A UNIVERSAL PROPERTY OF SEMIGROUP C^*-ALGEBRAS FOR FREE PRODUCTS OF SEMIGROUPS OF RATIONALS

R.N. GUMEROV, A.S. KUKLIN, E.V. LIPACHEVA

Abstract. The paper deals with the left reduced semigroup C^*-algebras $C^*_\lambda(Q)$ for non-abelian cancellative semigroups Q associated with finite tuples $(M_1, M_2, \ldots, M_n)$ of sequences M_i of arbitrary natural numbers. Such a semigroup Q is defined to be the free product of semigroups consisting of positive numbers in the ordered groups of rationals Q_{M_i} generated by the reciprocals for products of the terms in M_i. The C^*-algebra $C^*_\lambda(Q)$ is generated by the left regular representation of Q. It is shown that each semigroup Q is not left amenable but its full and left reduced semigroup C^*-algebras are isomorphic and nuclear. We establish that every C^*-algebra $C^*_\lambda(Q)$ can be characterized as a universal C^*-algebra defined by a countable set of isometries satisfying a countable family of polynomial relations. To prove this result, we make use of the universal property of C^*-algebra $C^*_\lambda(Q)$ considered as the inductive limit for the inductive sequence of Toeplitz — Cuntz algebras associated with the tuple $(M_1, M_2, \ldots, M_n)$.

Keywords: free product of semigroups, full semigroup C^*-algebra, left amenable semigroup, left reduced semigroup C^*-algebra, nuclear C^*-algebra, relation, set of generators, universal C^*-algebra, universal property

Mathematics Subject Classification: 16S10, 20M30, 46L05, 46M40, 47L40

1. Introduction

The paper is devoted to the left reduced semigroup C^*-algebras $C^*_\lambda(Q)$ for non-abelian cancellative semigroups Q, which are the free products of semigroups of rational numbers. These C^*-algebras are very natural objects for studying since they are generated by the left regular representations of semigroups Q. The main purpose of our work is to describe the C^*-algebras $C^*_\lambda(Q)$ as universal C^*-algebras defined by countable sets of isometries subject to countable families of polynomial relations.

The study of semigroup C^*-algebras was initiated by Coburn. His papers [7], [8] deal with the reduced semigroup C^*-algebra for the additive semigroup of nonnegative integers. Coburn proved that this C^*-algebra is the universal C^*-algebra generated by one isometry (see Example 2.1). The semigroup C^*-algebras for semigroups in the additive group of the real numbers were studied by Douglas in [12]. The motivation for these works came from the index theory and K-theory. In [36], Murphy considered the semigroup C^*-algebras for semigroups which are the positive cones in arbitrary ordered abelian groups. In particular, it was proved that these C^*-algebras are universal C^*-algebras generated by families of isometries indexed by elements of positive cones (see Example 2.2). Moreover, Murphy started to study the left

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reduced semigroup C^*-algebras for arbitrary left cancellative semigroups [37], [38]. Afterwards, the theory of semigroup C^*-algebras was developed by a number of authors, and there is now a large literature on the subject (see, for instance, [40], [32], [33] and the references therein).

In [2], Blackadar gave a systematic studying the universal C^*-algebras defined by generators and relations in connection with applications of such algebras in shape theory for C^*-algebras. One of the main ideas in this theory is to represent a general C^*-algebra as the inductive limit of an inductive system consisting of "nice" C^*-algebras and classify the C^*-algebras up to an equivalence of the associated inductive systems [13]. The universal C^*-algebras serve as tools for constructing $*$-homomorphisms of C^*-algebras and representing C^*-algebras as the inductive limits for inductive systems of C^*-algebras. For interesting applications of the universal C^*-algebras, we refer the reader to the book [34] and the references therein.

In this paper, for every finite tuple $(M_1, M_2, \ldots, M_n)$ consisting of arbitrary sequences of natural numbers M_i, we consider the cancellative semigroup Q which is a free product of semigroups. To construct the semigroup Q, we take for every M_i the additive group Q_{M_i} generated by the reciprocals for products of the terms in the sequence M_i. Then, we consider the semigroup $Q^+_{M_i}$ consisting of positive rational numbers in the group Q_{M_i}. Adjoining the neutral element to the free product of the semigroups $Q^+_{M_i}$, one obtains the cancellative non-abelian semigroup Q associated with the tuple $(M_1, M_2, \ldots, M_n)$. For brevity, we do not use the tuple $(M_1, M_2, \ldots, M_n)$ in the notation of this semigroup. The paper deals with properties of the semigroups Q and their semigroup C^*-algebras.

A part of motivation for our work comes from studies of the inductive systems of C^*-algebras in [16], [17], [18], [19], [20], [21], [14], [22]. The C^*-algebras $C^*_\lambda(Q)$ arise as the inductive limits for the inductive sequences of the Toeplitz — Cuntz algebras associated with the tuples $(M_1, M_2, \ldots, M_n)$ [14]. It is worth noting that the results on $*$-homomorphisms of semigroup C^*-algebras [16], [19], [22] are closely connected with the facts about mappings of the compact groups called the solenoids [6], [23], [24], [15], [25], [26], [27]. The results on the semigroup C^*-algebras for the positive cones in the groups of rationals in [16] are operator–algebraic analogs of those contained in [6], [23], [24], [26].

Our work is also motivated by Li's results on a connection between amenability of semigroups and nuclearity for semigroup C^*-algebras (see [33] and the references therein). We recall that for a discrete group G the following properties are equivalent: 1) G is amenable; 2) the full group C^*-algebra $C^*(G)$ is nuclear; 3) the reduced group C^*-algebra $C^*_\lambda(G)$ is nuclear; 4) the C^*-algebras $C^*(G)$ and $C^*_\lambda(G)$ are canonically isomorphic (see [33, Sect. 5.6.1], [3, Ch. 2, § 6]). But analogs of these properties for semigroups and their C^*-algebras are not equivalent. For example, the non-abelian free monoid on two generators is not left amenable, however its left reduced semigroup C^*-algebra is nuclear. In [33], these phenomena are explained. Moreover, there are analogs of the above mentioned criteria for G in the context of cancellative semigroups and their semigroup C^*-algebras. In our exposition we use the notion of the full semigroup C^*-algebra $C^*(S)$ which is introduced by Li for a left cancellative semigroup S. Namely, the C^*-algebra $C^*(S)$ is defined to be the C^*-algebra that is universal for $*$-representations of the left inverse hull of S by partial isometries (see Example 2.4).

In this paper, we show that for each tuple $(M_1, M_2, \ldots, M_n)$ the semigroup Q is not left amenable, but the left reduced semigroup C^*-algebra $C^*_\lambda(Q)$ and the full semigroup C^*-algebra $C^*(Q)$ are canonically isomorphic and nuclear. Thus, the semigroup Q provides another example of a semigroup with the indicated properties. We prove that both C^*-algebras $C^*_\lambda(Q)$ and $C^*(Q)$ for the semigroup Q associated with the tuple $(M_1, M_2, \ldots, M_n)$ are the universal C^*-algebras generated by a countable set of isometries which satisfy a countable set of polynomial relations defined by the tuple $(M_1, M_2, \ldots, M_n)$. To obtain this result, we use the fact that the left reduced semigroup C^*-algebra $C^*_\lambda(Q)$ is the inductive limit for the inductive sequence of the Toeplitz — Cuntz algebras associated with the tuple $(M_1, M_2, \ldots, M_n)$.

The paper is organized as follows. It consists of five sections. Section 1 is an introduction to the subject under our consideration. Section 2 contains notation and necessary information about semigroups and C^*-algebras. In Example 2.4, we briefly give the definitions of the full semigroup C^*-algebras and their left regular representations which were proposed by Li. The semigroups Q are introduced in Section 3. We show that these semigroups satisfy the independence condition and are not reversible. As a consequence, we obtain that the semigroups Q are not left amenable. In Section 4, we establish that the C^*-algebras $C^*_\lambda(Q)$ are nuclear. Moreover, we show that the left regular representations of the full semigroup C^*-algebras $C^*(Q)$ yield the isomorphisms between the semigroup C^*-algebras $C^*(Q)$ and $C^*_\lambda(Q)$. In Section 5 we give a description of these algebras as the universal C^*-algebras generated by sets of generators subject to relations.

2. Preliminaries

In what follows, S is a discrete semigroup with the left cancellation property, that is, $st = sr$ implies $t = r$ whenever $s, t, r \in S$. As usual, the symbol $l^2(S)$ stands for the complex Hilbert space of all square summable complex–valued functions on S. In the space $l^2(S)$, we consider the standard orthonormal basis $\{e_s \mid s \in S\}$ consisting of the functions $e_s : S \to \mathbb{C}$ given by $e_s(t) = \delta_{st}$, where $t \in S$ and the symbol δ_{st} denotes the Kronecker delta.

Let $\mathfrak{B}(l^2(S))$ be the C^*-algebra of all bounded linear operators on the Hilbert space $l^2(S)$. In $\mathfrak{B}(l^2(S))$, for every $s \in S$, we consider the isometry $T_s : l^2(S) \to l^2(S)$ defined by

$$T_s(e_t) = e_{st}, \quad t \in S. \tag{2.1}$$

The C^*-subalgebra of the algebra $\mathfrak{B}(l^2(S))$ generated by the set of isometries $\{T_s \mid s \in S\}$ is called *the left reduced semigroup C^*-algebra for the semigroup S*. It is denoted by $C^*_\lambda(S)$.

This paper is concerned with the left reduced semigroup C^*-algebras for the free products of finite families of semigroups. Therefore, we recall how to construct the free product of semigroups.

Throughout, for an integer $n \geqslant 2$, the set $\{1, 2, \ldots, n\}$ is denoted by $\overline{n}$.

Let $\{S_k \mid k \in \overline{n}\}$ be a family of disjoint semigroups. We denote by $S_1 * \ldots * S_n$ the set consisting of all non-empty finite words $s_1 s_2 \ldots s_l$, $l \in \mathbb{N}$, in the alphabet $\bigsqcup_{k=1}^{n} S_k$ satisfying the following property: any two adjacent letters in $s_1 s_2 \ldots s_l$ belong to distinct semigroups. The multiplication $*$ on the set $S_1 * \ldots * S_n$ is defined as follows:

$$s_1 s_2 \ldots s_l * t_1 t_2 \ldots t_m = \begin{cases} s_1 \ldots s_l t_1 t_2 \ldots t_m & \text{if } s_l \in S_i, t_1 \in S_j, i \neq j; \\ s_1 \ldots s_{l-1}(s_l \cdot t_1) t_2 \ldots t_m & \text{if } s_l, t_1 \in S_i \text{ for some } i, \end{cases} \tag{2.2}$$

where $s_1 s_2 \ldots s_l, t_1 t_2 \ldots t_m \in S_1 * \ldots * S_n$, $l, m \in \mathbb{N}$, $i, j \in \overline{n}$. It is straightforward to check that the set $S_1 * \ldots * S_n$ endowed with the multiplication $*$ is a non-abelian semigroup which is called *the free product of the semigroups S_k, $k \in \overline{n}$*.

The following question is discussed in this paper. Is the free product of semigroups left amenable? We recall that a semigroup S is said to be *left amenable* if there exists a left invariant mean on the Banach space of all bounded complex-valued functions on S with the uniform norm (see [33, Sect. 5.6.2]). For instance, every abelian semigroup is left amenable.

For studying the left amenability of semigroups in [33], Li made use of the left (right) reversibility of semigroups and the independence condition.

Let S be a cancellative semigroup, which means both left and right cancellative. The semigroup S is said to be *left (right) reversible*, if for every $s, t \in S$ we have $sS \cap tS \neq \emptyset$ ($Ss \cap St \neq \emptyset$).

Here $sS := \{sr \mid r \in S\}$ and $Ss := \{rs \mid r \in S\}$. For instance, every cancellative abelian semigroup is both left and right reversible. We also note that a cancellative semigroup, which is left or right reversible, embeds into a group [33, Sect. 5.4.1].

To define the independence condition for a left cancellative semigroup S, one uses so-called constructible right ideals of S [33, Def. 6.30]. We do not give this definition here, since the following fact is sufficient for our exposition. As follows from [33, Lms. 6.31, 6.32], the left cancellative monoids satisfying the independence condition are precisely the right LCM (Least Common Multiple) monoids. We recall that a left cancellative semigroup S with identity is called a right LCM monoid if for all $s, t \in S$ with $sS \cap tS \neq \emptyset$ there exists $r \in S$ such that $sS \cap tS = rS$. For instance, positive cones in totally ordered groups are right LCM monoids and, hence, they satisfy the independence condition. But the additive semigroup $\mathbb{N}^+ \setminus \{1\} = \{0, 2, 3, \ldots\}$ is not a right LCM monoid, so it does not satisfy the independence condition [33, Sect. 5.6.5].

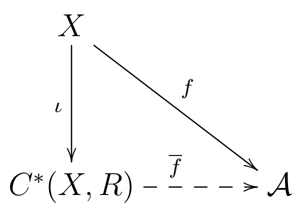
We note that the C^*-algebras associated to right LCM monoids are of great interest in the theory of semigroup C^*-algebras, and they have attracted a lot of attention in recent years (see, for example, [4, 5]).

Further, we define the universal C^*-algebra generated by a set of generators subject to polynomial relations (see [2, Section 1], [34, Chapter 3]). For a categorical approach to this notion we refer the reader to [35, 1, 29].

Let $X = \{x_i \mid i \in I\}$ be a set and $\mathcal{F}(X)$ be the complex involutive algebra of non-commutative polynomials in indeterminants from $X \sqcup X^*$, where $X^* = \{x_i^* \mid i \in I\}$. Let R be a subset in $\mathcal{F}(X)$. We call X and R the sets of *generators* and *relations* respectively. The pair (X, R) is said to be the $*$-*polynomial pair* [1].

For a C^*-algebra $\mathcal{A}$, a function $f : X \to \mathcal{A}$ is called a *representation* of the pair (X, R) if for every polynomial $p(x_{i_1}, \ldots, x_{i_s}, x_{j_1}^*, \ldots, x_{j_t}^*)$ in R the equality $p(f(x_{i_1}), \ldots, f(x_{i_s}), f(x_{j_1})^*, \ldots, f(x_{j_t})^*) = 0$ holds in $\mathcal{A}$, where $s, t \in \mathbb{N}$.

The universal C^*-algebra generated by a set of generators X subject to relations in R is a pair $(C^*(X, R), \iota)$ consisting of a C^*-algebra $C^*(X, R)$ and a representation $\iota : X \to C^*(X, R)$ satisfying the following *universal property*. For every C^*-algebra $\mathcal{A}$ and every representation $f : X \to \mathcal{A}$ there exists a unique $*$-homomorphism $\overline{f} : C^*(X, R) \to \mathcal{A}$ such that the diagram



commutes, that is, $f = \overline{f} \circ \iota$. The C^*-algebra $C^*(X, R)$ itself is often called the universal C^*-algebra generated by the pair (X, R).

We note that the relations can be of a very general nature and the universal C^*-algebra generated by a set subject to relations does not always exist. Blackadar introduced the concept of *an admissible pair* (X, R) and proved that for such a pair (X, R) the universal algebra $C^*(X, R)$ exists [2, Def. 1.2]. We also note that for every C^*-algebra $\mathcal{A}$ there exists a $*$-polynomial pair (X, R) such that $\mathcal{A} = C^*(X, R)$ [1, Thm. 2].

A description of any C^*-algebra $\mathcal{A}$ as a universal C^*-algebra $C^*(X, R)$ generated by a set of generators X satisfying a set of relations R allows us to construct $*$-homomorphisms from $\mathcal{A}$ to C^*-algebras. An interesting task that naturally arises in this case is to find simpler relations which are sufficient to characterize $\mathcal{A}$ as a universal C^*-algebra.

If a universal C^*-algebra $C^*(X, R)$ is unital with the unit 1 and $\iota(x) = 1$ for some $x \in X$, then the element x is also denoted by 1. As usual, we do not write the natural relations for 1 in the set R.

Now, we give several examples of universal C^*-algebras defined by sets of generators subject to relations.

Example 2.1. *Let $S = \mathbb{N}^+$ be the additive semigroup of non-negative integers. The C^*-algebra $C^*_\lambda(\mathbb{N}^+)$ is called the* Toeplitz algebra*. It is the C^*-subalgebra in the C^*-algebra $\mathfrak{B}(l^2(\mathbb{N}^+))$ generated by the right shift operator T_1 acting on the Hilbert space $l^2(\mathbb{N}^+)$.*

If V is an isometric non-unitary element in a C^*-algebra $\mathcal{A}$, then, by Coburn theorem [39, Thm. 3.5.18]*, there exists a unique unital isometric $*$-homomorphism $\varphi : C^*_\lambda(\mathbb{N}^+) \to \mathcal{A}$ such that $\varphi(T_1) = V$. In other words, the Toeplitz algebra $C^*_\lambda(\mathbb{N}^+)$ is isomorphic to the universal C^*-algebra*

$$C^*(\{1, v\}, \{v^*v = 1\})$$

generated by an isometry.

In the next example we consider a generalization of Example 2.1.

Example 2.2. *Let $S = G^+$ be a positive cone in an abelian totally ordered group G. In* [36]*, Murphy proved that the C^*-algebra $C^*_\lambda(G^+)$ is isomorphic to the universal C^*-algebra*

$$C^*(\{1, v_x \mid x \in G^+\}, \{v_x^*v_x = 1, v_xv_y = v_{x+y} \mid x, y \in G^+\}).$$

The following example of the C^*-algebra $C^*_\lambda(G^+)$ is given in [16], [28]*. Let $P = (p_1, p_2, \ldots)$ be a sequence of arbitrary prime numbers and $G = \mathbb{Q}_P$ be the additive totally ordered group of rational numbers defined by*

$$\mathbb{Q}_P = \left\{ \frac{m}{p_1 \ldots p_n} \middle| \; m \in \mathbb{N}, n \in \mathbb{N} \right\}.$$

Then, the C^*-algebra $C^*_\lambda(\mathbb{Q}_P^+)$ is isomorphic to the universal C^*-algebra (see [28]*)*

$$C^*(\{1, x_n \mid n \in \mathbb{N}\}, \{x_n^*x_n = 1, x_n = x_{n+1}^{p_n} \mid n \in \mathbb{N}\}).$$

Example 2.3. *Let $n \geqslant 2$ be an integer. Consider the free product $S = \mathbb{N} * \ldots * \mathbb{N}$ of n copies of the additive semigroup of natural numbers. Then the C^*-algebra $C^*_\lambda(\mathbb{N} * \ldots * \mathbb{N})$ is isomorphic to the universal C^*-algebra generated by n isometries with pairwise orthogonal ranges* [32]*. This universal C^*-algebra was defined and studied by Cuntz* [10], [11]*. It is called the* Toeplitz — Cuntz algebra *and is denoted by $\mathcal{TO}_n$, that is,*

$$\mathcal{TO}_n = C^*\left(\{u_1, \ldots, u_n\}, \{u_i^*u_i = 1, u_i^*u_j = 0 \mid i, j \in \overline{n}, i \neq j\}\right). \tag{2.3}$$

In [11, Lemma 3.1]*, it is shown that for any unital C^*-algebra $\mathcal{A}$ and its isometries $A_1, A_2, \ldots, A_n$ with pairwise orthogonal ranges satisfying the inequality*

$$A_1A_1^* + A_2A_2^* + \cdots + A_nA_n^* < 1,$$

there exists a unique unital isometric $*$-homomorphism from $\mathcal{TO}_n$ into $\mathcal{A}$ sending the generator u_i to the isometry A_i for every $i \in \overline{n}$ (see also [32]*).*

The next example contains the definitions of the full semigroup C^*-algebra and its left regular representations which are considered in Sections 4 and 5.

Example 2.4. *Let S be a left cancellative semigroup. We denote by $I_l(S)$ the smallest semigroup of partial isometries on the Hilbert space $l^2(S)$ containing all the isometries T_a (see (2.1)) and their adjoints T_a^*, $a \in S$, and that is closed under multiplication. The set $I_l(S)$ is an inverse semigroup, that is, for every element $a \in I_l(S)$ there exists a unique $b \in S$ such that $aba = a$*

and $bab = b$. *In this case, one writes* $b = a^{-1}$. *The semigroup* $I_l(S)$ *is called the left inverse hull attached to* S. *Li* [33, Def. 5.6.38] *defined the full semigroup* C^**-algebra of* S *as*

$$C^*(S) := C^*\left(\{v_a\}_{a\in I_l(S)}, \{v_a v_b = v_{ab}, v_a^* = v_{a^{-1}}, v_0 = 0 \quad \textit{if} \quad 0 \in I_l(S), a, b \in I_l(S)\}\right).$$

In other words, $C^*(S)$ *is the* C^**-algebra universal for* $*$*-representations of the semigroup* $I_l(S)$ *by partial isometries.*

Note that one has the canonical surjective $*$*-homomorphism*

$$C^*(S) \to C^*_\lambda(S) : v_a \mapsto T_a, \qquad a \in S,$$

which is called the left regular representation of the full semigroup C^**-algebra of* S. *For details we refer the reader to* [33, Sect. 5.6.6].

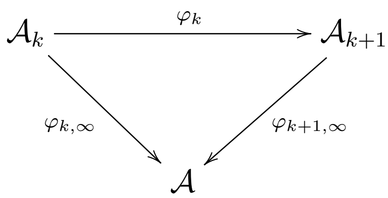
Since inductive sequences of C^*-algebras and their inductive limits are involved in our exposition, we recall necessary definitions and facts about these concepts (for detail see, for instance, [39, Ch. 6]).

A collection $\{\mathcal{A}_k, \varphi_k \mid k \in \mathbb{N}\}$ consisting of C^*-algebras $\mathcal{A}_k$ and $*$-homomorphisms $\varphi_k : \mathcal{A}_k \to \mathcal{A}_{k+1}$ is called the *inductive sequence of* C^**-algebras*. Usually, it is written as the diagram

$$\mathcal{A}_1 \xrightarrow{\varphi_1} \mathcal{A}_2 \xrightarrow{\varphi_2} \mathcal{A}_3 \xrightarrow{\varphi_3} \dots. \tag{2.4}$$

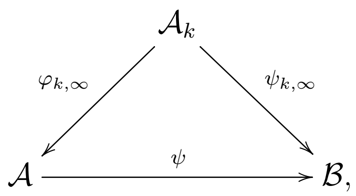
A pair $(\mathcal{A}, \{\varphi_{k,\infty} \mid k \in \mathbb{N}\})$, where $\mathcal{A}$ is a C^*-algebra and $\{\varphi_{k,\infty} : \mathcal{A}_k \to \mathcal{A} \mid k \in \mathbb{N}\}$ is a sequence of $*$-homomorphisms, is called the *inductive limit of the inductive sequence* (2.4) if the following conditions hold:

1) for every $k \in \mathbb{N}$ the diagram



commutes, that is, we have the equality $\varphi_{k,\infty} = \varphi_{k+1,\infty} \circ \varphi_k$;

2) (*the universal property of the inductive limit*) for every C^*-algebra $\mathcal{B}$ and every sequence of $*$-homomorphisms $\{\psi_{k,\infty} : \mathcal{A}_k \to \mathcal{B}\}_{k=1}^{+\infty}$ satisfying the relation $\psi_{k,\infty} = \psi_{k+1,\infty} \circ \varphi_k$ for every $k \in \mathbb{N}$, there is a unique $*$-homomorphism $\psi : \mathcal{A} \to \mathcal{B}$ such that the diagram



commutes for every $k \in \mathbb{N}$, that is, $\psi_{k,\infty} = \psi \circ \varphi_{k,\infty}$.

An inductive limit $(\mathcal{A}, \{\varphi_{k,\infty} \mid k \in \mathbb{N}\})$ exists for every inductive sequence (2.4). Usually, the C^*-algebra $\mathcal{A}$ itself is called an inductive limit of (2.4). Moreover, it follows from the universal property of the inductive limit that two inductive limits of the same inductive sequence (2.4) are isomorphic.

3. Free products for semigroups of rationals

In this section, for a finite tuple of sequences of arbitrary natural numbers, we construct semigroups of rational numbers and the free product for these semigroups. We establish certain properties of this free product. The next two sections are devoted to studying the left reduced semigroup C^*-algebras generated by the left regular representations of such free products.

Let $n \geqslant 2$ be an integer. Together with n infinite sequences of natural numbers

$$M_1 = (m_{11}, m_{21}, \ldots), \ldots, M_n = (m_{1n}, m_{2n}, \ldots), \tag{3.1}$$

we consider additive semigroups of rational numbers $\mathbb{Q}^+_{M_i}$, $i \in \overline{n}$, defined as

$$\mathbb{Q}^+_{M_i} = \left\{ \frac{l}{m_{1i} \ldots m_{ti}} \mid l \in \mathbb{N}, t \in \mathbb{N} \right\}.$$

Note that for $p \in \mathbb{Q}^+_{M_i}$ the set $p + \mathbb{Q}^+_{M_i}$ never contains p.

Further, for each $i \in \overline{n}$, we take the semigroup $\mathbb{Q}^+_{M_i} \times \{i\}$ whose binary operation defined by the formula

$$(p, i) + (q, i) = (p + q, i), \quad p, q \in \mathbb{Q}^+_{M_i}.$$

Then, let us construct the free product of the semigroups $\mathbb{Q}^+_{M_i} \times \{i\}$, $i \in \overline{n}$, and adjoin the neutral element 0, that is, we obtain the semigroup

$$(\mathbb{Q}^+_{M_1} \times \{1\}) * \ldots * (\mathbb{Q}^+_{M_n} \times \{n\}) \sqcup \{0\}.$$

This semigroup is denoted by Q and is called the *semigroup associated with tuple of sequences of natural numbers* (3.1).

It is worth noting that the semigroup Q is non-commutative and has the cancellation property, that is, both left and right cancellation properties. Moreover, one can see that Q is embedded into the group which is the free product of additive groups of rational numbers $\mathbb{Q}_{M_i}$, $i \in \overline{n}$. Some additional properties of the semigroup Q are summarized in the next statement.

Theorem 3.1. *Let Q be the semigroup associated with a tuple of sequences of natural numbers $(M_1, \ldots, M_n)$. Then the following properties are fulfilled:*

1. *Q satisfies the independence condition;*
2. *Q is neither left nor right reversible;*
3. *Q is not left amenable.*

Proof. (1) As was mentioned in Section 2, it suffices to show that the semigroup Q is a right LCM monoid. To this end, we assume that two distinct elements $a, b \in Q$ satisfy the condition $aQ \cap bQ \neq \emptyset$. Take an element $d \in aQ \cap bQ$. Then there exist elements $c_1, c_2 \in Q$ such that $d = a * c_1 = b * c_2$. Consequently, we have two possibilities, namely, $a = b * d_1$ for some $d_1 \in Q$ or $b = a * d_2$ for some $d_2 \in S$. The former yields the relation $aS = (b * d_1)S \subset bS$ which implies the equality $aS \cap bS = aS$. Similarly, the latter yields the equality $aS \cap bS = bS$.

(2) Let $a = (p, i)$ and $b = (q, j)$ with $p \in \mathbb{Q}^+_{M_i}$, $q \in \mathbb{Q}^+_{M_j}$, $i, j \in \overline{n}$ and $i \neq j$. Obviously, one has the equality $aS \cap bS = \emptyset$. Thus, the semigroup Q is not left reversible. It is similarly shown that the semigroup Q is not right reversible.

(3) By (1) and (2), the cancellative semigroup Q satisfies the independence condition and is not left reversible. Combining Theorem 5.6.42 and Lemma 5.6.43 in [33], we conclude that the semigroup Q is not left amenable, as required.

The proof is complete. □

4. Reduced semigroup C^*-algebras for free products of semigroups

This section deals with the left reduced semigroup C^*-algebra $C^*_\lambda(Q)$ for the semigroup Q associated with a tuple of sequences of natural numbers (3.1). We show that this C^*-algebra is nuclear, or equivalently, amenable and that the left regular representation of the full semigroup C^*-algebra $C^*(Q)$ yields the isomorphism between the C^*-algebras $C^*(Q)$ and $C^*_\lambda(Q)$.

Various properties of the C^*-algebra $C^*_\lambda(Q)$ and its automorphisms are studied in [14, 22]. It is shown that $C^*_\lambda(Q)$ is the C^*-subalgebra of the C^*-algebra $\mathfrak{B}(l^2(Q))$ of all bounded operators on the Hilbert space $l^2(Q)$ generated by the countable family of isometries

$$\begin{aligned} &T_{1,1} := T_{(1,1)}, T_{1,2} := T_{(1,2)}, \dots, T_{1,n} := T_{(1,n)}, \\ &T_{k+1,1} := T_{\left(\frac{1}{m_{11}\cdots m_{k1}},1\right)}, T_{k+1,2} := T_{\left(\frac{1}{m_{12}\cdots m_{k2}},2\right)}, \dots, T_{k+1,n} := T_{\left(\frac{1}{m_{1n}\cdots m_{kn}},n\right)} \end{aligned} \tag{4.1}$$

whenever $k \in \mathbb{N}$ (see (2.1)).

Moreover, the pair $(C^*_\lambda(S), \{\varphi_{k,\infty} \mid k \in \mathbb{N}\})$ is the inductive limit of the inductive sequence

$$\mathcal{TO}_n \xrightarrow{\varphi_1} \mathcal{TO}_n \xrightarrow{\varphi_2} \mathcal{TO}_n \xrightarrow{\varphi_3} \dots \tag{4.2}$$

which is called the *inductive sequence of the Toeplitz — Cuntz algebras associated with tuple of sequences of natural numbers* (3.1). Here the connecting injective $*$-homomorphisms φ_k and the injective $*$-homomorphisms $\varphi_{k,\infty}$ are given by the formulas

$$\begin{aligned} \varphi_k &: \mathcal{TO}_n \to \mathcal{TO}_n : u_i \longmapsto u_i^{m_{ki}}, \\ \varphi_{k,\infty} &: \mathcal{TO}_n \to C^*_\lambda(Q) : u_i \longmapsto T_{k,i} \end{aligned}$$

whenever $k \in \mathbb{N}$ and $i \in \overline{n}$. Thus, we have the equality

$$C^*_\lambda(Q) = \overline{\bigcup_{k=1}^{+\infty} \varphi_{k,\infty}(\mathcal{TO}_n)},$$

where the closure is taken with respect to the norm topology. For details, the reader is referred to [14].

We recall that a C^*-algebra A is said to be *nuclear* if, for every C^*-algebra B, there is only one C^*-norm on the algebraic tensor product $A \otimes B$.

Theorem 4.1. *Let Q be the semigroup associated with a tuple of sequences of natural numbers. Then the left reduced semigroup C^*-algebra $C^*_\lambda(Q)$ is nuclear.*

Proof. As is well known, the Toeplitz — Cuntz algebra $\mathcal{TO}_n$ is nuclear. Indeed, $\mathcal{TO}_n$ is an extension of the Cuntz algebra $\mathcal{O}_n$ by a two-sided ideal $\mathcal{J}$ in $\mathcal{TO}_n$ which is isomorphic to the C^*-algebra $\mathcal{K}$ of compact operators on an infinite-dimensional separable Hilbert space. That is, one has the exact sequence of C^*-algebras and their $*$-homomorphisms

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{TO}_n \longrightarrow \mathcal{O}_n \longrightarrow 0.$$

Both C^*-algebras $\mathcal{O}_n$ and $\mathcal{K}$ are nuclear, hence, the C^*-algebra $\mathcal{TO}_n$ is nuclear as well (see [10, Sections 2, 3], [39, Example 6.3.2, Theorem 6.5.3]).

Since the left reduced semigroup C^*-algebra $C^*_\lambda(Q)$ is the inductive limit of the inductive sequence consisting of the copies of the Toeplitz — Cuntz algebra $\mathcal{TO}_n$ and injective $*$-homomorphisms (see (4.2)), we conclude that the C^*-algebra $C^*_\lambda(Q)$ is nuclear [39, Thm. 6.3.10]. □

A C^*-algebra is nuclear if and only if it is amenable [9], [30]. We recall that a Banach algebra A is said to be *amenable* if every bounded derivation from A to a dual Banach A-bimodule is inner [31, Thm. 7.3.37 (II)]. As a consequence of Theorem 4.1 and the equivalence of nuclearity and amenability for C^*-algebras, we obtain

Colorally 4.1. *Let Q be the semigroup associated with a tuple of sequences of natural numbers. Then the left reduced semigroup C^*-algebra $C^*_\lambda(Q)$ is amenable.*

In the following statement we establish that the C^*-algebras $C^*_\lambda(Q)$ and $C^*(Q)$ are isomorphic.

Theorem 4.2. *Let Q be the semigroup associated with a tuple of sequences of natural numbers. Then the left regular representation*

$$C^*(Q) \to C^*_\lambda(Q) : v_a \longmapsto T_a, \quad a \in Q,$$

is an isomorphism of C^*-algebras.

Proof. As was noted in Section 3, the semigroup Q is embedded into a group. Therefore, we may use Theorem 5.6.44 from [33]. Since the C^*-algebra $C^*_\lambda(Q)$ is nuclear (Theorem 4.1) and the semigroup Q satisfies the independence condition (Theorem 3.1, item (2)), it follows from [33, Thm. 5.6.44] that the left regular representation $C^*(Q) \to C^*_\lambda(Q)$ is an isomorphism. □

Remark 4.1. *We note that the faithfulness of the left regular representation of the C^*-algebra $C^*(Q)$ can be proved without using the nuclearity of the C^*-algebra $C^*_\lambda(Q)$. Indeed, one can see that Theorem 4.4 and Corollary 3.8 in* [32] *imply the desired faithfulness (for detail see* [40], [32]*).*

In the next section, we obtain a natural and simple description of the C^*-algebras $C^*(Q)$ and $C^*_\lambda(Q)$ in terms of generators and relations.

5. Semigroup algebras as universal C^*-algebras

In this section, we continue studying properties of the left reduced semigroup C^*-algebras $C^*_\lambda(Q)$ and show that they can be described as universal C^*-algebras generated by countable sets of generators subject to countable families of relations.

More precisely, we prove that the C^*-algebra $C^*_\lambda(Q)$ for the semigroup Q associated with a tuple of sequences of natural numbers (3.1) is isomorphic to the universal C^*-algebra $C^*(X,R)$, where

$$X = \{1, x_{ki} \mid k \in \mathbb{N}, i \in \overline{n}\} \tag{5.1}$$

is a set of indeterminants satisfying the following set of relations

$$R = \left\{x^*_{ki}x_{ki} = 1, x^*_{ki}x_{kj} = 0, x_{ki} = x^{m_{ki}}_{k+1,i} \mid k \in \mathbb{N}, i,j \in \overline{n}, i \neq j\right\}. \tag{5.2}$$

It is worth noting that, by [2, Def. 1.1], the pair (X,R) is admissible, and the universal C^*-algebra $C^*(X,R)$ generated by (5.1) and (5.2) exists (see [2, Def. 1.2]).

To construct the desired isomorphism of the C^*-algebras, we introduce two $*$-homomorphisms between the C^*-algebras $C^*_\lambda(Q)$ and $C^*(X,R)$ in the following lemmas. Both the universal property of the C^*-algebra $C^*(X,R)$ and the universal property of the C^*-algebra $C^*_\lambda(Q)$ considered as an inductive limit are involved in our construction. Then, it is shown in the theorem that these $*$-homomorphisms are mutually inverse isomorphisms of the C^*-algebras.

Lemma 5.1. *There exists a unique unital $*$-homomorphism of C^*-algebras*

$$\varphi : C^*(X,R) \to C^*_\lambda(Q)$$

such that the following condition is satisfied:

$$\varphi(\iota(x_{ki})) = T_{k,i} \tag{5.3}$$

whenever $k \in \mathbb{N}$ and $i \in \overline{n}$.

Proof. We consider the function $g : X \to C^*_\lambda(Q)$ given by the formula

$$g(x) = \begin{cases} 1 & \text{if} \quad x = 1, \\ T_{k,i} & \text{if} \quad x = x_{ki}, \quad k \in \mathbb{N}, \quad i \in \overline{n}. \end{cases}$$

We are going to verify the following relations for the elements $g(x_{ki})$ of the left reduced semigroup C^*-algebra $C^*_\lambda(Q)$:

$$1)\ g(x_{ki})^*g(x_{ki}) = 1; \qquad 2)\ g(x_{ki})^*g(x_{kj}) = 0; \qquad 3)\ g(x_{ki}) = g(x_{k+1,i})^{m_{ki}}$$

whenever $k \in \mathbb{N}$ and $i,j \in \overline{n}$, $i \neq j$. The relation 1) is valid because every operator $T_{k,i}$ is an isometry on the Hilbert space $l^2(Q)$ (see (2.1)) and (4.1)). To prove that the relation 2) holds, it suffices to show that

$$T^*_{k,i}T_{k,j}e_s = 0 \tag{5.4}$$

for each vector e_s from the orthonormal basis $\{e_s \mid s \in Q\}$ of the Hilbert space $l^2(Q)$. To this end, we fix an element $s \in Q$. For the inner product $\langle \cdot, \cdot \rangle$ on the space $l^2(Q)$ and an element $t \in Q$, we have the equalities

$$\langle e_t, T_{k,i}^* T_{k,j} e_s \rangle = \langle T_{k,i} e_t, T_{k,j} e_s \rangle = \langle e_{(p,i)*t}, e_{(q,j)*s} \rangle,$$

where $p = q = 1$ if $k = 1$ and $p = \frac{1}{m_{1i} \cdots m_{(k-1)i}}$, $q = \frac{1}{m_{1j} \cdots m_{(k-1)j}}$ if $k \geqslant 2$ (see (4.1)). Since $i \neq j$, by the definition of the multiplication in the free semigroup Q (see (2.2)), we get

$$(p, i) * t \neq (q, j) * s,$$

which implies the equality

$$\langle e_{(p,i)*t}, e_{(q,j)*s} \rangle = 0.$$

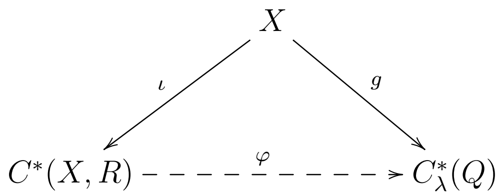
Therefore,

$$\langle e_t, T_{k,i}^* T_{k,j} e_s \rangle = 0$$

for every $t \in Q$. As a consequence, one has equality (5.4). Thus, the relation 2) holds. Finally, by (2.1), (2.2) and (4.1), we have the relation 3):

$$g(x_{k+1,i})^{m_{ki}} = T_{k+1,i}^{m_{ki}} = T_{\left(\frac{m_{ki}}{m_{1i} \cdots m_{ki}}, i\right)} = T_{k,i} = g(x_{k,i}).$$

It follows from the universal property of the universal C^*-algebra $C^*(X, R)$ that there is a unique unital $*$-homomorphism of C^*-algebras $\varphi : C^*(X, R) \to C_\lambda^*(Q)$ such that the diagram



commutes. This means that condition (5.3) is satisfied. The proof is complete. □

Further, we make use of the universal property of the C^*-algebra $C_\lambda^*(Q)$ which is treated as the inductive limit of inductive sequence of the Toeplitz — Cuntz algebras (4.2) associated with tuple of natural numbers (3.1).

Lemma 5.2. *There exists a unique unital $*$-homomorphism of C^*-algebras*

$$\psi : C_\lambda^*(Q) \to C^*(X, R)$$

such that the following condition is satisfied:

$$\psi(T_{k,i}) = \iota(x_{ki}) \tag{5.5}$$

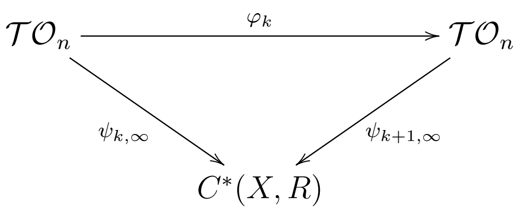
whenever $k \in \mathbb{N}$ and $i \in \overline{n}$. Moreover, the $*$-homomorphism ψ is injective.

Proof. Firstly, we claim that for every $k \in \mathbb{N}$ there exists a unique $*$-homomorphism

$$\psi_{k,\infty} : \mathcal{TO}_n \to C^*(X, R)$$

such that $\psi_{k,\infty}(u_i) = \iota(x_{ki})$ for each $i \in \overline{n}$. Indeed, the elements $\iota(x_{k1}), \ldots, \iota(x_{kn})$ of the C^*-algebra $C^*(X, R)$ satisfy the relations in the definition of the universal C^*-algebra $\mathcal{TO}_n$ (see (5.2) and (2.3)). Hence, the universal property of the C^*-algebra $\mathcal{TO}_n$ guarantees the existence of the desired $*$-homomorphism $\psi_{k,\infty}$.

Secondly, for every $k \in \mathbb{N}$, we consider the diagram

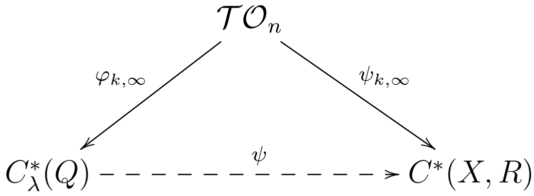


To show that this diagram is commutative, it is sufficient to check that the $*$-homomorphisms $\psi_{k+1,\infty} \circ \varphi_k$ and $\psi_{k,\infty}$ take the same values at the generators $u_1, \ldots, u_n$ of the Toeplitz — Cuntz algebra $\mathcal{TO}_n$. Checking this, for every $i \in \overline{n}$, we have the equalities

$$\psi_{k+1,\infty}(\varphi_k(u_i)) = \psi_{k+1,\infty}(u_i^{m_{ki}}) = \iota(x_{k+1,i})^{m_{ki}} = \iota(x_{ki}) = \psi_{k,\infty}(u_i).$$

Thus, we conclude that the above diagram commutes.

Thirdly, by the universal property of the inductive limit of inductive sequence of the Toeplitz — Cuntz algebras (4.2), there exists a unique unital $*$-homomorphism $\psi : C^*_\lambda(Q) \to C^*(X, R)$ making the diagram



commute for every $k \in \mathbb{N}$, that is, the equality

$$\psi \circ \varphi_{k,\infty} = \psi_{k,\infty} \tag{5.6}$$

holds.

We claim that the $*$-homomorphism ψ satisfies condition (5.5). Indeed, using relation (5.6), we get condition (5.5)

$$\psi(T_{k,i}) = \psi(\varphi_{k,\infty}(u_i)) = \psi_{k,\infty}(u_i) = \iota(x_{ki})$$

whenever $k \in \mathbb{N}$ and $i \in \overline{n}$.

Finally, since the C^*-algebra $C^*_\lambda(Q)$ is simple [14, Thm. 3], we conclude that the $*$-homomorphism ψ is injective. The proof is complete. □

Next, we obtain the description of the left reduced semigroup C^*-algebra $C^*_\lambda(Q)$ as the universal C^*-algebra generated by set of generators (5.1) subject to relations (5.2).

Theorem 5.1. *Let $n \geqslant 2$ be an integer and $(M_1, \ldots, M_n)$ be an n-tuple of infinite sequences $M_i = (m_{1i}, m_{2i}, \ldots)$ of natural numbers m_{ki}, $i \in \overline{n}$, $k \in \mathbb{N}$. Let*

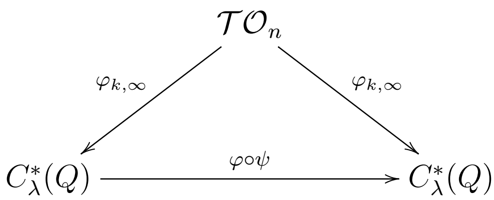
$$X = \{1, x_{ki} \mid k \in \mathbb{N}, i \in \overline{n}\}$$

be a set of noncommuting indeterminants satisfying the relations in the set

$$R = \left\{x^*_{ki}x_{ki} = 1, x^*_{ki}x_{kj} = 0, x_{ki} = x^{m_{ki}}_{k+1,i} \mid k \in \mathbb{N}, i, j \in \overline{n}, i \neq j\right\}.$$

Then the universal C^*-algebra $C^*(X, R)$ generated by the set of generators X subject to the relations in R is isomorphic to the left reduced semigroup C^*-algebra $C^*_\lambda(Q)$ for the semigroup Q associated with the tuple $(M_1, \ldots, M_n)$.

Proof. Let φ and ψ be the $*$-homomorphisms constructed in Lemmas 5.1 and 5.2. We are going to show that φ and ψ are mutually inverse isomorphisms between the C^*-algebras $C^*(X, R)$ and $C^*_\lambda(Q)$, that is, the compositions $\varphi \circ \psi$ and $\psi \circ \varphi$ are the identity mappings denoted by $Id_{C^*_\lambda(Q)}$ and $Id_{C^*(X,R)}$ respectively. To this end, for every $k \in \mathbb{N}$, we consider the diagram



(5.7)

We claim that it is commutative. Indeed, to prove equality

$$\varphi \circ \psi \circ \varphi_{k,\infty} = \varphi_{k,\infty}, \tag{5.8}$$

it suffices to show that the $*$-homomorphisms $\varphi \circ \psi \circ \varphi_{k,\infty}$ and $\varphi_{k,\infty}$ take the same values at the generators $u_1, \dots, u_n$ of the Toeplitz — Cuntz algebra $\mathcal{TO}_n$. By (5.3) and (5.5), for every $i \in \overline{n}$, we get the equalities

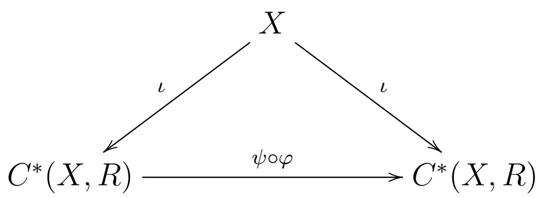
$$\varphi \circ \psi(\varphi_{k,\infty}(u_i)) = \varphi \circ \psi(T_{k,i}) = \varphi(\iota(x_{ki})) = T_{k,i} = \varphi_{k,\infty}(u_i).$$

Hence, equality (5.8) holds, as claimed.

By the universal property of the C^*-algebra $C^*_\lambda(Q)$ considered as the inductive limit for inductive sequence of Toeplitz — Cuntz algebras (4.2), there exists a unique $*$-homomorphism making diagram (5.7) commute for every $k \in \mathbb{N}$. Thus, we obtain the required equality

$$\varphi \circ \psi = Id_{C^*_\lambda(Q)}.$$

Further, let us take the diagram



(5.9)

By (5.3) and (5.5), we have the equalities

$$\psi \circ \varphi(\iota(x_{ki})) = \psi(T_{k,i}) = \iota(x_{ki})$$

whenever $k \in \mathbb{N}$ and $i \in \overline{n}$. Moreover, by Lemmas 5.1 and 5.2, we also have the equality $\psi \circ \varphi \circ \iota(1) = \iota(1)$. Thus, diagram (5.9) is commutative.

Now, the universal property of the universal C^*-algebra $C^*(X, R)$ yields the equality

$$\psi \circ \varphi = Id_{C^*(X,R)}.$$

The proof is complete. □

Remark 5.1. *There is another way of proving Theorem 5.1. Namely, one can use the well–known construction of the universal C^*-algebra $C^*(X, R)$* [2, Def. 1.2] *and the injective $*$-homomorphism ψ from Lemma 5.2. To show that ψ is an isomorphism of C^*-algebras, it suffices to prove that its image is dense in $C^*(X, R)$* [31, Corollary 4.7.84]. *Instead of this way of proving, we have preferred to make use of the universal properties of the objects.*

Combining Theorem 5.1 and Theorem 4.2, we obtain the following statement.

Colorally 5.1. *Under the hypotheses of Theorem 5.1, the full semigroup C^*-algebra $C^*(Q)$ is up to an isomorphism the universal C^*-algebra $C^*(X, R)$.*

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