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CRITERION OF FUNDAMENTAL PRINCIPLE

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Abstract. In the work we study the subspace of functions analytic in a convex domain and invariant with respect to the differentiation operator. We study the fundamental principle problem, namely, on representation of all functions in the invariant subspace by the series of exponential monomials. These exponential monomials are the eigenfunctions and generalized eigenfunctions of differentiation operator in the invariant subspace. We obtain a simple geometric criterion of fundamental principle. We also obtain a similar criterion for solvability of an interpolation problem in the spaces of entire functions of exponential type.

Keywords: invariant subspace, fundamental principle, exponential monomial, entire function, exponential series.

Mathematics Subject Classification: 30D10

1. Introduction

Let $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$ be a sequence of mutually different complex numbers λ_k and of their multiplicities n_k. We suppose that $|\lambda_k|$ are non–decreasing and $|\lambda_k| \to \infty$, $k \to \infty$. By the symbol $\Xi(\Lambda)$ we denote the set of limits of all converging subsequences of form $\left\{\overline{\lambda}_{k_j}/|\lambda_{k_j}|\right\}_{j=1}^{\infty}$ ($\overline{\lambda}$ is the complex conjugation). The set $\Xi(\Lambda)$ is closed and is a subset of unit circumference $S(0,1) = \{z \in \mathbb{C} : |z| = 1\}$. We introduce the set of exponential monomials

$$\mathcal{E}(\Lambda) = \{z^n e^{\lambda_k z}\}_{k=1,n=0}^{\infty,n_k-1}.$$

Let $D \subset \mathbb{C}$ be a convex domain and

$$H_D(\varphi) = \sup_{z\in D} \operatorname{Re}(ze^{-i\varphi}), \qquad \varphi \in [0, 2\pi]$$

be its support function. We let

$$J(D) = \{e^{i\varphi} \in S(0,1) : H_D(\varphi) = +\infty\}.$$

By the symbols $\operatorname{int} J(D)$, $\overline{J(D)}$ and $\partial J(D)$ we denote respectively the interior, closure and boundary of set $J(D)$ in the topology of circumnference $S(0,1)$. If the set $\overline{J(D)} \setminus J(D) = \partial J(D) \setminus J(D)$ is non–empty, then it consists either of one or two points. If D is a bounded domain, then $J(D) = \emptyset$. In the case of unbounded domain the following situations are possible:

1) $J(D) = S(0,1)$, that is, $D = \mathbb{C}$,
2) D is the half–plane $\{z \in \mathbb{C} : \operatorname{Re}(ze^{-i\varphi}) < a\}$ and $J(D) = S(0,1) \setminus \{e^{i\varphi}\}$,
3) D is the strip $\{z \in \mathbb{C} : b < \operatorname{Re}(ze^{-i\varphi}) < a\}$ and $J(D) = S(0,1) \setminus \{e^{i\varphi}, e^{i\varphi+\pi}\}$,
4) in other cases $J(D)$ is an arc of the unit circumference, which is supported by an angle of opening at least π.

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We call unbounded convex domains described in cases 1)–4) the domains of I–IV type, respectively.

Let $H(D)$ be the space of functions analytic in the domain D with the topology of uniform convergence on compact sets $K \subset D$. By the symbol $W(\Lambda, D)$ we denote the closure of linear hull of system $\mathcal{E}(\Lambda)$ in the space $H(D)$. If the system $\mathcal{E}(\Lambda)$ is incomplete in the space $H(D)$, then $W(\Lambda, D)$ is a non–trivial (that is, $W(\Lambda, D) \neq \{0\}$, $H(D)$) closed subspace in $H(D)$. It follows from the definition that it is invariant with respect to the differentiation operator. At the same time, the system $\mathcal{E}(\Lambda)$ is the set of eigenfunctions and generalized eigenfunctions of differentiation operator in $W(\Lambda, D)$, and Λ is its spectrum.

Let $W \subset H(D)$ be a non–trivial closed subspace invariant with respect to the differentiation operator, and $\Lambda = \{\lambda_k, n_k\}$ be its multiple spectrum. This is at most countable set with the only accumulation point ∞ [20, Ch. II, Sect. 7]. The family $\mathcal{E}(\Lambda)$ set of eigenfunctions and generalized eigenfunctions of differentiation operator in W. We say that the subspace W admits the spectral synthesis if it coincides with $W(\Lambda, D)$. We mention the spectral synthesis problem was completely solved in works [3] and [4]. If D is an unbounded convex domain, then the identity $W = W(\Lambda, D)$ always holds, that is, W admits the spectral synthesis [4, Thm. 8.2].

In the case, when the spectrum of subspace W is finite, it coincides with the space of solutions a homogeneous linear differential equation of finite order with constant coefficients. As a more general example of invariant subspace, the set of solutions to convolution equation $\mu(g(z+w)) \equiv 0$ (or systems of such equations) can serve, where μ is a linear continuous functional on the space $H(D)$. Particular cases of convolution equation are linear differential equations, difference, differential–difference equations with constant coefficients of finite and infinite orders as well as some types of integral equations.

The main problem in the theory of invariant subspaces if the fundamental principle problem, that is, the problem on representation of an arbitrary function in W be means of a series in the elements of system $\mathcal{E}(\Lambda)$. We say that the fundamental principle holds in a space W with the spectrum $\{\lambda_k, n_k\}$ if for each function $g \in W$ the representation holds

$$g(z) = \sum_{k=1,n=0}^{\infty,n_k-1} d_{k,n} z^n e^{\lambda_k z}, \qquad z \in D, \tag{1.1}$$

and the series converges uniformly on compact sets in the domain D. The notion of fundamental principle appeared in relation with its particular case of invariant subspace, which is the set of solutions to linear homogeneous differential equation with constant coefficients. It is known that each solution of such an equation is a linear combination of elementary solutions, which are the exponential monomials $z^n e^{\lambda_k z}$, the exponents of which are zeroes, possibly multiple, of a characteristic polynomial. The presence of such representation is called the Euler fundamental principle.

It is natural to consider the fundamental principle problem only for invariant subspaces, which admit the spectral synthesis, that is, for subspaces of form $W(\Lambda, D)$.

By means of the Laplace transform the fundamental principle problem is reduced to the dual problem on multiple interpolation in the space of entire functions of exponential type. The studies of both problems, made independently, have a rich history. Its main milestones are presented in works [1] and [5]. In the case of a bounded convex domain the problem of fundamentlal principle was completely solved in [5]–[7], [10]. For invariant subspaces admitting the spectral synthesis, a simple geometric criterion of fundamental principle was obtained [10, Thm. 3.2], which was formulated only in terms of condensation index S_Λ (which will be defined later), the maximal angular density of sequence Λ and the length of boundary of domain D.

The situation with unbounded convex domains is much worse. In [5] the criterion of fundamental principle was obtained for arbitrary convex domains. However, it has two disadvantages.

There are some restrictions for the multiplicities n_k of points λ_k. Moreover, it contains the following condition, which is equivalent to presence of fundamental principle. Namely, one requires the existence of family of entire functions vanishing at the points λ_k with the multiplicities at least n_k, the growth of which is close to the regular one and is related with D. The question on conditions for Λ and D, under which such a family exists, remains open. The problem on construction of such a family is rather complicated. Concerning unbounded domains, mostly only three particular cases were studied, namely, the domains of type I–III.

A complete solution of fundamental principle problem for nontrivial invariant subspaces of entire functions was given in [9]. It was proved that the presence of fundamental principle in each such subspace is equivalent to the finiteness of the condensation index S_Λ.

Invariant subspaces in the half–plane were studied in the case of a simple positive spectrum possessing a density. In [8] this problem was solved completely for an arbitrary convex domain D. The solution was found in terms of simple geometric characteristics of the sequence Λ and domain D. It contains a principally new point. It turned out that in the case of the right half–plane, the validity of fundamental principle does not require the measurability of sequence Λ and even the finiteness of its maximal density despite the support function of the half–plane is bounded in the positive direction. The necessary and sufficient condition of this situation is the vanishing of characteristics S_Λ. In [11] this result is extended for case of invariant subspaces with an almost real spectrum Λ (that is, $\Xi(\Lambda) = \{1\}$). Here significant difficulties were overcome, which were related with the multiplicities n_k of spectral points λ_k. We note that the results of [11] can be easily extended to the case of invariant subspaces with the spectrum Λ, for which $\Xi(\Lambda)$ is a single–point set.

In [12] by means of decomposition of invariant subspaces into the sum of two invariant subspaces and on the base of results in [9] and [11], a criterion of fundamental principle was obtained for invariants subspaces in the half–plane with an arbitrary spectrum. It is formulated only in terms of condensation index S_Λ. The same concerns the results of [13]. In this paper a criterion of fundamental principle was given for invariant subspaces in an arbitrary convex domain under the condition $\Xi(\Lambda) \subseteq J(D)$.

In [23] the results of [13] were extended to the case, when $\Xi(\Lambda)$ is located in the closure $\overline{J(D)}$ of set $J(D)$. We observe that this case differs principally from the case $\Xi(\Lambda) \subseteq J(D)$. There was obtained a simple geometric criterion of fundamental principle, which relies only on the notion of condensation index S_Λ. This criterion is based on the results in [9], [11] and [12].

If D is a convex domain of I–III type, then for each invariant subspace the inclusion $\Xi(\Lambda) \subseteq \overline{J(D)}$ holds. Thus, the fundamental principle problem for unbounded convex domains of I–III types was solved completely.

In the case of domains of IV type there is a result in work [5], which was mentioned above. Moreover, in this case simple geometric conditions necessary for fundamental principle were obtained. In [11] a restriction for the multiplicities n_k of points λ_k was established. In [15] necessary conditions were formulated in terms of the length of boundary of D and of maximal density of sequence Λ.

This work completes a long series of studies made in [5]–[16], [23]. Here we obtain a complete solution of fundamental principle problem for arbitrary invariant subspaces, which admit the spectral synthesis, in arbitrary convex domains in the complex plane. The criterion is formulated only in terms of the condensation index S_Λ, the maximal angular density of sequence Λ and the length of boundary of D. We also obtain the solvability criterion for the corresponding interpolation problem in the space of entire functions of exponential type.

The work consists of five sections. In the second section we provide some results from [5]–[16], [23], which are needed to prove the main result in the last section. The third section has an auxiliary nature. Here we construct special entire functions of exponential type, which vanish on a prescribed sequence Λ and has a growth close to the regular one. On this base, in the fourth section we study the problem on decomposition of the invariant subspace into the sum of invariant subspaces. We prove that under some conditions, each function in the invariant subspace in an arbitrary unbounded domain of IV type can be represented as a sum of functions from three invariant subspaces. Their spectra correspond to the bounded and unbounded parts of convex domain and to the boundary between these parts.

In the last section we prove a criterion of fundamental principle and a solvability criterion for the interpolation problem.

2. Preliminaries

We first of all provide the results from [14], in which the nature of convergence of series of form (1.1) is described. Let $\Lambda = \{\lambda_k, n_k\}$ and

$$m(\Lambda) = \overline{\lim_{k\to\infty}} \frac{n_k}{|\lambda_k|}, \qquad \sigma(\Lambda) = \overline{\lim_{j\to\infty}} \frac{\ln j}{|\xi_j|},$$

where $\{\xi_j\}$ is the sequence of point λ_k arranged in the ascending order of their absolute values, and each λ_k appears exactly n_k times,

$$m(\Lambda, \mu) = \sup \overline{\lim_{j\to\infty}} \frac{n_{k_j}}{\lambda_{k_j}},$$

where the supremum is taken over all subsequences $\{\lambda_{k_j}\}$ such that $\overline{\lambda}_{k_j}/|\lambda_{k_j}| \to \mu$. If $\mu \notin \Xi(\Lambda)$, we then let $m(\Lambda, \mu) = 0$.

Let $D \subset \mathbb{C}$ be a convex domain. Let us describe the space of sequences of coefficients $\{d_{k,n}\}_{k=1,n=0}^{\infty,n_k-1}$, for which series (1.1) converges in the domain D. By $\mathcal{K}(D) = \{K_p\}_{p=1}^{\infty}$ we denote a sequence of convex compact sets in the domain D, which strictly exhausts it, that is,

$$K_p \subset \operatorname{int} K_{p+1}, \qquad p \geqslant 1, \qquad D = \bigcup_{p=1}^{\infty} K_p.$$

Here the symbol "int" stands for the interior of a set. By the mentioned inclusion and the definition of support function for each $p \geqslant 1$ there exists $\nu_p > 0$ such that

$$H_{K_p}(\varphi) + \nu_p \leqslant H_{K_{p+1}}(\varphi), \qquad \varphi \in [0, 2\pi], \qquad \nu_p > \nu_{p+1}. \tag{2.1}$$

We let

$$Q_p(\Lambda) = \{d = \{d_{k,n}\} : \|d\|_p = \sup_{k,n} |d_{k,n}| p^n \exp(r_k H_{K_p}(-\varphi_k)) < \infty\}, \qquad \lambda_k = r_k e^{i\varphi_k},$$

$$Q(D, \Lambda) = \bigcap_{p=1}^{\infty} Q_p(\Lambda).$$

In the space $Q(D, \Lambda)$ we define the metrics

$$\rho(d, b) = \sum_{s=1}^{\infty} 2^{-s} \frac{\|d - b\|_s}{1 + \|d - b\|_s}.$$

With this metrics, $Q(D, \Lambda)$ becomes a Fréchet space.

In [14, Thm. 2.1, Lm. 2.6], the following results were proved.

Theorem 2.1. *Let D be a convex domain and $\Lambda = \{\lambda_k, n_k\}$. Suppose that $\sigma(\Lambda) = 0$, $m(\Lambda) < \infty$ and $m(\Lambda, \mu) = 0$ for $\mu \in \Xi(\Lambda) \setminus \overline{J(D)}$. Then the following statements are equivalent:*

1) *Series (1.1) converges at each point in the domain D.*
2) *The inclusion $d = \{d_{k,n}\} \in Q(D, \Lambda)$ holds.*

Lemma 2.1. *Let D be a convex domain, $\Lambda = \{\lambda_k, n_k\}$ and $\sigma(\Lambda) = 0$. Then for each $p \geqslant 1$ there exist $C_p > 0$ and an index $m(p)$ such that*

$$\sum_{k=1,n=0}^{\infty,m_k-1} |d_{k,n}| \sup_{z\in K_p} |z^n e^{z\lambda_k}| \leqslant C_p \|d\|_{m(p)}, \qquad d = \{d_{k,n}\} \in Q(D,\Lambda). \tag{2.2}$$

Now we reproduce the result from [16], which establishes the duality of the fundamental principle problem and the interpolation problem in a space of entire functions of exponential type.

Let $\mathbb{E}$ be the operator, which maps each element $d = \{d_{k,n}\} \in Q(D,\Lambda)$ into the sum of series (1.1), which converges uniformly on compact sets in the domain D.

We also introduce the spaces of complex sequences

$$\mathcal{R}_s(\Lambda) = \{b = \{b_{k,n}\} : \|b\|^s = \sup_{k,n} |b_{k,n}| s^{-n} \exp(-r_k H_{K_s}(-\varphi_k)) < \infty\}, \quad s \geqslant 1.$$

Let $\mathcal{R}(D,\Lambda)$ be the inductive limit of Banach spaces $\mathcal{R}_s(\Lambda)$. Then the equality of sets holds

$$\mathcal{R}(D,\Lambda) = \bigcup_{s=1}^{\infty} \mathcal{R}_s(\Lambda).$$

We consider the multiple interpolation problem

$$f^{(n)}(\lambda_k) = b_{k,n}, \qquad n = \overline{0, n_k - 1}, \qquad k \geqslant 1. \tag{2.3}$$

Let $\widehat{\mu}$ stand for the Laplace transform of a linear continuous functional $\mu \in H^*(D) : \widehat{\mu}(\lambda) = \mu(e^{\lambda z})$. The function $\widehat{\mu}$ is an entire and has an exponential type, that is, for some $A, B > 0$ the inequality holds

$$|\widehat{\mu}(\lambda)| \leqslant A e^{B|\lambda|}, \qquad \lambda \in \mathbb{C}. \tag{2.4}$$

It is known [21, Ch. III, Sect. 12, Thm. 12.3] that the Laplace transform makes an algebraic and topological isomorphism between the spaces $H^*(D)$ and P_D, where P_D is the inductive limit of Banach spaces

$$P_s = \left\{ f \in H(\mathbb{C}) : \|f\|_s = \sup_{re^{i\varphi}\in\mathbb{C}} |f(re^{i\varphi})| \exp\left(-rH_{K_s}(-\varphi)\right) < \infty \right\}.$$

On the space P_D we define a linear operator Σ, which maps each function f into the sequence $b = \{b_{k,n}\} = \{f^{(n)}(\lambda_k)\}$.

Let D be a convex domain, $\Lambda = \{\lambda_k, n_k\}$, the system $\mathcal{E}(\Lambda)$ be incomplete in $H(D)$ and $I(\Lambda, D)$ be the kernel of operator

$$\Sigma : P_D \to \mathcal{R}(D,\Lambda).$$

This is a closed subspace in P_D. It follow from the Hahn — Banach theorem that this subspace is non–trivial if and only if the system $\mathcal{E}(\Lambda)$ is incomplete in $H(D)$. The subspace $I(\Lambda, D) \subset P_D$ consists exactly of functions, which vanish (at least) at the points λ_k with the multiplicity at least n_k.

By the symbol $\mathbb{F}(\Lambda)$ we denote the set of all entire functions of exponential type f such that f vanishes (at least) at the points λ_k with the multiplicity at least n_k. Then $I(\Lambda, D) = P_D \cap \mathbb{F}(\Lambda)$.

The quotient space $P_D/I(\Lambda, D)$, as well as P_D, is the union of increasing sequence of Banach spaces $P_{s,0}$. The element $[f] \in P_D/I(\Lambda, D)$ belongs to $P_{s,0}$ if and only if each representative $g \in P_D$ of the equivalent class $[f]$ belongs to P_s. At the same time, the norm $\|[f]\|_s$ is equal to the infimum of norms $\|g\|_s$ over all representative $g \in P_s$ in the class $[f]$. In the standard way the operator Σ generates the operator

$$\Sigma_0 : P_D/I(\Lambda, D) \to \mathcal{R}(D,\Lambda).$$

It is injective. The following statement holds [16, Thm. 4.2].

Theorem 2.2. *Let D be a convex domain, $\Lambda = \{\lambda_k, n_k\}$, and the system $\mathcal{E}(\Lambda)$ be incomplete in $H(D)$. The following statements are equivalent*

1) *the fundamental principle holds in the space $W(\Lambda, D)$;*
2) *the operator $\mathbb{E} : Q(D, \Lambda) \to W(\Lambda, D)$ is an isomorphism;*
3) *the operator $\Sigma_0 : P_D/I(\Lambda, D) \to \mathcal{R}(D, \Lambda)$ is an isomorphism;*
4) *interpolation problem (2.3) is solvable in the space P_D for each right hand side $b = \{b_{k,n}\} \in \mathcal{R}(D, \Lambda)$.*

Now we are going to formulate the results, which provide the complete solution of fundamental principle problem for bounded domains, as well as for domains of type I–III.

By the symbols $B(z, r)$ and $S(z, r)$ we denote respectively the open circle and the circumference of radius $r > 0$ centered at the point $z \in \mathbb{C}$. Let $\Lambda = \{\lambda_k, n_k\}$, and $n(r, \Lambda)$ denote the number of points λ_k counting their multiplicities n_k in the circle $B(0, r)$. We let

$$\overline{n}(\Lambda) = \varlimsup_{r\to\infty} \frac{n(r,\Lambda)}{r}, \qquad \overline{n}_0(\Lambda, \delta) = \varlimsup_{r\to\infty} \frac{n(r,\Lambda) - n((1-\delta)r, \Lambda)}{\delta r}, \qquad \overline{n}_0(\Lambda) = \varlimsup_{\delta\to 0} \overline{n}_0(\Lambda, \delta).$$

The quantities $\overline{n}(\Lambda)$ and $\overline{n}_0(\Lambda)$ are called respectively the upper and maximal density of sequence Λ. We say that Λ has the density $n1(\Lambda)$ if there exists the limit

$$n(\Lambda) = \lim_{r\to\infty} \frac{n(r,\Lambda)}{r}.$$

As it is easy to see, in this case the identity $\overline{n}(\Lambda) = \overline{n}_0(\Lambda)$ holds. We let

$$q_\Lambda^m(z, \delta) = \prod_{\lambda_k \in B(\lambda_m, \delta|\lambda_m|), k\neq m} \left(\frac{z - \lambda_k}{3\delta|\lambda_k|}\right)^{n_k}, \qquad m \geqslant 1.$$

The absolute value of function $q_\Lambda^m(z, \delta)$ can be interpreted as a measure of accumulation of points $\lambda_k \in B(\lambda_m, \delta|\lambda_m|)$ at z. We observe that

$$|q_\Lambda^m(z, \delta)| \leqslant 1, \qquad z \in B(\lambda_m, \delta|\lambda_m|), \qquad m \geqslant 1.$$

In the case, when the circle $B(\lambda_m, \delta|\lambda_m|)$ contains no points λ_k, we let $q_\Lambda^m(z, \delta) \equiv 1$. Following [5], we introduce the quantity

$$S_\Lambda = \lim_{\delta\to 0} \varliminf_{m\to\infty} \frac{\ln |q_\Lambda^m(\lambda_m, \delta)|}{|\lambda_m|}.$$

We also let

$$S_\Lambda(\mu) = \inf_{\{\lambda_{k(j)}\}} \lim_{\delta\to 0} \varliminf_{j\to\infty} \frac{\ln |q_\Lambda^{k(j)}(\lambda_{k(j)}, \delta)|}{|\lambda_{k(j)}|},$$

where the infimum is take over all subsequences $\{\lambda_{k(j)}\}$ of the sequence $\{\lambda_k\}$ such that $\overline{\lambda_{k(j)}}/|\lambda_{k(j)}| \to \mu$, $j \to \infty$. If $\mu \notin \Xi(\Lambda)$, we then let $S_\Lambda(\mu) = 0$. We observe that $S_\Lambda \leqslant 0$ and $S_\Lambda(\mu) \leqslant 0$, $\mu \in \Xi(\Lambda)$.

Let D be a convex domain. For each $\varphi \in [0, 2\pi)$ such that $H_D(\varphi) < +\infty$, the intersection $T(\varphi) = \partial D \cap L(\varphi, D)$ of the boundary of domain D with the support straight line

$$L(\varphi, D) = \{z : \operatorname{Re}(ze^{-i\varphi}) = H_D(\varphi)\}$$

is either the point $z_D(\varphi)$, or a segment, or the empty set. The set $\Phi(D)$ of direction φ, for which $T(\varphi)$ is a segment or the empty set, is at most countable.

Let the set $S(0, 1) \setminus \overline{J(D)}$ be non–empty. Then two cases are possible.

1. D is a bounded domain and $\operatorname{int}(S(0, 1) \setminus J(D)) = S(0, 1)$.
2. D is an unbounded domain of IV type and there exist ψ_1, ψ_2 such that $0 < \psi_2 - \psi_1 \leqslant \pi$ and

$$\operatorname{int}(S(0, 1) \setminus J(D)) = \{e^{i\varphi} : \varphi \in (\psi_1, \psi_2)\}.$$

In the latter case we write $D \in \mathcal{D}(\psi_1, \psi_2)$.

Let $0 < \varphi_2 - \varphi_1 < \pi$, $\varphi_1, \varphi_2 \notin \Phi(D)$. If $D \in \mathcal{D}(\psi_1, \psi_2)$, we additionally suppose that $\psi_1 < \varphi_1 < \varphi_2 < \psi_2$. By the symbol $\Upsilon_D(\varphi_1, \varphi_2)$ we denote the length of arc on the boundary of domain D, which connects the points $z_D(\varphi_1)$ and $z_D(\varphi_2)$, while the passage from $z_D(\varphi_1)$ to $z_D(\varphi_2)$ along this arc is made in the positive direction.

By the symbol $\Lambda(\varphi_1, \varphi_2)$ we denote the sequence consisting of all pairs λ_k, n_k such that λ_k is located in the angle

$$\Gamma(\varphi_1, \varphi_2) = \{te^{i\varphi} : \varphi \in (\varphi_1, \varphi_2), t > 0\}.$$

In [10, Thm. 3.2], the following result was proved.

Theorem 2.3. *Let $\Lambda = \{\lambda_k, n_k\}$, D be a bounded convex domain and $\mathcal{E}(\Lambda)$ be incomplete in $H(D)$. Then the following statements are equivalent:*

1) *the fundamental principle holds in the space $W(\Lambda, D)$;*
2) *$S_\Lambda = 0$ and there exists $\gamma > 0$ such that for all $\varphi_1, \varphi_2 \notin \Phi(D)$ with the condition $0 < \varphi_2 - \varphi_1 < \gamma$ the inequality holds*

$$\overline{n}_0(\Lambda(-\varphi_2, -\varphi_1)) \leqslant \frac{\Upsilon_D(\varphi_1, \varphi_2)}{2\pi}.$$

Remark 2.1. *Suppose that Assertion 2) of Theorem 2.3 holds for some $\gamma > 0$. Then the definition of maximal density and the additivity of arc length imply that it holds for each $\gamma \in (0, \pi]$.*

Let $\varphi \in \mathbb{R}$, $a \leqslant +\infty$, and

$$\Pi(a, \varphi) = \{z \in \mathbb{C} : \operatorname{Re}(ze^{-i\varphi}) < a\}.$$

The set $\Pi(a, \varphi)$ is the half–plane when $a \in \mathbb{R}$. If $a = +\infty$, then $\Pi(a, \varphi) = \mathbb{C}$.

Let D be an unbounded convex domain and $D \neq \Pi(a, \varphi)$, $\varphi \in \mathbb{R}$, $a \leqslant +\infty$. Then $\partial J(D) = \{e^{i\varphi_1}, e^{i\varphi_2}\}$ is a two–point set.

The next result was obtained in [23, Thm. 3.2].

Theorem 2.4. *Let $\Lambda = \{\lambda_k, n_k\}$, D be a convex set, and the system $\mathcal{E}(\Lambda)$ be incomplete in $H(D)$. The following statements are equivalent.*

1) *Each function $g \in W(\Lambda, D)$ is represented by series (1.1), which converges uniformly on compact sets in the domain $D_0 = \Pi(H_D(\varphi_1), \varphi_1) \cap \Pi(H_D(\varphi_2), \varphi_2)$.*
2) *$\Xi(\Lambda) \subseteq \overline{J(D)}$, $\partial J(D) \subseteq \{e^{i\varphi_1}, e^{i\varphi_2}\}$, $S_\Lambda > -\infty$ and $S_\Lambda(\mu) = 0$, $\mu \in \partial J(D) \setminus J(D)$.*

Remark 2.2. *Each unbounded convex domain of type I–III is represented as $\Pi(a_1, \varphi_1) \cap \Pi(a_2, \varphi_2)$. Thus, in Theorem 2.4, the fundamental principle problem is completely solved for domains of type I–III.*

We shall also need some necessary and sufficient conditions of fundamental principle obtained in the above cited works. The following two results were proved in [11, Lm. 3.4, Thm. 4.1]. We call the sequence $\Lambda = \{\lambda_k, n_k\}$ almost real if $\operatorname{Re}\lambda_k > 0$ and $\operatorname{Im}\lambda_k / \operatorname{Re}\lambda_k \to 0$ as $k \to \infty$.

Lemma 2.2. *Let D be a convex domain such that $1 \notin J(D)$, $\Lambda = \{\lambda_k, n_k\}$ be an almost real sequence, and $\mathcal{E}(\Lambda)$ be incomplete in $H(D)$. Suppose that the fundamental principle holds in $W(\Lambda, D)$. Then $S_\Lambda = 0$.*

Theorem 2.5. *Let D be a convex domain such that $1 \in S(0,1) \setminus \overline{J(D)}$, $\Lambda = \{\lambda_k, n_k\}$ be an almost real sequence and $\mathcal{E}(\Lambda)$ be incomplete in $H(D)$. Suppose that the fundamental principle holds in $W(\Lambda, D)$. Then $m(\Lambda) = 0$.*

In [15, Thm. 3.1] a necessary condition for the fundamental principle was provided. Let us formulate it.

Theorem 2.6. *Let D be a convex domain, $\Lambda = \{\lambda_k, n_k\}$, and the system $\mathcal{E}(\Lambda)$ be incomplete in $H(D)$. Suppose that the fundamental principle holds in $W(\Lambda, D)$. Then for all $\varphi_1, \varphi_2 \notin \Phi(D)$ such that $0 < \varphi_2 - \varphi_1 < \pi$ and $\{e^{i\varphi} : \varphi \in [\varphi_1, \varphi_2]\} \subset S(0,1) \setminus \overline{J(D)}$ the estimate holds*

$$\overline{n}_0(\Lambda(-\varphi_2, -\varphi_1)) \leqslant \frac{\Upsilon_D(\varphi_1, \varphi_2)}{2\pi}.$$

We formulate one more result [12, Thm. 4.3], which can be easily transformed into the necessary condition for the fundamental principle.

Theorem 2.7. *Let $\Lambda = \{\lambda_k, n_k\}$, $S_\Lambda = -\infty$. Then there exist numbers $\{d_{k,n}\}$ and indices k_s, $1 = k_1 < k_2 < \ldots$, such that the series*

$$\sum_{s=1}^{\infty} \left(\sum_{k=k_s}^{k=k_{s+1}-1} \sum_{n=0}^{n_k-1} d_{k,n} z^n e^{\lambda_k z} \right) \tag{2.5}$$

converges uniformly on compact sets in the plane, while series (1.1) diverges at each point in the plane.

We shall also need one sufficient condition for the fundamental principle. It is a direct corollary of Theorem 3.6 and Proposition 2.10 in [12].

Theorem 2.8. *Let $\Lambda = \{\lambda_k, n_k\}$, D be a convex domain, $\mathcal{K}(D) = \{K_p\}$, and the system $\mathcal{E}(\Lambda)$ be incomplete in $H(D)$. Suppose that $\Xi(\Lambda) \subset S(0,1) \setminus \overline{J(D)}$, $S_\Lambda = 0$ and for all $p \geqslant 1$, $\delta \in (0,1)$ there exist $f \in I(\Lambda, D)$ and $R > 0$ such that*

$$\lambda_k \in \bigcup_{\ln|f(z)| \geqslant r H_{K_p}(-\varphi)} B(z, \delta r), \qquad z = re^{i\varphi}, \qquad |\lambda_k| > R.$$

Then $m(\Lambda) = 0$ and each function $g \in W(\Lambda, D)$ is represented by series (1.1), which converges at each point in the domain D.

3. Construction of special entire function

Let f be an entire function of exponential type. The indicator of f is the function

$$h_f(\varphi) = \varlimsup_{t\to\infty} \frac{\ln|f(te^{i\varphi})|}{t}, \qquad \varphi \in [0, 2\pi].$$

It is convex and positively homogeneous of order one since it coincides with the support function of some convex set $L \subset \mathbb{C}$, which is called the indicator diagram of f. By the symbol $\gamma(t, f)$ we denote the function associated with f in the Borel sense [20, Ch. I, Sect. 5]. The adjoint diagram K of function f is the convex hull of the set of special points $\gamma(t, f)$. Thus, $\gamma(t, f)$ is analytic outside the compact set K. By the Pólya theorem [20, Ch. I, Sect. 5, Thm. 5.4],

$$h_f(\varphi) = H_L(\varphi) = H_K(-\varphi), \qquad \varphi \in [0, 2\pi]. \tag{3.1}$$

Therefore, K is the compact set, which is the complex conjugation of the compact set L.

Let $\Lambda = \{\lambda_k, n_k\}$, D be a convex domain, and the system $\mathcal{E}(\Lambda)$ be incomplete in $H(D)$. Then the sequence Λ is a part of zero set of each entire function of exponential type in the space $I(\Lambda, D) \neq \emptyset$. This is why [18, Ch. I, Sect. 1, Thm. 1.1.5] the upper density $\overline{n}(\Lambda)$ is finite.

The linear density of the set $E = \cup B(z_i, \rho_i)$ is the quantity

$$p_E = \varlimsup_{r\to\infty} \frac{1}{r} \sum_{|z_i|<r} \rho_i.$$

Lemma 3.1. *Let f be an entire function of exponential type. For each $\delta \in (0,1)$ there exists a set $E = \cup B(z_i, \rho_i)$ and a number $C > 0$ such that $p_E \leqslant \delta$ and*

$$\ln |f(\xi)| \geqslant -C|\xi|, \qquad \xi \in \mathbb{C} \setminus E. \tag{3.2}$$

Proof. We represent the function f as

$$f(z) = az^m f_1(z),$$

where $f_1(0) = 1$. Since f_1 is of an exponential type, there exist $B > 0$, $R > 1$ such that

$$\ln |f_1(z)| \leqslant B|z|, \qquad z \in \mathbb{C} \setminus B(0, R).$$

Hence, for each $\mu \in (0,1)$ the theorem on lower bound for the absolute value of an analytic function [17, Ch. I, Thm. 11] implies the inequality

$$\ln |f_1(\lambda)| \geqslant -2eH(\mu)BR_l, \qquad H(\mu) = 3 + \ln \frac{3}{2\mu}, \qquad R_l = 2^l R, l \geqslant 1.$$

It holds in the circle $B(0, R_l)$, but outside the union V_l of finitely many exceptional circles with the sum of radii less than $4\mu R_l$. We have

$$\ln |f_1(\lambda)| \geqslant -4eH(\mu)B|\lambda|, \qquad \lambda \in B(0, R_l) \setminus (B(0, R_{l-1}) \cup V_{l,0}), \qquad l \geqslant 1,$$

where $V_{l,0}$ is the union of all circles in the set V_l, which intersect the annulus $B(0, R_l) \setminus B(0, R_{l-1})$. We let

$$C = 4eH(\mu)B + 1, \qquad E = \bigcup_i B(z_i, \rho_i) = B(0, R_0) \cup \left(\bigcup_l V_{l,0} \right), \qquad R_{0,0} = R.$$

Exceeding $R_{0,0}$ if it is needed, by the latter inequality we obtain (3.2). Let us estimate from above the linear density of set E.

Let $\mu \in (0, 1/16)$ and $B(z_i, \rho_i) \in V_{s,0}$, $s \geqslant l+1$. Then by construction $\rho_i \leqslant 4\mu R_s$. Since $B(z_i, \rho_i)$ intersects the annulus $B(0, R_s) \setminus B(0, R_{s-1})$, we have $|z_i| \geqslant R_{s-1} - 4\mu R_s$. Therefore,

$$|z_i| \geqslant R_{s-1} - 8\mu R_{s-1} > \frac{R_{s-1}}{2} \geqslant R_l.$$

Thus, if $|z_i| < R_l$, then

$$B(z_i, \rho_i) \subset \bigcup_{s=1}^{l+1} V_{s,0}.$$

Let $\delta \in (0,1)$, $\mu = \delta/32$ and $R_{l-1} < r < R_l$. Then

$$\frac{1}{r} \sum_{|z_i| < r} \rho_i \leqslant \frac{1}{R_{l-1}} \sum_{s=1}^{l+1} 4\mu R_s = \sum_{s=1}^{l+1} 4\mu \frac{R_s}{R_{l-1}} = \frac{\delta}{2} \sum_{s=1}^{l+1} 2^{s-l-1} < \delta.$$

The proof is complete. □

The proof of the next statement is based on the proof of Lemma 4.3 in [7].

Lemma 3.2. *Let $0 < \psi_2 - \psi_1 \leqslant \pi$, $D \in \mathcal{D}(\psi_1, \psi_2)$, $\mathcal{K}(D) = \{K_p\}$, $\Lambda = \{\lambda_k, n_k\}$ and the system $\mathcal{E}(\Lambda)$ be incomplete $H(D)$. Suppose that for all $\varphi_1, \varphi_2 \notin \Phi(D)$ such that $\psi_1 < \varphi_1 < \varphi_2 < \psi_2$ the inequality holds*

$$\overline{n}_0(\Lambda(-\varphi_2, -\varphi_1)) \leqslant \frac{\Upsilon_D(\varphi_1, \varphi_2)}{2\pi}. \tag{3.3}$$

Then for all $p \geqslant 1$, $\delta \in (0,1)$ there exists $\varepsilon_0 > 0$ such that α_1, α_2, $\psi_1 < \alpha_1 < \alpha_2 < \psi_2$, there exist a set $E = \cup B(z_i, \rho_i)$ with a linear density $p_E \leqslant \delta$ and a function $f_0 \in I(\Lambda(-\alpha_2, -\alpha_1), D)$, which obey the conditions

$$h_{f_0}(\varphi) \leqslant H_D(-\varphi) - \varepsilon_0, \qquad \varphi \in [0, 2\pi], \tag{3.4}$$

$$\ln|f_0(re^{i\varphi})| \geqslant rH_{K_p}(-\varphi), \qquad re^{i\varphi} \in \Gamma(-\alpha_2, -\alpha_1) \setminus E. \tag{3.5}$$

Proof. We consider the function

$$f_{0,0}(\lambda) = \prod_{\lambda_k \in \Gamma(-\alpha_2,-\alpha_1)} \left(1 - \frac{\lambda}{\lambda_k}\right)^{n_k} e^{n_k \frac{\lambda}{\lambda_k}}.$$

We seek the required function f_0 as the product $f_0 = f_{0,0}\varphi$. In order to do this, we first need to construct a special subharmonic function, and then to approximate by the logarithm of the absolute value of an analytic function, which will play the role of φ.

We define the needed subharmonic function by the identity

$$\psi_\beta(\lambda) = \beta \ln|\omega_1(\lambda)| + (1-\beta)\ln|\omega_2(\lambda)|, \qquad \lambda \in \mathbb{C}, \qquad \beta \in (0,1),$$

where $\omega_j = f_j/f_{0,0}$, $j = 1, 2$, $f_1 \in I(\Lambda, D)$, $f_2 \in \mathbb{F}(\Lambda(-\alpha_2, -\alpha_1))$. We proceed to finding f_j.

Since $\mathcal{E}(\Lambda)$ is incomplete in $H(D)$, we have $I(\Lambda, D) \neq \emptyset$. Let $f_1 \in I(\Lambda, D)$. The sequence Λ is a part of zero set of entire function f_1 of exponential type.

This is why [18, Ch. I, Thms. 2.2, 3.1, 3.3] f_Λ is an entire function of at most first order (and possibly, of infinite type at order one), which vanishes only at the points λ_k with the multiplicities n_k, $k \geqslant 1$. Then by the corollary of Theorem 12 in [17, Ch. I], the entire function $\omega_1 = f_1/f_{0,0}$ has the growth of first order.

Now we define f_2. Let $\alpha_{1,0}, \alpha_{2,0} \notin \Phi(D)$, $\psi_1 < \alpha_{1,0} < \alpha_1 < \alpha_2 < \alpha_{2,0} < \psi_2$, and K be a convex hull of arc $\gamma(\alpha_{1,0}, \alpha_{2,0})$ on the boundary ∂D, which connects the points $z_D(\alpha_{1,0})$ and $z_D(\alpha_{2,0})$; the passage along $\gamma(\alpha_{1,0}, \alpha_{2,0})$ from $z_D(\alpha_{1,0})$ to $z_D(\alpha_{2,0})$ is made in the positive direction. The boundary ∂K consists of the arc $\gamma(\alpha_{1,0}, \alpha_{2,0})$ and the interval $(z_D(\alpha_{2,0}), z_D(\alpha_{1,0}))$. It is easy to see that the relations hold

$$H_K(\varphi) \leqslant H_D(\varphi), \quad \varphi \in [0, 2\pi], \tag{3.6}$$

$$H_K(\varphi) = \begin{cases} H_D(\varphi), & \varphi \in [\alpha_{1,0}, \alpha_{2,0}] \\ \max\limits_{j=1,2} \operatorname{Re}(z_D(\alpha_{j,0})e^{-i\varphi}), & \varphi \in [\alpha_{2,0}, \alpha_{1,0} + 2\pi]. \end{cases} \tag{3.7}$$

Since the sequence $\Lambda_0 = \Lambda(-\alpha_{2,0}, -\alpha_{1,0})$ consists of the pairs λ_k, n_k such that $\lambda_k \in \Gamma(-\alpha_{2,0}, -\alpha_{1,0})$, and the boundary of the domain $D_0 = \operatorname{int} K$ contains the arc $\gamma(\alpha_{1,0}, \alpha_{2,0})$, inequality (3.3) implies

$$\overline{n}_0(\Lambda_0(-\varphi_2, -\varphi_1)) \leqslant \frac{\Upsilon_{D_0}(\varphi_1, \varphi_2)}{2\pi}, \qquad \varphi_1, \varphi_2 \notin \Phi(D_0), \qquad 0 < \varphi_2 - \varphi_1 < \pi.$$

Then by Theorem 2.1 in [10] (here we take into consideration that the definition of the support function in [10] differs from the definition of support function in this work) there exists the completion $\Lambda_{0,0}$ of sequence Λ_0, which a properly distributed set [17, Ch. II, Sect. 1], and at the same time the identity holds

$$n(\Lambda_{0,0}(-\varphi_2, -\varphi_1)) = \frac{\Upsilon_{D_0}(\varphi_1, \varphi_2)}{2\pi}, \qquad \varphi_1, \varphi_2 \notin \Phi(D_0), \qquad 0 < \varphi_2 - \varphi_1 < \pi, \tag{3.8}$$

where $n(\Lambda_{0,0}(\varphi, \psi))$ is the angular density of sequence $\Lambda_{0,0}$:

$$n(\Lambda_{0,0}(\varphi, \psi)) = \lim_{r \to \infty} \frac{n(r, \Lambda_{0,0}(\varphi, \psi))}{r}.$$

By (3.8) the canonical function $f_{2,0}$ of sequence $\Lambda_{0,0}$ has an exponential type and a completely regular growth and for some $a \in \mathbb{C}$ the adjoint diagram of function $f_2(z) = f_{2,0}(z)e^{az}$ coincides with $K = \overline{D_0}$ [17, Ch. II, Sect. 1, Eq. (2.07)]. As above, the function $\omega_2 = f_2/f_{0,0}$ has the growth order one. By Theorem 5 in [22], there exists an entire function φ_β, $C(\beta) > 0$ and $E_\beta \subset \mathbb{C}$ such that

$$|\ln|\varphi_\beta(\lambda)| - \psi_\beta(\lambda)| \leqslant C(\beta)\ln|\lambda|, \qquad \lambda \in \mathbb{C} \setminus E_\beta. \tag{3.9}$$

And E_β can be covered by circles $B(z_l(\beta),\rho_l(\beta))$, $l\geqslant 1$, where $|z_l(\beta)|\to\infty$, $l\to\infty$ and

$$\sum_{l=1}^{\infty}\rho_l(\beta)=r(\beta)<\infty.$$

We let $f_\beta=\varphi_\beta f_{0,0}$. We are going to show that as $f_0\in I(\Lambda(-\alpha_2,-\alpha_1),D)$ we can take f_β with an appropriate parameter β.

We first observe that by construction $f_\beta\in\mathbb{F}(\Lambda(-\alpha_2,-\alpha_1))$, $\beta\in(0,1)$. Let $\beta\in(0,1)$ and

$$v_\beta(\lambda)=\beta\ln|f_1(\lambda)|+(1-\beta)\ln|f_2(\lambda)|.$$

Since the indicator is uniformly continuous on the segment $[0,2\pi]$ [17, Ch. I, Thm. 28] for $\tau>0$ there exist α, $R>0$ such that

$$\frac{v_\beta(te^{i\varphi})}{t}\leqslant\beta h_{f_1}(\theta)+(1-\beta)h_{f_2}(\theta)+\tau,\qquad\varphi\in[\theta-\alpha,\theta+\alpha],\quad t\geqslant R,\quad\theta\in[0,2\pi].\tag{3.10}$$

By (3.9)

$$|\ln|f_\beta(\lambda)|-v_\beta(\lambda)|=|\ln|\varphi_\beta(\lambda)|-\psi_\beta(\lambda)|\leqslant C(\beta)\ln|\lambda|,\qquad\lambda\in\mathbb{C}\setminus E_\beta.\tag{3.11}$$

Since $r(\beta)<\infty$, using (3.10), (3.11) and applying the maximum principle for subharmonic function, it is easy to obtain the estimate

$$\frac{\ln|f_\beta(te^{i\varphi})|}{t}\leqslant\beta h_{f_1}(\varphi)+(1-\beta)h_{f_2}(\varphi)+2\tau,\qquad\varphi\in[0,2\pi],\quad t\geqslant R_1,$$

for some number $R_1\geqslant R$. By the arbitrariness of $\tau>0$ we hence get

$$h_{f_\beta}(\varphi)\leqslant\beta h_{f_1}(\varphi)+(1-\beta)h_{f_2}(\varphi),\qquad\varphi\in[0,2\pi].\tag{3.12}$$

Since $f_1\in P_D$, there exists $a>0$ such that

$$h_{f_1}(\varphi)\leqslant H_D(-\varphi)-a,\qquad\varphi\in[0,2\pi].$$

By construction, K is the adjoint diagram of function f_2. This is why, in accordance with (3.1),

$$h_{f_2}(\varphi)=H_K(-\varphi),\qquad\varphi\in[0,2\pi].\tag{3.13}$$

In view of the previous inequality, (3.12) and (3.6) we hence find

$$h_{f_\beta}(\varphi)\leqslant\beta(H_D(-\varphi)-a)+(1-\beta)H_D(-\varphi)=H_D(-\varphi)-\beta a,\qquad\varphi\in[0,2\pi].\tag{3.14}$$

This means that $f_\beta\in P_D$. Thus, $f_\beta\in I(\Lambda(-\alpha_2,-\alpha_1),D)$, $\beta\in(0,1)$.

It remains to choose a number $\beta\in(0,1)$, for which relations (3.4) are (3.5) are satisfied. We first observe that by (3.14) the function $f_0=f_\beta$ and $\varepsilon_0=\beta a$ satisfies inequality (3.4) for all $\beta\in(0,1)$ and α_1,α_2, $\psi_1<\alpha_1<\alpha_2<\psi_2$. In order to obtain (3.5), we need to estimate from below the functions $|f_1|$ and $|f_2|$.

We fix $p\geqslant 1$ and $\delta\in(0,1)$. By Lemma 3.1, there exists a set E_1 and a number $C>0$ such that $p_{E_1}\leqslant\delta$ and

$$\ln|f_1(\xi)|\geqslant -C|\xi|,\qquad\xi\in\mathbb{C}\setminus E_1.\tag{3.15}$$

Now we estimate $|f_2|$. Since f_2 has a completely regular growth, in view of (3.13), the identity holds [17, Ch. II, Thm. 2, Ch. III, Thm. 4]

$$\ln|f_2(re^{i\varphi})|=rH_K(-\varphi)+\alpha(r),\qquad re^{i\varphi}\in\mathbb{C}\setminus E_0,\quad\frac{\alpha(r)}{r}\to 0,\quad r\to\infty,\tag{3.16}$$

where $E_0=\cup B(w_l,r_l)$ is a set of zero linear density.

Let $\nu_p>0$ be the number in inequality (2.1). We choose $\beta\in(0,1)$ such that

$$C\beta\leqslant\frac{\nu_p}{4},\qquad\beta\max_{\varphi\in[0,2\pi]}H_{K_{p+1}}(\varphi)\leqslant\frac{\nu_p}{4}.\tag{3.17}$$

We note that the choice of number β depends only on the index p and δ.

According to (3.16) and (3.17), there exists $R_0 > 0$ such that

$$\ln|f_2(re^{i\varphi})| \geqslant rH_{K_{p+1}}(-\varphi) - \frac{\nu_p r}{4}, \qquad re^{i\varphi} \in \Gamma(-\alpha_2, -\alpha_1) \setminus (E_0 \cup B(0, R_0)). \tag{3.18}$$

At the same time we can suppose that

$$C(\beta)\ln r \leqslant \frac{\nu_p r}{4}, \qquad r \geqslant R_0. \tag{3.19}$$

Let us show that for the chosen β the function $f_0 = f_\beta$ possesses all needed properties. As it has been mentioned above, f_0 satisfies (3.4) with $\varepsilon_0 = \beta a$ (ε_0 depends only on p and δ). We let $E = E_\beta \cup E_1 \cup E_0 \cup B(0, R_0)$. By (3.11), (3.15), (3.18), (3.19) and (2.1) we have

$$\begin{aligned}\ln|f_0(z)| &\geqslant \beta \ln|f_1(z)| + (1-\beta)\ln|f_2(z)| - C(\beta)\ln r \\ &\geqslant (1-\beta)\left(rH_{K_{p+1}}(-\varphi) - \frac{\nu_p r}{4}\right) - C\beta r - \frac{\nu_p r}{4} \geqslant -\nu_p r + rH_{K_{p+1}}(-\varphi) \geqslant rH_{K_p}(-\varphi),\end{aligned}$$

where $z = re^{i\varphi} \in \Gamma(-\alpha_2, -\alpha_1) \setminus E$. Since $p_{E_0} = 0$ and $p_{E_\beta} = 0$, we obtain $p_E \leqslant \delta$. The proof is complete. □

Let $\tau \in (0, 1)$ and

$$\Gamma(\tau) = \{t\lambda : \lambda \in B(1, \tau), t \in \mathbb{R}\}, \qquad \Gamma_\varphi(\tau) = \{t\lambda : \lambda \in B(e^{i\varphi}, \tau), t > 0\}.$$

The next statement was proved in [11, Thm. 2.2]. We formulate it in a simpler form and for sequence of more particular form than in [11].

Lemma 3.3. *Let $\Lambda^0 = \{\lambda_k, n_k\}$ be a sequence such that $\lambda_k > 0$, $k \geqslant 1$, and $\overline{n}(\Lambda) < +\infty$. Then for all $\varepsilon > 0$, $\tau \in (0, 1)$ there exist $\gamma \in (0, 1)$, $f \in \mathbb{F}(\Lambda^0)$ and $R > 0$ such that*

$$\left|\ln|f(\lambda)| - \frac{\pi|\Im\lambda|}{\gamma})\right| \leqslant \varepsilon|\lambda|, \qquad \lambda \in (\mathbb{C} \setminus (\Gamma(\tau) \cup B(0, R))),$$

$$h_f(\varphi) \leqslant \frac{\pi|\sin\varphi|}{\gamma} + \varepsilon, \qquad \varphi \in [0, 2\pi].$$

Remark 3.1. *The function $f \in \mathbb{F}(\Lambda^0)$ in Theorem 2.2 in [11] is constructed as follows. First one constructe the sequence $\Lambda^{0,0} = \{\lambda_{j,0}, n_{j,0}\}$ of positive numbers, which completes the sequence Λ^0. Then the function f is defined as the product*

$$f = \prod_{j=1}^{\infty}\left(1 - \frac{\lambda^2}{|\lambda_{j,0}|^2}\right)^{n_{j,0}}. \tag{3.20}$$

Thus, the zero set of function f does not contain the zero, is real and symmetric with respect to the origin.

Let $d > 0$. Following [2] we say that the sequence $\{\varsigma_l\}$ is asymptotically d–close to $\{\xi_l\}$ if

$$\overline{\lim_{l\to\infty}} \frac{|\varsigma_l - \xi_l|}{|\xi_l|} \leqslant d.$$

We also sat that the set of circles $\cup B_i$ is centered with the sequence $\{\xi_l\}$ if each point ξ_l to at least one of circles B_i and each circle B_i contains at least one point ξ_l.

Lemma 3.4. *Let $0 < \psi_2 - \psi_1 \leqslant \pi$, $D \in \mathcal{D}(\psi_1, \psi_2)$, $\mathcal{K}(D) = \{K_p\}$, $\Lambda = \{\lambda_k, n_k\}$, the system $\mathcal{E}(\Lambda)$ be incomplete in $H(D)$, and $\Xi(\Lambda) \subseteq S(0, 1) \setminus \mathrm{int}J(D)$. Suppose that for all $\varphi_1, \varphi_2 \notin \Phi(D)$ such that $\psi_1 < \varphi_1 < \varphi_2 < \psi_2$ inequality (3.3) holds. Then for all $s \geqslant 1$, $\delta \in (0, 1)$ and θ_1, θ_2, $\psi_1 < \theta_1 < \theta_2 < \psi_2$, there exists a set E_0 with a linear density $p_{E_0} \leqslant \delta$ and a function $f \in I(\Lambda, D)$ such that*

$$\ln|f(re^{i\varphi})| \geqslant rH_{K_s}(-\varphi), \quad re^{i\varphi} \in \Gamma(-\theta_2, -\theta_1) \setminus E_0. \tag{3.21}$$

Proof. We fix $s \geqslant 1$, $\delta \in (0,1)$ and θ_1, θ_2, $\psi_1 < \theta_1 < \theta_2 < \psi_2$. We choose $\tau \in (0, \frac{1}{2})$ such that

$$\Gamma_{\psi_1}(\tau) \cap \Gamma(\theta_1, \theta_2) = \emptyset, \qquad \Gamma_{\psi_2}(\tau) \cap \Gamma(\theta_1, \theta_2) = \emptyset. \tag{3.22}$$

By Lemma 3.2 for $p = s+1$ and δ we find $\varepsilon_0 > 0$. Let

$$0 < \varepsilon < \min\left\{\frac{\varepsilon_0}{17}, \frac{\nu_s}{5}\right\},$$

where $\nu_s > 0$ is the number from inequality (2.1)

Let $\Lambda^j = \{\lambda_{l,j}, n_{l,j}\}$, $j = 1,2$, be the sequence, which consists of the points $|\lambda_k|e^{-i\psi_j}$, $\lambda_k \neq 0$. At the same time, $n_{l,j}$ coincides with the sum n_m of all points λ_m such that $|\lambda_m| = |\lambda_{l,j}|$. By Lemma 3.3 for ε and τ there exist $\gamma_j \in (0,1)$, $f_j \in \mathbb{F}(\Lambda^j)$ and $R_j > 0$ such that

$$\left|\ln|f_j(e^{-i\psi_j})\lambda| - \frac{\pi|\operatorname{Im}\lambda|}{\gamma_j}\right| \leqslant \varepsilon|\lambda|, \quad \lambda \in (\mathbb{C} \setminus (\Gamma(\tau) \cup B(0, R_j))), \tag{3.23}$$

$$h_{f_j}(\varphi) \leqslant \frac{\pi|\sin(\varphi + \psi_j)|}{\gamma_j} + \varepsilon, \quad \varphi \in [0, 2\pi], \quad j = 1,2. \tag{3.24}$$

At the same time, according to Remark 3.1 to Lemma 3.3 the zero set of function f_j is symmetric with respect to the origin and it is located on the straight line $\{te^{-i\psi_j}, t \in \mathbb{R}\}$.

Let $\Lambda_0^j = \{\lambda_{l,j,0}, n_{l,j,0}\}$ be the multiple zero set of function f_j. Since $f_j \in \mathbb{F}(\Lambda^j)$, for each $\lambda_k \neq 0$ the point $|\lambda_k|e^{-i\psi_j}$ is a term of Λ_0^j with the multiplicity not less than the sum of multiplicities n_m of all points λ_m such that $|\lambda_m| = |\lambda_k|$. By symmetric property of sequence Λ_0^j, the point $-|\lambda_k|e^{-i\psi_j}$ is also the its term with the same multiplicity as $|\lambda_k|e^{-i\psi_j}$.

We construct the function $f_{j,0}$ by means of function f_j using only the shifts of some of its zeroes. Let $d \in (0,1)$ and $\psi > 0$ be such that

$$|1 - e^{i\psi}| = d. \tag{3.25}$$

We first define the sequence $\Lambda_{0,+}^j$, the points of which are located in the angle

$$\Gamma_j = \Gamma(-\psi_j - \psi, -\psi_j + \psi).$$

Let $\lambda_{l,j,0}$ be located on the ray

$$\{te^{-i\psi_j}, t > 0\}.$$

If $|\lambda_{l,j,0}| \neq |\lambda_k|$ for each $\lambda_k \in \Gamma_j$, then the pair $\lambda_{l,j,0}, n_{l,j,0}$ is treated as an element of the sequence $\Lambda_{0,+}^j$. Otherwise, the pair λ_k, n_k is treated as an element of the sequence $\Lambda_{0,+}^j$ for all $\lambda_k \in \Gamma_j$ such that $|\lambda_{l,j,0}| = |\lambda_k|$. At the same time, if the sum n_0 of multiplicities n_k of all these points $\lambda_k \in \Gamma_j$ is strictly less than $n_{l,j,0}$, then we also treat the pair $\lambda_{l,j,0}$, $n_{l,j,0} - n_0$ as an element of the sequence $\Lambda_{0,+}^j$.

Thus, we have constructed the sequence $\Lambda_{0,+}^j$. Using this sequence, as in (3.20), we construct the entire function $f_{j,0}$. Its zero set is the union $\Lambda_{0,+}^j \cup \Lambda_{0,-}^j$ ($\Lambda_{0,-}^j$ is symmetric to $\Lambda_{0,+}^j$ with respect to the origin) and is located in the union of angles $\Gamma_j \cup (-\Gamma_j) = \Gamma_j \cup e^{i\pi}\Gamma_j$. In particular, $f_{j,0} \in \mathbb{F}(\Lambda(-\psi_j - \psi, -\psi_j + \psi))$.

By construction and (3.25), it is easy to establish a one–to–one correspondence between the points of sequences Λ_0^j and $\Lambda_{0,+}^j \cup \Lambda_{0,-}^j$ (counting the multiplicities) so that these sequences become d–asymptotically close one to the other.

The sequence Λ_0^j is the zero set of the entire function f_j of exponential type, and it does not contain the zero, see the remark to Lemma 3.3. This is why Λ_0^j has a finite upper density, and the following quantity is also finite

$$C = \sup_r \frac{n(r, \Lambda_0^j)}{r}.$$

Then by Theorem B in [2] (in view of symmetricity of the zeroes of function f_j and $f_{j,0}$ with respect to the origin) for each $d \in (0, \frac{1}{2})$ there exists $C_1 > 0$ (which depends only on C) and a set $E_j(d) = \cup B(y_{m,j}, q_{m,j})$ such that $E_j(d)$ is centered with $\Lambda_0^j \cup \Lambda_{0,+}^j \cup \Lambda_{0,-}^j$, has the linear density $p_{E_j(d)} \leqslant \sqrt[4]{d}$ and

$$|\ln |f_j(\lambda)| - \ln |f_{j,0}(\lambda)|| \leqslant C_1 \sqrt{d} |\lambda|, \qquad \lambda \in \mathbb{C} \setminus E_j(d), \qquad j = 1, 2. \tag{3.26}$$

By (3.24) and the properties of indicator [17, Ch. I, Sect. 18, Thm. 28], there exist $\mu \in (0, \frac{\tau}{2})$ and $r_0 > 0$ such that the inequality holds

$$\ln |f_j(t\lambda)| \leqslant 2\varepsilon |t|, \qquad \lambda \in B(e^{-i\psi_j}, 2\mu), \qquad |t| \geqslant r_0, \qquad j = 1, 2. \tag{3.27}$$

We choose $d \in (0, \frac{1}{2})$ so that

$$C_1 \sqrt{d} \leqslant \varepsilon, \qquad \sqrt[4]{d} \leqslant \frac{\mu}{6}. \tag{3.28}$$

Since $p_{E_j(d)} \leqslant \sqrt[4]{d}$, for some number $r_1 > r_0$ the inequality holds

$$\sum_{|y_{m,j}| < r} q_{m,j} \leqslant \frac{\mu r}{5}, \qquad r \geqslant r_1, \qquad j = 1, 2. \tag{3.29}$$

Let $r \geqslant r_1$ and $B(y_{m,j}, q_{m,j})$ intersect $B(0, r)$. If $|y_{m,j}| \geqslant r$, then according to (3.29) $q_{m,j} \leqslant \frac{\mu |y_{m,j}|}{5}$. This is why $(1 - \frac{\mu}{5})|y_{m,j}| < r$. We hence have $|y_{m,j}| < \frac{5r}{4}$. Thus, by (3.29), the sum of radii of all circles $B(y_{m,j}, q_{m,j})$, which intersect $B(0, r)$, does not exceed $\frac{\mu r}{4}$.

Let $\lambda \in E_j(d) \setminus B(0, r_1)$ and the circle $B(y_{m,j}, q_{m,j})$ contain λ. Then $B(y_{m,j}, q_{m,j})$ intersect $B(0, |\lambda|)$. This is why $q_{m,j} \leqslant \frac{\mu|\lambda|}{4}$. Since $E_j(d)$ is centered with the set $\Lambda_0^j \cup \Lambda_{0,+}^j \cup \Lambda_{0,-}^j$, this set contains a point λ_0 such that

$$|\lambda - \lambda_0| \leqslant |\lambda - y_{m,j}| + |y_{m,j} - \lambda_0| \leqslant \frac{\mu|\lambda|}{4} + \frac{\mu|\lambda|}{4} = \frac{\mu|\lambda|}{2}.$$

Therefore,

$$|\lambda| \leqslant \frac{|\lambda_0|}{(1 - \frac{\mu}{2})} \leqslant \frac{4|\lambda_0|}{3}$$

Since all points in the set $\Lambda_0^j \cup \Lambda_{0,+}^j \cup \Lambda_{0,-}^j$ are located in the union of angles $\Gamma_j \cup (-\Gamma_j)$, in view of the above inequality and (3.25), (3.28) we obtain

$$|\lambda - |\lambda_0| e^{-i\psi_j}| \leqslant |\lambda - \lambda_0| + |\lambda_0 - |\lambda_0| e^{-i\psi_j}| \leqslant \frac{\mu|\lambda|}{2} + \frac{\mu|\lambda_0|}{6} \leqslant \frac{4\mu|\lambda_0|}{6} + \frac{\mu|\lambda_0|}{6} = \frac{5\mu|\lambda_0|}{6}.$$

Thus, the inclusions hold

$$E_j(d) \setminus B(0, r_1) \subset \Gamma_{-\psi_j}\left(\frac{5\mu}{6}\right) \cup \Gamma_{\pi - \psi_j}\left(\frac{5\mu}{6}\right) \subset \Gamma_{-\psi_j}(\tau) \cup \Gamma_{\pi - \psi_j}(\tau). \tag{3.30}$$

In view of (3.23), (3.26) and (3.28) this yields

$$\ln |f_{j,0}(\lambda)| \geqslant \frac{\pi |\operatorname{Im}(\lambda e^{i\psi_j})|}{\gamma_j} - 2\varepsilon |\lambda|, \qquad \lambda \in \mathbb{C} \setminus (\Gamma_{-\psi_j}(\tau) \cup \Gamma_{\pi - \psi_j}(\tau) \cup B(0, r_2)), \tag{3.31}$$

where $j = 1, 2$ and $r_2 = \max\{r_1, R_1, R_2\}$. Moreover, by (3.26) and (3.28)

$$\ln |f_{j,0}(\lambda)| \leqslant \ln |f_j(\lambda)| + \varepsilon |\lambda|, \qquad \lambda \in \mathbb{C} \setminus E_j(d), \qquad j = 1, 2. \tag{3.32}$$

Let $\lambda \in E_j(d) \setminus B(0, 2r_1)$. By (3.30) there exists $t \in \mathbb{R}$ such that $\lambda \in B\left(te^{-i\psi_j}, \frac{5\mu|t|}{6}\right)$. Then

$$2r_1 \leqslant |\lambda| \leqslant |t| + \frac{5\mu|t|}{6} \leqslant 2|t|, \qquad |\lambda| \geqslant |t| - \frac{5\mu|t|}{6} \geqslant \frac{|t|}{2}.$$

Therefore, $|t| \geqslant r_1$. As it has been shown above, the sum of diameters of circles $B(y_{m,j}, q_{m,j})$, which intersect the circle $B(te^{-i\psi_j}, 2\mu|t|)$, does not exceed

$$2\frac{\mu(|t| + 2\mu|t|)}{4} \leqslant \mu|t|.$$

Therefore, there exists $\mu_0 \in \left(\frac{5\mu}{6}, 2\mu\right)$ such that the circumference $S(te^{-i\psi_j}, \mu_0|t|)$ is disjoint with $E_j(d)$. Then by (3.32) and (3.27)

$$\ln|f_{j,0}(\xi)| \leqslant 2\varepsilon|t| + \varepsilon|\xi| \leqslant 2\varepsilon|t| + \varepsilon|t|(1+\mu_0) \leqslant 4\varepsilon|t|, \qquad \xi \in S(te^{-i\psi_j}, \mu_0|t|).$$

By the maximum principle we hence obtain

$$\ln|f_{j,0}(\lambda)| \leqslant 4\varepsilon|t| \leqslant 8\varepsilon|\lambda|, \qquad \lambda \in E_j(d) \setminus B(0, 2r_1).$$

Then in view of (3.32) and (3.24) we have

$$h_{f_{j,0}}(\varphi) \leqslant \frac{\pi|\sin(\varphi+\psi_j)|}{\gamma_j} + 8\varepsilon, \qquad \varphi \in [0, 2\pi], \qquad j = 1, 2. \tag{3.33}$$

We consider the function

$$g(\lambda) = f_{1,0}(\lambda) f_{2,0}(\lambda) e^{a_1\lambda} e^{a_2\lambda}, \qquad a_1 = \frac{-i\pi}{\gamma_1} e^{i\psi_1}, \qquad a_2 = \frac{i\pi}{\gamma_1} e^{i\psi_2}.$$

By (3.33) for each $\varphi \in [0, 2\pi]$ we have

$$h_g(\varphi) \leqslant \frac{\pi|\sin(\varphi+\psi_1)|}{\gamma_1} + \frac{\pi|\sin(\varphi+\psi_2)|}{\gamma_2} + \frac{\pi\sin(\varphi+\psi_1)}{\gamma_1} - \frac{\pi\sin(\varphi+\psi_2)}{\gamma_2} + 16\varepsilon.$$

Therefore,

$$h_g(\varphi) \leqslant 16\varepsilon, \qquad \varphi \in [-\psi_2, -\psi_1]. \tag{3.34}$$

In the same way, using (3.31), we find

$$\ln|g(\lambda)| \geqslant -4\varepsilon|\lambda|, \qquad \lambda \in \Gamma(-\psi_2, -\psi_1) \setminus \left(\Gamma_{-\psi_1}(\tau) \cup \Gamma_{-\psi_2}(\tau) \cup B(0, r_2)\right). \tag{3.35}$$

We choose α_1, α_2

$$\psi_1 < \alpha_1 < \theta_1 < \theta_2 < \alpha_2 < \psi_2, \qquad \alpha_1 < \psi_1 + \psi, \qquad \alpha_2 > \psi_2 - \psi. \tag{3.36}$$

By Lemma 3.2 there exist a set E with the linear density $p_E \leqslant \delta$ and a function $f_0 \in I(\Lambda(-\alpha_2, -\alpha_1), D)$ obeying conditions (3.4) and (3.5).

Since $D \in \mathcal{D}(\psi_1, \psi_2)$ and $\Xi(\Lambda) \subseteq S(0,1) \setminus \operatorname{int} J(D)$, by two last inequalities in (3.36) outside the angle $\Gamma(-\alpha_2, -\alpha_1) \cup \Gamma_1 \cup \Gamma_2$ there exist just finitely many number of points λ_k. Suppose these are the points λ_k, $k = \overline{1, k_0}$. We let

$$f(\lambda) = g(\lambda) f_0(\lambda) G(\lambda), \qquad G(\lambda) = \prod_{k=1}^{k_0} (\lambda - \lambda_k)^{n_k}.$$

By construction, $f \in \mathbb{F}(\Lambda)$. Since $h_G \equiv 0$, by (3.34), (3.4), the choice of number $\varepsilon > 0$ and the definition of set $J(D)$ we have

$$h_f(\varphi) \leqslant h_g(\varphi) + h_{f_0}(\varphi) \leqslant H_D(-\varphi) - \varepsilon, \qquad \varphi \in [0, 2\pi].$$

This means that $f \in P_D$. Hence, $f \in I(\Lambda, D)$. We choose $R_0 \geqslant r_2$ such that

$$\ln|G(\lambda)| \geqslant -\varepsilon|\lambda|, \qquad \lambda \in \mathbb{C} \setminus B(0, R_0),$$

and we let $E_0 = E \cup B(0, R_0)$. Then $p_{E_0} \leqslant \delta$, and by (3.35), (3.22), (3.5), the choice of the number $\varepsilon > 0$ and (2.1) we obtain (3.24). The proof is complete. □

4. Decomposition of invariant subspace

In order to represent the functions from the invariant subspace as the sum of functions from other invariant subspaces, we need the Leontiev interpolating function.

Let f be an entire function of exponential type, $\gamma(t,f)$ be the function associated with f in the Borel sense, K be the adjoint diagram of function f and $0\in K$. We suppose that D is the convex domain, $g\in H(D)$, and $\sigma\in\mathbb{C}$ is such that the translation $K+\sigma$ is located in the domain D. The interpolating function for the function g is [19, Ch. I, Sect. 2]

$$\omega_f(\lambda,\sigma,g)=e^{-\sigma\lambda}\frac{1}{2\pi i}\int\limits_{\Omega}\gamma(t,f)\left(\int\limits_0^t g(t+\sigma-\eta)e^{\lambda\eta}d\eta\right)dt, \tag{4.1}$$

where Ω is the contour (simple closed continuous rectifiable curve), which envelopes the compact set K and is located in the domain $D-\sigma$.

Let us omit the restriction $0\in K$. We choose an arbitrary point $w\in K$ and let

$$f_w(z)=f(z)e^{-wz}.$$

Then [20, Ch. I, Thm. 5.3] the identity holds $\gamma(t,f_w)=\gamma(t+w,f)$. The adjoint diagram of function $f_w(z)$ coincides with the compact set $K_w=K-w$, which contains the origin. Then the formula (4.1) defines the function $\omega_{f_w}(\lambda,\sigma,g)$ for all $\sigma\in\mathbb{C}$ such that the compact set $K_w+\sigma$ lies in the domain D. If $K\subset D$, then for each $w\in K$ the compact set K_w+w lies in the domain D. This is why in this case for each $w\in K$ the function $\omega_{f_w}(\lambda,w,g)$ is well–defined.

We mention some properties of the function $\omega_{f_w}(\lambda,\sigma,g)$. It follows from (4.1) that it is entire and linear in the third variable. Let$K(\varepsilon)=K+B(0,\varepsilon)$ be the ε–extension of th compact set K, $\omega(\varepsilon)=\partial(K(\varepsilon))-w$ and $\omega_\sigma(\varepsilon)=\omega(\varepsilon)+\sigma\subset D$. Since

$$H_{\omega(\varepsilon)}(\varphi)=H_K(\varphi)+\varepsilon-\mathrm{Re}(we^{i\varphi}),$$

by (4.1) we have

$$\begin{aligned}|\omega_{f_w}(\lambda,\sigma,g)|&\leqslant\frac{1}{2\pi}|e^{-\sigma\lambda}|\sup_{z\in\omega(\varepsilon)}|e^{\lambda z}|\sup_{z\in\omega_\sigma(\varepsilon)}|g(z)|\int\limits_{\omega(\varepsilon)}|\gamma(t,f_w)||t||dt|\\&\leqslant\frac{\tau_\varepsilon}{2\pi}\exp\left(rH_{\omega(\varepsilon)}(-\varphi)-\mathrm{Re}(\sigma\lambda)\right)\sup_{z\in\omega_\sigma(\varepsilon)}|g(z)|\int\limits_{\partial K(\varepsilon)}|\gamma(t,f)||dt|\\&=A(f,\varepsilon)\exp\left(rH_K(-\varphi)+\varepsilon r-\mathrm{Re}((w+\sigma)\lambda)\right)\sup_{z\in\omega_\sigma(\varepsilon)}|g(z)|,\qquad\lambda=re^{i\varphi},\end{aligned}$$

where $A(f,\varepsilon)=(2\pi)^{-1}\tau_\varepsilon(f)d_\varepsilon$, d_ε is the diameter of domain $K(\varepsilon)$ and $\tau_\varepsilon(f)$ be the latter integral. In view of identity (4.1), for all $\lambda\in\mathbb{C}$ we hence find

$$|\omega_{f_w}(\lambda,\sigma,g)|\leqslant A(f,\varepsilon)\exp((h_f(\varphi)+\varepsilon)r-\mathrm{Re}((w+\sigma)\lambda))\sup_{z\in\omega_\sigma(\varepsilon)}|g(z)|. \tag{4.2}$$

In the case $\sigma=w$ we have $\omega_\sigma(\varepsilon)=\partial(K(\varepsilon))$.

Now we mention the main property of interpolating function. Let $\Lambda=\{\lambda_k,n_k\}$ be the multiple zero set of function f and

$$P(z)=\sum_{k=1}^{p}\sum_{n=0}^{n_k-1}a_{k,n}z^ne^{\lambda_k z}.$$

Then the identity hold [19, Ch. I, Sect. 2, Thm. 1.2.4]

$$\frac{1}{2\pi i}\int\limits_{S(\lambda_k,b_k)}\frac{\omega_{f_w}(\lambda,\sigma,P)}{f_w(\lambda)}e^{\lambda z}d\lambda=\sum_{n=0}^{n_k-1}a_{k,n}z^ne^{\lambda_k z},\qquad\sigma\in\mathbb{C},\qquad k=\overline{1,p}, \tag{4.3}$$

where $S(\lambda_k, b_k)$ is the circumference, which contains no points λ_s, $s \neq k$. It is easy to note that identity (4.3) is true also for each function $f \in \mathbb{F}(\Lambda)$.

The next statements are particular cases of Theorems 2.1.1 and 2.1.2 from [19].

Lemma 4.1. *Let $\Lambda = \{\lambda_k, n_k\}$, D be a convex domain and the system $\mathcal{E}(\Lambda)$ be incomplete in the space $H(D)$. We suppose that*

$$g(z) = \lim_{\mu\to\infty} P_\mu(z), \qquad P_\mu(z) = \sum_{k=1}^{\mu}\sum_{n=0}^{n_k-1} a_{k,n,\mu} z^n e^{\lambda_k z}, \tag{4.4}$$

and the convergence is uniform on compact sets in the domain D. Then there exist the limits

$$a_{k,n} = \lim_{\mu\to\infty} a_{k,n,\mu}, \qquad n = \overline{0, n_k-1}, \qquad k \geqslant 1.$$

Lemma 4.2. *Let $\Lambda = \{\lambda_k, n_k\}$, D be a convex domain and the system $\mathcal{E}(\Lambda)$ be incomplete in the space $H(D)$. Suppose that (4.4) is true and*

$$g(z) = \lim_{\mu\to\infty} Q_\mu(z), \qquad Q_\mu(z) = \sum_{k=1}^{\mu}\sum_{n=0}^{n_k-1} b_{k,n,\mu} z^n e^{\lambda_k z},$$

and the convergence is uniform on compact sets in the domain D. Then

$$\lim_{\mu\to\infty} a_{k,n,\mu} = \lim_{\mu\to\infty} b_{k,n,\mu}, \qquad n = \overline{0, n_k-1}, \qquad k \geqslant 1.$$

And vice versa, if the sequences $\{P_\mu\}$ and $\{Q_\mu\}$ converge uniformly on compact sets in the domain D and the last identities are true, then these sequences converge to the same function.

Let $\Lambda = \{\lambda_k, n_k\}$, $\Lambda_1 = \{\xi_p, m_p\}$ and $\Lambda_2 = \{\varsigma_j, l_j\}$. We write $\Lambda = \Lambda_1 \cup \Lambda_2$, if for each $k \geqslant 1$ there exists $p \geqslant 1$ such that $\lambda_k = \xi_p$ and $n_k = m_p$, or there exists $j \geqslant 1$ such that $\lambda_k = \varsigma_j$ and $n_k = l_j$. By means of interpolating function in [12, Thm. 3.4)] the following result was proved.

Theorem 4.1. *Let $\Lambda = \{\lambda_k, n_k\}$, D be a convex domain and the system $\mathcal{E}(\Lambda)$ is incomplete in the space $H(D)$. Then there exist sequences Λ_1 and Λ_2 such that $\Lambda = \Lambda_1 \cup \Lambda_2$, $\Xi(\Lambda_2) \subset S(0,1) \setminus \operatorname{int} J(D)$ and for each function $g \in W(\Lambda, D)$ the representation $g = g_1 + g_2$ holds, where $g_1 \in W(\Lambda_1, \mathbb{C})$ and $g_2 \in W(\Lambda_2, D)$. In particular, $\Lambda_1 = \emptyset$ and $g_1 = 0$ as $\Xi(\Lambda) \cap \operatorname{int} J(D) = \emptyset$, while $\Lambda_2 = \emptyset$ and $g_2 = 0$ as $\Xi(\Lambda) \cap \operatorname{int} J(D)$.*

Lemma 4.3. *Let $0 < \psi_2 - \psi_1 \leqslant \pi$, $D \in \mathcal{D}(\psi_1, \psi_2)$, $\Lambda = \{\lambda_k, n_k\}$, the system $\mathcal{E}(\Lambda)$ be incomplete in the space $H(D)$. Suppose that $S_\Lambda(\mu) = 0$, $\mu \in S(0,1) \setminus \overline{J(D)}$, and for all $\varphi_1, \varphi_2 \notin \Phi(D)$ such that $\psi_1 < \varphi_1 < \varphi_2 < \psi_2$ inequality (3.3) holds. Then for all θ_1, θ_2, $\psi_1 < \theta_1 < \theta_2 < \psi_2$, the fundamental principal holds in the space $W(\Lambda(-\theta_2, -\theta_1), D)$ and $m(\Lambda(-\theta_2, -\theta_1)) = 0$.*

Proof. Let $\psi_1 < \theta_1 < \theta_2 < \psi_2$ and $\Lambda_0 = \Lambda(-\theta_2, -\theta_1)$. Then $\Xi(\Lambda_0) \subset S(0,1) \setminus \overline{J(D)}$. It follows from the assumptions that $S_{\Lambda_0} = 0$. Let $\mathcal{K}(D) = \{K_p\}$. We fix $p \geqslant 1$, $\delta \in (0,1)$ and $\delta_0 \in \left(0, \frac{\delta}{16}\right)$. By Lemma 3.4 there exist a set $E_0 \cup B(z_i, \rho_i)$ with a linear density $p_{E_0} \leqslant \delta_0$ and a function $f \in I(\Lambda_0, D)$ such that

$$\ln|f(re^{i\varphi})| \geqslant rH_{K_p}(-\varphi), \qquad re^{i\varphi} \in \Gamma(-\theta_2, -\theta_1) \setminus E_0.$$

We choose $R > 0$ such that

$$\sum_{|z_i|<r} \rho_i < 2\delta_0 r, \qquad r \geqslant R.$$

As in Lemma 3.4 we show that the sum of radii of all circles $B(z_i, \rho_i)$, which intersect $B(0,r)$, is strictly less than $\frac{8\delta_0 r}{3}$.

Let $\lambda_k \in \Gamma(-\theta_2, -\theta_1) \setminus B(0, R)$. Then the sum of diameters of all circles $B(z_i, \rho_i)$, which intersect $B(\lambda_k, 8\delta_0|\lambda_k|)$, is strictly less than $8\delta_0|\lambda_k|$. Therefore, there exists $\mu \in (0, 8)$ such that the circumference $S(\lambda_k, \mu\delta_0|\lambda_k|)$ has no common points with the set E_0, that is,

$$\ln|f(z)| \geqslant rH_{K_p}(-\varphi), \qquad z = re^{i\varphi} \in \Gamma(-\theta_2, -\theta_1) \cap S(\lambda, \mu\delta_0|\lambda_k|).$$

Let $z \in \Gamma(-\theta_2, -\theta_1) \cap S(\lambda_k, \mu\delta_0|\lambda_k|)$. Then $\lambda_k \in B(z, 2\mu\delta_0 r) \subset B(z, \delta r)$.

Thus, all assumptions of Theorem 2.8 are satisfied for the sequence Λ_0. This is, in accordance with this theorem, $m(\Lambda_0) = 0$, and each function $g \in W(\Lambda_0, D)$ is represented by the series

$$g(z) = \sum_{\lambda_k, n_k \in \Lambda_0} d_{k,n} z^n e^{\lambda_k z}, \qquad z \in D. \tag{4.5}$$

Since the system $\mathcal{E}(\Lambda)$ is incomplete in $H(D)$, we have $\overline{n}(\Lambda_0) \leqslant \overline{n}(\Lambda) < \infty$. This yields that $\sigma(\Lambda_0) = 0$. Then by Theorem 2.1 the inclusion $d = \{d_{k,n}\} \in Q(D, \Lambda_0)$ holds. This is by Lemma 2.1 for each $p \geqslant 1$ there exist $C_p > 0$ and an index $m(p)$ such that

$$\sum_{\lambda_k, n_k \in \Lambda_0} |d_{k,n}| \sup_{z \in K_p} |z^n e^{z\lambda_k}| \leqslant C_p \|d\|_{m(p)}, \qquad d = \{d_{k,n}\} \in Q(D, \Lambda).$$

In particular, this means that series (4.5) converges uniformly on compact sets in the domain D for each function $g \in W(\Lambda_0, D)$, that is, the fundamental principle in the space $W(\Lambda_0, D)$. The proof is complete. □

Theorem 4.2. *Let $0 < \psi_2 - \psi_1 \leqslant \pi$, $D \in \mathcal{D}(\psi_1, \psi_2)$, $\Lambda = \{\lambda_k, n_k\}$, the system $\mathcal{E}(\Lambda)$ be incomplete in the space $H(D)$. Suppose that $S_\Lambda(\mu) = 0$, $\mu \in S(0,1) \setminus \overline{J(D)}$, and for all $\varphi_1, \varphi_2 \notin \Phi(D)$ such that $\psi_1 < \varphi_1 < \varphi_2 < \psi_2$ inequality (3.3) holds. Then for each function $g \in W(\Lambda, D)$ the representation $g = g_1 + g_2 + g_3$ holds, where $g_1 \in W(\Lambda_1, \mathbb{C})$, $g_2 \in W(\Lambda_2, D)$ and $g_3 \in W(\Lambda_3, D)$. At the same time, $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$, $\Xi(\Lambda_2) \subseteq \partial J(D)$ and g_3 is represented by series (1.1), which converges uniformly on compact sets in the domain D.*

In particular, $\Lambda_1 = \emptyset$ and $g_1 = 0$, when $\Xi(\Lambda) \cap \operatorname{int} J(D) = \emptyset$, $\Lambda_2 = \emptyset$ and $g_2 = 0$ as $\Xi(\Lambda) \subseteq \operatorname{int} J(D)$, $\Lambda_3 = \emptyset$ and $g_3 = 0$ as $\Xi(\Lambda) \subseteq \overline{J(D)}$.

Proof. By Theorem 4.1 there exist sequences Λ_1 and $\Lambda_{2,0}$ such that $\Lambda = \Lambda_1 \cup \Lambda_{2,0}$, $\Xi(\Lambda_{2,0}) \subset S(0,1) \setminus \operatorname{int} J(D)$ and for each function $g \in W(\Lambda, D)$ the representation $g = g_1 + g_{2,0}$ holds, where $g_1 \in W(\Lambda_1, \mathbb{C})$ and $g_{2,0} \in W(\Lambda_{2,0}, D)$. If $\Xi(\Lambda_{2,0}) \subseteq \partial J(D)$, then the proof is complete. Otherwise we should represent $g_{2,0}$ as the sum $g_2 + g_3$. In order to do this, we need to represent $\Lambda_{2,0}$ as the union $\Lambda_{2,1} \cup \Lambda_{3,1}$, where $\Lambda_{2,1}$ satisfies the condition $\Xi(\Lambda_{2,1}) \subseteq \partial J(D)$, and the sequence $\Lambda_{3,1}$ is partitioned into special finite groups.

Let $g_{2,0} \in W(\Lambda_{2,0}, D)$. Then

$$g_{2,0}(z) = \lim_{\mu \to \infty} P_\mu(z), \qquad P_\mu(z) = \sum_{k=1}^{\mu} \sum_{n=0}^{n_k - 1} a_{k,n,\mu} z^n e^{\lambda_k z},$$

and the convergence is uniform on copact sets in the domain D. If the pair λ_k, n_k is an element of the sequence Λ_1, then $a_{k,n,\mu} = 0$. We also let $a_{k,n,\mu} = 0$, $k > \mu$.

We choose sequences $\{\theta_{1,l}\}$, $\{\theta_{2,l}\}$ such that

$$\begin{gathered} \psi_1 < \dots < \theta_{1,l} < \dots < \theta_{1,1} < \theta_{2,1} < \dots < \theta_{2,l} < \ldots < \psi_2, \\ (\psi_1, \psi_2) = \bigcup_{l=1}^{\infty} (\theta_{1,l}, \theta_{2,l}). \end{gathered} \tag{4.6}$$

Since $\mathcal{K}(D) = \{K_p\}_{p=1}^{\infty}$ exhausts the domain D, there exists a subsequence of natural numbers $\{p(l)\}$, which obeys the conditions

$$p(l) \geqslant l + 1, \qquad H_{K_{p(l)}}(\varphi) + \frac{\nu_l}{4} \geqslant H_D(\varphi), \qquad \varphi \in [\theta_{1,l+1}, \theta_{2,l+1}], \qquad l \geqslant 1, \tag{4.7}$$

where ν_l is the number in (2.1). We choose the numbers $\delta_l \in \left(0, \frac{1}{30}\right)$ such that

$$\Gamma_{\theta_{1,l}}(18\delta_l) \cup \Gamma(\theta_{1,l}, \theta_{2,l}) \cup \Gamma_{\theta_{2,l}}(18\delta_l) \subset \Gamma(\theta_{1,l+1}, \theta_{2,l+1}), \qquad \delta_l > \delta_{l+1}, \qquad l \geqslant 1. \tag{4.8}$$

By Lemma 3.4 for each $l \geqslant 1$ there exists a set $E_l = \cup B(z_{j,l}, r_{j,l})$ with a linear density $p_{E_l} \leqslant \delta_l$ and a function $f_l \in I(\Lambda_{2,0}, D)$ such that

$$\ln |f_l(re^{i\varphi})| \geqslant rH_{K_{p(l)}}(-\varphi), \qquad re^{i\varphi} \in \Gamma(-\theta_{2,l+1}, -\theta_{1,l+1}) \setminus E_l. \tag{4.9}$$

Since $f_l \in I(\Lambda_{2,0}, D)$, there exists $\varepsilon_l > 0$ such that

$$h_{f_l}(\varphi) + \varepsilon_l < H_D(-\varphi), \qquad \varphi \in [0, 2\pi], \qquad l \geqslant 1. \tag{4.10}$$

Let T_l be the adjoint diagram of function f_l. It follows from (4.10) and (3.1) that $T_l \subset D$. Moreover, the closure of domain $T_{l,0} = T_l + B(0, \varepsilon_l)$ also lies in the domain D. If $T_l = K$, then the boundary $\partial T_{l,0}$ coincides with the set $\omega_w(\varepsilon_l)$ from inequality (4.2). By (4.10) there exists $R_l > 0$ such that

$$A(f_l, \varepsilon_l) \exp((h_{f_l}(\varphi) + \varepsilon_l)r) \leqslant \exp(H_D(-\varphi)r), \qquad r \geqslant R_l, \qquad l \geqslant 1, \tag{4.11}$$

where $A(f_l, \varepsilon_l)$ is the number, which is determined in the same way as in (4.2). Since P_μ converges uniformly on each compact set in the domain D, we can suppose that

$$\begin{gathered} \max_{z \in \partial T_{l,0}} |g_{2,0}(z)| \leqslant \exp\left(r\frac{\nu_l}{6}\right), \qquad \max_{z \in \partial T_{l,0}} |P_\mu(z)| \leqslant \exp\left(r\frac{\nu_l}{6}\right), \\ r \geqslant R_l, \qquad \mu \geqslant 1, \qquad l \geqslant 1. \end{gathered} \tag{4.12}$$

We note that here the first inequality is implied by the other one. We can suppose that

$$\frac{1}{r}\sum_{|z_{j,l}|<r} r_{j,l} < \frac{6\delta_l}{5}, \qquad r \geqslant R_l, \qquad 2R_l \leqslant R_{l+1}, \qquad l \geqslant 1, \tag{4.13}$$

$$n(r+1, \Lambda) \leqslant (\overline{n}(\Lambda) + 1)r, \qquad r \geqslant R_1, \tag{4.14}$$

$$\pi(\overline{n}(\Lambda) + 1)r^2 \leqslant \exp\left(r\frac{\nu_l}{6}\right), \qquad r \geqslant R_l, \qquad l \geqslant 1. \tag{4.15}$$

Let $r \geqslant R_l$ and $B(z_{j,l}, r_{j,l})$ intersect $B(0,r)$. Then, as in Lemma 3.4, we have $|z_{j,l}| < \dfrac{10r}{9}$. This is why by (4.13) the sum of radii of all circles $B(z_{j,l}, r_{j,l})$, which intersect $B(0,r)$, do not exceed $\dfrac{4\delta_l r}{3}$.

We construct the domains $\Omega_{l,m}$, $m = \overline{1, m(l)}$, $l \geqslant 1$, by induction. We note that for some indices l the sets $\Omega_{l,m}$, $m = \overline{1, m(l)}$, can be absent. In this case it is convenient to suppose that $m(l) = 0$ and $\cup_m \Omega_{l,m} = \emptyset$. We let

$$\Gamma_l = \Gamma(-\theta_{2,l}, -\theta_{1,l}) \cap B(0, R_{l+2}) \setminus B(0, R_{l+1}).$$

Let $l = 1$. If the set Γ_1 does not contain the points λ_k, then the set $\Omega_{1,m}$, $m = \overline{1, m(1)}$, is absent. Otherwise let $\lambda_{k(1,1)} \in \Gamma_1$ be one of points λ_k such that

$$|\lambda_{k(1,1)}| = \min\{|\lambda_k| : \lambda_k \in \Gamma_1\}.$$

Since $9\delta_l < \dfrac{1}{2}$, it follows from the said above that the sum of diameters of all circles $B(z_{j,1}, r_{j,1})$ and $B(z_{j,2}, r_{j,2})$, which intersect the circle $B(\lambda_{k(1,1)}, 9\delta_1|\lambda_{k(1,1)}|)$, does not exceed

$$2\frac{4\delta_1|\lambda_{k(1,1)}|}{3}\frac{3}{2} + 2\frac{4\delta_2|\lambda_{k(1,1)}|}{3}\frac{3}{2} < 8\delta_1|\lambda_{k(1,1)}|.$$

Therefore, the exists $\mu_{1,1} \in (0, 9)$ such that

$$S(\lambda_{k(1,1)}, \mu_{1,1}\delta_1|\lambda_{k(1,1)}|) \cap E_1 = \emptyset, \qquad S(\lambda_{k(1,1)}, \mu_{1,1}\delta_1|\lambda_{k(1,1)}|) \cap E_2 = \emptyset.$$

We let $\Omega_{1,1} = B(\lambda_{k(1,1)}, \mu_{1,1}\delta_1|\lambda_{k(1,1)}|)$. Suppose that we have constructed the domains $\Omega_{1,1}, \dots, \Omega_{1,m-1}$. If the set $\Gamma_1 \setminus \cup_{s=1}^{m-1}\Omega_{1,s}$ contains no points λ_k, we then let $m(1) = m-1$. Otherwise we define $\Omega_{1,m}$. Let $\lambda_{k(1,m)} \in \Gamma_1 \setminus \cup_{s=1}^{m-1}\Omega_{1,s}$ be one of points λ_k such that

$$|\lambda_{k(1,m)}| = \min\{|\lambda_k| : \lambda_k \in \Gamma_1 \setminus \cup_{s=1}^{m-1}\Omega_{1,s}\}.$$

As above, we find $\mu_{1,m} \in (0,9)$ by the conditions

$$S(\lambda_{k(1,m)}, \mu_{1,m}\delta_1|\lambda_{k(1,m)}|) \cap E_1 = \emptyset, \qquad S(\lambda_{k(1,1)}, \mu_{1,1}\delta_1|\lambda_{k(1,1)}|) \cap E_2 = \emptyset.$$

We let

$$\Omega_{1,m} = B(\lambda_{k(1,m)}, \mu_{1,m}\delta_1|\lambda_{k(1,m)}|) \setminus \bigcup_{s=1}^{m-1}\Omega_{1,s}.$$

Suppose that we have constructed the domains $\Omega_{i,m}$, $m = \overline{1, m(i)}$, $i = \overline{1, l-1}$. If the set Γ_l contains no points λ_k, then the set $\Omega_{l,m}$, $m = \overline{1, m(l)}$, is absent. Otherwise, as above, we construct the domains $\Omega_{l,m}$, $m = \overline{1, m(l)}$ as

$$\begin{aligned} \Omega_{l,1} &= B(\lambda_{k(l,1)}, \mu_{l,1}\delta_l|\lambda_{k(l,1)}|) \setminus \left(\bigcup_{s=1}^{m(l-1)} \Omega_{l-1,s}\right), \\ \Omega_{l,m} &= B(\lambda_{k(l,m)}, \mu_{l,m}\delta_l|\lambda_{k(l,m)}|) \setminus \left(\bigcup_{s=1}^{m-1} \Omega_{l,s} \bigcup_{s=1}^{m(l-1)} \Omega_{l-1,s}\right), \qquad m = \overline{2, m(l)}, \end{aligned} \tag{4.16}$$

where $\mu_{l,m} \in (0,9)$, $m = \overline{1, m(l)}$, satisfies the conditions

$$S(\lambda_{k(l,m)}, \mu_{l,m}\delta_l|\lambda_{k(l,m)}|) \cap E_l = \emptyset, \qquad S(\lambda_{k(l,m)}, \mu_{l,m}\delta_l|\lambda_{k(l,m)}|) \cap E_{l+1} = \emptyset. \tag{4.17}$$

Thus, the domains $\Omega_{l,m}$, $m = \overline{1, m(l)}$, $l \geqslant 1$, are well–defined. Let $\Lambda_{3,1}$ consist of all pairs λ_k, n_k such that

$$\lambda_k \in \Omega = \bigcup_{l=1}^{\infty} \bigcup_{m=1}^{m(l)\neq 0} \Omega_{l,m},$$

and $\Lambda_{2,1}$ be the sequence, which completes $\Lambda_{3,1}$ to $\Lambda_{2,0}$ ($\Lambda_{2,0} = \Lambda_{2,1} \cup \Lambda_{3,1}$). By construction, each point

$$\lambda_k \in \bigcup_{l=1}^{\infty} (\Gamma(-\theta_{2,l}, -\theta_{1,l}) \setminus B(0, R_{l+1}))$$

belongs to the set Ω. Therefore, by the identity in (4.6) we have

$$\Xi(\Lambda_{2,1}) \subseteq \partial J(D).$$

By construction, in view of (4.8), we also have

$$R_{l+1} \leqslant |\lambda_{k(l,m)}| \leqslant R_{l+2}, \qquad l \geqslant 1, \qquad m(l) \neq 0, \tag{4.18}$$

$$|\lambda_{k(l,m)} - z| < 9\delta_l|\lambda_{k(l,m)}|, \qquad z \in \Omega_{l,m}, \qquad m = \overline{1, m(l)}, \qquad l \geqslant 1, \qquad m(l) \neq 0, \tag{4.19}$$

$$\overline{\Omega_{l,m}} \subset \Gamma(\theta_{1,l+1}, \theta_{2,l+1}), \qquad m = \overline{1, m(l)}, \qquad l \geqslant 1, \qquad m(l) \neq 0. \tag{4.20}$$

Since $\delta_l \in \left(0, \frac{1}{30}\right)$, by the second relation in (4.6) the circles $B(\lambda_{k(l,m)}, \mu_{l,m}\ \delta_l|\lambda_{k(l,m)}|)$ and $B(\lambda_{k(s,j)}, \mu_{s,j}\delta_l|\lambda_{k(s,j)}|)$ are disjoint for all $s < l-1$ and $l \geqslant 3$. Therefore, the domains $\Omega_{l,m}$ and $\Omega_{s,j}$ are disjoint for all $s < l-1$ and $l \geqslant 3$. By construction the domains $\Omega_{l,m}$, $m = \overline{1, m(l)}$, $\Omega_{l-1,j}$, $j = \overline{1, m(l-1)}$, are pairwise disjoint. Thus, the domains $\Omega_{l,m}$, $m = \overline{1, m(l)}$, $l \geqslant 1$, are pairwise disjoint.

By construction, $\partial\Omega_{l,m}$ consists of arcs of some circumference in the set

$$S(\lambda_{k(l,m)}, \mu_{l,m}\delta_l|\lambda_{k(l,m)}|), \qquad m = \overline{1, m-1},$$
$$S(\lambda_{k(l-1,m)}, \mu_{l-1,m}\delta_{l-1}|\lambda_{k(l-1,m)}|), \qquad m = \overline{1, m(l-1)}.$$

At the same time, all centers of circumferences are different and

$$|\lambda_{k(l-1,m)}| \leqslant \ldots \leqslant |\lambda_{k(l-1,m(l-1))}| \leqslant |\lambda_{k(l,1)}| \leqslant \ldots \leqslant |\lambda_{k(l,m)}|.$$

Thus, the number of circumferences, whose arcs form the boundary $\partial\Omega_{l,m}$, does not exceed $n(|\lambda_{k(l,m)}| + 1, \Lambda)$. Therefore, in view of inclusion $\delta_l \in \left(0, \frac{1}{30}\right)$ we obtain

$$b_{l,m} \leqslant \pi n(|\lambda_{k(l,m)}| + 1, \Lambda)|\lambda_{k(l,m)}|, \qquad m = \overline{1, m(l)}, \qquad l \geqslant 1, \qquad m(l) \neq 0, \tag{4.21}$$

where $b_{l,m}$ is the length of boundary $\partial\Omega_{l,m}$. By (4.17), (4.20) and (4.9) we have

$$\begin{aligned} &\ln|f_l(re^{i\varphi})| \geqslant rH_{K_{p(l)}}(-\varphi), \qquad re^{i\varphi} \in \partial\Omega_{l,m}, \\ &m = \overline{1, m(l)}, \qquad l \geqslant 1, \qquad m(l) \neq 0. \end{aligned} \tag{4.22}$$

Let $w \in T_l$ and $f_{l,w}(z) = f_l(z)e^{-wz}$. By formula (4.1) we define the interpolating function $\omega_{f_{l,w}}(\lambda, w, P_\mu)$ for all $\mu \geqslant 1$. It follows (4.3) from that

$$\frac{1}{2\pi i} \int\limits_{S(\lambda_k, b_k)} \frac{\omega_{f_{l,w}}(\lambda, w, P_\mu)}{f_{l,w}(\lambda)} e^{\lambda z} d\lambda = \sum_{n=0}^{n_k-1} a_{k,n,\mu} z^n e^{\lambda_k z}, \qquad w \in T_l, \qquad k \geqslant 1, \qquad l \geqslant 1,$$

where $S(\lambda_k, b_k)$ is the circumference, which contains no points λ_s, $s \neq k$,

$$\frac{1}{2\pi i} \int\limits_{S(\xi_s, c_s)} \frac{\omega_{f_{l,w}}(\lambda, w, P_\mu)}{f_{l,w}(\lambda)} e^{\lambda z} d\lambda = 0,$$

where ξ_s is the zero of function $f_{l,w}$ different from all λ_k, and $S(\xi_s, c_s)$ is the circumference, which contains no zeroes of the function $f_{l,w}$ except for the point ξ_s.

We fix $l \geqslant 1$, $m(l) \neq 0$, $w \in T_l$. By the residue theorem we obtain

$$\frac{1}{2\pi i} \int\limits_{\partial\Omega_{l,m}} \frac{\omega_{f_{l,w}}(\lambda, w, P_\mu)}{f_{l,w}(\lambda)} e^{\lambda z} d\lambda = \sum_{\lambda_k \in \Omega_{l,m}} \sum_{n=0}^{n_k-1} a_{k,n,\mu} z^n e^{\lambda_k z} = p_{l,m,\mu}(z), \qquad m = \overline{1, m(l)}. \tag{4.23}$$

Since $\delta_l \in \left(0, \frac{1}{30}\right)$, in accordance with (4.18) and (4.19) we have $\partial\Omega_{l,m} \cap B(0, R_l) = \emptyset$. This is why by (4.2), (4.11) and (4.22) we obtain

$$|p_{l,m,\mu}(z)| \leqslant \frac{b_{l,m}}{2\pi} \max_{\lambda \in \partial\Omega_{l,m}} \left(\frac{\exp(rH_D(-\varphi) - \mathrm{Re}((2w - z)\lambda))}{\exp(rH_{K_{p(l)}}(-\varphi) - \mathrm{Re}(w\lambda))} \right) \max_{z \in \partial T_{l,0}} |P_\mu(z)|,$$

where $\lambda = re^{i\varphi}$. In view of (4.7), (4.20), (4.14), (4.15), (4.21) and (4.12) we hence have

$$\begin{aligned} |p_{l,m,\mu}(z)| &\leqslant b_{l,m} \max_{\lambda \in \partial\Omega_{l,m}} \left(\exp\left(\frac{\nu_l}{4} r - \mathrm{Re}((w - z)\lambda)\right)\right) \max_{z \in \partial T_{l,0}} |P_\mu(z)| \\ &\leqslant \max_{\lambda \in \partial\Omega_{l,m}} \left(\exp\left(\frac{\nu_l}{4} r - \mathrm{Re}((w - z)\lambda)\right)\right) \exp\left(\frac{\nu_l}{3}|\lambda_{k(l,m)}|\right), \\ m &= \overline{1, m(l)}, \quad \mu \geqslant 1. \end{aligned} \tag{4.24}$$

We choose $w \in T_l$ such that

$$\mathrm{Re}(w\lambda) = rH_{T_l}(-\varphi).$$

By (3.1) and (4.22) this implies

$$\mathrm{Re}(w\lambda) \geqslant rH_{K_{p(l)}}(-\varphi), \qquad \lambda \in re^{i\varphi} \in \Omega_{l,m}, \qquad m = \overline{1, m(l)}, \qquad m(l) \neq 0. \tag{4.25}$$

Let $\sigma \geqslant 1$. In view of (2.1) and the first inequality in (4.7) we have

$$\mathrm{Re}(z\lambda) = \mathrm{Re}(zre^{i\varphi}) \leqslant rH_{K_\sigma}(-\varphi) \leqslant rH_{K_{p(l)}}(-\varphi) - \nu_\sigma r, \qquad z \in K_\sigma, \qquad l \geqslant \sigma. \tag{4.26}$$

Thus, by (4.24)—(4.26), (4.19) and (2.1) we get

$$\begin{aligned} |p_{l,m,\mu}(z)| &\leqslant \max_{\lambda\in\partial\Omega_{l,m}} \left(\exp(\frac{\nu_l}{4}r - \nu_\sigma r)\right) \exp\left(\frac{\nu_l}{3}|\lambda_{k(l,m)}|\right) \\ &\leqslant \max_{\lambda\in\partial\Omega_{l,m}} \left(\exp(\frac{\nu_\sigma}{4}r - \nu_\sigma r)\right) \exp\left(\frac{\nu_\sigma}{3}|\lambda_{k(l,m)}|\right) \\ &\leqslant \exp\left(\left(\frac{\nu_\sigma}{3} - \frac{\nu_\sigma}{2}\right)|\lambda_{k(l,m)}|\right) = \exp\left(-\frac{\nu_\sigma}{6}|\lambda_{k(l,m)}|\right), \end{aligned} \tag{4.27}$$

$$z \in K_\sigma, \qquad m = \overline{1, m(l)}, \qquad l \geqslant \sigma, \qquad m(l) \neq 0, \qquad \mu \geqslant 1,$$

$$\begin{aligned} |p_{l,m,\mu}(z)| &\leqslant \max_{\lambda\in\partial\Omega_{l,m}} \left(\exp\left(\frac{\nu_l}{4}r - \left(H_{K_{p(l)}}(-\varphi) - H_{K_\sigma}(-\varphi)\right)r\right)\right) \\ &\quad \cdot \exp\left(\frac{\nu_l}{3}|\lambda_{k(l,m)}|\right) \leqslant C(\sigma), \end{aligned} \tag{4.28}$$

$$z \in K_\sigma, \qquad m = \overline{1, m(l)}, \qquad l = \overline{1, \sigma - 1}, \qquad m(l) \neq 0, \qquad \mu \geqslant 1.$$

We represent the polynomials P_μ as

$$P_\mu(z) = P_{\mu,1}(z) + P_{\mu,2}(z), \qquad \mu \geqslant 1,$$

$$P_{\mu,1}(z) = \sum_{k=1}^{\infty}\sum_{n=0}^{n_k-1} a_{k,n,\mu,1}z^n e^{\lambda_k z}, \qquad a_{k,n,\mu,1} = 0, \qquad k > \mu,$$

$$a_{k,n,\mu,1} = a_{k,n,\mu}, \qquad n = \overline{0, n_k - 1}, \qquad \lambda_k \in \Omega, \qquad a_{k,n,\mu,1} = 0, \qquad n = \overline{0, n_k - 1}, \qquad \lambda_k \notin \Omega.$$

Since the domains $\Omega_{l,m}$ are pairwise disjoint, the identity holds

$$P_{\mu,1}(z) = \sum_{l=1}^{\infty}\sum_{m=1}^{m(l)\neq 0}\sum_{\lambda_k\in\Omega_{l,m}}\sum_{n=0}^{n_k-1} a_{k,n,\mu,1}z^n e^{\lambda_k z} = \sum_{l=1}^{\infty}\sum_{m=1}^{m(l)\neq 0} p_{l,m,\mu}(z), \qquad \mu \geqslant 1.$$

By (4.27) and (4.28) we obtain

$$|P(\mu, 1)(z)| \leqslant \sum_{l=1}^{\sigma-1}\sum_{m=1}^{m(l)\neq 0} C(\sigma) + \sum_{l=\sigma}^{\infty}\sum_{m=1}^{m(l)\neq 0} \exp\left(-\frac{\nu_\sigma}{6}|\lambda_{k(l,m)}|\right), \qquad z \in K_\sigma, \qquad \mu \geqslant 1.$$

Since $\overline{n}(\Lambda) < +\infty$, the last series converges. Thus, the sequence of functions $\{|P_{\mu,1}|\}$ is uniformly bounded on each compact set K_σ. Since the sequence $\{K_\sigma\}$ exhausts the domain D, by the Montel theorem there exists a subsequence $\{P_{\mu(j),1}\}_{j=1}^{\infty}$, which converges uniformly on each compact set in the domain D. Let

$$g_{3,1}(z) = \lim_{j\to\infty} P_{\mu(j),1}(z), \quad z \in D. \tag{4.29}$$

Since $\{P_\mu\}$ converges uniformly on compact sets in the domain D, the sequence $P_{\mu(j),2} = P_{\mu(j)} - P_{\mu(j),1}$ also converges on compact sets in the domain D to some function $g_{2,1}$. It is obvious that $g_{2,0} = g_{2,1} + g_{3,1}$ and $g_{2,1} \in W(\Lambda_{2,1}, D)$, $g_{3,1} \in W(\Lambda_{3,1}, D)$.

Let $l \geqslant 1$, $m(l) \neq 0$, $m = \overline{1, m(l)}$, $w \in T_l$. Using the residues, we obtain

$$\begin{aligned} p_{l,m}(z) &= \frac{1}{2\pi i}\int_{\partial\Omega_{l,m}} \frac{\omega_{f_{l,w}}(\lambda, w, g_{2,0})}{f_{l,w}(\lambda)} e^{\lambda z} d\lambda \\ &= \sum_{\lambda_k\in\Omega_{l,m}}\sum_{n=0}^{n_{k,0}-1} a_{k,n,0}z^n e^{\lambda_k z} + \sum_{\xi_s\in\Omega_{l,m}}\sum_{n=0}^{q_s-1} b_{s,n}z^n e^{\xi_s z}, \end{aligned} \tag{4.30}$$

where $n_{k,0} \geqslant n_k$ is the multiplicity of zero λ_k of the function $f_{l,w}$ and ξ_s is the zero of function $f_{l,w}$ of multiplicity q_s, which differs from all λ_k. By Lemma 4.1 the limits

$$a_{k,n} = \lim_{\mu\to\infty} a_{k,n,\mu} = \lim_{\mu\to\infty} a_{k,n,\mu,1}, \qquad n = \overline{0, n_k - 1}, \qquad \lambda_k \in \Omega, \tag{4.31}$$

are well–defined. Hence, in view of (4.29), (4.2), (4.23) and (4.30), we see that

$$\begin{aligned} p_{l,m}(z) &= \frac{1}{2\pi i}\int\limits_{\partial\Omega_{l,m}} \frac{\omega_{f_{l,w}}(\lambda, w, g_{2,0})}{f_{l,w}(\lambda)} e^{\lambda z} d\lambda \\ &= \frac{1}{2\pi i}\lim_{j\to\infty}\int\limits_{\partial\Omega_{l,m}} \frac{\omega_{f_{l,w}}(\lambda, w, P_{\mu(j),1})}{f_{l,w}(\lambda)} e^{\lambda z} d\lambda \\ &= \lim_{j\to\infty} p_{l,m,\mu(j)}(z) = \sum_{\lambda_k\in\Omega_{l,m}}\sum_{n=0}^{n_k-1} a_{k,n} z^n e^{\lambda_k z}, \qquad z \in \mathbb{C}. \end{aligned} \tag{4.32}$$

This means that

$$a_{k,n,0} = a_{k,n}, \qquad n = \overline{0, n_k - 1}, \qquad a_{k,n,0} = 0, \qquad n \geqslant n_k, \qquad \lambda_k \in \Omega, \qquad b_{s,n} = 0.$$

Using (4.30), (4.24)—(4.26), (4.19) and (2.1), as in (4.27), (4.28), for $\sigma \geqslant 1$ we obtain

$$|p_{l,m}(z)| \leqslant \exp\left(-\frac{\nu_\sigma}{6}|\lambda_{k(l,m)}|\right), \qquad z \in K_\sigma, \qquad m = \overline{1, m(l)}, \qquad l \geqslant \sigma, \qquad m(l) \neq 0, \tag{4.33}$$

$$|p_{l,m}(z)| \leqslant C_0(\sigma), \qquad z \in K_\sigma, \qquad m = \overline{1, m(l)}, \qquad l = \overline{1, \sigma - 1}, \qquad m(l) \neq 0. \tag{4.34}$$

By (4.32) this implies that the series

$$\sum_{l=1}^{\infty}\sum_{m=1}^{m(l)\neq 0}\sum_{\lambda_k\in\Omega_{l,m}}\sum_{n=0}^{n_k-1} a_{k,n} z^n e^{\lambda_k z} = \sum_{l=1}^{\infty}\sum_{m=1}^{m(l)\neq 0} p_{l,m}(z) \tag{4.35}$$

converges uniformly on each compact set in the domain D. Then according to (4.29), (4.31) and Lemma 4.2 it converges to the function $g_{3,1}$.

We represent $g_{3,1}$ as the sum $g_{2,2} + g_3$. In order to represent $\Lambda_{3,1}$ as the union $\Lambda_{2,2} \cup \Lambda_3$, where $\Lambda_{2,2}$ satisfies the condition $\Xi(\Lambda_{2,2}) \subseteq \partial J(D)$, and g_3 is represented as

$$g_3(z) = \sum_{\lambda_k, n_k\in\Lambda_3} a_{k,n} z^n e^{\lambda_k z}, \quad z \in D, \tag{4.36}$$

and the series converges uniformly on compact sets in the domain D.

For each $l \geqslant 2$ by the symbol Ω_l we denote the union of all domais $\Omega_{s,m}$, each of which contains at least one point $\lambda_k \in \Gamma(-\theta_{2,l-1}, -\theta_{1,l-1})$. By (4.8) and (4.1) the inclusion $\Omega_l \subset \Gamma(-\theta_{2,l}, -\theta_{1,l})$, $l \geqslant 2$, hold. By construction, each point

$$\lambda_k \in \bigcup_{l=1}^{\infty} (\Gamma(-\theta_{2,l-1}, -\theta_{1,l-1}) \setminus B(0, R_l))$$

belongs to the set Ω_l. We consider the functions

$$g_{3,l}(z) = \sum_{\Omega_{s,m}\subset\Omega_l}\sum_{\lambda_k\in\Omega_{s,m}}\sum_{n=0}^{n_k-1} a_{k,n} z^n e^{\lambda_k z} = \sum_{\Omega_{s,m}\subset\Omega_l} p_{s,m}(z), \quad l \geqslant 2. \tag{4.37}$$

According to (4.33) and (4.34) we have

$$\sum_{\substack{\Omega_{s,m},\\ 1\leqslant s\leqslant\sigma-1}} |p_{s,m}(z)| + \sum_{\substack{\Omega_{s,m},\\ s\geqslant\sigma}} |p_{s,m}(z)| \leqslant \sum_{\substack{\Omega_{s,m},\\ 1\leqslant s\leqslant\sigma-1}} C_0(\sigma) + \sum_{\substack{\Omega_{s,m},\\ s\geqslant\sigma}} \exp\left(-\frac{\nu_\sigma}{6}|\lambda_{k(l,m)}|\right), \quad z \in K_\sigma, \quad \sigma \geqslant 1.$$

Since $\overline{n}(\Lambda) < +\infty$, the latter series converges. Thus, the series (4.37) converges uniformly on compact sets in the domain D. Therefore, $g_{3,l} \in W(\Lambda(-\theta_{2,l}, -\theta_{1,l}), D)$, $l \geqslant 2$.

By Lemma 4.3 for each $l \geqslant 2$ the fundamental principle holds in the space $W(\Lambda(-\theta_{2,l}, -\theta_{1,l}), D)$ and $m(\Lambda(-\theta_{2,l}, -\theta_{1,l})) = 0$. This is why

$$g_{3,l}(z) = \sum_{k=1,n=0}^{\infty,n_k-1} d_{k,n,l} z^n e^{\lambda_k z}, \qquad z \in D, \qquad l \geqslant 2,$$

and the series converges uniformly on compact sets in the domain D. We note that $d_{k,n,l} = 0$, if $\lambda_k \notin \Omega_l$, and by Lemma 4.2

$$d_{k,n,l} = a_{k,n}, \qquad \lambda_k \in \Omega_l, \qquad l \geqslant 2. \tag{4.38}$$

Since $\overline{n}(\Lambda) < +\infty$, we have $\sigma(\Lambda(-\theta_{2,l}, -\theta_{1,l})) \leqslant \sigma(\Lambda) = 0$. Then by Theorem 2.1 the inclusion $d_l = \{d_{k,n,l}\} \in Q(D, \Lambda)$, $l \geqslant 2$, holds. Therefore,

$$\sup_{k,n} |d_{k,n,l}| p^n \exp(r_k H_{K_p}(-\varphi_k)) < C_{p,l}, \qquad p \geqslant 1, \qquad l \geqslant 2.$$

By (2.1) we then get

$$\begin{aligned} |d_{k,n,l}| l^n \exp(r_k H_{K_l}(-\varphi_k)) &\leqslant |d_{k,n,l}|(l+1)^n \exp(r_k (H_{K_{l+1}}(-\varphi_k) - \nu_l)) \\ &\leqslant C_{l+1,l} \exp(-r_k \nu_l), \qquad n = \overline{0, n-1}, \qquad k \geqslant 1, \qquad l \geqslant 2. \end{aligned}$$

For each $l \geqslant 2$ we choose $R_{l,0} \geqslant R_l$ such that

$$|d_{k,n,l}| l^n \exp(r_k H_{K_l}(-\varphi_k)) \leqslant 1, \qquad n = \overline{0, n-1}, \qquad |\lambda_k| \geqslant \frac{1}{3} R_{l,0}. \tag{4.39}$$

We can suppose that $R_{2,0} < \ldots < R_{l,0} < \ldots$. For each $l \geqslant 2$ by the symbol $\Omega_{l,0}$ we denote the union of all domains $\Omega_{s,m} \subset \Omega_l$, each contains at least one point $\lambda_k \in \Gamma(-\theta_{2,l-1}, -\theta_{1,l-1}) \cap B(0, R_{l+1,0}) \setminus B(0, R_{l,0})$. We note that according to the construction of set Ω_l, each point $\lambda_k \in \Gamma(-\theta_{2,l-1}, -\theta_{1,l-1}) \cap B(0, R_{l+1,0}) \setminus B(0, R_{l,0})$ belongs to the set $\Omega_{l,0}$.

Let $\Omega_{s,m} \subset \Omega_{l,0}$. Since $\delta_l \in \left(0, \frac{1}{30}\right)$, by (4.19) we have $\Omega_{s,m} \cap B(0, R_{l,0}/3) = \emptyset$. This is why by (4.39) we get

$$|d_{k,n,l}| l^n \exp(r_k H_{K_l}(-\varphi_k)) \leqslant 1, \qquad n = \overline{0, n-1}, \qquad \lambda_k \in \Omega_{l,0}, \qquad l \geqslant 2. \tag{4.40}$$

We let

$$\Omega_0 = \bigcup_{l=2}^{\infty} \Omega_{l,0}.$$

Let Λ_3 be the sequence of all pairs λ_k, n_k such that $\lambda_k \in \Omega_0$ and $\Lambda_{2,2}$ be the sequence completing Λ_3 to $\Lambda_{3,1}$, that is, $\Lambda_{3,1} = \Lambda_{2,2} \cup \Lambda_3$. We observe that $\Lambda_{2,2}$ consists of all pairs λ_k, n_k such that $\lambda_k \in \Omega \setminus \Omega_0$. By construction, $\Xi(\Lambda_{2,2}) \subseteq \partial J(D)$. We consider the series

$$\sum_{\lambda_k \in \Omega_0} a_{k,n} z^n e^{\lambda_k z}. \tag{4.41}$$

By (4.38) and (4.40) we have

$$|a_{k,n,0}| l^n \exp(r_k H_{K_l}(-\varphi_k)) \leqslant 1, \qquad n = \overline{0, n-1}, \qquad \lambda_k \in \Omega_{l,0}, \qquad l \geqslant 2.$$

By (2.1) this implies

$$\sup_{k,n} |a_{k,n}| p^n \exp(r_k H_{K_p})(-\varphi_k) \leqslant 1, \qquad \lambda_k \in \Omega_0, \qquad l > p, \qquad p \geqslant 1.$$

Thus, $\{a_{k,n}\} \in Q(D, \Lambda_3)$. Since $\sigma(\Lambda_3) \leqslant \sigma(\Lambda) = 0$, by Lemma 2.1 series (4.41) converges uniformly on compact sets in the domain D. Let g_3 be the sum of series (4.41). Then the function $g_3 \in W(\Lambda_3, D)$ is represented by series (1.1), which converges uniformly on compact sets in the domain D.

We consider the series

$$\sum_{\Omega_{s,m}\subset\Omega_0}\sum_{\lambda_k\in\Omega_{s,m}}\sum_{n=0}^{n_k-1}a_{k,n}z^ne^{\lambda_k z}=\sum_{\Omega_{s,m}\subset\Omega_0}p_{s,m}(z).$$

It follows from (4.33) and (4.34) that the latter series converges uniformly on compact sets in the domain D. By Lemma 4.2 its sum coincides with the sum of series (4.41), that is, it is equal to g_3. We let $g_{2,2}=g_{3,1}-g_3$. Then

$$g_{2,2}(z)=\sum_{\Omega_{s,m}\subset\Omega}p_{s,m}(z)-\sum_{\Omega_{s,m}\subset\Omega_0}p_{s,m}(z)=\sum_{\Omega_{s,m}\subset\Omega\setminus\Omega_0}p_{s,m}(z).$$

Therefore, $g_{2,2}\in W(\Lambda_{2,2},D)$. We let $\Lambda_2=\Lambda_{2,1}\cup\Lambda_{2,2}$ and $g_2=g_{2,1}+g_{2,2}$. Then $\Xi(\Lambda_2)\subseteq\partial J(D)$ and $g_2\in W(\Lambda_2,D)$. The proof is complete. □

5. Criterion of fundamental principle and interpolating problem

Theorem 5.1. *Let D be a convex domain, $\Lambda=\{\lambda_k,n_k\}$ and the system $\mathcal{E}(\Lambda)$ be incomplete in the space $H(D)$. The following statements are equivalent.*

1) *The fundamental principle holds in the space $W(\Lambda,D)$.*
2) *$m(\Lambda,\mu)=0$, $\mu\in\Xi(\Lambda)\setminus\overline{J(D)}$ and each function $g\in W(\Lambda,D)$ is represented by series (1.1), which converges at each point in the domain D.*
3) *Each function $g\in W(\Lambda,D)$ is represented by series (1.1). At the same time, $d=\{d_{k,n}\}\in Q(D,\Lambda)$, and for each $p\geqslant 1$ there exist $C_p>0$ and an index $m(p)$ such that (2.2) holds.*
4) *The operator $\mathbb{E}:Q(D,\Lambda)\to W(\Lambda,D)$ is an isomorphism.*
5) *The operator Σ_0: $P_D/I(\Lambda,D)\to\mathcal{R}(D,\Lambda)$ is an isomorphism.*
6) *Interpolating problem (3.2) is solvable in the space P_D for each right hand side $b=\{b_{k,n}\}\in\mathcal{R}(D,\Lambda)$.*
7) *$S_\Lambda>-\infty$, $S_\Lambda(\mu)=0$, $\mu\in\Xi(\Lambda)\setminus J(D)$ and all $\varphi_1,\varphi_2\notin\Phi(D)$ such that for all $0<\varphi_2-\varphi_1<\pi$ and $\{e^{i\varphi}:\varphi\in[\varphi_1,\varphi_2]\}\subset S(0,1)\setminus\overline{J(D)}$ the inequality holds*

$$\overline{n}_0(\Lambda(-\varphi_2,-\varphi_1))\leqslant\frac{\Upsilon_D(\varphi_1,\varphi_2)}{2\pi}.\tag{5.1}$$

Proof. Let Assertion 1) hold. Then each function $g\in W(\Lambda,D)$ is represented by series (1.1), which converges at each point in the domain D. Suppose that $m(\Lambda,\mu)>0$ for some $\mu\in\Xi(\Lambda)\setminus\overline{J(D)}$. Then there exists a subsequence $\{\lambda_{k(j)}\}$ such that

$$\lambda_{j,0}=\lambda_{k(j)}\mu,\quad \operatorname{Re}\lambda_{j,0}>0,\quad j\geqslant 1,\quad \lim_{j\to\infty}\frac{n_{k(j)}}{|\lambda_{k(j)}|}>0.\tag{5.2}$$

We let $\Lambda_1=\{\lambda_{k(j)},n_{k(j)}\}$, $\Lambda_0=\{\lambda_{j,0},n_{k(j)}\}$ and $D_0=\{z\overline{\mu}:z\in D\}$. Then $\mu\in\Xi(\Lambda_1)\setminus\overline{J(D)}$ and Λ_0 is an almost real sequence. Then by assumption the system $\mathcal{E}(\Lambda_1)\subset\mathcal{E}(\Lambda)$ is incomplete in the space $H(D)$. This is why the system $\mathcal{E}(\Lambda_0)$ is incomplete in the space $H(D_0)$. According to Assertion 1), the fundamental principle holds in the space $W(\Lambda_1,D)\subset W(\Lambda,D)$. Then, as it is easy to see, the fundamental principle also holds in the space $W(\Lambda_0,D_0)$. Since $\mu\in\Xi(\Lambda_1)\setminus\overline{J(D)}$, we have $\mu\overline{\mu}=1\in\Xi(\Lambda_0)\setminus\overline{J(D_0)}$. Thus, all assumptions of Theorem 2.5 are satisfied, and hence, $m(\Lambda_0)=0$. This contradicts (5.2).

We have shown that $m(\Lambda,\mu)=0$, $\mu\in\Xi(\Lambda)\setminus\overline{J(D)}$. This is why Assertion 2) is true.

Suppose that Assertion 2) holds. Then the system $\mathcal{E}(\Lambda)$ is incomplete in the space $H(D)$ and $\overline{n}(\Lambda)<\infty$. This implies that $m(\Lambda)\leqslant\overline{n}(\Lambda)<\infty$ and $\sigma(\Lambda)=0$. Then in view of Theorem 2.1 each function $g\in W(\Lambda,D)$ is represented by series (1.1), and $d=\{d_{k,n}\}\in Q(D,\Lambda)$. Therefore, according to Lemma 2.1, for each $p\geqslant 1$ there exist $C_p>0$ and an index $m(p)$ such that (2.2) holds. This means that Assertion 3) is true.

Let Assertion 3) holds. Then each function $g \in W(\Lambda, D)$ is represented by series (1.1) and inequality (2.2) holds. It implies that series (1.1) converges uniformly on compact sets in the domain D. In other words, Assertion 1) holds.

Assertions 1), 4), 5) and 6) are equivalent by Theorem 2.2. It remains to prove the equivalence of Assertions 1) and 7).

Let Assertion 1) holds. Then by Theorem 2.6 inequality (5.1) holds.

Suppose that $S_\Lambda = -\infty$. Then by Theorem 2.7 there exist numbers $\{d_{k,n}\}$ and indices k_s, $1 = k_1 < k_2 < \ldots$, such that

a) series (2.5) converges uniformly on compact sets in the plane,
b) series (1.1) converges at each point in the plane.

It follows from Item a) that the sum g of series (2.5) belongs to $W(\Lambda, D)$. In view of Lemma 4.2, it follows from Item b) that g is not represented by series (1.1). This contradicts Assertion 1). Thus, $S_\Lambda > -\infty$.

Suppose that $S_\Lambda(\mu) \neq 0$ for some $\mu \in \Xi(\Lambda) \setminus J(D)$. Then there exists a subsequence $\{\lambda_{k(j)}\}$ such that

$$\lambda_{j,0} = \lambda_{k(j)}\mu, \qquad \operatorname{Re}\lambda_{j,0} > 0, \qquad j \geqslant 1, \qquad \lim_{\delta\to 0}\lim_{j\to\infty} \frac{\ln|q_\Lambda^{k(j)}(\lambda_{k(j)}, \delta)|}{|\lambda_{k(j)}|} \neq 0. \tag{5.3}$$

As above, the fundamental principle holds in the space $W(\Lambda_0, D_0)$. Then by Lemma 2.2 the identity $S_{\Lambda_0} = 0$ holds. This contradicts (5.3). Thus, $S_\Lambda(\mu) = 0$, $\mu \in \Xi(\Lambda) \setminus J(D)$, and Assertion 7) holds.

Finally, let Assertion 7) holds. We consider separately different classes of convex domains. Let D be a bounded domain. Then by Theorem 2.3 the fundamental principle holds in the space $W(\Lambda, D)$.

Let D be an unbounded convex domain of type I–III. Then for each invariant subspace $W(\Lambda, D)$ the inclusion $\Xi(\Lambda) \subseteq \overline{J(D)} = S(0,1)$ holds. Hence, by Theorem 2.3 the fundamental principle holds in the space $W(\Lambda, D)$.

Now let $0 < \psi_2 - \psi_1 \leqslant \pi$ and $D \in \mathcal{D}(\psi_1, \psi_2)$. Then by Theorem 4.2 for each function $g \in W(\Lambda, D)$ the representation $g = g_1 + g_2 + g_3$ holds, where $g_1 \in W(\Lambda_1, \mathbb{C})$, $g_2 \in W(\Lambda_2, D)$ and $g_3 \in W(\Lambda_3, D)$. At the same time, $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$, $\Xi(\Lambda_2) \subseteq \partial J(D)$ and

$$g_3(z) = \sum_{k=1,n=0}^{\infty,n_k-1} d_{k,n,3} z^n e^{\lambda_k z}, \qquad z \in D, d_{k,n,3} = 0, \qquad \lambda_k, n_k \notin \Lambda_3,$$

and the series converges uniformly on compact sets in the domain D. In particular, $\Lambda_1 = \emptyset$ and $g_1 = 0$, as $\Xi(\Lambda) \cap \operatorname{int} J(D) = \emptyset$, $\Lambda_2 = \emptyset$ and $g_2 = 0$, as $\Xi(\Lambda) \subseteq \operatorname{int} J(D)$, $\Lambda_3 = \emptyset$ and $g_3 = 0$ as $\Xi(\Lambda) \subseteq \overline{J(D)}$.

We note that by (5.1) we have identity $m(\Lambda, \mu) = 0$, $\mu \in \Xi(\Lambda) \setminus \overline{J(D)}$. Moreover, as above, $m(\Lambda) \leqslant \overline{n}(\Lambda) < \infty$ and $\sigma(\Lambda) = 0$. Then by Theorem 2.1 we have $d_3 = \{d_{k,n,3}\} \in Q(D, \Lambda)$.

If $g_1 \neq 0$, then by Theorem 2.4

$$g_1(z) = \sum_{k=1,n=0}^{\infty,n_k-1} d_{k,n,1} z^n e^{\lambda_k z}, \qquad z \in \mathbb{C}, \qquad d_{k,n,1} = 0, \qquad \lambda_k, n_k \notin \Lambda_1,$$

and the series converges uniformly on each compact set in the plane. Theorem 2.1 ensures the inclusion $d_1 = \{d_{k,n,1}\} \in Q(\mathbb{C}, \Lambda) \subset Q(D, \Lambda)$. If $g_2 \neq 0$, then by Theorem 2.4

$$g_2(z) = \sum_{k=1,n=0}^{\infty,n_k-1} d_{k,n,2} z^n e^{\lambda_k z}, \qquad z \in D_0 \supset D, \qquad d_{k,n,2} = 0, \qquad \lambda_k, n_k \notin \Lambda_2,$$

and the series converges uniformly on compact sets in the domain D_0. By Theorem 2.1 we have $d_1 = \{d_{k,n,2}\} \in Q(D_0, \Lambda) \subset Q(D, \Lambda)$.

We let

$$d_{k,n} = d_{k,n,1} + d_{k,n,2} + d_{k,n,3}.$$

Then $d = \{d_{k,n}\} \in Q(D, \Lambda)$. Therefore, in view of Lemma 2.1

$$g(z) = g_1(z) + g_2(z) + g_3(z) = \sum_{k=1,n=0}^{\infty,n_k-1} d_{k,n,3} z^n e^{\lambda_k z}, \qquad z \in D,$$

and the series converges uniformly on compact sets in the domain D. Thus, Assertion 1) holds. The proof is complete. □

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