overline{D_+(R)}$, $\mu(\overline{D_+(R)}) = S$, $F$ is an exceptional set for the measure $\mu$ constructed by means of functions*

$$A(r) \equiv S, \quad \varphi(\alpha) = 2\frac{1+\eta}{\eta}\alpha, \quad \eta \in (0,1).$$

*Then for $z \notin F$ the inequality*

$$u(z) := \iint\limits_{\overline{D_+(R)}} l(z,\zeta)\, d\mu(\zeta) > -MNS$$

*holds, where*

$$M = \frac{8}{\ln 3} \int\limits_0^\infty \frac{1}{1+u} \ln \left(1 + \frac{2}{u}\right) du, \qquad N = 2\frac{1+\eta}{\eta}.$$

**Remark 2.2.** *Similar theorem for the half–plane $\overline{\mathbb{C}}_+$ was proved by Grishin in his thesis* [5, Thm. 18]. *The proof for the semi–annulus $\overline{D_+(R)}$ is similar to the arguing in* [5], *however, we did not meet the Grishin theorem in other sources except for his thesis, in which the proof is provided briefly. For the sake of completeness of exposition, we provide the proof of Theorem 2.1 not claiming the authorship and using the ideas from* [5].

*Proof.* Since the kernel $l(z,\zeta)$ is negative, we obtain $u(z) \leqslant 0$. Since for

$$\zeta - z = \alpha r e^{i\psi}, \quad z = re^{i\theta}$$

the inequality

$$|\zeta| \leqslant (1+\alpha)r$$

holds, by Lemma 2.2

$$\begin{aligned} |u(z)| &\leqslant \frac{4}{\ln 3} \iint \frac{1+\alpha}{\sin\theta + \alpha} \ln\left(1 + \frac{2\sin\theta}{\alpha}\right) d\mu(\zeta) \\ &= \frac{4}{\ln 3} \int\limits_0^\infty \frac{1+\alpha}{\sin\theta+\alpha} \ln\left(1+\frac{2\sin\theta}{\alpha}\right) d\mu(B(z,\alpha r)). \end{aligned}$$

We denote

$$f(\alpha) = \frac{1+\alpha}{\sin\theta+\alpha} \ln\left(1+\frac{2\sin\theta}{\alpha}\right).$$

The function $f(\alpha)$ decreases on the interval $(0,+\infty)$, that is, $f'(\alpha) < 0$.

We observe that for $\alpha \geqslant \dfrac{1}{N}$ and $z \notin F$ the identity $\mu(B(z,\alpha r)) = S$ holds, since by construction of the set $F$ for such $\alpha$ we have

$$\mu(B(z,\alpha r)) \geqslant SN\alpha \geqslant S.$$

This implies

$$|u(z)| \leqslant \frac{4}{\ln 3} \int\limits_0^{\frac{1}{N}} f(\alpha)\, d\mu(B(z,\alpha r)).$$

For $z \notin F$ the inequality

$$\mu(B(z,\alpha r)) \leqslant \min\{NS\alpha; S\}$$

holds. This is why for all $z \notin F$

$$\begin{aligned} \int\limits_\varepsilon^{\frac{1}{N}} f(\alpha)\, d\mu(B(z,\alpha r)) &= \mu(B(z,\alpha r)) f(\alpha)\Big|_\varepsilon^{\frac{1}{N}} - \int\limits_\varepsilon^{\frac{1}{N}} \mu(B(z,\alpha r))\, df(\alpha) \\ &\leqslant Sf\left(\frac{1}{N}\right) - \mu(B(z,\varepsilon r)) f(\varepsilon) - NS \int\limits_\varepsilon^{\frac{1}{N}} \alpha\, df(\alpha) \\ &\leqslant NS\varepsilon f(\varepsilon) + NS \int\limits_\varepsilon^{\frac{1}{N}} f(\alpha)\, d\alpha. \end{aligned}$$

Passing to the limit as $\varepsilon \to +0$, we obtain

$$\begin{aligned} |u(z)| &\leqslant \frac{4NS}{\ln 3} \int\limits_0^{\frac{1}{N}} \frac{1+\alpha}{\sin\theta+\alpha} \ln\left(1+\frac{2\sin\theta}{\alpha}\right) d\alpha \\ &\leqslant \frac{8NS}{\ln 3} \int\limits_0^{\frac{1}{N}} \frac{1}{\sin\theta+\alpha} \ln\left(1+\frac{2\sin\theta}{\alpha}\right) d\alpha \end{aligned}$$

$$\leqslant \frac{8NS}{\ln 3} \int_0^\infty \frac{1}{1+u} \ln\left(1+\frac{2}{u}\right) du.$$

The proof is complete. □

## 3. Preliminaries

We first provide known formulas for the Green function of semi–annulus $D_+(R_1, R_2)$.

**Lemma 3.1.** *Let $G(z,\zeta)$ be the Green function for the semi–annulus $D_+(R_1, R_2)$ and let $R_1 = qR$, $R_2 = \dfrac{R}{q}$. Then*

$$G(z,\zeta) = \ln\left| \frac{\sigma\left(\ln \dfrac{z\overline{\zeta}}{q^2R^2}\right)\sigma\left(\ln \dfrac{z}{\overline{\zeta}}\right)}{\sigma\left(\ln \dfrac{z\zeta}{q^2R^2}\right)\sigma\left(\ln \dfrac{z}{\zeta}\right)} \right|, \tag{3.1}$$

*where*

$$\ln\omega = \ln|\omega| + i\arg\omega, \qquad 0 < \arg z, \arg\zeta < \pi, \qquad \arg\overline{\zeta} = -\arg\zeta,$$
$$\arg\omega_1\omega_2 = \arg\omega_1 + \arg\omega_2, \qquad \arg\frac{\omega_1}{\omega_2} = \arg\omega_1 - \arg\omega_2,$$

*$\sigma(u)$ is the Weierstrass sigma–function with the main periods $\omega_1 = -4\ln q$, $\omega_2 = 2\pi i$. The relations hold*

$$\begin{aligned} G(z,\zeta) =& 2\sum_{m=1}^\infty \frac{1}{m(1-q^{4m})}\left(\frac{\tau}{r}\right)^m \left(1-\frac{q^{2m}r^{2m}}{R^{2m}}\right)\left(1-\frac{q^{2m}R^{2m}}{\tau^{2m}}\right)\sin m\theta\sin m\varphi,\\ & qR < \tau < r < \frac{R}{q}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} G(z,\zeta) =& 2\sum_{m=1}^\infty \frac{1}{m(1-q^{4m})}\left(\frac{r}{\tau}\right)^m \left(1-\frac{q^{2m}R^{2m}}{r^{2m}}\right)\left(1-\frac{q^{2m}\tau^{2m}}{R^{2m}}\right)\sin m\theta\sin m\varphi,\\ & qR < r < \tau < \frac{R}{q}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} \frac{\partial G(z,t)}{\partial n} =& \frac{2}{t}\sum_{m=1}^\infty \frac{1}{1-q^{4m}}\left(\frac{t}{r}\right)^m \left(1-\frac{q^{2m}R^{2m}}{t^{2m}}\right)\left(1-\frac{q^{2m}r^{2m}}{R^{2m}}\right)\sin m\theta,\\ & qR < |t| < r < \frac{R}{q}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} \frac{\partial G(z,t)}{\partial n} =& \frac{2}{t}\sum_{m=1}^\infty \frac{1}{1-q^{4m}}\left(\frac{r}{t}\right)^m \left(1-\frac{q^{2m}t^{2m}}{R^{2m}}\right)\left(1-\frac{q^{2m}R^{2m}}{r^{2m}}\right)\sin m\theta,\\ & qR < r < |t| < \frac{R}{q}, \end{aligned} \tag{3.5}$$

$$\frac{\partial G\left(z, qRe^{i\varphi}\right)}{\partial n} = \frac{4}{qR}\sum_{m=1}^\infty \frac{1}{1-q^{4m}}\left(\frac{qR}{r}\right)^m\left(1-\frac{q^{2m}r^{2m}}{R^{2m}}\right)\sin m\theta\sin m\varphi, \tag{3.6}$$

$$\frac{\partial G\left(z, \dfrac{1}{q}Re^{i\varphi}\right)}{\partial n} = \frac{4q}{R}\sum_{m=1}^\infty \frac{1}{1-q^{4m}}\left(\frac{qr}{R}\right)^m\left(1-\frac{q^{2m}R^{2m}}{r^{2m}}\right)\sin m\theta\sin m\varphi. \tag{3.7}$$

*Proof.* We can obtain the expansion for the kernel in formula (3.1) for

$$R_1 = qR, \qquad R_2 = \frac{R}{q}, \qquad q \in (0,1), \qquad z = re^{i\theta}, \qquad \zeta = \tau e^{i\varphi}$$

by using the theory of elliptic functions, see, for instance, [1, Ch. III, Sect. 12]. Formulas (3.2)–(3.7) were provied in thesis by Grishin [5], see [7], [9]. □

The next theorem is the analogue of Littlewood theorem [18] for the unbounded semi–annulus $D_+(R)$.

**Theorem 3.1.** *Let $\mu$ be a positive measure concentrated on (open) semi–annulus $D_+(R)$ such that*

$$\iint\limits_{D_+(R)} \frac{d\mu(\zeta)}{1+|\zeta|^2} < \infty.$$

*Let*

$$v(z) = \frac{1}{2\pi} \iint\limits_{D_+(R)} L(z,\zeta)\, d\mu(\zeta).$$

*Then for almost all $x \in I_R = (-\infty, -R) \cup (R, +\infty)$ in the sense of the Lebesgue measure the limit exists*

$$\lim_{y \to +0} v(x+iy) = 0.$$

**Remark 3.1.** *For the case, when the measure $\mu$ is concentrated on the open upper half–plane $\mathbb{C}_+$, a similar theorem was proved in thesis by Grishin* [5, Thm. 26]*. We have not meet in other sources. This is why, for sake of completeness of presentation, we provide the proof of Theorem 3.1 by using arguing in* [5].

*Proof.* We treat the measure $\mu$ as continued as zero measure to the semi–annulus $D_+\left(\frac{R}{2}, R\right]$. Let

$$E = E_\delta = \Big\{ x \in I_R : \varliminf_{y \to +0} v(x+iy) < -\delta \Big\}.$$

Since $v(z) \leqslant 0$ (because $L(z,\zeta) \leqslant 0$ and $\mu$ is a positive measure), the statement of theorem is equivalent to fact that for each $\delta > 0$ the set $E_\delta$ has the zero measure. Let $R < a < b < +\infty$, $\Delta > 0$,

$$B = [a/2, 2b] \times [0, 2\Delta], \qquad B_1 = [a,b] \times [0,\Delta],$$
$$v(z) = \frac{1}{2\pi} \iint\limits_{B} L(z,\zeta)\, d\mu(\zeta) + \frac{1}{2\pi} \iint\limits_{D_+(R)\setminus B} L(z,\zeta)\, d\mu(\zeta) = v_1(z) + v_2(z).$$

For $z \in B_1$, $\zeta \notin B$ the inequalities hold

$$-L(z,\zeta) = \frac{1}{2\operatorname{Im}\zeta} \ln\left(1 + \frac{4\operatorname{Im} z \operatorname{Im}\zeta}{|z-\zeta|^2}\right) \leqslant \frac{2\operatorname{Im} z}{|z-\zeta|^2} \leqslant \frac{C\operatorname{Im} z}{1+|\zeta|^2},$$

where $C = C(a, b, \Delta, R)$. Then for $z \in B_1$

$$|v_2(z)| \leqslant C \operatorname{Im} z \iint\limits_{D_+(R)} \frac{d\mu(\zeta)}{1+|\zeta|^2}.$$

This is why

$$v_2(x+iy) \overset{y \to +0}{\rightrightarrows} 0 \quad \text{for} \quad x \in [a,b].$$

Then

$$v_1(z) = \frac{1}{2\pi} \iint\limits_B L(z,\zeta)\, d\mu(\zeta) = \iint\limits_B l(z,\zeta)\, d\mu_1(\zeta),$$

where

$$d\mu_1(\zeta) = \frac{1}{2\pi|\zeta|}\, d\mu(\zeta), \qquad \mu_1(B) \leqslant \frac{1}{2\pi a}\, \mu(B).$$

By Theorem 2.1 for each $\eta \in (0,1)$ there exists a system of circles $F$ with the sum of radii not exceeding $2\eta(b+\Delta)$ such that the inequality holds

$$v_1(z) \geqslant -2M\frac{1+\eta}{\eta}\mu_1(B).$$

For each $\eta > 0$ we can choose $\Delta > 0$ so that the inequality

$$2M\frac{1+\eta}{\eta}\mu_1(B) < \frac{\sigma}{2}$$

holds. We denote by $E_1$ the projection of $F$ onto the real axis,

$$E_+ = E_1 \cap (R, +\infty), \qquad E_- = E_1 \cap (-\infty, -R).$$

Then

$$\operatorname{mes} E_+ < \operatorname{mes} E_1 < 2\eta(b+\Delta) < 2\eta(b+1).$$

For $x \notin E_+$ the inequalities hold

$$v(x+iy) > -\frac{\sigma}{2}, \qquad \varliminf_{y\to+0} v(x+iy) \geqslant -\frac{\sigma}{2}.$$

This is why $E \cap [a,b] \subset E_+$. Since $\operatorname{mes} E_+ < 2\eta(b+1)$, and $\eta$ is an arbitrary number, we have $\operatorname{mes}(E\cap[a,b]) = 0$. Since $a$ and $b$ are arbitrary number, we have $\operatorname{mes}(E\cap(R,+\infty)) = 0$.

In the same way we prove that $\operatorname{mes}(E\cap(-\infty,-R)) = 0$. And hence, $\operatorname{mes} E = 0$. The proof is complete. □

**Theorem 3.2.** *Let $v$ be the same function as in Theorem 3.1. Then for each segment $[a,b] \subset I_R$ the limit is well–defined*

$$\lim_{y\to+0} \int\limits_a^b v(x+iy)\, dx = 0.$$

**Remark 3.2.** *For an open half–plane $\mathbb{C}_+$ a similar theorem was proved in thesis by Grishin* [5, Thm. 27]. *For the sake of completeness of exposition, we provide the proof of Theorem 3.2 using the arguing from* [5].

*Proof.* Let $R < a < b < +\infty$ and the functions $v_1$ and $v_2$ be the same as in the proof of Theorem 3.1. Since

$$v_2(x+iy) \overset{y\to+0}{\rightrightarrows} 0 \quad \text{for} \quad x \in [a,b],$$

we have

$$\lim_{y\to+0} \int\limits_a^b v_2(x+iy)\, dx = 0.$$

We denote $\zeta = \xi + i\eta$. Then

$$\int\limits_a^b |L(x+iy,\zeta)|\, dx = \frac{1}{2\eta} \int\limits_a^b \ln \frac{(x-\xi)^2 + (y+\eta)^2}{(x-\xi)^2+(y-\eta)^2}\, dx$$

$$< \frac{1}{2\eta}\int_R^{+\infty} \ln\frac{(x-\xi)^2+(y+\eta)^2}{(x-\xi)^2+(y-\eta)^2}\,dx < \frac{1}{2\eta}\int_{-\infty}^{+\infty} \ln\frac{(x-\xi)^2+(y+\eta)^2}{(x-\xi)^2+(y-\eta)^2}\,dx$$

$$= \frac{1}{2\eta}\int_{-\infty}^{+\infty} \ln\frac{u^2+(y+\eta)^2}{u^2+(y-\eta)^2}\,du = \frac{\pi}{\eta}(y+\eta-|y-\eta|) \leqslant 2\pi.$$

This is why

$$\int_a^b |v_1(x+iy)|\,dx \leqslant \frac{1}{2\pi}\iint_B \int_a^b |L(x+iy,\zeta)|\,dx d\mu(\zeta) \leqslant \mu(B).$$

Then

$$\overline{\lim_{y\to+0}}\int_a^b v(x+iy)\,dx \leqslant \mu(B).$$

Since $\lim\limits_{\Delta\to+0}\mu(B)=0$, this completes the proof. □

We shall need the next theorem.

**Theorem 3.3.** *Let $\mu$ be a measure on $I_R$ such that*

$$\int_{I_R} \frac{d|\mu|(t)}{1+t^2} < \infty, \quad |\mu| = \mu^+ + \mu^-,$$

*where $\mu=\mu^+-\mu^-$ is the Jordan decomposition of measure $\mu$. Let*

$$v(z) = \frac{y}{\pi}\int_{I_R}\frac{d\mu(t)}{(t-x)^2+y^2}, \quad z = x+iy.$$

1) *If at a point $x_0$ there exists a finite or infinite (of certain sign) derivative $\mu'(x_0)$, we have*

$$\lim_{y\to+0} v(x_0+iy) = \mu'(x_0).$$

2) *Let $[a,b]\subset I_R$, $\mu(\{a\})=\mu(\{b\})=0$. Then*

$$\lim_{y\to+0}\int_a^b v(x+iy)\,dx = \mu([a,b]).$$

*Proof.* Statement 1) is the well–known Fatou theorem. For the case of half–plane, its proof can be found, for instance, in [17, Ch. VI].

2) A similar statement for the half–plane was proved by Grishin [5, Thm. 29]. Another proof for the half–plane was given in [9, Thm. 3.3].

Let $a>R$. We fix $\delta>0$ so that $[a-\delta,b+\delta]\subset I_R$. We let $I_\delta = I_R\setminus[a-\delta,b+\delta]$. We represent the function $v$ as the sum

$$v(z) = \frac{y}{\pi}\int_{a-\delta}^{b+\delta}\frac{d\mu(t)}{(t-x)^2+y^2} + \frac{y}{\pi}\int_{I_\delta}\frac{d\mu(t)}{(t-x)^2+y^2} = v_1(z)+v_2(z).$$

As $x=\operatorname{Re} z\in[a,b]$, $t\in I_\delta$, the estimate holds

$$\frac{t^2+1}{(t-x)^2+y^2} \leqslant K(a,b,\delta),$$

where $K(a,b,\delta)>0$ is a constant. This is why

$$\lim_{y\to+0}\int\limits_a^b |v_2(x+iy)|\,dx \leqslant \lim_{y\to+0}\int\limits_a^b \left(K(a,b,\delta)\frac{y}{\pi}\int\limits_{-\infty}^{\infty}\frac{d|\mu|(t)}{1+t^2}\right)dx = 0.$$

We estimate the integral of function $v_1$

$$\int\limits_a^b v_1(x+iy)\,dx = \int\limits_{a-\delta}^{b+\delta}\frac{1}{\pi}\left(\arctan\frac{b-t}{y} - \arctan\frac{a-t}{y}\right)d\mu(t).$$

The integrand $f(t,y;a,b)$ in the right hand side of latter identity is bounded. Moreover,

$$\lim_{y\to+0} f(t,y;a,b) = \begin{cases} 0, & t\in\mathbb{R}\setminus[a,b],\\ 1, & t\in(a,b),\\ \frac{1}{2}, & t\in\{a;b\}.\end{cases}$$

Since $\mu(\{a\})=\mu(\{b\})=0$, the function $f(t,y;a,b)$ $|\mu|$–almost everywhere converges as $y\to+0$ to the characteristic function $\chi_{[a,b]}$ of an interval $[a,b]$. Then by the Lebesgue theorem on dominated convergence

$$\lim_{y\to+0}\int\limits_a^b v_1(x+iy)\,dx = \int\limits_a^b d\mu(t) = \mu([a,b]).$$

In the same way we consider the case $a<-R$. The proof is complete. □

## 4. Subharmonic functions on annulus

We begin with necessary definitions and exposition of properties of subharmonic functions, which will be employed in what follows. The proof of the next theorem can be found in [12].

**Theorem 4.1.** *Let $D$ be a domain with the Green function $G(z,\zeta)$. Let $v$ be a subharmonic function on the domain $D$, $\mu$ be the Riesz measure of function $v$. Let $h$ be the best harmonic majorant of function $v$ in the domain $D$. The representation*

$$v(z) = -\iint\limits_D G(z,\zeta)\,d\mu(\zeta) + h(z) \tag{4.1}$$

*holds with a harmonic function $h$ in the domain $D$ if and only if the function $v$ has a harmonic majorant in the domain $D$. If identity (4.1) holds, then $h$ is the best harmonic majorant function $v$ in the domain $D$.*

We provide the theorem on positive harmonic majorant [5, Thm. 31].

**Theorem 4.2.** *Let $D$ be a domain possessing the Green function. Let $v$ be a subharmonic function on the domain $D$, which has a harmonic majorant in this domain. Let $h$ be the best possible harmonic majorant of function $v$ in the domain $D$. The function $v$ possesses a positive harmonic majorant in the domain $D$ if and only if the function $h$ has a positive harmonic majorant in the domain $D$. In the case of existence of positive harmonic majorants for these functions, the best positive harmonic majorants of the functions $v$ and $h$ coincide.*

**Remark 4.1.** *Apart of the thesis by A.F. Grishin, we do not know the formulate and proof of this theorem in other sources. Due to its importance in our studies (in this one and further ones), we provide the proof of Theorem 4.2 using the arguing from [5] and not claiming the authorship.*

*Proof.* By Theorem 4.1, the function $v$ admits representation (4.1). Let there exist best positive harmonic majorants $H_v$ and $H_h$ for the functions $v$ and $h$ in the domain $D$, respectively. Then $H_v \leqslant H_h$. Since the domain $D$ possesses the Green function, there exists a sequence of finitely–connected bounded domains $D_m$, which exhaust the domain $D$, that is, $\bigcup\limits_{m=1}^{\infty} D_m = D$, and such that

1) $\overline{D_m} \subset D_{m+1}$,
2) $\overline{D_m}$ is a compact set,
3) the boundary of $D_m$ is the union of finitely many analytic Jordan curves.

Let the boundary $\mathcal{L}$ of domain $E$ consist of finitely many closed analytic Jordan curves. As it is known [3, Ch. 6, Sect. 3], if the function $u$ is harmonic in the domain $E$, bounded and continuous in the closure $\overline{E}$, then the Green formula holds

$$u(z) = \frac{1}{2\pi} \int\limits_{\mathcal{L}} \frac{\partial G(z,\zeta)}{\partial n} u(\zeta)\, ds, \tag{4.2}$$

where $G(z,\zeta)$ is the Green function of domain $E$, $n$ is the inward normal at the point $\zeta \in \mathcal{L}$, $s$ is the arc length on $\mathcal{L}$, $\zeta = \zeta(s)$ is the parametrization of $\mathcal{L}$.

Let $G_m(z,\zeta)$ and $G(z,\zeta)$ be the Green functions of domains $D_m$ and $D$. Then [3, Ch. 6, Sect. 3] $G_m(z,\zeta) \uparrow G(z,\zeta)$ as $m \to \infty$, $z \in D$ (it is clear that the limiting relation holds for $m$ such that $z \in D_m$).

Let $h_v$ be the best harmonic majorant of function $v$ in the domain $D$. According to formula (4.2) we have

$$h_v(z) = \frac{1}{2\pi} \int\limits_{\mathcal{L}_m} \frac{\partial G_m(z,\zeta_1)}{\partial n} h_v(\zeta_1)\, ds, \qquad \mathcal{L}_m = \partial D_m.$$

Applying identity (4.1), we obtain

$$h_v(z) = \frac{1}{2\pi} \int\limits_{\mathcal{L}_m} \frac{\partial G_m(z,\zeta_1)}{\partial n} v(\zeta_1)\, ds + \iint\limits_{D} \frac{1}{2\pi} \int\limits_{\mathcal{L}_m} \frac{\partial G_m(z,\zeta_1)}{\partial n} G(\zeta_1,\zeta)\, ds d\mu(\zeta).$$

By formula (4.2) we obtain

$$\frac{1}{2\pi} \int\limits_{\mathcal{L}_m} \frac{\partial G_m(z,\zeta_1)}{\partial n}\, G(z,\zeta_1)\, ds = \begin{cases} G(z,\zeta) - G_m(z,\zeta), & \zeta \in D_m, \\ G(z,\zeta), & \zeta \notin D_m. \end{cases}$$

Moreover, the inequality

$$\frac{1}{2\pi} \int\limits_{\mathcal{L}_m} \frac{\partial G_m(z,\zeta_1)}{\partial n} v(\zeta_1)\, ds \leqslant \frac{1}{2\pi} \int\limits_{\mathcal{L}_m} \frac{\partial G_m(z,\zeta_1)}{\partial n} H_v(\zeta_1)\, ds = H_v(z)$$

holds. This implies the inequality

$$\begin{aligned} h_v(z) &\leqslant H_v(z) + \frac{1}{2\pi} \iint\limits_{D_m} (G(z,\zeta) - G_m(z,\zeta))\, d\mu(\zeta) + \frac{1}{2\pi} \iint\limits_{D \setminus D_m} G(z,\zeta)\, d\mu(\zeta) \\ &\leqslant H_v(z) + \frac{1}{2\pi} \iint\limits_{D_{m_0}} (G(z,\zeta) - G_m(z,\zeta))\, d\mu(\zeta) + \frac{1}{2\pi} \iint\limits_{D \setminus D_{m_0}} G(z,\zeta)\, d\mu(\zeta) \end{aligned}$$

if $m_0 \leqslant m$.

Since

$$G(z,\zeta) - G_m(z,\zeta) \leqslant G(z,\zeta), \qquad \lim_{m\to\infty} (G(z,\zeta) - G_m(z,\zeta)) = 0,$$

by the Lebesgue theorem on dominated convergence

$$h_v(z) \leqslant H_v(z) + \frac{1}{2\pi} \iint\limits_{D\setminus D_{m_0}} G(z,\zeta)\,d\mu(\zeta) = H_v(z) + \frac{1}{2\pi} \iint\limits_{D} \chi_{D\setminus D_{m_0}}(\zeta)\,G(z,\zeta)\,d\mu(\zeta).$$

Applying again the Lebesgue theorem on dominated convergence, we obtain

$$h_v(z) \leqslant H_v(z).$$

Thus, the function $h_v(z)$ has a positive harmonic majorant and $H_v(z)$ is one of such majorants. Then $H_v(z) \geqslant H_h(z)$. In view of the aforementioned inequality $H_v(z) \leqslant H_h(z)$ we obtain $H_v(z) = H_h(z)$. The proof is complete. □

The next theorem is the famous Martin theorem on representation of positive harmonic functions. Its proof can be found in [15].

**Theorem 4.3.** *Let $D$ be a domain possessing the Green function, and let $\Gamma$ be the Martin boundary of domain $D$, $M(z,\zeta)$ be the Martin function of domain $D$, $\zeta \in \Gamma$. Let $u(z)$ be a positive harmonic function in the domain $D$. Then there exists a unique finite positive measure $\lambda$ on $\Gamma$ concentrated on the set of minimal points in the boundary $\Gamma$ such that*

$$u(z) = \int\limits_{\Gamma} M(z,\zeta)\,d\lambda(\zeta).$$

**Remark 4.2.** *The interested reader can read about Martin function and Martin boundary in* [15].

We shall need the Grishin theorem on best positive harmonic majorant [5, Thm. 33].

**Theorem 4.4.** *Let $D$ be a domain having the Green function. Let $u(z)$ be a harmonic function on the domain $D$ having a positive harmonic measure in this domain. Then there exist a unique measure $\nu$ on $\Gamma$ concentrated on the set of minimal points having a finite total variation such that*

$$u(z) = \int\limits_{\Gamma} M(z,\zeta)\,d\nu(\zeta).$$

*If $\Gamma = \Gamma_+ \cup \Gamma_-$ is the Hahn decomposition of the set $\Gamma$ for the measure $\nu$, and $\nu = \nu_+ - \nu_-$ is the Jordan decomposition of this measure, then the best positive harmonic majorant $H(z)$ of the function $u(z)$ satisfies the formula*

$$H(z) = \int\limits_{\Gamma_+} M(z,\zeta)\,d\nu(\zeta) = \int\limits_{\Gamma} M(z,\zeta)\,d\nu_+(\zeta).$$

**Remark 4.3.** *This theorem and its proof were provided in the thesis by A.F. Grishin. We did not meet them in other sources. This is why for the sake of completeness of presentation and in view of its importance in the theory of functions, we provide the proof of this theorem based on the ideas from the Grishin's thesis not claiming the authorship.*

*Proof.* Since the function $u(z)$ has a positive harmonic majorant in the domain $D$, it is represented by the difference of two positive harmonic functions. By Theorem 4.3 there exists a measure $\nu$ concentrated on the set of minimal boundary points $\Gamma_1$ having a finite total variation such that

$$u(z) = \int\limits_{\Gamma} M(z,\zeta)\,d\nu(\zeta).$$

Let us prove the uniqueness of such measure. If this is not the case, there exists a nonzero measure $\nu$ concentrated on the set $\Gamma_1$ and having a finite total variation such that

$$\int\limits_{\Gamma} M(z,\zeta)\,d\nu(\zeta)=0.$$

If $\nu=\nu_+-\nu_-$ is the Jordan decomposition for this measure, then each of measures $\nu_+$ and $\nu_-$ is concentrated on the set $\Gamma_1$. Then

$$\int\limits_{\Gamma} M(z,\zeta)\,d\nu_+(\zeta)=\int\limits_{\Gamma} M(z,\zeta)\,d\nu_-(\zeta).$$

By Theorem 4.3, $\nu_+=\nu_-$. Since these are mutually singular measures, we have $\nu_+=\nu_-=0$, $\nu=0$. We have obtained the contradiction that proves the uniqueness of measure $\nu=0$.

Let $H(z)$ be the best positive harmonic majorant of the function $u(z)$. By Theorem 4.3, there exists a positive measure $\nu_1$ concentrated on $\Gamma$ such that

$$H(z)=\int\limits_{\Gamma} M(z,\zeta)\,d\nu_1(\zeta).$$

Then

$$H(z)-u(z)=\int\limits_{\Gamma} M(z,\zeta)\,d(\nu_1(\zeta)-\nu(\zeta)).$$

By the above proven uniqueness $\nu_1-\nu$ is a positive measure. Let us show that the measure $\nu_2=\nu_1-\nu_+$ is negative. If this is not the case, then there exists a compact set $E_3\subset\Gamma$ such that the restriction $\nu_3$ of measure $\nu_2$ on $E_3$ is a positive measure and $\nu_3(E_3)=\nu_2(E_3)>0$. We have $\nu_1-\nu_3=\nu_1-\nu_2=\nu_+$ on $E_3$, $\nu_1-\nu_3=\nu_1$ on $CE_3$. Thus, $\nu_1-\nu_3$ is a positive measure and the function

$$H_1(z)=\int\limits_{\Gamma} M(z,\zeta)\,(d\nu_1(\zeta)-d\nu_3(\zeta))$$

is a positive harmonic function in $D$.

Since $\nu_3(E_3)>0$, the inequality $H_1(z)<H(z)$ holds for $z\in D$. Moreover,

$$H_1(z)-u(z)=\int\limits_{\Gamma} M(z,\zeta)\,d(\nu_1(\zeta)-\nu(\zeta)-\nu_3(\zeta)).$$

We consider the measure $\nu_1-\nu-\nu_3=\nu_1-\nu_+-\nu_3+\nu_-$. On the set $E_3$ this measure coincides with the measure $\nu_1-\nu_+-\nu_3+\nu_-=\nu_-$ and this is why it is positive. On the set $CE_3$ this measure coincides with the measure $\nu_1-\nu$ and is also positive. This is why $\nu_1-\nu-\nu_3$ is a positive measure, and $H_1(z)-u(z)$ is a positive function. Then $H_1(z)$ is a positive harmonic majorant of the function $u(z)$. Since $H_1(z)<H(z)$, we obtain the contradiction. The proof of negativity of the measure $\nu_2$ is complete.

Since $\nu_1$ is the positive measure, the measure $\nu_2$ is concentrated on the set $\Gamma_+$ as it is implied by the definition of $\nu_2$.

If $\nu_2(\Gamma_+)<0$, then

$$(\nu_1-\nu)(\Gamma_+)=(\nu_1-\nu_++\nu_-)(\Gamma_+)=(\nu_1-\nu_+)(\Gamma_+)=\nu_2(\Gamma_+)<0.$$

This contradicts to the fact that $\nu_1-\nu$ is a positive measure. Thus, $\nu_2$ is a negative measure concentrated on the set $\Gamma_+$. And $\nu_2(\Gamma_+)=0$, this is why $\nu_2=0$, $\nu_1=\nu_+$, and the proof is complete. □

The proof of the next theorem can be found in [15].

**Theorem 4.5.** *Let $D$ be a simply connected domain, the Euclidean boundary $\Gamma$ of which is the Jordan curve. Then the Martin boundary of the domain is homeomorphic to $\Gamma$ and each point of the boundary is minimal. If $\zeta \in \Gamma$, then the Martin function $M(z,\zeta)$ of domain $D$ associated with this point reads*

$$M(z,\zeta) = \lim_{\substack{\zeta_1 \to \zeta \\ \zeta_1 \in D}} \frac{G(z,\zeta)}{G(z_0,\zeta_1)},$$

*where $G(z,\zeta)$ is the Green function of domain $D$, and $z_0$ is a point such that $v(z_0) > -\infty$.*

**Theorem 4.6.** *Let $D$ be a simply connected domain enveloped by a bounded Jordan curve $\Gamma$. Let at a point $\zeta \in \Gamma$ the curve $\Gamma$ possesses the tangential line, and the Green function $G(z,\zeta)$ of domain $D$ has the normal derivative. Then*

$$M(z,\zeta) = \frac{\partial G(z,\zeta)/\partial n_\zeta}{\partial G(z_0,\zeta)/\partial n_\zeta}.$$

*Proof.* This theorem is easily implied by the previous one. □

Let $SK(R_1,R_2)$ be the class of harmonic functions on $D_+(R_1,R_2)$ having a positive harmonic majorant in this semi–annulus, $G(z,\zeta)$ be the Green function of semi–annulus $D_+(R_1,R_2)$. The next theorem was proved in [9, Thm. 7].

**Theorem 4.7.** *Let $v \in SK(R_1,R_2)$, $\mu$ be the Riesz measure of function $v(z)$, $G(z,\zeta)$ be the Green function of semi–annulus $D_+(R_1,R_2)$. Then there exist real numbers $a_j$, $j = 1,\dots,4$, uniquely determined by the function $v(z)$ of measure $\nu_j$, $j = 1,2$, on the interval $(0,\pi)$, and the measure $\nu$ on the set $I(R_1,R_2) = (-R_2,-R_1) \cup (R_1,R_2)$. Moreover, if $z_0$ such that $v(z_0) > -\infty$, then*

- *the measure $\tilde{\mu}$, $d\tilde{\mu}(\zeta) \overset{\text{def}}{=} G(z_0,\zeta)\,d\mu(\zeta)$ is finite on $D_+(R_1,R_2)$,*
- *the measures $\tilde{\nu}_j$, $d\tilde{\nu}_j(\zeta) \overset{\text{def}}{=} \dfrac{\partial G(z_0,R_je^{i\varphi})}{\partial n_\zeta}\,d\nu_j(\zeta)$, $j = 1,2$, have a bounded total variation on the interval $(0,\pi)$,*
- *the measure $\tilde{\nu}$, $d\tilde{\nu}(t) \overset{\text{def}}{=} \dfrac{\partial G(z_0,t)}{\partial n_\zeta}\,d\nu(t)$, has a bounded total variation on $I(R_1,R_2)$.*

*For $z \in D_+(R_1,R_2)$ the identity*

$$v(z) = -\iint\limits_{D_+(R_1,R_2)} G(z,\zeta)\,d\mu(\zeta) + \sum_{j=1}^{2} \frac{1}{2\pi} \int\limits_0^\pi \frac{\partial G(z,R_je^{i\varphi})}{\partial n_\zeta}\,d\nu_j(\varphi) + \frac{1}{2\pi} \int\limits_{I(R_1,R_2)} \frac{\partial G(z,t)}{\partial n_t}\,d\nu(t)$$
$$+ a_1M(z,R_1) + a_2M(z,-R_1) + a_3M(z,R_2) + a_4M(z,-R_2)$$

*holds, where $M(z,\zeta)$ is the Martin function of semi–annulus $D_+(R_1,R_2)$ associated with the boundary point $\zeta$, and the integrals are treated as improper with singularities at the integration limits. The formulas*

$$\nu_1([\alpha,\beta]) = \lim_{r\to R_1+0} R_1 \int\limits_\alpha^\beta v(re^{i\varphi})\,d\varphi, \quad \nu_2([\alpha,\beta]) = \lim_{r\to R_2-0} R_2 \int\limits_\alpha^\beta v(re^{i\varphi})\,d\varphi,$$

$$\text{if} \quad 0 < \alpha < \beta < \pi, \quad \nu_j(\{\alpha\}) = \nu_j(\{\beta\}) = 0, \quad j = 1,2,$$

$$\nu([a,b]) = \lim_{y\to+0} \int\limits_a^b v(x+iy)\,dx \quad \text{if} \quad [a,b] \in I(R_1,R_2), \quad \nu(\{a\}) = \nu(\{b\}) = 0,$$

$$d\nu_j(\varphi) = R_jv(R_je^{i\varphi})d\varphi + d\sigma_j(\varphi)d\varphi, \quad j = 1,2, \quad d\nu(t) = v(t)dt + d\sigma(t),$$

*hold, where almost everywhere*

$$v(R_1e^{i\varphi}) = \lim_{r\to R_1+0} v(re^{i\varphi}), \qquad v(R_2e^{i\varphi}) = \lim_{r\to R_2-0} v(re^{i\varphi}), \qquad v(t) = \lim_{y\to+0} v(t+iy),$$

*and* $\sigma_j$, $j=1,2$, $\sigma$ *are measures singular with respect to the Lebesgue measure (non–increasing bounded functions on* $(0,\pi)$ *and on* $I$, *which obey* $\sigma_j'=0$, $\sigma'=0$).

*If, in addition, the function* $v$ *is subharmonic and has a positive harmonic majorant on a wider semi–annulus*

$$D_+(R_1', R_2'), \qquad 0 < R_1' < R_1 < R_2 < R_2',$$

*then*

$$\begin{aligned} v(z) = &- \iint\limits_{D_+(R_1,R_2)} G(z,\zeta)d\mu(\zeta) + \sum_{j=1}^{2} \frac{1}{2\pi} \int_0^\pi \frac{\partial G(z, R_je^{i\varphi})}{\partial n_\zeta} v(R_je^{i\varphi})d\varphi \\ &+ \frac{1}{2\pi} \int\limits_{I(R_1,R_2)} \frac{\partial G(z,t)}{\partial n_t} d\nu(t). \end{aligned} \tag{4.3}$$

We fix $R>0$ and consider a class of subharmonic functions on $D_+(R)$ having a positive harmonic majorants in each semi–annulus $D_+(R_1,R_2)$, $R_1 > R$. We denote this class by $SK(R)$.

**Theorem 4.8.** *Let* $v \in SK(R)$ *and* $\mu$ *be its Riesz measure. Then*

1) *there exists a measure* $\nu$ *on* $I_R = (-\infty,-R)\cup(R,+\infty)$ *having a bounded total variation on each segment* $[a,b]\subset I_R$, *and, if* $\nu(\{a\}) = \nu(\{b\}) = 0$, *then*

$$\nu([a,b]) = \lim_{y\to+0} \int_a^b v(x+iy)\,dx,$$

2) *for almost all* $x\in I_R$ *in the sense of Lebesgue measure the limit*

$$v(x) = \lim_{y\to+0} v(x+iy)\,dx$$

*is well–defined, and* $v(x)$ *belongs to the class* $L_1$ *of functions integrable by the Lebesgue measure on each segment* $[a,b]\subset I_R$,

3) $d\nu(x) = v(x)dx + d\sigma(x)$, *where* $\sigma$ *is a measure on* $I_R$ *singular with respect to the Lebesgue measure,*
4) *if the function* $v$ *is bounded from above on the semi–annulus* $D_+(x_0, x_0+\delta)\subset D_+(R)$, *then the measure* $\sigma$ *is negative on* $[-x_0-\delta,-x_0]\cup[x_0,x_0+\delta]$,
5) *if* $d\mu_1(\zeta) = 2\pi \operatorname{Im}\zeta d\mu(\zeta)$, *then the measure* $\mu_1$ *is finite on each compact set* $K\subset \overline{D_+(R)}\setminus L_R$,
6) *for each* $r>R$

$$\int_0^\pi |v(re^{i\theta})|\sin\theta d\theta < \infty.$$

*Proof.* We consider the semi–annulus $D_+(R_1,4R_1)$, $R_1>R$. By Theorem 3.1 there exists a finite measure $\nu$ on $I(R_1,4R_1)$ such that for $z\in D_+(R_1,4R_1)$ the identity

$$\begin{aligned} v(z)=&-\iint\limits_{D_+(R_1,4R_1)} G(z,\zeta)\,d\mu(\zeta)+\frac{1}{2\pi}\int\limits_0^{\pi}\frac{\partial G(z,R_1e^{i\varphi})}{\partial n_\zeta}v(R_1e^{i\varphi})\,d\varphi\\ &+\frac{1}{2\pi}\int\limits_0^{\pi}\frac{\partial G(z,4R_1e^{i\varphi})}{\partial n_\zeta}v(4R_1e^{i\varphi})\,d\varphi+\frac{1}{2\pi}\int\limits_{I(R_1,4R_1)}\frac{\partial G(z,t)}{\partial n_t}\,d\nu(t)\end{aligned}\tag{4.4}$$

holds. And if $[a,b]\in I(R_1,4R_1)$, $\nu(\{a\})=\nu(\{b\})=0$, then

$$\nu([a,b])=\lim_{y\to+0}\int\limits_a^b v(x+iy)\,dx.$$

The written formula shows that the measure $\nu$ is independent of $R_1$ and is uniquely determined on the set $I_R$. Since the measure $\nu$ has a bounded complete variation on each segment $I(R_1,4R_1)\subset I_R$, by the Lebesgue theorem

$$d\nu(t)=\varphi(t)\,dt+d\sigma(t),$$

where $\varphi$ belongs to $L_1$ on each segment $I(R_1,4R_1)\subset I_R$, and is a measure $\sigma$ singular with respect to the Lebesgue measure.

Let $v(z_0>-\infty)$, $z_0\in D_+(R_1,R_2)$, $G(z,\zeta)$ be the Green function of semi–annulus $D_+(R_1,4R_1)$. Then the measure $\mu_2$, $d\mu_2(\zeta)=G(z_0,\zeta)d\mu(\zeta)$, is a finite measure on $D_+(R_1,4R_1)$. Since the function $\dfrac{\operatorname{Im}\zeta}{G(z_0,\zeta)}$ is a bounded function on $D_+\left(\frac{3}{2}R_1,\frac{7}{2}R_1\right)$, the measure

$$d\mu_1(\zeta)=2\pi\operatorname{Im}\zeta\,d\mu(\zeta)=2\pi\frac{\operatorname{Im}\zeta}{G(z_0,\zeta)}\,d\mu_2(\zeta)$$

is a bounded measure on $D_+(R_1,4R_1)$. Since $R_1>R$ is arbitrary, the measure $\mu_1$ is finite on each compact set $K\subset\overline{D_+(R)}\setminus L_R$.

We treat the measure $\mu_1$ as the measure on the entire plane, and its restriction to the semi–annulus $D_+(R)$ is equal $2\pi\operatorname{Im}\zeta\,d\mu(\zeta)$, while its restriction to $\mathbb{C}\setminus D_+(R)$ vanishes.

It follows from Theorems 3.1 and 4.6 that the measures

$$d\nu_1(\varphi)=\frac{\partial G(z_0,R_1e^{i\varphi})}{\partial n}v(R_1e^{i\varphi})\,d\varphi,\qquad d\nu_2(\varphi)=\frac{\partial G(z_0,4R_1e^{i\varphi})}{\partial n}v(4R_1e^{i\varphi})\,d\varphi,$$

has a bounded total variation on the segment $[0,\pi]$. This is why

$$\int\limits_0^{\pi}\frac{\partial\,G(z_0,R_1e^{i\varphi})}{\partial n}|v(R_1e^{i\varphi})|\,d\varphi<\infty,\qquad \int\limits_0^{\pi}\frac{\partial\,G(z_0,4R_1e^{i\varphi})}{\partial n}|v(4R_1e^{i\varphi})|\,d\varphi<\infty.$$

Since

$$\frac{\sin\varphi}{\partial\,G(z_0,R_1e^{i\varphi})/\partial n}\leqslant M(z_0,\pm R_1)\quad\text{for}\quad\varphi\in(0,\pi),$$

we have

$$\int\limits_0^{\pi}|v(R_1e^{i\theta})|\sin\theta\,d\theta<\infty.$$

Let $G_1(z,\zeta)$ be the Green function of the upper half plane $\mathbb{C}_+$. Then the quotient $\dfrac{G(z,\zeta)}{G_1(z,\zeta)}$ is bounded for $z,\zeta\in D_+(R_1,4R_1)$. Moreover, $\dfrac{G_1(z,\zeta)}{\operatorname{Im}\zeta}=-L(z,\zeta)$.

Then

$$v_1(z)=\iint\limits_{D_+(R_1,4R_1)} G(z,\zeta)\,d\mu(\zeta)=-\frac{1}{2\pi}\iint\limits_{D_+(R_1,4R_1)} L(z,\zeta)\frac{G(z,\zeta)}{G_1(z,\zeta)}\,d\mu_1(\zeta)$$
$$\leqslant -\frac{M}{2\pi}\iint\limits_{D_+(R_1,4R_1)} L(z,\zeta)\,d\mu_1(\zeta).$$

Then by Theorem 3.1 $\lim\limits_{y\to+0} v_1(x+iy)=0$ almost everywhere.

Since for each fixed $x\in I(R_1,4R_1)$ the relations

$$\frac{\partial G(x+iy,R_1e^{i\varphi})/\partial n}{\partial G(z_0,R_1e^{i\varphi})/\partial n}\overset{y\to+0}{\rightrightarrows}0,\qquad \frac{\partial G(x+iy,4R_1e^{i\varphi})/\partial n}{\partial G(z_0,4R_1e^{i\varphi})/\partial n}\overset{y\to+0}{\rightrightarrows}0,\quad \varphi\in(0,\pi),$$

hold for the functions

$$v_2(z)=\frac{R_1}{2\pi}\int_0^\pi \frac{\partial G(z,R_1e^{i\varphi})}{\partial n}v(R_1e^{i\varphi})\,d\varphi=\frac{R_1}{2\pi}\int_0^\pi \frac{\partial G(z,R_1e^{i\varphi})/\partial n}{\partial G(z_0,R_1e^{i\varphi})/\partial n}\,d\nu_1(\varphi),$$

$$v_3(z)=\frac{4R_1}{2\pi}\int_0^\pi \frac{\partial G(z,4R_1e^{i\varphi})}{\partial n}v(4R_1e^{i\varphi})\,d\varphi=\frac{4R_1}{2\pi}\int_0^\pi \frac{\partial G(z,4R_1e^{i\varphi})/\partial n}{\partial G(z_0,R_1e^{i\varphi})/\partial n}\,d\nu_2(\varphi),$$

the identities $\lim\limits_{y\to+0} v_j(x+iy)=0$, $j=2,3$, hold.

Since

$$\frac{\partial G(x+iy,t)}{\partial n}-\frac{\partial G_1(x+iy,t)}{\partial n}\overset{y\to+0}{\rightrightarrows}0\quad\text{for}\quad x\in I\left(\frac{3R_1}{2},\frac{7R_1}{2}\right),\quad t\in I(R_1,4R_1),$$

the function

$$v_4(z)=\frac{1}{2\pi}\int\limits_{I(R_1,4R_1)}\left(\frac{\partial G(z,t)}{\partial n}-\frac{\partial G_1(z,t)}{\partial n}\right)d\nu(t)$$

satisfies the identity

$$\lim_{y\to+0} v_4(x+iy)=0,\quad x\in I\left(\frac{3R_1}{2},\frac{7R_1}{2}\right).$$

Let

$$v_5(z)=\frac{1}{2\pi}\int\limits_{I(R_1,4R_1)}\frac{\partial G_1(z,t)}{\partial n}\,d\nu(t)$$
$$=\frac{1}{2\pi}\int\limits_{I(R_1,4R_1)}\frac{\partial G_1(z,t)}{\partial n}\,\varphi(t)\,dt+\frac{1}{2\pi}\int\limits_{I(R_1,4R_1)}\frac{\partial G_1(z,t)}{\partial n}\,d\sigma(t).$$

It follows from Theorem 3.3 that for almost all $x\in I(R_1,4R_1)$ the identity holds

$$\lim_{y\to+0} v_5(x+iy)=\varphi(x).$$

It follows from the obtained identities for $v_j$, $j=\overline{1,5}$, and (4.4) that for almost all $x\in I(R)$ the identity holds

$$\lim_{y\to+0} v(x+iy)=\varphi(x).$$

It remains to prove Assertion 4 of the theorem. Without loss of generality, we can suppose that $x_0 = 2R_1$, $\delta = R_1$.

Since the measures $d\sigma(t)$ and $v(t)dt$ are mutually singular, we obtain

$$d\nu_+(t) = v_+(t) + d\sigma_+(t), \qquad d\nu_-(t) = v_-(t) + d\sigma_-(t).$$

By Theorem 4.2 the best harmonic majorants of the functions $v(z)$ and

$$v_7(z) = v(z) + v_1(z)$$

in the semi–annulus $D_+(R_1, R_2)$ coincides. By Theorem 4.4, the best harmonic majorant of function $v_7(z)$ in the semi–annulus $D_+(R_1, R_2)$ reads

$$H(z) = \sum_{j=1}^{2} \frac{R_j}{2\pi} \int_0^{\pi} \frac{\partial G(z, R_j e^{i\varphi})}{\partial n_\zeta} v_+(R_j e^{i\varphi})\, d\varphi$$
$$+ \frac{1}{2\pi} \int\limits_{I(R_1,R_2)} \frac{\partial G(z,t)}{\partial n_t}\, d\nu_+(t) + \frac{1}{2\pi} \int\limits_{I(R_1,R_2)} \frac{\partial G(z,t)}{\partial n_t}\, d\sigma_+(t).$$

Since by the assumption of theorem $v(z) \leqslant K$ in the semi–annulus $D_+(R_1, R_2)$, and $H(z)$ is the best harmonic majorant of the function $v(z)$ in this semi–annulus, we have $H(z) \leqslant K$. In particular,

$$u(z) = \frac{1}{2\pi} \int\limits_{I(R_1,R_2)} \frac{\partial G(z,t)}{\partial n_t} d\sigma_+(t) \leqslant K.$$

It follows from the Fatou theorem that for almost everywhere for all $x \in I(R_1, R_2)$ the identity holds

$$\lim_{y \to +0} u(x + iy) = \sigma'_+(x) = 0.$$

By Theorem 4.7, if $[a, b] \subset I(R_1, R_2)$, $\sigma(\{a\}) = \sigma(\{b\}) = 0$, then the identity holds

$$\sigma_+([a, b]) = \int_a^b u(x + iy)\, dx.$$

Since $u(z) \leqslant K$, by the Lebesgue theorem on dominated convergence $\sigma_+([a, b]) = 0$. Thus, $\sigma_+$ is the zero measure on $I(R_1, R_2)$, and $\sigma = \sigma_+ - \sigma_- = \sigma_-$ is the negative measure on $I(R_1, R_2)$. The proof is complete. □

Now we introduce an important in the theory of subharmonic functions on semi–annulus notion of total measure of subharmonic function. The total measure with be defined not on the entire set of subharmonic functions on the semi–annulus, but only on the class $SK(R)$.

**Definition 4.1.** *Let $v \in SK(R)$. The total measure of function $v$ is a measure $\lambda$ such that*

$$d\lambda(\zeta) = 2\pi \operatorname{Im} \zeta\, d\mu(\zeta) - d\nu(t), \qquad t = \operatorname{Re} \zeta, t \in I(R) = (-\infty, -R] \cup [R, +\infty),$$

*where $\mu$ is the Riesz measure of function $v$, $d\nu(t) = v(t)dt + d\sigma(t)$ is the boundary measure of function $v$, and $\sigma$ is a singular boundary measure of function $v$.*

**Remark 4.4.** *The measure $\lambda$ possesses the following properties*

1) *$\lambda$ is a finite measure on each compact set $K \subset \overline{D_+(R)}$;*
2) *$\lambda$ is a non–negative measure on $D_+(R)$;*
3) *$\lambda$ vanishes on the complement $\mathbb{C} \setminus \overline{D_+(R)}$.*

*And vice versa, if a measure $\lambda$ satisfied conditions 1) — 3), then there exists a function $v \in SK(R)$ with the total measure equalling to $\lambda$.*

We treat $\lambda$ as a measure on the entire plane. Its restriction to $\mathbb{C}\setminus\overline{D_+(R)}$ is the zero measure, the restriction to $D_+(R)$ is a positive measure, and its restriction to $I(R)$ is equal to $-\nu$, where $\nu$ is the boundary measure. Generally speaking, the boundary measure $\nu$ is a sign–changing measure (charge).

In the next theorem we establish with what accuracy the functions in the class $SK(R)$ are determined by their total measure.

**Theorem 4.9.** *Let $v_1, v_2 \in SK(R)$. The total measures of functions $v_1$ and $v_2$ coincide if and only if there exists a function $g$ analytic on the set $\mathbb{C}\setminus C(0,R)$ taking real values on $I(R)$ such that*

$$v_1(z) - v_2(z) = \operatorname{Im} g(z), \quad z \in D_+(R). \tag{4.5}$$

*Proof.* Let identity (4.5) holds. Since $\Delta g(z) \equiv 0$, the functions $v_1$ and $v_2$ have the same Riesz measures. Since

$$\lim_{y\to+0} \int\limits_{[a,b]\cap I(R)} \operatorname{Im} g(x+iy)\,dx = 0,$$

the functions $v_1$ and $v_2$ haver the same boundary measures. Therefore, these functions have the same total measure and same boundary measures.

Vice versa, let the functions $v_1$ and $v_2$ have same total measures. Then they have same Riesz measures and same boundary measures. Let $R_1 > R$ be an arbitrary number. By Theorem (3.1)

$$\begin{aligned} v_1(z) - v_2(z) = &\int\limits_0^\pi \frac{\partial G(z, Re^{i\varphi})}{\partial n_\zeta} (v_1(Re^{i\varphi}) - v_2(Re^{i\varphi}))\,d\varphi \\ &+ \int\limits_0^\pi \frac{\partial G(z, R_1e^{i\varphi})}{\partial n_\zeta} (v_1(R_1e^{i\varphi}) - v_2(R_1e^{i\varphi}))\,d\varphi. \end{aligned}$$

The right hand side involves a harmonic function on the semi–annulus $D_+(R, R_1)$, which is continuously continued by zero to $I(R,R_1) = (-R_1, -R) \cup (R, R_1)$. Since $R_1$ is an arbitrary number, $v_1 - v_2$ is a harmonic function on $D_+(R)$, which is continuously continued by zero to $I(R)$. By the Schwarz reflection principle this function is continued as harmonic one to $\{z : |z| > R,\ \operatorname{Im} z < 0\}$. Thus, there exists a harmonic function $h$ on $\mathbb{C}\setminus B(R)$ vanishing on $I(R)$ such that $v_1 - v_2 = h$ for $\operatorname{Im} z > 0$. Let $-h_1$ be the function harmonically conjugated with the function $h$. Then $g = h + ih_1$ is the analytic function on $\mathbb{C}\setminus B(R)$ taking real values on $I(R)$ and $h = \operatorname{Im} g$. The proof is complete. □

We denote by $JS(R)$ the class of harmonic functions on $D_+(R)$ such that $\varlimsup\limits_{z\to t} v(z) \leqslant 0$ for each $t \in I(R)$.

**Remark 4.5.** *For subharmonic function in the class $JS(R)$ the total measure is a non–negative measure.*

**Proposition 4.1.** *$JS(R) \subset SK(R)$.*

*Proof.* Let

$$v \in JS(R), \qquad P = \{z : \operatorname{Re} z \in [a,b] \subset I(R),\ 0 < \operatorname{Im} z < T\}.$$

Since

$$\varlimsup_{y\to+0} v(x+iy) \leqslant 0,$$

the function $v(a+iy)$ is bounded from above for $y \in (0,T)$. This is true for $v(b+iy)$. The function $v(x+iT)$ is bounded from above for $x \in [a,b]$. The above semi–continuity implies that

$$\varlimsup_{\substack{z\in P\\ z\to\zeta}} v(z) \leqslant \begin{cases} v(\zeta), & \operatorname{Im}\zeta > 0,\\ 0, & \operatorname{Im}\zeta > 0.\end{cases}$$

In particular, there exists a constant $M$ such that

$$\varlimsup_{\substack{z\in P\\ z\to\zeta}} v(z) \leqslant M, \quad \zeta \in \partial P.$$

The maximum principle implies that $v(z) \leqslant M$ on $P$. The proof is complete. □

Now we are in position to give the definition of our main object, which is the $\delta$–subharmonic function.

Usually, as the $\delta$–subharmonic function, one calls the function, which can be represented as the difference of subharmonic functions. However, such definition needs a correction. The corrected definition of $\delta$–subharmonic function was given in [6].

**Definition 4.2.** *The function $v$ on the domain $G \subset \mathbb{C}$ is called $\delta$–subharmonic if the following three conditions are satisfied.*

1) *There exists a set $F$ of zero capacity such that on the set $G \setminus F$ the representation $v(z) = v_1(z) - v_2(z)$ holds, where $v_1(z)$ and $v_2(z)$ are subharmonic functions on the domain $G$. By means of this representation we determined the Riesz measure $\mu$ the function $v$ by the formula $\mu = \mu_1 - \mu_2$, where $\mu_1$ and $\mu_2$ are the Riesz measures of functions $v_1(z)$ and $v_2(z)$. The determining set $H$ of a function $v$ is the set of points $z \in G$, for which there exists $\delta > 0$ such that the inequality holds*

$$\int_0^\delta \frac{|\mu|(B(z,t))}{t}\,dt < \infty.$$

2) *For each point $z \in H$ the identity holds*

$$v(z) = \lim_{\delta\to 0} \frac{1}{2\pi\delta} \int_0^{2\pi} w(z+\delta e^{i\varphi})\,d\varphi.$$

3) $v(z) = 0$ *for* $z \in G \setminus H$.

Now we introduce the class $\delta S(R)$ of delta–subharmonic functions on $D_+(R)$ as the difference $\delta S(R) = JS(R) - JS(R)$. Here the difference is understood with the above made remark taken into consideration.

**Proposition 4.2.** $\delta S(R) = SK(R) - SK(R)$.

*Proof.* Let $v \in \delta S(R)$. Then

$$v = v_1 - v_2, \qquad v_1, v_2 \in JS(R) \subset SK(R).$$

This implies that

$$\delta S(R) \subset SK(R) - SK(R).$$

Now let $v \in SK(R) - SK(R)$, $\lambda$ be its total measure, $\lambda = \lambda_+ - \lambda_-$ be the Jordan composition of measure $\lambda$. Let the functions $v_1, v_2 \in SK(R)$ be such that $\lambda_+$ and $\lambda_-$ are their total measures; such functions exist by Remark 4.4. Then $v_1, v_2 \in JS(R)$. We have

$$v(z) = v_1(z) - v_2(z) + \operatorname{Im} g,$$

where $g$ is as in Theorem 4.5. Thus, we have obtained that

$$v \in \delta S(R), \quad SK(R) - SK(R) \subset JS(R) - JS(R) = \delta S(R).$$

The proof is complete. □

## 5. Fourier coefficients of function $v \in \delta S(R)$ in vicinity of $L_R$

The Fourier coefficients of functions $v \in \delta S(R)$ in the vicinity of $L_R$ by the identity

$$c_k(r, v) = \frac{2}{\pi} \int\limits_0^\pi v((r-R)e^{i\theta}) \sin k\theta \, d\theta, \quad R < r \leqslant 2R, \quad k \in \mathbb{N}.$$

We shall employ the generalized Carleman formula for the semi–annulus $D_+((r-R), 4(r-R))$, which can be obtained from formula (16) in [10]. We formulate it as a theorem, but first we introduce a notation. Let $\lambda$ be an arbitrary measure in the closed half–plane $\overline{\mathbb{C}_+}$. We define the measure $\lambda_m$, $m \in \mathbb{N}$, by the identity

$$d\lambda_m(\zeta) = \frac{\sin m\varphi}{\operatorname{Im} \zeta} \tau^m \, d\lambda(\zeta), \quad \zeta = \tau e^{i\varphi},$$

where $\dfrac{\sin m\varphi}{\sin\varphi}$ for $\varphi = 0, \pi$ is defined by the continuity.

**Theorem 5.1.** *Let $v \in \delta S(R)$, $\lambda$ be the total measure of function $v$. Then for each $r$, $R < r \leqslant 2R$, the formula holds*

$$\begin{aligned}
&\int\limits_0^\pi v(2(r-R)e^{i\theta}) \sin m\theta \, d\theta \\
&\qquad = \frac{2^{m-2}}{(r-R)(4^m+1)} \int\limits_0^\pi \left(4v\left((r-R)e^{i\varphi}\right) + v(4(r-R)e^{i\varphi})\right) \sin m\varphi \, d\varphi \\
&\qquad - \frac{2^m}{2m(4^m+1)(r-R)^m} \int\limits_{(r-R)}^{2(r-R)} \left(1 - \frac{(r-R)^{2m}}{\tau^{2m}}\right) d\lambda_m(\tau) \qquad (5.1)\\
&\qquad - \frac{8^m r^m}{2m(4^m+1)} \int\limits_{2(r-R)}^{4(r-R)} \left(\frac{1}{\tau^{2m}} - \frac{1}{16^m (r-R)^{2m}}\right) d\lambda_m(\tau).
\end{aligned}$$

We transform formula (5.1). In order to do this, we integrate by parts two last terms in the right hand side and we obtain

$$\begin{aligned}
&\int\limits_0^\pi v(2(r-R)e^{i\theta}) \sin m\theta \, d\theta \\
&\qquad = \frac{2^{m-2}}{(r-R)(4^m+1)} \int\limits_0^\pi \left(4v\left((r-R)e^{i\varphi}\right) + v(4(r-R)e^{i\varphi})\right) \sin m\varphi \, d\varphi \qquad (5.2)\\
&\qquad + \frac{2^m (r-R)^m}{4^m+1} \left( \int\limits_{(r-R)}^{2(r-R)} \frac{\lambda_m(\tau)}{\tau^{2m+1}} \, d\tau - 4^m \int\limits_{2(r-R)}^{4(r-R)} \frac{\lambda_m(\tau)}{\tau^{2m+1}} \, d\tau \right).
\end{aligned}$$

In particular, for $m=1$ formula (5.2) reads

$$\begin{aligned}\int\limits_0^\pi v(2(r-R)e^{i\theta})\sin\theta\,d\theta =& \frac{1}{10(r-R)}\int\limits_0^\pi \left(4v\left((r-R)e^{i\varphi}\right)+v(4(r-R)e^{i\varphi})\right)\sin\varphi\,d\varphi \\ &+\frac{2(r-R)}{5}\left(\int\limits_{(r-R)}^{2(r-R)}\frac{\lambda(\tau)}{\tau^3}\,d\tau-4\int\limits_{2(r-R)}^{4(r-R)}\frac{\lambda(\tau)}{\tau^3}\,d\tau\right).\end{aligned}\tag{5.3}$$

If $v=v_+-v_-$, and $\lambda=\lambda_+-\lambda_-$ is the Jordan decomposition of measure $\lambda$, formula (5.3) can be represented as

$$\begin{aligned}&\int\limits_0^\pi v_+(2(r-R)e^{i\theta})\sin\theta\,d\theta \\ &+\frac{1}{10(r-R)}\int\limits_0^\pi \left(4v_-\left((r-R)e^{i\varphi}\right)+v_-(4(r-R)e^{i\varphi})\right)\sin\varphi\,d\varphi \\ &+\frac{2(r-R)}{5}\left(\int\limits_{(r-R)}^{2(r-R)}\frac{\lambda_-(\tau)}{\tau^3}\,d\tau+4\int\limits_{2(r-R)}^{4(r-R)}\frac{\lambda_+(\tau)}{\tau^3}\,d\tau\right)=\int\limits_0^\pi v_-(2(r-R)e^{i\theta})\sin\theta\,d\theta \\ &+\frac{1}{10(r-R)}\int\limits_0^\pi \left(4v_+\left((r-R)e^{i\varphi}\right)+v_+(4(r-R)e^{i\varphi})\right)\sin\varphi\,d\varphi \\ &+\frac{2(r-R)}{5}\left(\int\limits_{(r-R)}^{2(r-R)}\frac{\lambda_+(\tau)}{\tau^3}\,d\tau+4\int\limits_{2(r-R)}^{4(r-R)}\frac{\lambda_-(\tau)}{\tau^3}\,d\tau\right).\end{aligned}\tag{5.4}$$

We rewrite formula (5.2) in terms of Fourier coefficients

$$\begin{aligned}c_m(2r,v)=&\frac{2^{m-2}}{(r-R)(4^m+1)}\left(4c_m\left(r,v\right)+c_m(4r,v)\right) \\ &+\frac{2^{m+1}(r-R)^m}{\pi(4^m+1)}\left(\int\limits_{(r-R)}^{2(r-R)}\frac{\lambda_m(\tau)}{\tau^{2m+1}}\,d\tau-4^m\int\limits_{2(r-R)}^{4(r-R)}\frac{\lambda_m(\tau)}{\tau^{2m+1}}\,d\tau\right).\end{aligned}\tag{5.5}$$

## 6. Delta–subharmonic functions of finite $\gamma$–type on $D_+(R)$ in vicinity of $L_R$

Let $v\in\delta S(R)$, $v=v_+-v_-$, $\lambda$ be the total measure of function $v$, $\lambda=\lambda_+-\lambda_-$ be the Jordan decomposition of measure $\lambda$. We denote

$$\begin{aligned}m_R(r,v)=(r-R)&\int\limits_0^\pi v_+(2(r-R)e^{i\theta})\sin\theta\,d\theta \\ &+\frac{1}{10}\int\limits_0^\pi \left(4v_-\left((r-R)e^{i\varphi}\right)+v_-(4(r-R)e^{i\varphi})\right)\sin\varphi\,d\varphi,\end{aligned}$$

$$N_R(r,v)=\frac{2(r-R)^2}{5}\left(\int\limits_{(r-R)}^{2(r-R)}\frac{\lambda_-(\tau)}{\tau^3}\,d\tau+4\int\limits_{2(r-R)}^{4(r-R)}\frac{\lambda_+(\tau)}{\tau^3}\,d\tau\right),$$
$$T_R(r,v)=m_R(r,v)+N_R(r,v),\quad r>R.$$

By (5.4) we obtain

$$T_R(r,v)=T_R(r,-v). \tag{6.1}$$

**Definition 6.1.** *A strictly positive continuous increasing and unbounded function $\gamma(r)$ defined on $R_+$ is called the growth function.*

**Definition 6.2.** *Let $\gamma$ be a growth function. A function $v\in\delta S(R)$ is called the function of finite $\gamma$–type in the vicinity of semi–circumference $L_R$ if there exist constants $A$ and $B>0$ such that*

$$T_R(r,v)\leqslant A\gamma\left(\frac{B}{r-R}\right) \tag{6.2}$$

*for all $r$, $R<r\leqslant 2R$.*

The class of given $\delta$–subharmonic functions of finite $\gamma$–type in the vicinity of circumference $L_R$ is denoted by $\delta S_{L_R}(R,\gamma)$. By $JS_{L_R}(R,\gamma)\subset JS(R)$ we denote the corresponding class of subharmonic function. It is clear that $JS_{L_R}(R,\gamma)\subset\delta S_{L_R}(R,\gamma)$.

**Definition 6.3.** *A positive measure $\lambda$ on $D_+(R)$ has a finite $\gamma$–density in the vicinity of semi–circumference $L_R$ if for some positive $A$ and $B$ for all $r$, $R<r\leqslant 2R$, the inequality holds*

$$\lambda(r-R)\leqslant A\gamma\left(\frac{B}{r-R}\right). \tag{6.3}$$

We note that inequality (6.3) implies

$$\int\limits_{(r-R)}^{2(r-R)}\frac{\lambda(\tau)}{\tau^3}\,d\tau\leqslant A\gamma\left(\frac{B}{r-R}\right) \tag{6.4}$$

for some positive $A$ and $B$ for all $r$, $R<r\leqslant 2R$. And vice versa, inequality (6.4) implies inequality (6.3).

**Remark 6.1.** *Let $v\in J\delta_{L_R}(R,\gamma)$, $\lambda$ be the total measure of function $v$, $\lambda=\lambda_+-\lambda_-$ be the Jordan decomposition of measure $\lambda$, $|\lambda|=\lambda_++\lambda_-$. Then the measure $|\lambda|$ has a finite $\gamma$–density in the vicinity of semi–circumference $L_R$.*

This is the corollary of Definitions 6.2 and 6.3.

Now we formulate and prove the criterion of belonging of a delta–subharmonic function $v$ to the class $J\delta_{L_R}(R,\gamma)$.

**Theorem 6.1.** *Let $v\in J\delta(R)$, $\lambda=\lambda_+-\lambda_-$ be the total measure of function $v$, $\gamma$ be a growth function. The following two statements are equivalent:*

1) $J\delta_{L_R}(R,\gamma)$;
2) *the measure $\lambda_+$ (or the measure $\lambda_-$) has a finite $\gamma$–density in the vicinity of circumference $L_R$ and for all $r$, $R<r\leqslant 2R$, for some positive $A$ and $B$ the inequality holds*

$$|c_k(2r,v)|\leqslant\frac{A}{r-R}\gamma\left(\frac{B}{r-R}\right),\quad k\in\mathbb{N}. \tag{6.5}$$

*Proof.* We prove by the scheme 1) $\Longrightarrow$ 2) $\Longrightarrow$ 1). We begin with the implication 1) $\Longrightarrow$ 2). The fact that the measures $\lambda_+(v)$ and $\lambda_-(v)$ have the finite $\gamma_2$-density in the vicinity of circumference $L_R$ is immediately implied by the definition of class $\delta S_{L_R}(R,\gamma)$ and identity (6.1). Then it follows from Remark 6.1 that the measure $|\lambda|$ has a finite $\gamma$–density in the vicinity of circumference $L_R$, that is,

$$|\lambda|(r-R) \leqslant A\,\gamma\left(\frac{B}{r-R}\right) \tag{6.6}$$

for some positive $A$ and $B$ for all $r$, $R<r\leqslant 2R$.

It follows from inequality (6.2) that

$$m(r,v) \leqslant A\,\gamma\left(\frac{B}{r-R}\right),$$

for some $A$ and $B$ for all $r$, $R<r\leqslant 2R$. Relation (6.1) also implies the inequality

$$m(r,-v) \leqslant A\,\gamma\left(\frac{B}{r-R}\right).$$

Hence, for all $r$, $R<r\leqslant 2R$,

$$\begin{aligned}
&\int\limits_0^{\pi} |v(2(r-R)e^{i\varphi})|\sin\varphi\,d\varphi = \frac{m(r,v)+m(r,-v)}{r-R} \leqslant \frac{A}{r-R}\gamma\left(\frac{B}{r-R}\right),\\
&|c_k(2r,v)| \leqslant \frac{2k}{\pi}\int\limits_0^{\pi} |v(2(r-R)e^{i\varphi})|\sin\varphi\,d\varphi \leqslant \frac{Ak}{r-R}\gamma\left(\frac{B}{r-R}\right)
\end{aligned} \tag{6.7}$$

We also note the inequality

$$\begin{aligned}
|\lambda_m(r)| &= \left|\iint\limits_{C(0,r)} d\lambda_m(\zeta)\right| = \left|\iint\limits_{C(0,r)} \frac{\sin m\varphi}{\sin\varphi}\tau^{m-1}\,d\lambda(\zeta)\right|\\
&\leqslant m\iint\limits_{C(0,r)} \tau^{m-1}\,d|\lambda|(\zeta) \leqslant m r^{m-1}|\lambda|(r).
\end{aligned}$$

By (5.5), (6.4), (6.7), (6) we obtain

$$\begin{aligned}
|c_m(2r,v)| \leqslant &\frac{2^{m-2}}{(r-R)(4^m+1)}\left(4\,|c_m(r,v)| + |c_m(4r,v)|\right)\\
&+\frac{2^{m+1}(r-R)^m}{\pi(4^m+1)}\left(\int\limits_{(r-R)}^{2(r-R)}\frac{|\lambda_m(\tau)|}{\tau^{2m+1}}\,d\tau + 4^m\int\limits_{2(r-R)}^{4(r-R)}\frac{|\lambda_m(\tau)|}{\tau^{2m+1}}\,d\tau\right)\\
\leqslant &\frac{2^m mA}{(r-R)((4^m+1)}\gamma\left(\frac{B}{r-R}\right)\\
&+\frac{m2^{m+1}(r-R)^m}{\pi(4^m+1)}\left(\int\limits_{(r-R)}^{2(r-R)}\frac{|\lambda|(\tau)d\tau}{\tau^{m+2}} + 4^m\int\limits_{2(r-R)}^{4(r-R)}\frac{|\lambda|(\tau)d\tau}{\tau^{m+2}}\right)\\
\leqslant &\frac{2^m mA}{(r-R)((4^m+1)}\gamma\left(\frac{B}{r-R}\right)
\end{aligned}$$

$$+\frac{m2^{m+1}(r-R)^m|\lambda|(4(r-R))}{\pi(4^m+1)}\left(\int\limits_{(r-R)}^{2(r-R)}\frac{d\tau}{\tau^{m+2}}+4^m\int\limits_{2(r-R)}^{4(r-R)}\frac{d\tau}{\tau^{m+2}}\right)$$
$$\leqslant\frac{2^m mA}{(r-R)((4^m+1)}\gamma\left(\frac{B}{r-R}\right)$$
$$+\frac{m2^{m+1}(r-R)^m|\lambda|(4(r-R))}{\pi(4^m+1)(m+1)}\left(\frac{1}{(r-R)^{m+1}}+\frac{2^m}{(r-R)^{m+1}}\right)$$
$$\leqslant\frac{A}{r-R}\gamma\left(\frac{B}{r-R}\right).$$

The proof of the implication 1) $\Longrightarrow$ 2) is complete.

We proceed to proving the implication 2) $\Longrightarrow$ 1). It follows from inequality (6.5) and formula (5.4) that if one of measures $\lambda_+(v)$ or $\lambda_-(v)$ has finite $\gamma$–density in the vicinity of circumference $L_R$, then the other measure also has a finite $\gamma$–density in the vicinity of circumference $L_R$. Therefore, the measure $|\lambda|$ has a finite $\gamma$–density in the vicinity of circumference $L_R$. Then we estimate $v_+(z)$ by employing formula (4.3) for $R_1=r-R$, $R_2=4(r-R)$. Using the expansion into the Fourier series (3.6), (3.7), for $q=\frac{1}{2}$, $R=2(r-R)=|z|$, we obtain

$$\left|\frac{1}{2\pi}\int\limits_0^\pi\frac{\partial G\left(z,4(r-R)e^{i\varphi}\right)}{\partial n}v\left(4(r-R)e^{i\varphi}\right)\,d\varphi+\frac{1}{2\pi}\int\limits_0^\pi\frac{\partial G\left(z,(r-R)e^{i\varphi}\right)}{\partial n}v\left((r-R)e^{i\varphi}\right)\,d\varphi\right|$$
$$=\frac{1}{\pi(r-R)}\sum_{m=1}^\infty\frac{1}{2^m}\frac{4^m}{1+4^m}\left|2\int\limits_0^\pi v\left((r-R)e^{i\varphi}\right)\sin m\varphi d\varphi+\frac{1}{2}\int\limits_0^\pi v\left(4(r-R)e^{i\varphi}\right)\sin m\varphi d\varphi\right|$$
$$\leqslant\frac{1}{r}\sum_{m=1}^\infty\frac{1}{2^m}\left(|\,c_m\left(r,v\right)|+\frac{1}{4}|\,c_m(4r,v)|\right)\leqslant\frac{A}{r-R}\gamma\left(\frac{B}{r-R}\right),$$

for some $A,B>0$. By this inequality and formula (4.3) we obtain

$$v_+(z)\leqslant\frac{1}{2\pi}\int\limits_{I(r,4r)}\frac{\partial G(z,t)}{\partial n_t}\,d|\nu|(t)+\frac{A}{r-R}\gamma\left(\frac{B}{r-R}\right).\tag{6.8}$$

Now, using the orthogonality of the system of polynomials $\{\sin k\theta\}$, $k=1,2,\ldots,$ on the segment $[0,\pi]$ and formulas (3.4), (3.5), we find

$$\int\limits_0^\pi v_+(2re^{i\theta})\sin\theta\,d\theta\leqslant\frac{1}{2\pi}\int\limits_{I(r-R,4(r-R))}\frac{\partial G(z,t)}{\partial n_t}\,d|\nu|(t)\sin\theta\,d\theta+\frac{A}{r-R}\gamma\left(\frac{B}{r-R}\right)$$
$$\leqslant\frac{2\pi}{5}\left(\int\limits_{(r-R)}^{2(r-R)}\left(\frac{1}{r-R}-\frac{r-R}{t^2}\right)\,d|\nu|(t)\right.$$
$$\left.+\int\limits_{2(r-R)}^{4(r-R)}\left(\frac{4(r-R)}{t^2}-\frac{1}{4(r-R)}\right)\,d|\nu|(t)\right)$$
$$+\frac{A}{r-R}\gamma\left(\frac{B}{r-R}\right)=\frac{16\pi(r-R)}{5}\int\limits_{2(r-R)}^{4(r-R)}\frac{|\nu|(t)}{t^3}\,dt$$

$$
\begin{aligned}
&-\frac{4\pi(r-R)}{5}\int\limits_{(r-R)}^{2(r-R)}\frac{|\nu|(t)}{t^3}\,dt+\frac{A}{r-R}\gamma\left(\frac{B}{r-R}\right)\\
\leqslant&\frac{16\pi(r-R)}{5}\int\limits_{2(r-R)}^{4(r-R)}\frac{|\nu|(t)}{t^3}\,dt+\frac{A}{r-R}\gamma\left(\frac{B}{r-R}\right)\\
\leqslant&\frac{A}{r-R}\gamma\left(\frac{B}{r-R}\right).
\end{aligned}
$$

This implies that the function $v$ belongs to the space $J\delta_{L_R}\Big(R,\gamma\Big)$. The proof is complete. □

For true subharmonic functions the following corollary of the above theorem holds.

**Theorem 6.2.** *Let $\gamma$ be a growth function, $v \in JS(R)$. The following two statements are equivalent:*

1) $v \in JS_{L_R}(R,\gamma)$;
2) *for some $r > R$ for all positive $A$, $B$ the inequality holds*

$$|c_k(2r,v)| \leqslant \frac{A}{r-R}\gamma\left(\frac{B}{r-R}\right), \quad k \in \mathbb{N}.$$

*Proof.* Indeed, if $v \in JS(R)$, then $\lambda_-(v) \equiv 0$ and hence, this measure has a finite $\gamma$–density. □

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Konstantin Gennadievich Malyutin,
Kursk State University,
Radischev str. 33,
305000, Kursk, Russia
E-mail: malyutinkg@gmail.com

Nguyen Van Quynh,
Hanoi University of Industry,
No. 298 Cau Dien Street,
Tay Tuu Ward,
Hanoi, Vietnam
E-mail: nguyenvanquynh@haui.edu.vn

Alena Alexandrovna Naumova,
Kursk State University,
Radischev str. 33,
305000, Kursk, Russia
E-mail: aliona.filatowa2013@yandex.ru