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PROBLEM OF DETERMINING CONVOLUTION KERNEL FOR HYPERBOLIC INTEGRO–DIFFERENTIAL EQUATION IN BOUNDED DOMAIN

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Abstract. We consider the inverse problem on determining the kernel of an integral term in an integro–differential equation. The problem of determining the memory kernel in the wave process is reduced to a nonlinear Volterra integral equation of the first kind of convolution type, then over determination condition it brings to the Volterra integral equation of the second kind. The method of contraction maps proves the unique solvability of the problem in the space of continuous functions with weight norms, and an estimate of the conditional stability of the solution is obtained.

Keywords: Integro–differential equation, inverse problem, kernel, spectral problem, fixed point theorem, Gronwall inequality.

Mathematics Subject Classification: 35A01, 35B65, 35R09

1. Introduction and statement of problem

Inverse problems for integro–differential equations arise in many areas of applied research, such as electrodynamics, acoustics, quantum scattering theory, geophysics, astronomy, etc. The presence of an integral term in the equation is due to the need to take into consideration the dependence of the instantaneous values of the characteristics of the described object on their respective previous values, that is, the impact of the prehistory on the current state of the system. Mathematically, convolution type integrals are added to the right–hand sides of the corresponding equations, which describe the delay phenomenon. The theory of inverse problems for integro–differential equations of hyperbolic type is a rapidly developing and relatively new area of modern mathematical physics. Methods for solving inverse problems for second order partial differential equations of various types can be found in [12], [25], [26] and there is an extensive bibliography in these publications.

The first results in the theory of inverse problems for integro–differential equations date back to the 90s of the last century and are presented in the studies by Italian mathematicians A. Lorenzi, E. Sinestrari, and E. Paparoni [18], [20], [16]. At present, one–dimensional and multidimensional inverse problems are the objects of study of many scientists to determine the kernel of the integral term of integro–differential equations. In [21], [22], one-dimensional inverse problems of finding a kernel with distributed sources of perturbations were studied; [23], [29], [31], [30], [27], [3], [24] are devoted to the study of inverse problems of integro–differential equations with a delta function on the right hand side or on the boundary condition. In these articles, the basic method for proving the existence and uniqueness theorems for the solution to the inverse problem is the contracting mapping principle. Stability estimates were also obtained.

J.Sh. Safarov, D.K. Durdiev, A.A. Rahmonov, Problem of determining convolution kernel for hyperbolic integro–differential equation in bounded domain.

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In [4], [11], [2], for multidimensional inverse problems of finding the kernel in second-order hyperbolic integro–differential equations, theorems of unique local solvability were proved in the class of analytic functions in space variables and continuous - in time variables. The studies in [15], [5], [10], [28], [9], [8], [7] are devoted to the global unique solvability of two–dimensional inverse problems when the kernel of the integral term weakly depends on the space variable. The papers [17], [6], [1] deal with numerical methods for solving inverse problems of determining the kernel in hyperbolic integro–differential equations. In [14], inverse problems of determining memory kernels in a heat flow were studied for various types of observations.

In this paper, we study the inverse problem, which consists in finding a solution and a one–dimensional convolution kernel of the integral term in a non–homogeneous integro–differential equation with a wave operator in the main part, from the conditions that make up the direct problem and some additional condition. As the condition, we consider the trace of the solution of the forward problem at a fixed point in the plane for the entire time interval.

Consider the following integro–differential equation:

$$w_{tt} - Lw = \int_0^t k(t-\theta)Lw(x,\theta)\,d\theta, \qquad (x,t) \in Q, \tag{1.1}$$

with the initial conditions

$$w|_{t=0} = \varphi(x), \qquad w_t|_{t=0} = \eta(x), \qquad x \in \overline{\Omega}, \tag{1.2}$$

and boundary conditions

$$w(x,t) = 0, \quad (x,t) \in \partial Q. \tag{1.3}$$

where

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} - c(x)$$

is the uniformly elliptic operator, $n \geqslant 1$, and the coefficients satisfy the conditions

$$a_{ij}(x) = a_{ji}(x), \qquad c(x) \geqslant 0, \qquad x \in \bar{\Omega}.$$

By $Q := \Omega \times (0,T]$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, and $\partial Q := \partial\Omega \times [0,T]$, $\varphi(x)$ and $\psi(x)$ are given functions. The direct problem is to find the function $u(x,t)$ by (1.1), (1.2), (1.3) with known $k(t)$.

Definition 1.1. *A classical solution of the initial and boundary value problem (1.1), (1.2), (1.3) is a function $u(x,t)$, which is twice continuously differentiable in the closed cylinder Q and satisfies all the conditions problem (1.1), (1.2), (1.3) in the usual sense.*

The inverse problem consists in determining the unknown coefficient $k(t)$, $t > 0$, from the available additional data on the solution to the direct problem at some point $x_0 \in \Omega$,

$$\left(w(x,t) + \int_0^t k(t-\theta) w(x,\theta)\,d\theta \right)\Big|_{x=x_0} = h_1(t), \qquad 0 \leqslant t \leqslant T, \tag{1.4}$$

where $h_1(t)$ is the given function. For simplicity, we assume that

$$k(0) = k'(0) = 0. \tag{1.5}$$

Definition 1.2. *The solution to the inverse problem (1.1), (1.2), (1.3), (1.4) is $u(x,t)$ and $k(t)$ from classes $C^2\big(Q\big) \cap C^1\big(\overline{Q}\big)$ and $C[0,T]$, respectively, which satisfying relations (1.1), (1.2), (1.3), (1.4).*

First, we study the direct problem.

2. Direct problem

We introduce a new function $u(x,t)$ by the relation

$$u(x,t) := \left(w(x,t) + \int_0^t k(t-\theta) w(x,\theta)\, d\theta \right) e^{-\frac{k(0)t}{2}}.$$

It can readily be verified [15] that the function $w(x,t)$ is expressed via $u(x,t)$ by the formula

$$w(x,t) = e^{\frac{k(0)t}{2}} u(x,t) + \int_0^t r(t-\theta) e^{\frac{k(0)\theta}{2}} u(x,\theta)\, d\theta,$$

where

$$r(t) = -k(t) - \int_0^t k(t-\theta) r(\theta)\, d\theta, \qquad t \geqslant 0.$$

It follow from formula (1.5) that $r(0) = r'(0) = 0$. For the new functions $u(x,t)$ and $r(t)$ equations (1.1), (1.2), (1.3) become

$$u_{tt} - Lu + \int_0^t R(t-\theta) u(x,\theta) d\theta = 0, \tag{2.1}$$

$$u\,|_{t=0} = \varphi(x), \quad u_t\,|_{t=0} = \eta(x) - \frac{r(0)}{2}\varphi(x) =: \psi(x), \quad x \in \overline{\Omega} \tag{2.2}$$

$$u \equiv 0, \qquad (x,t) \in \partial Q. \tag{2.3}$$

The additional condition (1.4) reads as

$$u(x_0,t) = h_1(t) e^{\frac{r(0)t}{2}} =: h(t). \tag{2.4}$$

We begin our study of the forward problem by considering the following spectral problem

$$Lv + \lambda^2 v = 0, \qquad x \in \Omega, \tag{2.5}$$

$$v\big|_{\partial\Omega} = 0. \tag{2.6}$$

It is known [13] that spectral problem (2.5), (2.6) in $L_2(\Omega)$ has a complete set of orthonormal eigenfunctions $v_m(x)$, $m \geqslant 1$, and the corresponding eigenvalues λ_m are positive and form a countable set.

The solution to problem (2.1), (2.2), (2.3) is sought in the form of a Fourier series

$$u(x,t) = \sum_{m=1}^{\infty} A_m(t) v_m(x), \tag{2.7}$$

where $v_m(x)$ are the eigenfunctions of problem (2.5), (2.6) and $A_m(t)$ are Fourier coefficients defined by the formula

$$A_m(t) = \int_\Omega u(x,t) v_m(x)\, dx. \tag{2.8}$$

Substituting (2.7) into equations (1.1), (1.2), we get the problem for the Fourier coefficients $A_m(t)$

$$A_m''(t) + \lambda_m^2 A_m(t) + \int_0^t k(t-\theta) A_m(\theta)\, d\theta = 0, \tag{2.9}$$

$$A_m(0) = \varphi(x), \qquad A_m'(0) = \psi_m. \tag{2.10}$$

where φ_m, ψ_m, are the Fourier coefficients of function $\varphi(x)$, $\psi(x)$

$$\varphi(x) = \sum_{m=1}^{\infty} \varphi_m v_m(x), \qquad \psi(x) = \sum_{m=1}^{\infty} \psi_m v_m(x).$$

Second order integro–differential equation (2.9) is equivalent to

$$A_m(t) = \frac{1}{\lambda_m} \int_0^t \sin \lambda_m(t-s) G_m(s)\, ds + \left(C_m \cos \lambda_m t + D_m \sin \lambda_m t\right), \tag{2.11}$$

where

$$G_m(t) := \int_0^t k(t-\theta) A_m(\theta)\, d\theta,$$

and C_m and D_m are arbitrary constants. Hence, using initial conditions (1.2), we obtain

$$A_m(0) = C_m, \qquad A'_m(0) = \lambda_m D_m. \tag{2.12}$$

It follows from formulas (2.10) and (2.12) that

$$C_m = \varphi_m, \qquad D_m = \frac{1}{\lambda_m}\psi_m.$$

In terms of these notation, we rewrite equation (2.11) as

$$A_m(t) = \varphi_m \cos \lambda_m t + \frac{\psi_m}{\lambda_m} \sin \lambda_m t + \frac{1}{\lambda_m} \int_0^t \sin \lambda_m(t-s) \int_0^s k(s-\theta) A_m(\theta)\, d\theta ds. \tag{2.13}$$

Thus, we obtained an integral equation of Volterra type of the second kind for the function $A_m(t)$. The theory of integral equations states this equation is uniquely solvable and the solution can be obtained by the method of successive approximations.

Now we are going to prove two lemmas, which will be employed in the proof of our main result.

Lemma 2.1. *Let $A_m(t)$, $A^1_m(t)$, $A^2_m(t)$ be the solutions of Equation (2.13) with the functions $k(t)$, $k^1(t)$, $k^2(t)$. The estimates hold*

$$|A_m(t)| \leqslant \left(|\varphi_m| + \frac{1}{\lambda_m}|\psi_m|\right) e^{\frac{\|k\| T^2}{2\lambda_1}}, \tag{2.14}$$

$$|A''_m(t)| \leqslant \left(|\varphi_m| + \frac{1}{\lambda_m}|\psi_m|\right)\left(\lambda_m^2 + \|k\| T\left(1 + \frac{T}{2}\lambda_m\right) e^{\frac{\|k\| T^2}{2\lambda_1}}\right), \tag{2.15}$$

$$|A^1_m(t) - A^2_m(t)| \leqslant \frac{1}{\lambda_m}\left(|\varphi_m| + \frac{1}{\lambda_m}|\psi_m|\right) e^{\frac{(\|k^2\| + \|k^1\|) T^2}{2\lambda_1}} \frac{T^2}{2} \|k^1 - k^2\|, \tag{2.16}$$

where

$$\|k\| = \max_{0 \leqslant t \leqslant T} |k(t)|.$$

Proof. Formula (2.12) yields

$$|A_m(t)| = \left| \varphi_m \cos \lambda_m t + \frac{1}{\lambda_m} \psi_m \sin \lambda_m t + \frac{1}{\lambda_m} \int_0^t \sin \lambda_m (t-s) G_m(s)\, ds \right|$$

$$\leqslant |\varphi_m| + \frac{1}{\lambda_m}|\psi_m| + \frac{1}{\lambda_m}\left| \int_0^t \sin \lambda_m(t-s) \int_0^s k(s-\theta) A_m(\theta)\, d\theta ds \right|$$

$$\leqslant |\varphi_m| + \frac{1}{\lambda_m}|\psi_m| + \frac{1}{\lambda_m}\|k\| \int\limits_0^t (t-\theta)|A_m(\theta)|\,d\theta.$$

By Grönwall integral inequality we arrive at (2.14). Differentiating equation (2.13) twice with respect to t and using estimate (2.14), we obtain (2.15).

For the solutions $A_m^1(t)$, $A_m^2(t)$ to equation (2.13) corresponding to functions $k^1(t)$, $k^2(t)$ we have

$$\begin{aligned}
|A_m^1(t) - A_m^2(t)| \leqslant & \left| \frac{1}{\lambda_m} \int\limits_0^t \sin\lambda_m(t-\alpha) \int\limits_0^\alpha \left(k^1(\tau)A_m^1(\alpha-\tau) - k^2(\tau)A_m^2(\alpha-\tau)\right)\,d\tau d\alpha \right| \\
\leqslant & \frac{1}{\lambda_m} \int\limits_0^t |\sin\lambda_m(t-\alpha)| \int\limits_0^\alpha \Big(\left|k^1(\tau)\right| \left|A_m^1(\alpha-\tau) - A_m^2(\alpha-\tau)\right| \\
& \qquad + \left|A_m^2(\alpha-\tau)\right| \left|k^1(\tau) - k^2(\tau)\right| \Big)\,d\tau d\alpha \qquad (2.17)\\
\leqslant & \frac{1}{\lambda_m} \int\limits_0^t |\sin\lambda_m(t-\alpha)| \int\limits_0^\alpha \left|A_m^2(\alpha-\tau)\right| \left|k^1(\tau) - k^2(\tau)\right|\,d\tau d\alpha \\
& + \frac{1}{\lambda_m} \int\limits_0^t |\sin\lambda_m(t-\alpha)| \int\limits_0^\alpha \left|k^1(\tau)\right| \left|A_m^1(\alpha-\tau) - A_m^2(\alpha-\tau)\right|\,d\tau d\alpha,
\end{aligned}$$

where we have used the obvious inequality

$$|\varphi_k^1\varphi_s^1 - \varphi_k^2\varphi_s^2| \leqslant |\varphi_k^1 - \varphi_k^2||\varphi_s^1| + |\varphi_k^2||\varphi_s^1 - \varphi_s^2|. \qquad (2.18)$$

Now we estimate the right side of inequality (2.18). To estimate the first term, we use formula (2.14)

$$\begin{aligned}
& \frac{1}{\lambda_m} \int\limits_0^t |\sin\lambda_m(t-\alpha)| \int\limits_0^\alpha \left|A_m^2(\alpha-\tau)\right| \left|k^1(\tau) - k^2(\tau)\right| d\tau\,d\alpha \\
& \qquad \leqslant \frac{1}{\lambda_m}\frac{T^2}{2}\left(|\varphi_m| + \frac{1}{\lambda_m}|\psi_m|\right) e^{\frac{\|k\|T^2}{2\lambda_m}} \|k^1 - k^2\|, \qquad (2.19)
\end{aligned}$$

where

$$\|k^1 - k^2\| = \max_{0\leqslant t\leqslant T} |k^1(t) - k^2(t)|.$$

To estimate the second term in (2.17), we change the order of integration

$$\begin{aligned}
& \frac{1}{\lambda_m} \int\limits_0^t |\sin\lambda_m(t-\alpha)| \int\limits_0^\alpha \left|k^1(\tau)\right| \left|A_m^1(\alpha-\tau) - A_m^2(\alpha-\tau)\right| d\tau\,d\alpha \\
& \qquad \leqslant \frac{1}{\lambda_m} \int\limits_0^t \left|A_m^1(\tau) - A_m^2(\tau)\right| \int\limits_\tau^t \left|k^1(\alpha-\tau)\right| d\alpha\,d\tau \qquad (2.20)\\
& \qquad = \frac{1}{\lambda_m} \int\limits_0^t \left|A_m^1(\tau) - A_m^2(\tau)\right| k_1(t-\tau)\,d\tau,
\end{aligned}$$

where

$$k_1(t) = \int_0^t \left|k^1(\alpha)\right| d\alpha.$$

Substituting (2.19) and (2.20) into (2.17), we obtain

$$|A_m^1(t) - A_m^2(t)|$$
$$\leqslant \frac{1}{\lambda_m}\left(\frac{T^2}{2}\left(|\varphi_m| + \frac{1}{\lambda_m}|\psi_m|\right) e^{\frac{\|k\|T^2}{2\lambda_m}} \|k^1 - k^2\| + \int_0^t \left|A_m^1(\tau) - A_m^2(\tau)\right| k_1(\tau)\, d\tau\right).$$

Applying the Grönwall lemma to this inequality, we obtain (2.16). The proof is complete. □

The main result of this section is the following theorem.

Theorem 2.1. *Let $k(t) \in C[0,T]$ and*

1) $a_{ij} \in C^{[\frac{N}{2}]+2}(\bar{\Omega}) \cap C^{1,\gamma}(\Omega)$, $c(x) \in C^{[\frac{N}{2}]+1}(\bar{\Omega}) \cap C^{0,\gamma}(\Omega)$, $0 < \gamma \leqslant 1$;
2) $\varphi \in H^{[\frac{N}{2}]+3}(\Omega)$, $\psi \in H^{[\frac{N}{2}]+2}(\Omega)$;
3) $\varphi, L\varphi, \ldots, L^{[\frac{N+4}{4}]}\varphi \in H_0^1(\Omega)$, $\psi, L\psi, \ldots, L^{[\frac{N+2}{4}]}\psi \in H_0^1(\Omega)$.

Then there is a unique classical solution to problem (1.1), (1.2), (1.3), where $H^m(\Omega)$ is the Sobolev space.

Proof. Differentiating formally the series (2.7) with respect to x and t, we obtain

$$u_{tt}(x,t) = \sum_{m=1}^{\infty} A_m''(t) v_m(x), \tag{2.21}$$

$$Lu(x,t) = \sum_{m=1}^{\infty} A_m(t) L v_m(x) = -\sum_{m=1}^{\infty} \lambda_m^2 A_m(t) v_m(x). \tag{2.22}$$

Let us prove the convergence of series (2.7), (2.21), (2.22).

It follows from (2.7) that

$$|u(x,t)| = \left|\sum_{m=1}^{\infty} A_m(t) v_m(x)\right| \leqslant \sum_{m=1}^{\infty} |A_m(t)||v_m(x)|.$$

Using estimate (2.14), we obtain

$$\begin{aligned} |u(x,t)| &\leqslant \sum_{m=1}^{\infty} |v_m(x)| \left(|\varphi_m| + \frac{1}{\lambda_m}|\psi_m|\right) e^{\frac{\|k\|t^2}{2\lambda_m}} \\ &\leqslant e^{\frac{\|k\|t^2}{2\lambda_1}} \left(\sum_{m=1}^{\infty} |\varphi_m||v_m(x)| + \sum_{m=1}^{\infty} \frac{|\psi_m|}{\lambda_m}|v_m(x)|\right). \end{aligned} \tag{2.23}$$

In the same way for series (2.21) and (2.22) we get

$$\begin{aligned} |u_{tt}(x,t)| &\leqslant \sum_{m=1}^{\infty} \lambda_m^2 |\varphi_m v_m(x)| + \sum_{m=1}^{\infty} \lambda_m |\psi_m v_m(x)| \\ &+ \|k\| t e^{\frac{\|k\|t^2}{2\lambda_1}} \left(\sum_{m=1}^{\infty} \left(1 + \frac{t}{2}\lambda_m\right)|\varphi_m v_m(x)| + \sum_{m=1}^{\infty} \left(\frac{1}{\lambda_m} + \frac{t}{2}\right)|\psi_m v_m(x)|\right) \end{aligned} \tag{2.24}$$

and

$$|Lu(x,t)| \leqslant e^{\frac{\|k\|t^2}{2\lambda_1}} \left(\sum_{m=1}^{\infty} \lambda_m^2 |\varphi_m||v_m(x)| + \sum_{m=1}^{\infty} \lambda_m |\psi_m v_m(x)|\right). \tag{2.25}$$

Under Conditions 1)–3), the convergence of the series in the right hand sides of inequalities (2.23), (2.24), (2.25) is implied by Theorem 8, see [13]. Then series (2.7), (2.21) and (2.22) converge uniformly and absolutely in $\overline{Q}$. Thus, the function $u(x,t)$ defined by series (2.7) solves problem (1.1), (1.2), (1.3) in $\overline{Q}$.

We proceed to proving the uniqueness of this solution. For $\varphi(x) \equiv 0$, $\psi(x) \equiv 0$ we obtain $\varphi_m \equiv 0$ and $\psi_m \equiv 0$. Then formula (2.13) implies that $A_m \equiv 0$ since A_m is a solution to the homogeneous equation

$$A_m(t) = \frac{1}{\lambda_m} \int_0^t A_m(\tau) d\tau \int_0^{t-\tau} \sin \lambda_m (t - \tau - s) k(s)\, ds.$$

Substituting $A_m \equiv 0$ into equation (2.8), we obtain

$$\int_\Omega u(x,t) v_m(x) dx = 0.$$

Since the system v_m is complete in space $L_2(\Omega)$, we have $u(x,t) = 0$ almost everywhere in Ω for each $t \in [0,T]$. Since the function u is smooth, we conclude that $u(x,t) \equiv 0$ in $\overline{Q}$. The proof is complete. □

3. Solvability of inverse problem

In what follows we rewrite formula (2.13) as

$$A_m(t) - (R_m A_m)(t) = \Phi_m(t), \tag{3.1}$$

where

$$\Phi_m(t) = \varphi_m \cos \lambda_m t + \frac{1}{\lambda_m} \psi_m \sin \lambda_m t,$$

$$(R_m A_m)(t) = \frac{1}{\lambda_m} \int_0^t r_m(t - \theta) A_m(\theta) d\theta,$$

$$r_m(t) = \int_0^t \sin \lambda_m (t - s) k(s) ds.$$

If we use formula (2.7), then the additional condition (2.4) becomes

$$\sum_{m=1}^{\infty} A_m(t) v_m(x_0) = h(t).$$

Substituting (3.1) into the above formula, we obtain the Volterra integral equation

$$\int_0^t k(s) M(k)(t - s) ds = f(t), \tag{3.2}$$

where

$$M(k)(t) = \sum_{m=1}^{\infty} \frac{1}{\lambda_m} v_m(x_0) \int_0^t A_m(\theta) \sin \lambda_m (t - \theta) d\theta,$$

$$f(t) = h(t) - \sum_{m=1}^{\infty} \Phi_m(t) v_m(x_0).$$

We rewrite equation (3.2) as

$$h(t) = \int_0^t k(s)M(k)(t-s)ds + F(t), \tag{3.3}$$

where

$$F(t) := \sum_{m=1}^{\infty} \Phi_m(t) v_m(x_0).$$

This implies

$$h(0) = \sum_{m=1}^{\infty} \varphi_m v_m(x_0) = \varphi(x_0). \tag{3.4}$$

To obtain the Volterra integral equation of the second kind for the function $k(t)$, we differentiate equation (3.3) three times

$$h'(t) = \int_0^t k(s)M'(k)(t-s)ds + F'(t),$$

where

$$M'(k)(t) = \sum_{m=1}^{\infty} v_m(x_0) \int_0^t A_m(\theta) \cos \lambda_m (t-\theta) d\theta.$$

Hence,

$$h'(0) = \sum_{m=1}^{\infty} \psi_m v_m(x_0) = \psi(x_0), \tag{3.5}$$

$$h''(t) = \int_0^t k(s)M''(k)(t-s)ds + F''(t),$$

where

$$M''(k)(t) = \sum_{m=1}^{\infty} v_m(x_0) A_m(t) - \sum_{m=1}^{\infty} \lambda_m v_m(x_0) \int_0^t A_m(\theta) \sin \lambda_m (t-\theta) d\theta$$

$$= h(t) - \sum_{m=1}^{\infty} \lambda_m v_m(x_0) \int_0^t A_m(\theta) \sin \lambda_m (t-\theta) d\theta.$$

It follow from (3.4) that

$$h''(0) = L\varphi(x_0). \tag{3.6}$$

Furthermore,

$$h'''(t) = k(t) \sum_{m=1}^{\infty} v_m(x_0) \varphi_m + \int_0^t k(s)M'''(k)(t-s)ds + F'''(t),$$

where

$$M'''(k)(t) = h'(t) - \sum_{m=1}^{\infty} \lambda_m^2 v_m(x_0) \int_0^t A_m(\theta) \cos \lambda_m (t-\theta) d\theta. \tag{3.7}$$

Then we obtain the integral equation for the function $k(t)$

$$k(t) = \frac{1}{h(0)}\left(h'''(t) - F'''(t) - \int_0^t k(s)M'''(k)(t-s)ds\right), \qquad t \in [0,T]. \tag{3.8}$$

Theorem 3.1. *Let $h(t) \in C^3[0,T]$, $|h(0)| > 0$ and*

1) $\varphi(x) \in H^{\left[\frac{N}{2}\right]+4}(\Omega)$, $\psi(x) \in H^{\left[\frac{N}{2}\right]+3}(\Omega)$;

2) $\varphi,\ L\varphi, \dots, L^{\left[\frac{N+4}{4}\right]}\varphi \in H_0^1(\Omega)$, $\psi,\ L\psi, \dots, L^{\left[\frac{N+2}{4}\right]}\psi \in H_0^1(\Omega)$,

and in addition, conditions (3.4), (3.5), (3.6) be satisfied. Then there is a unique solution to the inverse problem (1.1), (1.2), (1.3), (1.4) satisfying equation (3.8).

Proof. We rewrite the equation (3.8) as

$$k = Bk,$$

where the operator B reads as

$$Bk = k_0 - \frac{1}{h(0)}\int_0^t M'''(k)(t-\tau)k(\tau)d\tau,$$

$$k_0(t) = \frac{1}{h(0)}\left(h'''(t) - F'''(t)\right).$$

It is easy to see the assumptions of the theorem implies that the function $F'''(t)$ is continuous. Similarly, the kernel $M'''(k)$ is a continuous function.

By C_σ we denote the Banach space of continuous functions generated by the family of weight norms

$$\|k\|_\sigma = \max_{t\in[0,T]} |k(t)e^{-\sigma t}|, \quad \sigma \geqslant 0.$$

It is obvious that for $\sigma = 0$ this space coincides with the space of continuous functions with the usual norm. This norm is denoted by $\|k\|$. Because of the inequality

$$e^{-\sigma t}\|k\| \leqslant \|k\|_\sigma \leqslant \|k\|,$$

the norms $\|k\|_\sigma$ and $\|k\|$ are equivalent for each fixed $T \in (0,\infty)$. The number σ will be chosen later. Let

$$P_\sigma(k_0, \|k_0\|) := \{k \in C_\sigma : \|k - k_0\|_\sigma \leqslant \|k_0\|\}$$

be the ball of radius $\|k_0\|$ centered at the point k_0 of some weighted space $C_\sigma(\sigma \geqslant 0)$. It is easy to see that for $P_\sigma(k_0, \|k_0\|)$ the estimate holds

$$\|k\|_\sigma \leqslant \|k_0\|_\sigma + \|k_0\| \leqslant 2\|k_0\|.$$

Let $k(t) \in P_\sigma(k_0, \|k_0\|)$. Let us show that, under an appropriate choice of $\sigma > 0$, operator B transforms a ball into a ball, that is, $B \in P_\sigma(k_0, \|k_0\|)$. In order to do this, we verify the assumptions of Banach fixed point theorem [19]. We have

$$\begin{aligned}
\|Bk - k_0\|_\sigma &= \max_{t\in[0,T]} |(Bk - k_0)e^{-\sigma t}| \\
&= \max_{t\in[0,T]} \left|\frac{1}{h(0)}\int_0^t M'''(k)(t-\tau)k(\tau)e^{-\sigma\tau}e^{-\sigma(t-\tau)}d\tau\right| \\
&\leqslant \frac{2T}{|h(0)|}\left(\|h'\| + T\sum_{m=1}^{\infty}\lambda_m v_m(x_0)\left(|\varphi_m| + \frac{1}{\lambda_m}|\psi_m|\right)e^{\frac{\|k\|T^2}{2\lambda_m}}\right)\frac{\|k_0\|}{\sigma} \\
&=: \frac{\|k_0\|}{\sigma}\alpha_0.
\end{aligned}$$

Under Conditions 1)–2), the convergence of the series

$$\sum_{m=1}^{\infty} \lambda_m v_m(x_0) \left(|\varphi_m| + \frac{1}{\lambda_m} |\psi_m| \right) e^{\frac{\|k\| T^2}{2\lambda_m}}$$

is implied by [13, Thm. 8]. Choosing

$$\sigma \geqslant \alpha_0$$

we see that B maps the ball $P_\sigma(k_0, \|k_0\|)$ into the ball $P_\sigma(k_0, \|k_0\|)$.

We have

$$\begin{aligned}
\|Bk^1 - Bk^2\|_\sigma &= \max_{t\in[0,T]} |(Bk^1 - Bk^2) e^{-\sigma t}| \\
&= \max_{t\in[0,T]} \left| \frac{1}{h(0)} \int_0^t \Big(M'''(k^1)(t-\tau)k^1(\tau) \right. \\
&\qquad \left. - M'''(k^2)(t-\tau)k^2(\tau) \Big) e^{-\sigma\tau} e^{-\sigma(t-\tau)}\, d\tau \right| \\
&\leqslant \max_{t\in[0,T]} \frac{1}{|h(0)|} \int_0^t \Bigg| \Bigg(h'(t-\tau) \\
&\qquad - \int_0^t \sum_{m=1}^{\infty} \lambda_m^2 v_m(x_0) A_m^1(\alpha-\tau) \cos\lambda_m(t-\alpha)\, d\alpha \Bigg) k^1(\tau) \\
&\qquad - \Bigg(h'(t-\tau) \\
&\qquad - \int_0^t \sum_{m=1}^{\infty} \lambda_m^2 v_m(x_0) A_m^2(\alpha-\tau) \cos\lambda_m(t-\alpha) d\alpha \Bigg) \\
&\qquad \cdot k^2(\tau) e^{-\sigma\tau} e^{-\sigma(t-\tau)} \Bigg| \, d\tau \\
&\leqslant \max_{t\in[0,T]} \frac{1}{|h(0)|} \int_0^t \Big(\big|h'(t-\tau)\big| \big|k^1(\tau) - k^2(\tau)\big| e^{-\sigma\tau} e^{-\sigma(t-\tau)} \\
&\qquad + \int_0^t \sum_{m=1}^{\infty} \lambda_m^2 v_m(x_0) \Big| \cos\lambda_m(t-\alpha) \big[A_m^1(\alpha-\tau) k^1(\tau) \\
&\qquad - A_m^2(\alpha-\tau) k^2(\tau) \big] \Big| e^{-\sigma\tau} e^{-\sigma(t-\tau)} d\alpha \Big) d\tau.
\end{aligned} \tag{3.9}$$

The first term in the right hand side of (3.9) is estimated as

$$\max_{t\in[0,T]} \frac{1}{|h(0)|} \int_0^t \Big(\big|h'(t-\tau)\big| \big|k^1(\tau) - k^2(\tau)\big| e^{-\sigma\tau} e^{-\sigma(t-\tau)} \Big)\, d\tau \leqslant \frac{\|h'\| T}{|h(0)|} \cdot \frac{\|k^1 - k^2\|}{\sigma}. \tag{3.10}$$

To estimate the second term, we use (2.18)

$$\begin{aligned}
&\max_{t\in[0,T]} \frac{1}{|h(0)|} \int_0^t \int_0^t \Big(\sum_{m=1}^{\infty} \lambda_m^2 v_m(x_0) \Big| \cos\lambda_m(t-\alpha)\big[A_m^1(\alpha-\tau)k^1(\tau) \\
&\qquad - A_m^2(\alpha-\tau)k^2(\tau)\big]\Big| e^{-\sigma\tau} e^{-\sigma(t-\tau)}\, d\alpha \Big)\, d\tau \\
&\leqslant \max_{t\in[0,T]} \frac{1}{|h(0)|} \int_0^t \int_0^t \Big(\sum_{m=1}^{\infty} \lambda_m^2 v_m(x_0) \big| \cos\lambda_m(t-\alpha)\big| \\
&\qquad \cdot \Big(\big|A_m^1(\alpha-\tau) - A_m^2(\alpha-\tau)\big| \, \big|k^1(\tau)\big| \\
&\qquad + \big|A_m^2(\alpha-\tau)\big| \, \big|k^1(\tau)-k^2(\tau)\big| \Big) e^{-\sigma\tau} e^{-\sigma(t-\tau)} d\alpha \Big)\, d\tau \\
&\leqslant \frac{T^2}{|h(0)|} \sum_{m=1}^{\infty} \lambda_m^2 v_m(x_0) \Big(T^2 \|k_0\| e^{\int_0^t |k_1(\theta)|\, d\theta} + 1 \Big) \Big(|\varphi_m| + \frac{1}{\lambda_m} |\psi_m| T \Big) e^{\frac{\|k\|T^2}{2\lambda_1}} \frac{\|k^1-k^2\|}{\sigma}.
\end{aligned} \tag{3.11}$$

Here we have used estimates (2.14) and (2.16). Substituting (3.10) and (3.11) in (3.9), we obtain

$$\begin{aligned}
\|Bk^1 - Bk^2\|_\sigma &= \max_{t\in[0,T]} \big|(Bk^1 - Bk^2)e^{-\sigma t}\big| \\
&\leqslant \frac{T}{|h(0)|} \Big(\|h'\| + T \sum_{m=1}^{\infty} \lambda_m^2 v_m(x_0) \Big(T^2 \|k_0\| e^{\int_0^t |k_1(\theta)|\, d\theta} + 1 \Big) \\
&\qquad \cdot \Big(|\varphi_m| + \frac{1}{\lambda_m} |\psi_m| T \Big) e^{\frac{\|k\|T^2}{2\lambda_1}} \Big) \frac{\|k^1-k^2\|}{\sigma} =: \frac{\alpha_1}{\sigma} \|k^1 - k^2\|.
\end{aligned}$$

The Conditions 1)–2) imply the convergence of the series

$$\sum_{m=1}^{\infty} \lambda_m^2 v_m(x_0) \Big(T^2 \|k_0\| e^{\int_0^t |k_1(\theta)|\, d\theta} + 1 \Big) \Big(|\varphi_m| + \frac{1}{\lambda_m} |\psi_m| T \Big) e^{\frac{\|k\|T^2}{2\lambda_1}}$$

by [13, Thm. 8]. The obtained estimates yield that if the number σ is chosen by condition $\sigma > \max\{\alpha_0, \alpha_1\}$, then the operator B is contracting on $P_\sigma(k_0, \|k_0\|)$. Then, according to the Banach principle, equation (3.3) also has a unique solution in $P_\sigma(k_0, \|k_0\|)$ for each fixed $T > 0$ [19]. □

4. Estimate of conditional stability

Let $K(k^0)$ be the set of functions $k(t) \in C[0,T]$ the inequality $\|k(t)\|_{C[0,T]} \leqslant k^0$ satisfying for $t \in [0,T]$ with a fixed positive constant k^0. This constant was defined in (4.2).

Theorem 4.1. *Let the conditions of Theorem 3.1 be satisfied and let k_1 and k_2 be two solutions to the inverse problem (1.1), (1.2), (1.3), (1.4) corresponding to two data sets $\{\varphi^1, \psi^1, h^1\}$ and $\{\varphi^2, \psi^2, h^2\}$. Then the stability estimate*

$$\|k^1 - k^2\|_{C[0,T]} \leqslant C\Big(\|\tilde\varphi\|_{H^{[\frac{N}{2}]+4}(\Omega)} + \|\tilde\psi\|_{H^{[\frac{N}{2}]+3}(\Omega)} \Big), \tag{4.1}$$

holds, where $C = C(k^0, T)$ is some positive constant.

Proof. Since the conditions of Theorem 3.1 are satisfied, the solution to equation (3.7) belongs to the set $P_\sigma(k_0, \|k_0\|)$ and

$$\max_{t\in[0,T]} |k(t)| \leqslant 2\|k_0\| := k^0. \tag{4.2}$$

Let ϕ^j, $j = 1, 2$, be a vector of solutions to (4.2) with the data sets $\{\varphi^j(x), \psi^j(x), h^j(t)\}$, $j = 1, 2$, respectively, that is, they solve the equations $\phi^j = A\phi^j$, $j = 1, 2$. The known functions $\psi^j(x)$, $j = 1, 2$, are involved in the terms of these integral equations in the appropriate way via the functions $M^j(k(t))$, $j = 1, 2$. In what follows, we denote the difference of two functions, the notation of which differ only by the superscript, by the same letter with the accent $\widetilde{\ }$; for example, $\widetilde{u} = u^1 - u^2$, $\widetilde{h} = h^1 - h^2$ etc. By (3.8) we obtain the equation for the function $\widetilde{k}(t)$

$$\widetilde{k}(t) = \frac{1}{h(0)}\left(\widetilde{h}'''(t) - \widetilde{F}'''(t) - \int\limits_0^t \left(k^1(s)\widetilde{M}'''(k)(t-s) + \widetilde{k}(s)M^{2'''}(k)(t-s)\right) ds\right], \qquad t \in [0, T],$$

where

$$\widetilde{M}'''[k](t) = \widetilde{h}'(t) - \sum_{m=1}^{\infty} \lambda_m^2 v_m(x_0) \int\limits_0^t \widetilde{A}_m(\theta) \cos\lambda_m(t-\theta)\, d\theta.$$

We note that the functions involved in this equation can be estimated by apriori information about the problem data. Using this apriori information and applying the Grönwall inequality, we obtain the estimate (4.1). The proof is complete. □

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