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NEW HERMITE — HADAMARD TYPE INTEGRAL INEQUALITIES VIA WEIGHTED OPERATORS

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Abstract. In this article we demonstrate new integral inequalities of the Hermite — Hadamard type for differentiable functions, which are (h, m)-convex of the second type. We show that many known results are particular cases of ours.

Keywords: convexity, weighted integrals, Hermite–Hadamard inequality, (h, m)-convex modified functions.

Mathematics Subject Classification: 26A51, 26D10, 26A51, 26D15

1. Introduction

Convex functions play a key role in mathematics and are used in many disciplines, from the optimization to the function theory. They form the foundation of various theoretical developments including integral inequalities such as the Hermite — Hadamard — Fejer inequality. Paper [29] briefly overviews convex functions and presents new results related to generalized integral operators in these areas.

Definition 1. *A function $\digamma : [\mu_1, \mu_2] \to \mathbb{R}$, is considered convex if the condition*

$$\digamma\left(\epsilon x + (1-\epsilon) y\right) \leqslant \epsilon \digamma\left(x\right) + (1-\epsilon) \digamma\left(y\right)$$

holds for all $x, y \in (\mu_1, \mu_2)$ and $\epsilon \in [0, 1]$.

It is worth emphasizing the importance of the Hermite — Hadamard inequality in many areas covering both theoretical and applied sciences. The multitude of studies conducted in recent few tens years related to this inequality serves as a testament to its widespread usefulness.

This inequality gives an estimate of the average value of a convex function of a real variable $\digamma$ defined on the interval $[\mu_1, \mu_2]$. In the literature it is given as follows:

$$\digamma\left(\frac{\mu_2 + \mu_1}{2}\right) \leqslant \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \digamma(\epsilon) d\epsilon \leqslant \frac{\digamma(\mu_1) + \digamma(\mu_2)}{2}.$$

Bessenyei and Páles in [4] investigated the classical Hermite — Hadamard inequality in the context of generalized higher–order convex functions. In [3], Alomari et al. provided some inequalities of Hadamard type for quasi–convex functions. In their study, Xi et al. [40] obtained Hermite — Hadamard inequalities for certain classes of convex functions using classical Euler poly–gamma, beta and gamma functions.

Klaricic et al. in [24] established numerous Hadamard type inequalities for differentiable m-convex and (α, m)-convex functions. In [5] the authors obtained new generalizations of Hadamard — type inequalities through fractional integral operators for functions whose absolute values of second derivatives are convex and take values at intermediate points. Shuang

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and Qi in [38] established new integral inequalities of the Hermite — Hadamard type for extended s-convex functions. In [11], Butt et al. introduced the concept of polynomial–harmonic exponential convex functions and thereby obtained new integral inequalities. Wu et al. [42] introduced novel Hermite — Hadamard type inequalities incorporating k-Riemann — Liouville fractional integrals. New versions of the Hermite — Hadamard inequalities for convex mappings using generalized fractional operators were obtained by Hyder et al. in [15]. Applications of the Hadamard integral inequality for fractional Hadamard integrals using GA-convexity are discussed by Latif in [26].

In [37], Set et al. obtained new inequalities of Hermite — Hadamard type by utilizing Atangana — Baleanu integral operators. Butt et al. in [12] utilized the Hermite — Hadamard inequality in the formulation of the symmetric quantum generalization. In [25], Korus at al. they obtained several extensions of the Hermite — Hadamard inequality using equations involving weighted integral operators containing fractional operators. In [6], Bayraktar and Nápoles established numerous inequalities for (h, m)-convex maps related to weighted integrals. Jarad et al. in [19] defined the weighted fractional operators on some spaces. In [13], Butt at al. introduced trapezoid inequalities for an F-convex function utilizing Katugampola fractional integral operators.

In [8], the concepts of first and second type modified (h, m)-convex function in the interval $I = [0, +\infty)$ were given; the classes $N^{s,1}_{h,m}(I)$ and $N^{s,2}_{h,m}(I)$ were respectively defined.

Definition 2. *Let $h : [0,1] \to (0,1]$ and $F : I \to I$. If the inequality*

$$F(\epsilon\mu_1 + m(1-\epsilon)\mu_2) \leqslant h^s(\epsilon)F(\mu_1) + m(1 - h^s(\epsilon))F(\mu_2)$$

holds for all $\epsilon \in [0,1]$, where $s \in [-1,1]$, $m \in [0,1]$ and $\mu_1, \mu_2 \in I$, then $F \in N^{s,1}_{h,m}(I)$.

Definition 3. *Let $h : [0,1] \to (0,1]$ and $F : I \to I$. If the inequality*

$$F(\epsilon\mu_1 + m(1-\epsilon)\mu_2) \leqslant h^s(\epsilon)F(\mu_1) + m(1 - h(\epsilon))^s F(\mu_2)$$

holds for all $\epsilon \in [0,1]$, where $s \in [-1,1]$, $m \in [0,1]$ and $\mu_1, \mu_2 \in I$, then $F \in N^{s,2}_{h,m}(I)$.

Remark 1. *In [6], [8], several special cases of the generalized class (h, m)-convexity related to known classes of convex functions in the literature were presented. For example, here are some of them:*

1. $(\epsilon, 1, 1)$, *then F is a classical convex function on $[0, +\infty)$([10]).*
2. $(\epsilon, m, 1)$, *then F is a m-convex function on $[0, +\infty)$([39]).*
3. $(\epsilon, 1, -1)$ *then F is a Godunova — Levin convex function on $[0, +\infty)$.*
4. (ϵ, m, s) *and $s \in (0, 1]$, then F is a (s, m)-convex function on $[0, +\infty)$ ([41]).*

The theory of fractional calculus plays an important role in solving mathematical g problems which model various complex systems and process a fractal structure (in living and non–living systems), [28], [33], [22]. We introduce the definition of the Riemann — Liouville fractional integral, where the parameters satisfy $0 \leqslant \mu_1 < \epsilon < \mu_2 \leqslant \infty$.

Definition 4. *Let $F \in L[\mu_1, \mu_2]$. Then the Riemann — Liouville fractional integrals of order $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ are defined by (right and left respectively):*

$$^{\alpha}I_{\mu_1+}F(x) = \frac{1}{\Gamma(\alpha)} \int_{\mu_1}^{x} (x-\epsilon)^{\alpha-1} F(\epsilon)\, d\epsilon, \qquad x > \mu_1$$

$$^{\alpha}I_{\mu_2-}F(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\mu_2} (\epsilon-x)^{\alpha-1} F(\epsilon)\, d\epsilon, \qquad x < \mu_2.$$

To study complex systems, it is necessary to create non-local compatible models for data of various types; in such cases, generalized fractional integral operators are an important tool for analyzing these systems. A number of recent works defined new generalized fractional integrals with various kernels, see, for instance, [15], [1], [16]. Following these works, we introduce the weighted integral operators, on which our work will rely, as discussed in [6].

Definition 5. *Let a function $w : [0,1] \to I$ has a piecewise continuous derivative w' on I and let $\digamma \in L[\mu_1, \mu_2]$. The weighted fractional integrals, denoted by (right and left respectively), are defined as follows:*

$$J^w_{\mu_1+}\digamma(\mu_2) = \int_{\mu_1}^{\mu_2} w'\left(\frac{\epsilon-\mu_1}{\mu_2-\mu_1}\right)\digamma(\epsilon)d\epsilon, \qquad J^w_{\mu_2-}\digamma(\mu_1) = \int_{\mu_1}^{\mu_2} w'\left(\frac{\mu_2-\epsilon}{\mu_2-\mu_1}\right)\digamma(\epsilon)d\epsilon. \tag{1.1}$$

Remark 2. *To gain a better grasp of the extent covered by Definition 5, let us explore certain scenarios of the kernel function w':*

1. *As $w(\epsilon)$ to 1, we get the Riemann definite integral.*
2. *As $w'(\epsilon) = \frac{(\mu_2-\mu_1)^{\alpha-1}\epsilon^{\alpha-1}}{\Gamma(\alpha)}$, we have the Riemann — Liouville integral operators.*
3. *With the correct choice of the kernel $w\prime$, we can obtain generalized integral operators from [35], in addition, integral operators from [18], [21] and right fractional integrals of a function $\digamma$ with respect to another function h on an interval (see [2]). From this definition one can also obtain fractional k-Riemann — Liouville integrals, see [27], by choosing the appropriate kernel $w\prime$.*

The structure of the article is as follows. In Section 1, we obtain a version of the Hermite — Hadamard inequality for the generalized integral operator defined above, see Definition 5. In Section 2, using the proven identities in terms of modified (h,m)-convex functions, we obtain several new generalized weighted integral inequalities for the integral operators described in Definition 5. It is shown that special cases of these results are inequalities of the Trapezoid and Midpoint Hermite — Hadamard type. Finally, Section 3 provides conclusions and future research prospects.

Certainly, there exist other well–known integral operators, both fractional and non–fractional, which can be derived as specific instances of the aforementioned ones. However, we leave the exploration of these to interested readers. This paper explores various forms of the Hermite — Hadamard inequality within the context of (h,m)-convex modified functions, utilizing generalized operators described in Definition 5.

2. Analogue of Hermite — Hadamard inequality

At the outset, we establish an integral inequality analogous to the Hermite-Hadamard inequality for the weighted generalized integral operator formulated above; this is established in the following theorem.

Theorem 2.1. *Let $\digamma : [\mu_1, \mu_2] \to \mathbb{R}$ be a positive function and $\digamma \in L[\mu_1, \mu_2]$. If $\digamma$ is a convex function on $[\mu_1, \mu_2]$, then the generalized integral operators (1.1) satisfies the inequalities*

$$\begin{aligned} \digamma\left(\frac{\mu_2+\mu_1}{2}\right)(w(1)-w(0)) &\leqslant \frac{\kappa+1}{2(\mu_2-\mu_1)}\left(J^w_{\frac{\mu_1+\kappa\mu_2}{\kappa+1}+}\digamma(\mu_2) + J^w_{\frac{\mu_2+\mu_1\kappa}{\kappa+1}-}\digamma(\mu_1)\right) \\ &\leqslant \frac{\digamma(\mu_1)+\digamma(\mu_2)}{2}(w(1)-w(0)), \end{aligned} \tag{2.1}$$

where $w \in C[0,1]$ with piecewise continuous derivative on I and $\kappa \in \mathbb{N}$.

Proof. By using the convexity property of function $\digamma$, we have

$$\digamma\left(\frac{1-\epsilon}{\kappa+1}\mu_1 + \frac{\kappa+\epsilon}{\kappa+1}\mu_2\right) \leqslant \frac{1-\epsilon}{\kappa+1}\digamma(\mu_1) + \frac{\kappa+\epsilon}{\kappa+1}\digamma(\mu_2)$$

$$F\left(\frac{\kappa+\epsilon}{\kappa+1}\mu_1+\frac{1-\epsilon}{\kappa+1}\mu_2\right)\leqslant\frac{\kappa+\epsilon}{\kappa+1}F(\mu_1)+\frac{1-\epsilon}{\kappa+1}F(\mu_2).$$

By summing these inequalities we obtain

$$F\left(\frac{1-\epsilon}{\kappa+1}\mu_1+\frac{\kappa+\epsilon}{\kappa+1}\mu_2\right)+F\left(\frac{\kappa+\epsilon}{\kappa+1}\mu_1+\frac{1-\epsilon}{\kappa+1}\mu_2\right)\leqslant F(\mu_1)+F(\mu_2)$$

We multiply the resulting inequality by $w'(\epsilon)$ and integrating with respect to ϵ over $[0,1]$, we obtain

$$\int_0^1 w'(\epsilon)\left(F\left(\frac{1-\epsilon}{\kappa+1}\mu_1+\frac{\kappa+\epsilon}{\kappa+1}\mu_2\right)+F\left(\frac{\kappa+\epsilon}{\kappa+1}\mu_1+\frac{1-\epsilon}{\kappa+1}\mu_2\right)\right)d\epsilon$$
$$\leqslant(F(\mu_1)+F(\mu_2))\int_0^1 w'(\epsilon)d\epsilon.$$

In the resulting integrals we make a change of variables

$$\frac{1-\epsilon}{\kappa+1}\mu_1+\frac{\kappa+\epsilon}{\kappa+1}\mu_2=z,$$

and we get

$$\begin{aligned}\int_0^1 w'(\epsilon)F\left(\frac{1-\epsilon}{\kappa+1}\mu_1+\frac{\kappa+\epsilon}{\kappa+1}\mu_2\right)d\epsilon =&\frac{\kappa+1}{\mu_2-\mu_1}\int_{\frac{\mu_1+\kappa\mu_2}{\kappa+1}}^{\mu_2} w'\left(\frac{z-\frac{\mu_1+\kappa\mu_2}{\kappa+1}}{\frac{\mu_2-\mu_1}{\kappa+1}}\right)F(z)\,dz\\ =&\frac{\kappa+1}{\mu_2-\mu_1}\int_{\frac{\mu_1+\kappa\mu_2}{\kappa+1}}^{\mu_2} w'\left(\frac{z-\frac{\mu_1+\kappa\mu_2}{\kappa+1}}{\mu_2-\frac{\mu_1+\kappa\mu_2}{\kappa+1}}\right)F(z)\,dz\\ =&\frac{\kappa+1}{\mu_2-\mu_1}J^w_{\frac{\mu_1+\kappa\mu_2}{\kappa+1}+}F(\mu_2).\end{aligned}$$

In the same way for the second integral we have

$$\frac{\kappa+\epsilon}{\kappa+1}\mu_1+\frac{1-\epsilon}{\kappa+1}\mu_2=z,$$

and

$$\begin{aligned}\int_0^1 w'(\epsilon)F\left(\frac{\kappa+\epsilon}{\kappa+1}\mu_1+\frac{1-\epsilon}{\kappa+1}\mu_2\right)d\epsilon =&\frac{\kappa+1}{\mu_2-\mu_1}\int_{\mu_1}^{\frac{\mu_2+\mu_1\kappa}{\kappa+1}} w'\left(\frac{\frac{\mu_1\kappa+\mu_2}{\kappa+1}-z}{\frac{\mu_2-\mu_1}{\kappa+1}}\right)F(z)\,dz\\ =&\frac{\kappa+1}{\mu_2-\mu_1}\int_{\mu_1}^{\frac{\mu_2+\mu_1\kappa}{\kappa+1}} w'\left(\frac{\frac{\mu_1\kappa+\mu_2}{\kappa+1}-z}{\frac{\mu_1\kappa+\mu_2}{\kappa+1}-\mu_1}\right)F(z)\,dz\\ =&\frac{\kappa+1}{\mu_2-\mu_1}J^w_{\frac{\mu_2+\mu_1\kappa}{\kappa+1}-}F(\mu_1).\end{aligned}$$

Thus, we have obtained an analogue of the trapezoidal Hadamard inequality for our operator:

$$\frac{\kappa+1}{2(\mu_2-\mu_1)}J^w_{\frac{\mu_1+\kappa\mu_2}{\kappa+1}+}F(\mu_2)+\frac{\kappa+1}{\mu_2-\mu_1}J^w_{\frac{\mu_2+\mu_1\kappa}{\kappa+1}-}F(\mu_1)\leqslant\frac{F(\mu_1)+F(\mu_2)}{2}(w(1)-w(0))$$

or

$$\frac{\kappa+1}{2(\mu_2-\mu_1)}\left(J^w_{\frac{\mu_1+\kappa\mu_2}{\kappa+1}+}F(\mu_2)+J^w_{\frac{\mu_2+\mu_1\kappa}{\kappa+1}-}F(\mu_1)\right)\leqslant\frac{F(\mu_1)+F(\mu_2)}{2}(w(1)-w(0)). \tag{2.2}$$

To obtain the Midpoint Hadamard inequality, we use the Jensen inequality

$$F\left(\frac{v+u}{2}\right)\leqslant\frac{F(v)+F(u)}{2}$$

for the convex functions

$$F\left(\frac{\kappa+\epsilon}{\kappa+1}\mu_1+\frac{1-\epsilon}{\kappa+1}\mu_2\right)\leqslant\frac{\kappa+\epsilon}{\kappa+1}F(\mu_1)+\frac{1-\epsilon}{\kappa+1}F(\mu_2)$$
$$F\left(\frac{1-\epsilon}{\kappa+1}\mu_1+\frac{\kappa+\epsilon}{\kappa+1}\mu_2\right)\leqslant\frac{1-\epsilon}{\kappa+1}F(\mu_1)+\frac{\kappa+\epsilon}{\kappa+1}F(\mu_2).$$

We sum these inequalities and dividing by 2 and we get

$$\frac{F\left(\frac{\kappa+\epsilon}{\kappa+1}\mu_1+\frac{1-\epsilon}{\kappa+1}\mu_2\right)+F\left(\frac{1-\epsilon}{\kappa+1}\mu_1+\frac{\kappa+\epsilon}{\kappa+1}\mu_2\right)}{2}\leqslant\frac{F(\mu_1)+F(\mu_2)}{2}.$$

Since

$$\begin{aligned}F\left(\frac{\mu_2+\mu_1}{2}\right)&=F\left(\frac{\frac{\kappa+\epsilon}{\kappa+1}\mu_1+\frac{1-\epsilon}{\kappa+1}\mu_2+\frac{1-\epsilon}{\kappa+1}\mu_1+\frac{\kappa+\epsilon}{\kappa+1}\mu_2}{2}\right)\\&\leqslant\frac{F\left(\frac{\kappa+\epsilon}{\kappa+1}\mu_1+\frac{1-\epsilon}{\kappa+1}\mu_2\right)+F\left(\frac{1-\epsilon}{\kappa+1}\mu_1+\frac{\kappa+\epsilon}{\kappa+1}\mu_2\right)}{2}\leqslant\frac{F(\mu_1)+F(\mu_2)}{2},\end{aligned}$$

we have

$$2F\left(\frac{\mu_2+\mu_1}{2}\right)\leqslant F\left(\frac{\kappa+\epsilon}{\kappa+1}\mu_1+\frac{1-\epsilon}{\kappa+1}\mu_2\right)+F\left(\frac{1-\epsilon}{\kappa+1}\mu_1+\frac{\kappa+\epsilon}{\kappa+1}\mu_2\right).$$

Let us multiply the resulting inequality by $w\prime(\epsilon)$ and perform procedures similar to the previous step, as a result we have

$$F\left(\frac{\mu_2+\mu_1}{2}\right)(w(1)-w(0))\leqslant\frac{\kappa+1}{2(\mu_2-\mu_1)}\left(J^w_{\frac{\mu_1+\kappa\mu_2}{\kappa+1}+}F(\mu_2)+J^w_{\frac{\mu_2+\mu_1\kappa}{\kappa+1}-}F(\mu_1)\right).\tag{2.3}$$

The inequalities (2.2) and (2.3) imply (2.1). The proof is complete. □

3. Results

The next lemma plays an important role in obtaining our results.

Lemma 3.1. *Let F be a real function defined on some interval $[\mu_1,\mu_2]\subset\mathbb{R}$ and differentiable on (μ_1,μ_2). If $F\in L[\mu_1,\mu_2]$ and $w(\epsilon)$ is a differentiable function on $[\mu_1,\mu_2]$, then*

$$\int_0^1 w(1-\epsilon)F\left(\frac{\epsilon}{\kappa+1}\mu_1+\frac{1-\epsilon+\kappa}{1+\kappa}\mu_2\right)d\epsilon=\int_0^1 w(z)F\left(\mu_1\frac{1-z}{1+\kappa}+\mu_2\frac{z+\kappa}{1+\kappa}\right)dz.$$

Proof. It is sufficient to use the change of variables $z=1-\epsilon$ in the integral of the left side, from which the integral of the right side is easily obtained. □

Lemma 3.2. *Let $0<m\leqslant 1$; $F:[\mu_1,\mu_2]\to\mathbb{R}$, $F\in C^1(\mu_1,\mu_2)$ and $0<\mu_1<x<\mu_2$. If $F\in L[\mu_1,\mu_2]$ and $w'\geqslant 0$, then, for $\kappa\in\mathbb{N}$,*

$$\begin{aligned}&\frac{\kappa+1}{x-\mu_1}\left(w(1)\left(F(x)+F(\mu_1)\right)-w(0)\left(F\left(\frac{\mu_1+\kappa x}{\kappa+1}\right)+F\left(\frac{\kappa\mu_1+x}{\kappa+1}\right)\right)\right)\\&\quad-\left(\frac{\kappa+1}{x-\mu_1}\right)^2\left(J^w_{x-}F\left(\frac{\mu_1+\kappa x}{\kappa+1}\right)+J^w_{\mu_1+}F\left(\frac{\kappa\mu_1+x}{\kappa+1}\right)\right)\\&=\int_0^1 w(\epsilon)\left(F'\left(\mu_1\frac{1-\epsilon}{\kappa+1}+\frac{\kappa+\epsilon}{\kappa+1}x\right)-F'\left(\mu_1\frac{\epsilon+\kappa}{\kappa+1}+\frac{1-\epsilon}{\kappa+1}x\right)\right)d\epsilon.\end{aligned}$$

Proof. Using the well–known property of the integral, we can write

$$\begin{aligned}&\int_0^1 w(\epsilon)\left(F'\left(\frac{1-\epsilon}{\kappa+1}\mu_1+\frac{\kappa+\epsilon}{\kappa+1}x\right)-F'\left(\frac{\kappa+\epsilon}{\kappa+1}\mu_1+\frac{1-\epsilon}{\kappa+1}x\right)\right)d\epsilon\\&\quad=\int_0^1 w(\epsilon)F'\left(\frac{1-\epsilon}{\kappa+1}\mu_1+\frac{\kappa+\epsilon}{\kappa+1}x\right)d\epsilon-\int_0^1 w(\epsilon)F'\left(\frac{\kappa+\epsilon}{\kappa+1}\mu_1+\frac{1-\epsilon}{\kappa+1}x\right)d\epsilon\\&\quad=I_1-I_2.\end{aligned}$$

By employing integration by parts and performing a variable substitution, we obtain

$$\begin{aligned}I_1&=\int_0^1 w(\epsilon)F'\left(\frac{1-\epsilon}{\kappa+1}\mu_1+\frac{\kappa+\epsilon}{\kappa+1}x\right)d\epsilon\\&=\frac{\kappa+1}{x-\mu_1}\left(w(1)F(x)-w(0)F\left(\frac{\mu_1+\kappa x}{\kappa+1}\right)\right)\\&\quad-\frac{\kappa+1}{x-\mu_1}\int_0^1 w'(\epsilon)F\left(\frac{1-\epsilon}{\kappa+1}\mu_1+\frac{\kappa+\epsilon}{\kappa+1}x\right)d\epsilon\\&=\frac{\kappa+1}{x-\mu_1}\left(w(1)F(x)-w(0)F\left(\frac{\mu_1+\kappa x}{\kappa+1}\right)\right)-\left(\frac{\kappa+1}{x-\mu_1}\right)^2 J^w_{x-}F\left(\frac{\mu_1+\ \kappa x}{\kappa+1}\right)\end{aligned}$$

Similarly, we have

$$\begin{aligned}I_2&=\int_0^1 w(\epsilon)F'\left(\frac{\kappa+\epsilon}{\kappa+1}\mu_1+\frac{1-\epsilon}{\kappa+1}x\right)d\epsilon\\&=-\frac{\kappa+1}{x-\mu_1}\left(w(1)F(\mu_1)-w(0)F\left(\frac{\kappa\mu_1+x}{\kappa+1}\right)\right)+\left(\frac{\kappa+1}{x-\mu_1}\right)^2 J^w_{\mu_1+}F\left(\frac{\kappa\mu_1+x}{\kappa+1}\right)\end{aligned}$$

Taking the differenceI_1-I_2, we obtain the desired result. This completes the proof. □

Remark 3. *With $w(\epsilon)=\epsilon$, $\kappa=1$ and $x=\mu_2$, the above result cover Lemma 2.1 in* [3].

Remark 4. *Putting*

$$\kappa=0,\qquad w(\epsilon)=\frac{(x-\mu_1)^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\epsilon^{\frac{\alpha}{k}},\qquad \mu_1=l_1,\qquad x=\frac{l_2}{\mu_2},\qquad 0<\mu_2<1,$$

we cover Lemma 2 in [32].

Remark 5. *Taking into account the Lemma 3.1, we can reformulate Lemma 3.2 in such a way that if we put $x=\mu_2$, $\kappa=0$ and $w(\epsilon)=\epsilon^{\alpha}$, we obtain the Lemma 2 of* [34], *under above assumptions if $w(\epsilon)=\epsilon$ we obtain the Lemma 2.1 of* [14].

Hermite — Hadamard inequalities can be represented in weighted integral forms as follows.

Theorem 3.1. *Let a function $F:[\mu_1,\mu_2]\to\mathbb{R}$ and $F\in C^1(\mu_1,\mu_2)$. If $F'\in L[\mu_1,\mu_2]$ and $|F'|\in N^{s,2}_{h,m}[\mu_1,\mu_2]$, then the following inequality for the weighted integral holds:*

$$\begin{aligned}&\left|\frac{\kappa+1}{x-\mu_1}\left(w(1)\left(F(x)+F(\mu_1)\right)-w(0)\left(F\left(\frac{\mu_1+\kappa x}{\kappa+1}\right)+F\left(\frac{\kappa\mu_1+x}{\kappa+1}\right)\right)\right)\right.\\&\qquad\left.-\left(\frac{\kappa+1}{x-\mu_1}\right)^2\left(J^w_{x-}F\left(\frac{\mu_1+\kappa x}{\kappa+1}\right)+J^w_{\mu_1+}F\left(\frac{\kappa\mu_1+x}{\kappa+1}\right)\right)\right|\\&\quad\leqslant|F'(\mu_1)|\int_0^1 w(\epsilon)\left(h^s\left(\frac{1-\epsilon}{\kappa+1}\right)+h^s\left(\frac{\kappa+\epsilon}{\kappa+1}\right)\right)d\epsilon\\&\qquad+m\left|F'\left(\frac{x}{m}\right)\right|\int_0^1 w(\epsilon)\left(\left(1-h\left(\frac{1-\epsilon}{\kappa+1}\right)\right)^s+\left(1-h\left(\frac{\kappa+\epsilon}{\kappa+1}\right)\right)^s\right)d\epsilon,\end{aligned}\tag{3.1}$$

where w and κ are defined above in Theorem 2.1.

Proof. Employing the definition of (h,m)-convexity of F, we have

$$\begin{aligned} \left|F'\left(\frac{1-\epsilon}{\kappa+1}\mu_1+\frac{\kappa+\epsilon}{\kappa+1}x\right)\right| &\leqslant |F'(\mu_1)|\, h^s\left(\frac{1-\epsilon}{\kappa+1}\right)+m\left|F'\left(\frac{x}{m}\right)\right|\left(1-h\left(\frac{1-\epsilon}{\kappa+1}\right)\right)^s, \\ \left|F'\left(\frac{\kappa+\epsilon}{\kappa+1}\mu_1+\frac{1-\epsilon}{\kappa+1}x\right)\right| &\leqslant |F'(\mu_1)|\, h^s\left(\frac{\kappa+\epsilon}{\kappa+1}\right)+m\left|F'\left(\frac{x}{m}\right)\right|\left(1-h\left(\frac{\kappa+\epsilon}{\kappa+1}\right)\right)^s. \end{aligned} \tag{3.2}$$

By Lemma 3.2 we find

$$\begin{aligned} &\Bigg|\frac{\kappa+1}{x-\mu_1}\left(w(1)\left(F(x)+F(\mu_1)\right)-w(0)\left(F\left(\frac{\mu_1+\kappa x}{\kappa+1}\right)+F\left(\frac{\kappa\mu_1+x}{\kappa+1}\right)\right)\right) \\ &\qquad-\left(\frac{\kappa+1}{x-\mu_1}\right)^2\left(J^w_{x-}F\left(\frac{\mu_1+\kappa x}{\kappa+1}\right)+J^w_{\mu_1+}F\left(\frac{\kappa\mu_1+x}{\kappa+1}\right)\right)\Bigg| \\ &\leqslant\left|\int_0^1 w(\epsilon)F'\left(\frac{1-\epsilon}{\kappa+1}\mu_1+\frac{\kappa+\epsilon}{\kappa+1}x\right)d\epsilon\right|+\left|\int_0^1 w(\epsilon)F'\left(\frac{\kappa+\epsilon}{\kappa+1}\mu_1+\frac{1-\epsilon}{\kappa+1}x\right)d\epsilon\right|. \end{aligned} \tag{3.3}$$

Taking into account inequalities (3.2), we obtain

$$\begin{aligned} &\Bigg|\frac{\kappa+1}{x-\mu_1}\left(w(1)\left(F(x)+F(\mu_1)\right)-w(0)\left(F\left(\frac{\mu_1+\kappa x}{\kappa+1}\right)+F\left(\frac{\kappa\mu_1+x}{\kappa+1}\right)\right)\right) \\ &\qquad-\left(\frac{\kappa+1}{x-\mu_1}\right)^2\left(J^w_{x-}F\left(\frac{\mu_1+\kappa x}{\kappa+1}\right)+J^w_{\mu_1+}F\left(\frac{\kappa\mu_1+x}{\kappa+1}\right)\right)\Bigg| \\ &\leqslant|F'(\mu_1)|\int_0^1 w(\epsilon)h^s\left(\frac{1-\epsilon}{\kappa+1}\right)d\epsilon+m\left|F'\left(\frac{x}{m}\right)\right|\int_0^1 w(\epsilon)\left(1-h\left(\frac{1-\epsilon}{\kappa+1}\right)\right)^s d\epsilon \\ &\quad+|F'(\mu_1)|\int_0^1 w(\epsilon)h^s\left(\frac{\kappa+\epsilon}{\kappa+1}\right)d\epsilon+m\left|F'\left(\frac{x}{m}\right)\right|\int_0^1 w(\epsilon)\left(1-h\left(\frac{\kappa+\epsilon}{\kappa+1}\right)\right)^s d\epsilon. \end{aligned}$$

From this last inequality, after a simple algebraic work, the desired result is obtained. The proof is complete. □

Corollary 1. *For $x=\mu_2,\kappa=1,m=1,\ s=1,h(\epsilon)=\epsilon$ and $w(\epsilon)=\epsilon^\alpha$ by (3.1) we get the Trapezoid inequality*

$$\begin{aligned} &\left|\frac{F(\mu_2)+F(\mu_1)}{2}-\frac{1}{\mu_2-\mu_1}\left(J^\alpha_{\mu_2-}F\left(\frac{\mu_2+\mu_1}{2}\right)+J^\alpha_{\mu_1+}F\left(\frac{\mu_2+\mu_1}{2}\right)\right)\right| \\ &\qquad\leqslant\frac{\mu_2-\mu_1}{4}\,\frac{|F'(\mu_1)|+|F'(\mu_2)|}{\alpha+1}. \end{aligned} \tag{3.4}$$

Proof. Indeed, for the left hand side of (3.1) with $\kappa=1$ we get

$$\begin{aligned} &\Bigg|\frac{\kappa+1}{x-\mu_1}\left(w(1)\left(F(x)+F(\mu_1)\right)-w(0)\left(F\left(\frac{\mu_1+\kappa x}{\kappa+1}\right)+F\left(\frac{\kappa\mu_1+x}{\kappa+1}\right)\right)\right) \\ &\qquad-\left(\frac{\kappa+1}{x-\mu_1}\right)^2\left(J^w_{x-}F\left(\frac{\mu_1+\kappa x}{\kappa+1}\right)+J^w_{\mu_1+}F\left(\frac{\kappa\mu_1+x}{\kappa+1}\right)\right)\Bigg| \\ &=\left|2\frac{F(\mu_2)+F(\mu_1)}{\mu_2-\mu_1}-\left(\frac{2}{\mu_2-\mu_1}\right)^2\left(J^\alpha_{\mu_2-}F\left(\frac{\mu_2+\mu_1}{2}\right)+J^\alpha_{\mu_1+}F\left(\frac{\mu_2+\mu_1}{2}\right)\right)\right|, \end{aligned}$$

and for the right side hand we get

$$|F'(\mu_1)|\int_0^1 w(\epsilon)\left(h^s\left(\frac{1-\epsilon}{\kappa+1}\right)+h^s\left(\frac{\kappa+\epsilon}{\kappa+1}\right)\right)d\epsilon$$

$$+ m\left|F'\left(\frac{x}{m}\right)\right| \int_0^1 w(\epsilon)\left(\left(1-h\left(\frac{1-\epsilon}{\kappa+1}\right)\right)^s + \left(1-h\left(\frac{\kappa+\epsilon}{\kappa+1}\right)\right)^s\right) d\epsilon.$$
$$= |F'(\mu_1)| \int_0^1 \epsilon^\alpha \left(\frac{1-\epsilon}{2}+\frac{1+\epsilon}{2}\right) d\epsilon + |F'(\mu_2)| \int_0^1 \epsilon^\alpha \left(1-\frac{1-\epsilon}{2}+1-\frac{1+\epsilon}{2}\right) d\epsilon$$
$$= \frac{|F'(\mu_1)|+|F'(\mu_2)|}{\alpha+1}.$$

We multiply both sides by $\frac{\mu_2-\mu_1}{4}$ and obtain (3.4). The proof is complete. □

Remark 6. *For $\alpha = 1$, by Corollary 1 we obtain a well–known in the literature upper bound estimate of Hadamard–type, which is called the Trapezoid inequality:*

$$\left|\frac{F(\mu_2)+F(\mu_1)}{2} - \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} F(\epsilon)d\epsilon\right| \leqslant \frac{\mu_2-\mu_1}{8}\left(|F'(\mu_1)|+|F'(\mu_2)|\right).$$

Remark 7. *Taking into account the Lemma 3.1, for m-convex functions, that is, $h(\epsilon)=\epsilon$, $s=1$ and $w(\epsilon)=\epsilon$, the above result implies the case $q=1$ of Theorem 2.1 in [24]. Under above assumptions for $m=1$ we obtain Theorem 2.2 in [14].*

Theorem 3.2. *Let $F:[\mu_1,\mu_2]\to\mathbb{R}$ and $F\in C^1(\mu_1,\mu_2)$. If $F'\in L(\mu_1,\mu_2)$ and $|F'|^q \in N^{s,2}_{h,m}[\mu_1,\mu_2]$, then for $q>1$ the inequality holds*

$$\begin{aligned} &\left|\frac{\kappa+1}{x-\mu_1}\left(w(1)\left(F(x)+F(\mu_1)\right) - w(0)\left(F\left(\frac{\mu_1+\kappa x}{\kappa+1}\right)+F\left(\frac{\kappa\mu_1+x}{\kappa+1}\right)\right)\right)\right.\\ &\qquad \left. -\left(\frac{\kappa+1}{x-\mu_1}\right)^2\left(J^w_{x-}F\left(\frac{\mu_1+\kappa x}{\kappa+1}\right)+J^w_{\mu_1+}F\left(\frac{\kappa\mu_1+x}{\kappa+1}\right)\right)\right| \\ &\qquad \leqslant \left(\int_0^1 (w(\epsilon))^p d\epsilon\right)^{\frac{1}{p}}\left((\mathbf{Q}_1)^{\frac{1}{q}}+(\mathbf{Q}_2)^{\frac{1}{q}}\right), \end{aligned} \tag{3.5}$$

where w and κ are defined above in Theorem 2.1,

$$\mathbf{Q}_1 = |F'(\mu_1)|^q \int_0^1 h^s\left(\frac{1-\epsilon}{\kappa+1}\right) d\epsilon + m\left|F'\left(\frac{x}{m}\right)\right|^q \int_0^1 \left(1-h\left(\frac{1-\epsilon}{\kappa+1}\right)\right)^s d\epsilon$$
$$\mathbf{Q}_2 = |F'(\mu_1)|^q \int_0^1 h^s\left(\frac{\kappa+\epsilon}{\kappa+1}\right) d\epsilon + m\left|F'\left(\frac{x}{m}\right)\right|^q \int_0^1 \left(1-h\left(\frac{\kappa+\epsilon}{\kappa+1}\right)\right)^s d\epsilon.$$

Proof. By using the well–known Hölder integral inequality, in view of $|F'|^q \in N^{s,2}_{h,m}[\mu_1,\mu_2]$, for the right hand side of (3.3) we get

$$\begin{aligned} &\left|\int_0^1 w(\epsilon)F'\left(\frac{1-\epsilon}{\kappa+1}\mu_1+\frac{\kappa+\epsilon}{\kappa+1}x\right)d\epsilon\right| + \left|\int_0^1 w(\epsilon)F'\left(\frac{\kappa+\epsilon}{\kappa+1}\mu_1+\frac{1-\epsilon}{\kappa+1}x\right)d\epsilon\right| \\ &\leqslant \left(\int_0^1 (w(\epsilon))^p d\epsilon\right)^{\frac{1}{p}}\left(\int_0^1 \left|F'\left(\frac{1-\epsilon}{\kappa+1}\mu_1+\frac{\kappa+\epsilon}{\kappa+1}x\right)\right|^q d\epsilon\right)^{\frac{1}{q}} \\ &\quad + \left(\int_0^1 (w(\epsilon))^p d\epsilon\right)^{\frac{1}{p}}\left(\int_0^1 \left|F'\left(\frac{\kappa+\epsilon}{\kappa+1}\mu_1+\frac{1-\epsilon}{\kappa+1}x\right)\right|^q d\epsilon\right)^{\frac{1}{q}} \\ &\leqslant \left(\int_0^1 (w(\epsilon))^p d\epsilon\right)^{\frac{1}{p}} \\ &\quad \cdot\left(\left(|F'(\mu_1)|^q \int_0^1 h^s\left(\frac{1-\epsilon}{\kappa+1}\right)d\epsilon + m\left|F'\left(\frac{x}{m}\right)\right|^q \int_0^1 \left(1-h\left(\frac{1-\epsilon}{\kappa+1}\right)\right)^s d\epsilon\right)^{\frac{1}{q}}\right. \end{aligned}$$

$$+\left(|F'(\mu_1)|^q\int_0^1 h^s\left(\frac{\kappa+\epsilon}{\kappa+1}\right)d\epsilon+m\left|F'\left(\frac{x}{m}\right)\right|^q\int_0^1\left(1-h\left(\frac{\kappa+\epsilon}{\kappa+1}\right)\right)^s d\epsilon\right)^{\frac{1}{q}}\Bigg).$$

Taking into account inequalities (3.2), we get (3.5). The proof is complet. □

Corollary 2. *For $x=\mu_2$, $\kappa=0$, $m=1$, $s=1$, $h(\epsilon)=\epsilon$ and $w(\epsilon)=\epsilon^{\alpha}$, by (3.5) for all $\alpha>0$ we get*

$$\begin{aligned}\left|\frac{F(\mu_2)+F(\mu_1)}{2}-\frac{\Gamma(\alpha+1)}{2(\mu_2-\mu_1)^{\alpha}}\left(J^{\alpha}_{\mu_2-}F(\mu_1)+J^{\alpha}_{\mu_1+}F(\mu_2)\right)\right|\\ \leqslant\frac{2(\mu_2-\mu_1)}{(\alpha p+1)^{\frac{1}{p}}}\left(\frac{|F'(\mu_1)|^q+|F'(\mu_2)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}\tag{3.6}$$

Proof. By (3.5) we have

$$\begin{aligned}\left|\frac{F(\mu_2)+F(\mu_1)}{\mu_2-\mu_1}-\left(\frac{1}{\mu_2-\mu_1}\right)^2\left(J^{w}_{x-}F(\mu_1)+J^{w}_{\mu_1+}F(\mu_2)\right)\right|\\ \leqslant\left(\int_0^1(w(\epsilon))^p\,d\epsilon\right)^{\frac{1}{p}}\left((\mathbf{Q}_1)^{\frac{1}{q}}+(\mathbf{Q}_2)^{\frac{1}{q}}\right).\end{aligned}$$

Here

$$\begin{aligned}&\int_0^1(w(\epsilon))^p\,d\epsilon=\frac{1}{\alpha p+1},\\ &\mathbf{Q}_1=|F'(\mu_1)|^q\int_0^1 h^s\left(\frac{1-\epsilon}{\kappa+1}\right)d\epsilon+m\left|F'\left(\frac{x}{m}\right)\right|^q\int_0^1\left(1-h\left(\frac{1-\epsilon}{\kappa+1}\right)\right)^s d\epsilon\\ &\quad=|F'(\mu_1)|^q\int_0^1(1-\epsilon)\,d\epsilon+|F'(\mu_2)|^q\int_0^1\epsilon d\epsilon\\ &\quad=\frac{|F'(\mu_1)|^q+|F'(\mu_2)|^q}{2},\\ &\mathbf{Q}_2=|F'(\mu_1)|^q\int_0^1 h^s\left(\frac{\kappa+\epsilon}{\kappa+1}\right)d\epsilon+m\left|F'\left(\frac{x}{m}\right)\right|^q\int_0^1\left(1-h\left(\frac{\kappa+\epsilon}{\kappa+1}\right)\right)^s d\epsilon\\ &\quad=\frac{|F'(\mu_1)|^q+|F'(\mu_2)|^q}{2}.\end{aligned}$$

For the weighted fractional integrals we obtain

$$\begin{aligned}J^{w}_{\mu_1+}F(\mu_2)&=\int_{\mu_1}^{\mu_2}w'\left(\frac{\epsilon-\mu_1}{\mu_2-\mu_1}\right)F(\epsilon)d\epsilon=\alpha\int_{\mu_1}^{\frac{\mu_2+\mu_1}{2}}\left(\frac{\epsilon-\mu_1}{\mu_2-\mu_1}\right)^{\alpha-1}F(\epsilon)d\epsilon\\ &=\frac{\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha-1}\Gamma(\alpha)}\int_{\mu_1}^{\frac{\mu_2+\mu_1}{2}}(\epsilon-\mu_1)^{\alpha-1}F(\epsilon)d\epsilon=\frac{\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha-1}}J^{\alpha}_{\mu_1+}F(\mu_2).\\ J^{w}_{\mu_2-}F(\mu_1)&=\frac{\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha-1}}J^{\alpha}_{\mu_2-}F(\mu_1).\end{aligned}$$

Thus, we get

$$\left|\frac{F(\mu_2)+F(\mu_1)}{\mu_2-\mu_1}-\frac{\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha+1}}\left(J^{\alpha}_{\mu_1+}F(\mu_2)+J^{\alpha}_{\mu_2-}F(\mu_1)\right)\right|$$

$$\leqslant 2\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{|F'(\mu_1)|^q+|F'(\mu_2)|^q}{2}\right)^{\frac{1}{q}}.$$

Multiplying the last inequality by $\frac{\mu_2-\mu_1}{2}$, we get (3.6). □

Corollary 3. *For $x=\mu_2$, $\kappa=0$, $m=1$, $s=1, h(\epsilon)=\epsilon$ and $w(\epsilon)=(1-\epsilon)^\alpha$, by (3.5) for all $\alpha>0$ we get the Midpoint inequality*

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{2(\mu_2-\mu_1)^\alpha}\left(J^\alpha_{\mu_2-}F(\mu_1)+J^\alpha_{\mu_1+}F(\mu_2)\right)-F\left(\frac{\mu_2+\mu_1}{2}\right)\right|\\&\leqslant\frac{2(\mu_2-\mu_1)}{(\alpha p+1)^{\frac{1}{p}}}\left(\frac{|F'(\mu_1)|^q+|F'(\mu_2)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}\tag{3.7}$$

Proof. By (3.5) we have

$$\left|2\frac{F\left(\frac{\mu_2+\mu_1}{2}\right)}{\mu_2-\mu_1}-\left(\frac{1}{\mu_2-\mu_1}\right)^2\left(J^w_{x-}F(\mu_1)+J^w_{\mu_1+}F(\mu_2)\right)\right|\leqslant\left(\int_0^1(w(\epsilon))^p\,d\epsilon\right)^{\frac{1}{p}}\left((\mathbf{Q}_1)^{\frac{1}{q}}+(\mathbf{Q}_2)^{\frac{1}{q}}\right).$$

Here,

$$\begin{aligned}&\int_0^1(w(\epsilon))^p\,d\epsilon=\frac{1}{\alpha p+1},\\&=|F'(\mu_1)|^q\int_0^1 h^s\left(\frac{1-\epsilon}{\kappa+1}\right)d\epsilon+m\left|F'\left(\frac{x}{m}\right)\right|^q\int_0^1\left(1-h\left(\frac{1-\epsilon}{\kappa+1}\right)\right)^s d\epsilon\\&\quad=|F'(\mu_1)|^q\int_0^1(1-\epsilon)\,d\epsilon+|F'(\mu_2)|^q\int_0^1\epsilon d\epsilon\\&\quad=\frac{|F'(\mu_1)|^q+|F'(\mu_2)|^q}{2},\\&\mathbf{Q}_2=|F'(\mu_1)|^q\int_0^1 h^s\left(\frac{\kappa+\epsilon}{\kappa+1}\right)d\epsilon+m\left|F'\left(\frac{x}{m}\right)\right|^q\int_0^1\left(1-h\left(\frac{\kappa+\epsilon}{\kappa+1}\right)\right)^s d\epsilon\\&\quad=\frac{|F'(\mu_1)|^q+|F'(\mu_2)|^q}{2},\end{aligned}$$

For $J^w_{\mu_1+}F(\mu_2)$ and $J^w_{\mu_2-}F(\mu_1)$ we obtain

$$\begin{aligned}J^w_{\mu_1+}F(\mu_2)&=\int_{\mu_1}^{\mu_2}w'\left(\frac{\epsilon-\mu_1}{\mu_2-\mu_1}\right)F(\epsilon)d\epsilon=\alpha\int_{\mu_1}^{\frac{\mu_2+\mu_1}{2}}\left(\frac{\epsilon-\mu_1}{\mu_2-\mu_1}\right)^{\alpha-1}F(\epsilon)d\epsilon\\&=\frac{\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha-1}\Gamma(\alpha)}\int_{\mu_1}^{\frac{\mu_2+\mu_1}{2}}(\epsilon-\mu_1)^{\alpha-1}F(\epsilon)d\epsilon=\frac{\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha-1}}J^\alpha_{\mu_1+}F(\mu_2),\\J^w_{\mu_2-}F(\mu_1)&=\frac{\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha-1}}J^\alpha_{\mu_2-}F(\mu_1).\end{aligned}$$

Thus, we get

$$\begin{aligned}&\left|\frac{\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha+1}}\left(J^\alpha_{\mu_1+}F(\mu_2)+J^\alpha_{\mu_2-}F(\mu_1)\right)-\frac{2}{\mu_2-\mu_1}F\left(\frac{\mu_2+\mu_1}{2}\right)\right|\\&\qquad\leqslant 2\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{|F'(\mu_1)|^q+|F'(\mu_2)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

Multiplying the last inequality by $\frac{\mu_2-\mu_1}{2}$, we get (3.7). □

Corollary 4. *For $x=\mu_2$, $\kappa=1$, $m=1$, $s=1, h(\epsilon)=\epsilon$ and $w(\epsilon)=\epsilon^{\alpha}$ by (3.5) for all $\alpha>0$ we get*

$$\begin{aligned}
&\left|\frac{F(\mu_2)+F(\mu_1)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha}}\left(J^{\alpha}_{\mu_1+}F\left(\frac{\mu_2+\mu_1}{2}\right)+J^{\alpha}_{\mu_2-}F\left(\frac{\mu_2+\mu_1}{2}\right)\right)\right|\\
&\leqslant\frac{\mu_2-\mu_1}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\left(\frac{|F'(\mu_1)|^q+3|F'(\mu_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|F'(\mu_1)|^q+|F'(\mu_2)|^q}{4}\right)^{\frac{1}{q}}\right).
\end{aligned}\tag{3.8}$$

Proof. By (3.5) we have

$$\begin{aligned}
&\left|2\frac{F(\mu_2)+F(\mu_1)}{\mu_2-\mu_1}-\left(\frac{2}{\mu_2-\mu_1}\right)^2\left(J^{w}_{x-}F\left(\frac{\mu_2+\mu_1}{2}\right)+J^{w}_{\mu_1+}F\left(\frac{\mu_2+\mu_1}{2}\right)\right)\right|\\
&\qquad\leqslant\left(\int_0^1(w(\epsilon))^p\,d\epsilon\right)^{\frac{1}{p}}\left((\mathbf{Q}_1)^{\frac{1}{q}}+(\mathbf{Q}_2)^{\frac{1}{q}}\right).
\end{aligned}$$

Here

$$\begin{aligned}
&\int_0^1(w(\epsilon))^p\,d\epsilon=\frac{1}{\alpha p+1},\\
&\mathbf{Q}_1=|F'(\mu_1)|^q\int_0^1 h^s\left(\frac{1-\epsilon}{\kappa+1}\right)d\epsilon+m\left|F'\left(\frac{x}{m}\right)\right|^q\int_0^1\left(1-h\left(\frac{1-\epsilon}{\kappa+1}\right)\right)^s d\epsilon\\
&\quad=\frac{|F'(\mu_1)|^q}{2}\int_0^1(1-\epsilon)\,d\epsilon+\frac{|F'(\mu_2)|^q}{2}\int_0^1(1+\epsilon)\,d\epsilon\\
&\quad=\frac{|F'(\mu_1)|^q+3|F'(\mu_2)|^q}{4},\\
&\mathbf{Q}_2=\frac{3|F'(\mu_1)|^q+|F'(\mu_2)|^q}{4}.
\end{aligned}$$

Since

$$J^{w}_{\mu_1+}F(\mu_2)=\int_{\mu_1}^{\mu_2}w'\left(\frac{\epsilon-\mu_1}{\mu_2-\mu_1}\right)F(\epsilon)d\epsilon,$$

for $J^{w}_{\mu_1+}F\left(\frac{\mu_2+\mu_1}{2}\right)$ we get

$$\begin{aligned}
J^{w}_{\mu_1+}F\left(\frac{\mu_2+\mu_1}{2}\right)&=\alpha\int_{\mu_1}^{\frac{\mu_2+\mu_1}{2}}\left(\frac{\epsilon-\mu_1}{\frac{\mu_2+\mu_1}{2}-\mu_1}\right)^{\alpha-1}F(\epsilon)d\epsilon\\
&=\frac{\Gamma(\alpha+1)}{\left(\frac{\mu_2+\mu_1}{2}-\mu_1\right)^{\alpha-1}\Gamma(\alpha)}\int_{\mu_1}^{\frac{\mu_2+\mu_1}{2}}(\epsilon-\mu_1)^{\alpha-1}F(\epsilon)d\epsilon\\
&=\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha-1}}J^{\alpha}_{\mu_1+}F\left(\frac{\mu_2+\mu_1}{2}\right).
\end{aligned}$$

Doing the same thing, we obtain

$$J^{w}_{\mu_2-}F\left(\frac{\mu_2+\mu_1}{2}\right)=\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha-1}}J^{\alpha}_{\mu_2-}F\left(\frac{\mu_2+\mu_1}{2}\right).$$

Thus, we have

$$\begin{aligned}
&\left|2\frac{F(\mu_2)+F(\mu_1)}{\mu_2-\mu_1}-\frac{2^{\alpha+1}\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha+1}}\left(J^{\alpha}_{\mu_1+}F\left(\frac{\mu_2+\mu_1}{2}\right)+J^{\alpha}_{\mu_2-}F\left(\frac{\mu_2+\mu_1}{2}\right)\right)\right|\\
&\leqslant\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\left(\frac{|F'(\mu_1)|^q+3|F'(\mu_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|F'(\mu_1)|^q+|F'(\mu_2)|^q}{4}\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Multiplying the last inequality by $\frac{\mu_2-\mu_1}{4}$, we obtain (3.8). The proof is complete. □

Remark 8. *The above result implies Corollary 2.10 for $m=1$ in [17] and Corollary 2.7(i) in [31].*

Corollary 5. *For $x=\mu_2$, $\kappa=1$, $m=1$, $s=1, h(\epsilon)=\epsilon$ and $w(\epsilon)=(1-\epsilon)^{\alpha}$ by (3.5) for all $\alpha>0$ we get*

$$\begin{aligned} &\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha}}\left(J^{\alpha}_{\mu_1+}F\left(\frac{\mu_2+\mu_1}{2}\right)+J^{\alpha}_{\mu_2-}F\left(\frac{\mu_2+\mu_1}{2}\right)\right)-F\left(\frac{\mu_2+\mu_1}{2}\right)\right| \\ &\leqslant \frac{\mu_2-\mu_1}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\left(\frac{|F'(\mu_1)|^q+3\,|F'(\mu_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\,|F'(\mu_1)|^q+|F'(\mu_2)|^q}{4}\right)^{\frac{1}{q}}\right). \end{aligned} \tag{3.9}$$

Proof. By (3.5) we have

$$\begin{aligned} &\left|-\frac{4F\left(\frac{\mu_2+\mu_1}{2}\right)}{\mu_2-\mu_1}+\left(\frac{2}{\mu_2-\mu_1}\right)^2\left(J^{w}_{x-}F\left(\frac{\mu_2+\mu_1}{2}\right)+J^{w}_{\mu_1+}F\left(\frac{\mu_2+\mu_1}{2}\right)\right)\right| \\ &\qquad\leqslant \left(\int_0^1 (w(\epsilon))^p\,d\epsilon\right)^{\frac{1}{p}}\left((\mathbf{Q}_1)^{\frac{1}{q}}+(\mathbf{Q}_2)^{\frac{1}{q}}\right), \end{aligned}$$

Here

$$\begin{aligned} \int_0^1 (w(\epsilon))^p\,d\epsilon &= \int_0^1 (1-\epsilon)^{\alpha p}\,d\epsilon = \frac{1}{\alpha p+1}, \\ \mathbf{Q}_1 &= |F'(\mu_1)|^q\int_0^1 h^s\left(\frac{1-\epsilon}{\kappa+1}\right)d\epsilon + m\left|F'\left(\frac{x}{m}\right)\right|^q\int_0^1\left(1-h\left(\frac{1-\epsilon}{\kappa+1}\right)\right)^s d\epsilon \\ &= \frac{|F'(\mu_1)|^q}{2}\int_0^1 (1-\epsilon)\,d\epsilon + \frac{|F'(\mu_2)|^q}{2}\int_0^1 (1+\epsilon)\,d\epsilon \\ &= \frac{|F'(\mu_1)|^q+3\,|F'(\mu_2)|^q}{4}, \\ \mathbf{Q}_2 &= |F'(\mu_1)|^q\int_0^1 h^s\left(\frac{\kappa+\epsilon}{\kappa+1}\right)d\epsilon + m\left|F'\left(\frac{x}{m}\right)\right|^q\int_0^1\left(1-h\left(\frac{\kappa+\epsilon}{\kappa+1}\right)\right)^s d\epsilon \\ &= |F'(\mu_1)|^q\int_0^1 \frac{1+\epsilon}{2}d\epsilon + |F'(\mu_2)|^q\int_0^1 \frac{1-\epsilon}{2}d\epsilon \\ &= \frac{3\,|F'(\mu_1)|^q+|F'(\mu_2)|^q}{4}. \end{aligned}$$

For the weight fraction integral we obtain

$$\begin{aligned} J^{w}_{\mu_1+}F\left(\frac{\mu_2+\mu_1}{2}\right) &= -\alpha\int_{\mu_1}^{\frac{\mu_2+\mu_1}{2}}\left(1-\frac{\epsilon-\mu_1}{\frac{\mu_2+\mu_1}{2}-\mu_1}\right)^{\alpha-1}F(\epsilon)d\epsilon \\ &= \frac{-2^{\alpha-1}\alpha}{(\mu_2-\mu_1)^{\alpha-1}}\int_{\mu_1}^{\frac{\mu_2+\mu_1}{2}}\left(\frac{\mu_2+\mu_1}{2}-\epsilon\right)^{\alpha-1}F(\epsilon)d\epsilon \\ &= -\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha-1}}J^{\alpha}_{\mu_1+}F\left(\frac{\mu_2+\mu_1}{2}\right), \\ J^{w}_{\mu_2-}F\left(\frac{\mu_2+\mu_1}{2}\right) &= -\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha-1}}J^{\alpha}_{\mu_2-}F\left(\frac{\mu_2+\mu_1}{2}\right). \end{aligned}$$

Thus, we have

$$\begin{aligned}&\left|\frac{2^{\alpha+1}\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha+1}}\left(J^{\alpha}_{\mu_1+}F\left(\frac{\mu_2+\mu_1}{2}\right)+J^{\alpha}_{\mu_2-}F\left(\frac{\mu_2+\mu_1}{2}\right)\right)-\frac{4F\left(\frac{\mu_2+\mu_1}{2}\right)}{\mu_2-\mu_1}\right|\\&\leqslant\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\left(\frac{|F'(\mu_1)|^q+3\,|F'(\mu_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\,|F'(\mu_1)|^q+|F'(\mu_2)|^q}{4}\right)^{\frac{1}{q}}\right).\end{aligned}$$

Multiplying the last inequality by $\frac{\mu_2-\mu_1}{4}$, we obtain (3.9). The proof is complete. □

Remark 9. *By (3.9) for $\alpha=1$ we obtain Corollary 2.10 for $m=1$ in [17] and Corollary 3.9 in [9], as well as the case $x=\mu_1$ and $y=\mu_2$ of Theorem 3.11 in [30], and Theorem 6 in [36].*

Theorem 3.3. *Let $F:[\mu_1,\mu_2]\to\mathbb{R}$ and $F\in C^1(\mu_1,\mu_2)$. If $F'\in L(\mu_1,\mu_2)$ and $|F'|^q\in N^{s,2}_{h,m}[\mu_1,\mu_2]$, then for all $q\geqslant 1$ the inequality holds*

$$\begin{aligned}&\left|\frac{\kappa+1}{x-\mu_1}\left(w(1)\left(F(x)+F(\mu_1)\right)-w(0)\left(F\left(\frac{\mu_1+\kappa x}{\kappa+1}\right)+F\left(\frac{\kappa\mu_1+x}{\kappa+1}\right)\right)\right)\right.\\&\qquad\left.-\left(\frac{\kappa+1}{x-\mu_1}\right)^2\left(J^{w}_{x-}F\left(\frac{\mu_1+\kappa x}{\kappa+1}\right)+J^{w}_{\mu_1+}F\left(\frac{\kappa\mu_1+x}{\kappa+1}\right)\right)\right|\\&\qquad\leqslant\left(\int_0^1 w(\epsilon)d\epsilon\right)^{1-\frac{1}{q}}\left((\mathbf{P}_1)^{\frac{1}{q}}+(\mathbf{P}_2)^{\frac{1}{q}}\right),\end{aligned}\tag{3.10}$$

where w and κ are defined above in Theorem 2.1,

$$\mathbf{P}_1=|F'(\mu_1)|^q\int_0^1 w(\epsilon)h^s\left(\frac{1-\epsilon}{\kappa+1}\right)d\epsilon+m\left|F'\left(\frac{x}{m}\right)\right|^q\int_0^1 w(\epsilon)\left(1-h\left(\frac{1-\epsilon}{\kappa+1}\right)\right)^s d\epsilon$$
$$\mathbf{P}_2=|F'(\mu_1)|^q\int_0^1 w(\epsilon)h^s\left(\frac{\kappa+\epsilon}{\kappa+1}\right)d\epsilon+m\left|F'\left(\frac{x}{m}\right)\right|^q\int_0^1 w(\epsilon)\left(1-h\left(\frac{\kappa+\epsilon}{\kappa+1}\right)\right)^s d\epsilon.$$

Proof. By using the well–known Power mean integral inequality and since $|F'|^q\in N^{s,2}_{h,m}[\mu_1,\mu_2]$, for the right hand side in (3.3) we get

$$\begin{aligned}&\left|\int_0^1 w(\epsilon)F'\left(\frac{1-\epsilon}{\kappa+1}\mu_1+\frac{\kappa+\epsilon}{\kappa+1}x\right)d\epsilon\right|+\left|\int_0^1 w(\epsilon)F'\left(\frac{\kappa+\epsilon}{\kappa+1}\mu_1+\frac{1-\epsilon}{\kappa+1}x\right)d\epsilon\right|\\&\leqslant\left(\int_0^1 w(\epsilon)d\epsilon\right)^{1-\frac{1}{q}}\left(\int_0^1 w(\epsilon)\left|F'\left(\frac{1-\epsilon}{\kappa+1}\mu_1+\frac{\kappa+\epsilon}{\kappa+1}x\right)\right|^q d\epsilon\right)^{\frac{1}{q}}\\&\quad+\left(\int_0^1 w(\epsilon)d\epsilon\right)^{1-\frac{1}{q}}\left(\int_0^1 w(\epsilon)\left|F'\left(\frac{\kappa+\epsilon}{\kappa+1}\mu_1+\frac{1-\epsilon}{\kappa+1}x\right)\right|^q d\epsilon\right)^{\frac{1}{q}}\\&\leqslant\left(\int_0^1 w(\epsilon)d\epsilon\right)^{1-\frac{1}{q}}\left(\left(|F'(\mu_1)|^q\int_0^1 w(\epsilon)h^s\left(\frac{1-\epsilon}{\kappa+1}\right)d\epsilon\right.\right.\\&\quad\left.+m\left|F'\left(\frac{x}{m}\right)\right|^q\int_0^1 w(\epsilon)\left(1-h\left(\frac{1-\epsilon}{\kappa+1}\right)\right)^s d\epsilon\right)^{\frac{1}{q}}\\&\quad\left.+\left(|F'(\mu_1)|^q\int_0^1 w(\epsilon)h^s\left(\frac{\kappa+\epsilon}{\kappa+1}\right)d\epsilon\ +m\left|F'\left(\frac{x}{m}\right)\right|^q\int_0^1 w(\epsilon)\left(1-h\left(\frac{\kappa+\epsilon}{\kappa+1}\right)\right)^s d\epsilon\right)^{\frac{1}{q}}\right).\end{aligned}$$

Taking into account the inequalities (3.2), we get (3.10). The proof is complete. □

Corollary 6. *For $x=\mu_2$, $\kappa=1$, $m=1$, $s=1, h(\epsilon)=\epsilon$ and $w(\epsilon)=\epsilon^{\alpha}$, by (3.10) for all $\alpha>0$ we get*

$$\left|\frac{F(\mu_2)+F(\mu_1)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha}}\left(J^{\alpha}_{\mu_1+}F\left(\frac{\mu_2+\mu_1}{2}\right)+J^{\alpha}_{\mu_2-}F\left(\frac{\mu_2+\mu_1}{2}\right)\right)\right| \leqslant \frac{\mu_2-\mu_1}{4(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{\frac{1}{q}}\left((\mathbf{P}_{11})^{\frac{1}{q}}+(\mathbf{P}_{12})^{\frac{1}{q}}\right), \tag{3.11}$$

where

$$\mathbf{P}_{11}=\frac{|F'(\mu_1)|^q+(2\alpha+3)\,|F'(\mu_2)|^q}{2},\qquad \mathbf{P}_{12}=\frac{(2\alpha+3)\,|F'(\mu_1)|^q+|F'(\mu_2)|^q}{2}.$$

Proof. By (3.10) we have

$$\left|2\frac{F(\mu_2)+F(\mu_1)}{\mu_2-\mu_1}-\left(\frac{2}{\mu_2-\mu_1}\right)^2\left(J^{w}_{x-}F\left(\frac{\mu_2+\mu_1}{2}\right)+J^{w}_{\mu_1+}F\left(\frac{\mu_2+\mu_1}{2}\right)\right)\right| \leqslant\left(\int_0^1 w(\epsilon)d\epsilon\right)^{1-\frac{1}{q}}\left((\mathbf{P}_1)^{\frac{1}{q}}+(\mathbf{P}_2)^{\frac{1}{q}}\right).$$

Here,

$$\int_0^1 w(\epsilon)d\epsilon=\frac{1}{\alpha+1},$$

$$\begin{aligned}\mathbf{P}_1&=|F'(\mu_1)|^q\int_0^1 w(\epsilon)h^s\left(\frac{1-\epsilon}{\kappa+1}\right)d\epsilon+m\left|F'\left(\frac{x}{m}\right)\right|^q\int_0^1 w(\epsilon)\left(1-h\left(\frac{1-\epsilon}{\kappa+1}\right)\right)^s d\epsilon\\&=\frac{|F'(\mu_1)|^q}{2}\int_0^1(1-\epsilon)\,\epsilon^{\alpha}d\epsilon+\frac{|F'(\mu_2)|^q}{2}\int_0^1(1+\epsilon)\,\epsilon^{\alpha}d\epsilon\\&=\frac{|F'(\mu_1)|^q+(2\alpha+3)\,|F'(\mu_2)|^q}{2\,(\alpha+1)\,(\alpha+2)}=\frac{1}{(\alpha+1)\,(\alpha+2)}\frac{|F'(\mu_1)|^q+(2\alpha+3)\,|F'(\mu_2)|^q}{2}\\&=\frac{1}{(\alpha+1)\,(\alpha+2)}\mathbf{P}_{11},\end{aligned}$$

$$\mathbf{P}_2=\frac{(2\alpha+3)\,|F'(\mu_1)|^q+|F'(\mu_2)|^q}{2\,(\alpha+1)\,(\alpha+2)}=\frac{1}{(\alpha+1)\,(\alpha+2)}\mathbf{P}_{12}.$$

For the weight fractional integrals we obtain

$$\begin{aligned}J^{w}_{\mu_1+}F\left(\frac{\mu_2+\mu_1}{2}\right)&=\alpha\int_{\mu_1}^{\frac{\mu_2+\mu_1}{2}}\left(\frac{\epsilon-\mu_1}{\frac{\mu_2+\mu_1}{2}-\mu_1}\right)^{\alpha-1}F(\epsilon)d\epsilon\\&=\frac{\Gamma(\alpha+1)}{\left(\frac{\mu_2+\mu_1}{2}-\mu_1\right)^{\alpha-1}\Gamma(\alpha)}\int_{\mu_1}^{\frac{\mu_2+\mu_1}{2}}(\epsilon-\mu_1)^{\alpha-1}F(\epsilon)d\epsilon\\&=\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha-1}}J^{\alpha}_{\mu_1+}F\left(\frac{\mu_2+\mu_1}{2}\right),\end{aligned}$$

$$J^{w}_{\mu_2-}F\left(\frac{\mu_2+\mu_1}{2}\right)=\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha-1}}J^{\alpha}_{\mu_2-}F\left(\frac{\mu_2+\mu_1}{2}\right).$$

Thus,

$$\left|2\frac{F(\mu_2)+F(\mu_1)}{\mu_2-\mu_1}-\frac{2^{\alpha+1}\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha+1}}\left(J^{\alpha}_{\mu_2-}F\left(\frac{\mu_2+\mu_1}{2}\right)+J^{\alpha}_{\mu_1+}F\left(\frac{\mu_2+\mu_1}{2}\right)\right)\right|$$
$$\leqslant\left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{1}{(\alpha+1)(\alpha+2)}\right)^{\frac{1}{q}}(\mathbf{P}_{11}+\mathbf{P}_{12})$$
$$=\frac{1}{\alpha+1}\left(\frac{1}{\alpha+2}\right)^{\frac{1}{q}}(\mathbf{P}_{11}+\mathbf{P}_{12}).$$

Multiplying the last inequality by $\frac{\mu_2-\mu_1}{4}$, we obtain (3.11). The proof is complete. □

Remark 10. *The above result for $\alpha=1$ and $q=1$ implies Corollary 3 in* [20] *and Theorem 2.2 in* [14].

Corollary 7. *For $x=\mu_2$, $\kappa=1$, $m=1$, $s=1$, $h(\epsilon)=\epsilon$ and $w(\epsilon)=(1-\epsilon)^{\alpha}$ by (3.10) for all $\alpha>0$ we get*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha}}\left[J^{\alpha}_{\mu_1+}F\left(\frac{\mu_2+\mu_1}{2}\right)+J^{\alpha}_{\mu_2-}F\left(\frac{\mu_2+\mu_1}{2}\right)\right]-F\left(\frac{\mu_2+\mu_1}{2}\right)\right|$$
$$\leqslant\frac{\mu_2-\mu_1}{4(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{q}\left((\mathbf{P}_{21})^{\frac{1}{q}}+(\mathbf{P}_{22})^{\frac{1}{q}}\right), \tag{3.12}$$

where

$$\mathbf{P}_{21}=\frac{(\alpha+1)|F'(\mu_1)|^q+(\alpha+3)|F'(\mu_2)|^q}{2},\qquad \mathbf{P}_{22}=\frac{(\alpha+3)|F'(\mu_1)|^q+(\alpha+1)|F'(\mu_2)|^q}{2}.$$

Proof. By (3.10) we have

$$\left|-\frac{4}{\mu_2-\mu_1}F\left(\frac{\mu_2+\mu_1}{2}\right)-\left(\frac{2}{\mu_2-\mu_1}\right)^2\left(J^{w}_{x-}F\left(\frac{\mu_2+\mu_1}{2}\right)+J^{w}_{\mu_1+}F\left(\frac{\mu_2+\mu_1}{2}\right)\right)\right|$$
$$\leqslant\left(\int_0^1 w(\epsilon)d\epsilon\right)^{1-\frac{1}{q}}\left((\mathbf{P}_1)^{\frac{1}{q}}+(\mathbf{P}_2)^{\frac{1}{q}}\right).$$

Here,

$$\mathbf{P}_1=|F'(\mu_1)|^q\int_0^1 w(\epsilon)h^s\left(\frac{1-\epsilon}{\kappa+1}\right)d\epsilon+m\left|F'\left(\frac{x}{m}\right)\right|^q\int_0^1 w(\epsilon)\left(1-h\left(\frac{1-\epsilon}{\kappa+1}\right)\right)^s d\epsilon$$
$$=|F'(\mu_1)|^q\int_0^1 w(\epsilon)h\left(\frac{1-\epsilon}{2}\right)d\epsilon+|F'(\mu_2)|^q\int_0^1 w(\epsilon)\left(1-h\left(\frac{1-\epsilon}{2}\right)\right)^s d\epsilon$$
$$=\frac{|F'(\mu_1)|^q}{2}\int_0^1(1-\epsilon)^{\alpha+1}d\epsilon+\frac{|F'(\mu_2)|^q}{2}\int_0^1(1-\epsilon)^{\alpha}(1+\epsilon)\,d\epsilon$$

and since

$$\int_0^1(1-\epsilon)^{\alpha}(1+\epsilon)\,d\epsilon=\int_0^1\epsilon^{\alpha}(2-\epsilon)\,d\epsilon=\frac{\alpha+3}{(\alpha+1)(\alpha+2)},$$

for $\mathbf{P}_1$ we have

$$\mathbf{P}_1=\frac{|F'(\mu_1)|^q}{2(\alpha+2)}+\frac{|F'(\mu_2)|^q(\alpha+3)}{2(\alpha+1)(\alpha+2)}=\frac{1}{(\alpha+1)(\alpha+2)}\mathbf{P}_{21}.$$

Doing the same thing for $\mathbf{P_2}$ from (3.10), we obtain

$$\mathbf{P_2}=\frac{(\alpha+3)|F'(\mu_1)|^q+(\alpha+1)|F'(\mu_2)|^q}{2(\alpha+2)(\alpha+3)}=\frac{1}{(\alpha+1)(\alpha+2)}\mathbf{P}_{22}.$$

Thus, we get

$$\left| -\frac{4}{\mu_2-\mu_1}F\left(\frac{\mu_2+\mu_1}{2}\right)+\frac{2^{\alpha+1}\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha+1}}\left(J^{\alpha}_{\mu_1+}F\left(\frac{\mu_2+\mu_1}{2}\right)+J^{\alpha}_{\mu_2-}F\left(\frac{\mu_2+\mu_1}{2}\right)\right)\right|$$
$$\leqslant \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{1}{(\alpha+1)(\alpha+2)}\right)^{q}\left((\mathbf{P}_{21})^{\frac{1}{q}}+(\mathbf{P}_{22})^{\frac{1}{q}}\right)$$
$$=\frac{1}{\alpha+1}\left(\frac{1}{\alpha+2}\right)^{q}\left((\mathbf{P}_{21})^{\frac{1}{q}}+(\mathbf{P}_{22})^{\frac{1}{q}}\right).$$

Multiplying the last inequality by $\frac{\mu_2-\mu_1}{4}$, we obtain (3.12). □

Corollary 8. *For $x=\mu_2$, $\kappa=0$, $m=1$, $s=1, h(\epsilon)=\epsilon$ and $w(\epsilon)=\epsilon^{\alpha}$ by (3.10) for all $\alpha>0$ we get*

$$\begin{aligned}&\left|\frac{F(\mu_2)+F(\mu_1)}{2} -\frac{\Gamma(\alpha+1)}{2(\mu_2-\mu_1)^{\alpha}}\left(J^{\alpha}_{\mu_2-}F(\mu_1)+J^{\alpha}_{\mu_1+}F(\mu_2)\right)\right| \\ &\leqslant \frac{\mu_2-\mu_1}{2(\alpha+1)}\left((\mathbf{P}_{31})^{\frac{1}{q}}+(\mathbf{P}_{32})^{\frac{1}{q}}\right),\end{aligned} \tag{3.13}$$

where

$$\mathbf{P}_{31}=\frac{|F'(\mu_1)|^q+(\alpha+1)|F'(\mu_2)|^q}{\alpha+2}, \qquad \mathbf{P}_{32}=\frac{(\alpha+1)|F'(\mu_1)|^q+|F'(\mu_2)|^q}{\alpha+2}.$$

Proof. By (3.10) we have

$$\left|\frac{F(x)+F(\mu_1)}{\mu_2-\mu_1}-\left(\frac{1}{\mu_2-\mu_1}\right)^{2}\left(J^{w}_{x-}F(\mu_1)+J^{w}_{\mu_1+}F(\mu_2)\right)\right|$$
$$\leqslant\left(\int_0^1 w(\epsilon)d\epsilon\right)^{1-\frac{1}{q}}\left((\mathbf{P}_1)^{\frac{1}{q}}+(\mathbf{P}_2)^{\frac{1}{q}}\right).$$

Here

$$\begin{aligned}\mathbf{P}_1&=|F'(\mu_1)|^q\int_0^1 w(\epsilon)h^s\left(\frac{1-\epsilon}{\kappa+1}\right)d\epsilon+m\left|F'\left(\frac{x}{m}\right)\right|^q\int_0^1 w(\epsilon)\left(1-h\left(\frac{1-\epsilon}{\kappa+1}\right)\right)^s d\epsilon\\&=|F'(\mu_1)|^q\int_0^1\epsilon^{\alpha}(1-\epsilon)\,d\epsilon+|F'(\mu_2)|^q\int_0^1\epsilon^{\alpha+1}d\epsilon\\&=\frac{|F'(\mu_1)|^q+(\alpha+1)|F'(\mu_2)|^q}{(\alpha+1)(\alpha+2)}=\frac{1}{\alpha+1}\mathbf{P}_{31},\\ \mathbf{P}_2&=|F'(\mu_1)|^q\int_0^1 w(\epsilon)h^s\left(\frac{\kappa+\epsilon}{\kappa+1}\right)d\epsilon+m\left|F'\left(\frac{x}{m}\right)\right|^q\int_0^1 w(\epsilon)\left(1-h\left(\frac{\kappa+\epsilon}{\kappa+1}\right)\right)^s d\epsilon\\&=\left(\frac{(\alpha+1)|F'(\mu_1)|^q+|F'(\mu_2)|^q}{(\alpha+1)(\alpha+2)}\right)^{\frac{1}{q}}=\frac{1}{\alpha+1}\mathbf{P}_{32},\\ J^{w}_{\mu_1+}F(\mu_2)&=\int_{\mu_1}^{\mu_2}w'\left(\frac{\epsilon-\mu_1}{\mu_2-\mu_1}\right)F(\epsilon)d\epsilon=\alpha\int_{\mu_1}^{\frac{\mu_2+\mu_1}{2}}\left(\frac{\epsilon-\mu_1}{\mu_2-\mu_1}\right)^{\alpha-1}F(\epsilon)d\epsilon\\&=\frac{\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha-1}\Gamma(\alpha)}\int_{\mu_1}^{\frac{\mu_2+\mu_1}{2}}(\epsilon-\mu_1)^{\alpha-1}F(\epsilon)d\epsilon=\frac{\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha-1}}J^{\alpha}_{\mu_1+}F(\mu_2)\\ J^{w}_{\mu_2-}F(\mu_1)&=\frac{\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha-1}}J^{\alpha}_{\mu_2-}F(\mu_1).\end{aligned}$$

Thus,

$$\left|\frac{F(x)+F(\mu_1)}{\mu_2-\mu_1}-\left(\frac{1}{\mu_2-\mu_1}\right)^2\left(J^w_{x-}F(\mu_1)+J^w_{\mu_1+}F(\mu_2)\right)\right|\leqslant\left(\frac{1}{\alpha+1}\right)\left((\mathbf{P}_{31})^{\frac{1}{q}}+(\mathbf{P}_{32})^{\frac{1}{q}}\right).$$

Multiplying the last inequality by $\frac{\mu_2-\mu_1}{2}$, we obtain (3.13). The proof is complete. □

Remark 11. *The above result for $\alpha=1$ implies the case $w(\epsilon)=(1-\epsilon)^\alpha$ in Theorem 1 in* [7] *and Theorem 2.2 in* [23].

Corollary 9. *For $x=\mu_2$, $\kappa=0$, $m=1$, $s=1$, $h(\epsilon)=\epsilon$ and $w(\epsilon)=(1-\epsilon)^\alpha$ by (3.10) for all $\alpha>0$ we get*

$$\begin{aligned}\left|\frac{\Gamma(\alpha+1)}{2(\mu_2-\mu_1)^\alpha}\left(J^\alpha_{\mu_2-}F(\mu_1)+J^\alpha_{\mu_1+}F(\mu_2)\right)-F\left(\frac{\mu_2+\mu_1}{2}\right)\right|&\\ \leqslant\frac{\mu_2-\mu_1}{\alpha+1}\left((\mathbf{P}_{41})^{\frac{1}{q}}+(\mathbf{P}_{42})^{\frac{1}{q}}\right),&\end{aligned}\tag{3.14}$$

where

$$\mathbf{P}_{41}=\frac{(\alpha+1)|F'(\mu_1)|^q+|F'(\mu_2)|^q}{\alpha+2},\qquad\mathbf{P}_{42}=\frac{|F'(\mu_1)|^q+(\alpha+1)|F'(\mu_2)|^q}{\alpha+2}.$$

Proof. By (3.10) we have

$$\begin{aligned}\left|-\frac{1}{\mu_2-\mu_1}F\left(\frac{\mu_2+\mu_1}{2}\right)+\left(\frac{1}{\mu_2-\mu_1}\right)^2\left(J^w_{x-}F(\mu_1)+J^w_{\mu_1+}F(\mu_2)\right)\right|&\\ \leqslant\left(\int_0^1 w(\epsilon)d\epsilon\right)^{1-\frac{1}{q}}\left((\mathbf{P}_1)^{\frac{1}{q}}+(\mathbf{P}_2)^{\frac{1}{q}}\right).&\end{aligned}$$

Here,

$$\begin{aligned}&\int_0^1 w(\epsilon)d\epsilon=\frac{1}{\alpha+1},\\ &\mathbf{P}_1=|F'(\mu_1)|^q\int_0^1 w(\epsilon)h^s\left(\frac{1-\epsilon}{\kappa+1}\right)d\epsilon+|F'(\mu_2)|^q\int_0^1 w(\epsilon)\left(1-h\left(\frac{1-\epsilon}{\kappa+1}\right)\right)^s d\epsilon\\ &\quad=|F'(\mu_1)|^q\int_0^1(1-\epsilon)^{\alpha+1}d\epsilon+|F'(\mu_2)|^q\int_0^1\epsilon(1-\epsilon)^\alpha d\epsilon\\ &\quad=\frac{|F'(\mu_1)|^q}{\alpha+2}+\frac{|F'(\mu_2)|^q}{(\alpha+1)(\alpha+2)}=\frac{1}{\alpha+1}\frac{(\alpha+1)|F'(\mu_1)|^q+|F'(\mu_2)|^q}{\alpha+2}\\ &\quad=\frac{1}{\alpha+1}\mathbf{P}_{41},\\ &\mathbf{P}_2=\left(\frac{1}{\alpha+1}\right)^{\frac{1}{q}}\frac{|F'(\mu_1)|^q+(\alpha+1)|F'(\mu_2)|^q}{\alpha+2}=\frac{1}{\alpha+1}\mathbf{P}_{42},\\ &J^w_{\mu_1+}F(\mu_2)=\int_{\mu_1}^{\mu_2}w'\left(\frac{\epsilon-\mu_1}{\mu_2-\mu_1}\right)F(\epsilon)d\epsilon=\alpha\int_{\mu_1}^{\frac{\mu_2+\mu_1}{2}}\left(\frac{\epsilon-\mu_1}{\mu_2-\mu_1}\right)^{\alpha-1}F(\epsilon)d\epsilon\\ &\qquad=\frac{\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha-1}\Gamma(\alpha)}\int_{\mu_1}^{\frac{\mu_2+\mu_1}{2}}(\epsilon-\mu_1)^{\alpha-1}F(\epsilon)d\epsilon=\frac{\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha-1}}J^\alpha_{\mu_1+}F(\mu_2),\\ &J^w_{\mu_2-}F(\mu_1)=\frac{\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha-1}}J^\alpha_{\mu_2-}F(\mu_1).\end{aligned}$$

Thus,

$$\left| \frac{2F\left(\frac{\mu_2+\mu_1}{2}\right)}{\mu_2-\mu_1} - \frac{\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha+1}} \left(J^{\alpha}_{\mu_2-}F(\mu_1) + J^{\alpha}_{\mu_1+}F(\mu_2) \right) \right| \leqslant \left(\frac{1}{\alpha+1} \right) \left((\mathbf{P}_{41})^{\frac{1}{q}} + (\mathbf{P}_{42})^{\frac{1}{q}} \right).$$

Multiplying the last inequality by $\frac{\mu_2-\mu_1}{2}$, we obtain (3.14). The proof is complete. □

4. Conclusions

In this paper we establish, various extensions and generalizations of the classical Hermite — Hadamard type inequality in the context of weight integral operators. Throughout our work, we have observed that various results presented in the literature are special cases of our findings, demonstrating the breadth of their applicability. However, we did not want to conclude the study without mentioning two additional aspects of the breadth of our results. Referring to the used integral operator, we note that the weight function may encompass many known cases.

For example, by Theorem 3.1, we can obtain the following result for the Bullen type inequality.

Corollary 10. *Under the conditions of Theorem 3.1, for $x = \mu_2$, $\kappa = 1$, $m = 1$, $s = 1$, $h(\epsilon) = \epsilon$ and $w(\epsilon) = \epsilon - \frac{1}{2}$ for convex functions the inequality holds*

$$\left| \frac{1}{2}\left(\frac{F(\mu_2)+F(\mu_1)}{2} + F\left(\frac{\mu_2+\mu_1}{2}\right) \right) - \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} F(\epsilon)d\epsilon \right| \leqslant \frac{\mu_2-\mu_1}{2}\frac{|F'(\mu_1)|+|F'(\mu_2)|}{8}.$$

Our results can yield other inequalities for various classes of convex functions.

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