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ON DEGENERATE ELLIPTIC OPERATORS OF NON–DIVERGENT FORM IN BOUNDED DOMAIN

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Abstract. In this work we prove several inequalities, which provide a lower bound for the norm of an elliptic operator in non–divergent form in a bounded domain with power degeneration along the entire boundary. Earlier similar operators were studied in the case when they were initially defined in divergent form or reduced to such a form. In contrast, the coefficients of the operators we study are generally non–differentiable and cannot be reduced to divergent form. Only in the final section of the paper, in order to study the solvability of the corresponding differential equations, the differentiability is assumed for the coefficients of operator, and the corresponding adjoint operator is studied.

We first study degenerate elliptic operators of general form and prove an inequality for them in which the sum of norm of the action of operator and the norm of the function itself with some power weight in the space L_2 is bounded from below by the norm of function itself in a weighted Sobolev type space. We then consider the case, in which the elliptic operators are weakly positive. For such operators, we prove an inequality in which the real part of the scalar product of the action of operator and the function itself is bounded from below. In the final section we assume that weakly positive elliptic operators have strong degeneration along the entire boundary of the domain. For such operators involving a parameter λ, we first prove an inequality in which the norm of the action of operator is bounded from below by the norm of function itself in the underlying functional space. This inequality is then proved for the adjoint operator, and as a consequence, a result on the unique solvability of the corresponding differential equation is established.

The technique developed in this paper is based on the extension of some known results for elliptic operators with constant coefficients to the case of operators with degeneracy using auxiliary integral inequalities.

Keywords: elliptic operator, power degeneration, non–divergent form, bounded domain.

Mathematics Subject Classification: 35J70, 46E35, 47F10

1. Introduction

In this paper we study elliptic non–divergence operators with power degeneracy along the entire boundary of a bounded domain. We first prove several inequalities, which provide a lower bound for the norm of studied operator. While deriving these inequalities, we make no assumptions about the differentiability of the coefficients of operator. Only in the final section of the paper, in the case of strongly degenerate weakly positive operators, the differentiability of the coefficients of operator is assumed, and the corresponding adjoint operator is studied, in order to study the solvability of the corresponding differential equations.

It should be noted that at present, a well–developed approach for studying such operators is available in the case when the operator has a divergent form, or is reduced to such a form.

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In this case, the theory of sesquilinear forms in a Hilbert space and various generalizations of the Lax — Milgram theorem (see, for example, Theorem of [11, Thm. 2.0.1]) work well. The results of theory of weighted spaces of differentiable functions of several real variables are applied (embedding theorems, direct and inverse trace theorems, etc.). These studies were initiated by Kudryavtsev [8] and then they were continued by many Soviet and foreign mathematicians, see [2], [5], [6], [10], [11] and the references therein. Note that all these works are related to the case of elliptic operators of divergence type.

Our goal in this paper is to study degenerate elliptic operators of non–divergent form, the coefficients of which are non–differentiable and therefore cannot be reduced to the divergent form. We develop a special technique, which allows us to generalize some known results for regular (i.e., non–degenerate) elliptic operators of non–divergent form to the case of degenerate elliptic operators of this form.

We note that regular elliptic differential operators of non–divergence type were well studied in [3], [9, Ch. 2], [14, Ch. 5], [16].

2. Formulation of results

Let Ω be a bounded domain in n–dimensional Euclidean space $\mathbb{R}^n$ with a closed $(n-1)$–dimensional boundary $\partial\Omega$. We denote by $\rho(x)$ the regularized distance from a point $x\in\Omega$ to $\partial\Omega$, that is, a function of class $C^\infty(\Omega)$ with the following properties

$$\frac{1}{\varkappa}\operatorname{dist}\{x,\partial\Omega\}\leqslant\rho(x)\leqslant\varkappa\operatorname{dist}\{x,\partial\Omega\},\qquad x\in\Omega,\tag{2.1}$$

$$\left|\rho^{(k)}(x)\right|\leqslant\varkappa\rho^{1-|k|}(x),\qquad x\in\Omega,\qquad |k|\leqslant 2r.\tag{2.2}$$

Let r be a natural number and α, β be real numbers. We introduce the following functional spaces $L_{2;\beta}(\Omega)$, $W^{2r}_{2;\alpha}(\Omega)$ respectively with the following finite norms

$$\|u;L_{2;\beta}(\Omega)\|=\left\{\int\limits_\Omega\rho^{2\beta}(x)|u(x)|^2dx\right\}^{\frac12},$$

$$\left\|u;W^{2r}_{2;\alpha}(\Omega)\right\|=\left\{\left\|u;L^{2r}_{2;\alpha}(\Omega)\right\|^2+\int\limits_\Omega|u(x)|^2dx\right\}^{\frac12},\tag{2.3}$$

where

$$\left\|u;L^{2r}_{2;\alpha}(\Omega)\right\|=\left\{\sum_{|k|=2r}\int\limits_\Omega\rho^{2\alpha}(x)|u^{(k)}(x)|^2dx\right\}^{\frac12}.$$

We note that $L_{2;0}(\Omega)=L_2(\Omega)$. The closure of class $C_0^\infty(\Omega)$ by the norm (2.3) is denoted by $\mathring{W}^{2r}_{2;\alpha}(\Omega)$.

We formulate the main results.

Theorem 2.1. *Let the coefficients $b_k(x)$ of operator*

$$L[u](x)=\sum_{|k|\leqslant 2r}\rho^{\alpha|k|}(x)b_k(x)u^{(k)}(x),\quad u\in C_0^\infty(\Omega),\tag{2.4}$$

be bounded and satisfy the following conditions:

I) *ellipticity condition:*

$$M_0^{-1}|\xi|^{2r}\leqslant\left|\sum_{|k|=2r}b_k(x)\xi^k\right|\quad(x\in\Omega,\xi\in\mathbb{R}^n)\,;\tag{2.5}$$

II) *for each sufficiently small positive number* ν *there exists a natural number* $m(\nu) > 0$ *such that*

$$|b_k(y) - b_k(z)| < \nu, \quad |k| = 2r, \tag{2.6}$$

for each $y \in \Omega$ *and each*

$$z \in J_{1,m}(y) = \left\{ z \in \mathbb{R}^n : |z - y| < \frac{1}{m^2 \varkappa} \rho(y) \right\}, \qquad m \in \mathbb{N}, \qquad m \geqslant m(\nu), \tag{2.7}$$

where $\varkappa$ *is a constant from condition* (2.1);

III) *the numbers* $\alpha = \alpha_{2r}$, α_j, $0 \leqslant \alpha_j \leqslant 2r - 1$, *are such that*

$$\alpha_j \geqslant \alpha - 2r + j \quad \text{for all} \quad 0 \leqslant j \leqslant 2r - 1. \tag{2.8}$$

Then there exist positive numbers c, K *such that*

$$c \left\| v; L^{2r}_{2,\alpha}(\Omega) \right\|^2 \leqslant \int_\Omega |L[v(z)]|^2 \, dz + K \left\| v; L_{2,\alpha-2r}(\Omega) \right\|^2 \tag{2.9}$$

for all $v \in C_0^\infty(\Omega)$.

Degenerate differential operator (2.4) is called *weakly positive* if the condition holds

$$\operatorname{Re} \sum_{|k|=2r} b_k(x) \xi^k \geqslant 0, \quad \xi \in \mathbb{R}^n, \quad x \in \Omega. \tag{2.10}$$

Theorem 2.2. *Let the operator*

$$L_0[u(x)] = \sum_{|k|=2r} \rho^\alpha(x) b_k(x) u^{(k)}(x), \quad u \in C_0^\infty(\Omega), \tag{2.11}$$

be weakly positive and its coefficients $b_k(x)$ *satisfy the assumptions of Theorem 2.1. Then for all* $v \in C_0^\infty(\Omega)$ *the inequality holds*

$$\begin{aligned} \operatorname{Re} \int_\Omega L_0[v(z)] \cdot \overline{v(z)} dz \geqslant &- \varepsilon \left\| v; L^{2r}_{2,\alpha}(\Omega) \right\| \left\| v; L_2(\Omega) \right\| \\ &- \delta \left\| v; L_{2,\alpha-2r}(\Omega) \right\| \left\| v; L_2(\Omega) \right\| - K_1(\varepsilon, \delta) \left\| v; L_{2,\alpha-2r}(\Omega) \right\|^2, \end{aligned} \tag{2.12}$$

where ε, δ *are sufficiently small positive numbers and* $K_1(\varepsilon, \delta)$ *is some positive number depending only on* ε, δ.

Theorem 2.3. *Let* $\alpha - 2r \geqslant 0$ *(the case of strong degeneration) and all coefficients* $b_k(x)$ *of operator* (2.4) *satisfy the assumptions of Theorems 2.1 and 2.2. Then there exist numbers* $\varkappa_0 > 0$, $\lambda_0 \geqslant 0$ *such that for* $\lambda \geqslant \lambda_0$ *the inequality holds*

$$\left\| L[v] + \lambda v; L_2(\Omega) \right\| \geqslant \varkappa_0 \left\| v; W^{2r}_{2,\alpha}(\Omega) \right\| \tag{2.13}$$

for all $v \in C_0^\infty(\Omega)$.

For operator (2.4) we introduce the formally adjoint operator $L'[v]$ by the identity

$$L'[v](x) = \sum_{|k|\leqslant 2r} \left(\rho^{\alpha_{|k|}}(x) \overline{b_k(x)} v(x) \right)^{(k)}, \qquad v \in C_0^\infty(\Omega).$$

Theorem 2.4. *Let all coefficients* $b_k(x)$ *of operator* (2.4) *satisfy the assumptions of Theorem 2.3 and let*

IV) *the coefficients* $b_k(x)$, $|k| \leqslant 2r$, *have all derivatives up to* $|k|$*th order, which satisfy the condition*

$$\left| b_k^{(l)}(x) \right| \leqslant C_l \rho^{-|l|}(x), \quad x \in \Omega, \quad |l| \leqslant |k|, \tag{2.14}$$

where C_l *is some positive number.*

Then there exist numbers $\varkappa_0 > 0$, $\lambda_0 \geqslant 0$ such that for $\lambda \geqslant \lambda_0$ the inequality

$$\|L'[v] + \lambda v; L_2(\Omega)\| \geqslant \varkappa_0 \left\|v; W^{2r}_{2,\alpha}(\Omega)\right\| \tag{2.15}$$

holds for all $v \in C_0^\infty(\Omega)$.

Under the assumptions of Theorem 2.4 the following inequalities hold as well

$$\|L[u]; L_2(\Omega)\| \leqslant M_9 \left\|u; W^{2r}_{2;\alpha}(\Omega)\right\|, \qquad \|L^*[u]; L_2(\Omega)\| \leqslant M_{10} \left\|u; W^{2r}_{2;\alpha}(\Omega)\right\|$$

for all $v \in C_0^\infty(\Omega)$. This is why, by continuity, we can define the operators $\mathbb{L}$, $\mathbb{L}^*$ on the domains $D(\mathbb{L}) = \mathring{W}^{2r}_{2;\alpha}(\Omega)$, $D(\mathbb{L}^*) = \mathring{W}^{2r}_{2;\alpha}(\Omega)$, respectively, such that

$$\mathbb{L}[u] = L[u], \qquad \mathbb{L}^*[u] = L'[u] \qquad \text{for all} \quad u \in C_0^\infty(\Omega).$$

Corollary 2.1. *Let the assumptions of Theorem 2.4 be satisfied. Then for each $f \in L_2(\Omega)$ the equation*

$$\mathbb{L}[u] + \lambda u = f$$

has a unique solution in the space $\mathring{W}^{2r}_{2;\alpha}(\Omega)$, and the inequality

$$\left\|u; W^{2r}_{2,\alpha}(\Omega)\right\| \leqslant \frac{1}{\varkappa_0} \|f; L_2(\Omega)\|$$

holds, where $\varkappa_0$ is the same number as in Theorem 2.4.

3. Auxiliary lemmas

In this section we prove some auxiliary lemmas, which will be used in the proofs of main theorems in other sections.

We denote by $B_m(0)$ the open ball in the space $\mathbb{R}^n$ of radius $\frac{1}{m}$, $m = 1, 2, \dots$, centered at the origin.

Lemma 3.1. *For each natural number m there exists a function φ_m with the properties*

1. *$\varphi_m(x) \in C_0^\infty(\mathbb{R}^n)$;*
2. *$\varphi_m(x) = 1$ if $x \in B_{m+1}(0)$;*
3. *$\varphi_m(x) = 0$ if $|x| \geqslant \frac{1}{m}$;*
4. *$0 \leqslant \varphi_m(x) \leqslant 1$ for all $x \in \mathbb{R}^n$;*
5. *$\left|\varphi_m^{(k)}(x)\right| \leqslant C_3[2m(m+1)]^{|k|}$, where C_3 is some positive number independent of k.*

Proof. To construct a function with required properties, we employ the averaging technique, see, for instance, [12]. Let $\omega(x)$ be a some averaging kernel, that is, some non–negative function $\omega(x) \in C_0^\infty(\mathbb{R}^n)$ such that

$$\operatorname{supp} \omega(x) \subset \overline{B_1} = \{x \in \mathbb{R}^n : |x| \leqslant 1\}, \qquad \int\limits_{\mathbb{R}^n} \omega(x)dx = 1.$$

Let $u \in L_p(\Omega)$, $1 \leqslant p < \infty$. We continue it by zero outsied Ω. The mean function for $u(x)$ is the function $u_h(x)$ defined by the identity

$$u_h(x) = \frac{1}{h^n} \int\limits_{\mathbb{R}^n} \omega\left(\frac{x-y}{h}\right) u(y)dy = \frac{1}{h^n} \int\limits_{I_h(x)} \omega\left(\frac{x-y}{h}\right) u(y)dy,$$

where $I_h(x)$ is an open ball of radius h in $\mathbb{R}^n$ centered at the point x.

Let γ be some positive number. We denote by $I_\gamma(0)$ an open ball of radius γ in $\mathbb{R}^n$ centered at the origin, and by $\theta_\gamma(x)$ we denote the characteristic function of ball $I_\gamma(0)$. Let h be a sufficiently small positive number. We construct the averaging for the function $\theta_\gamma(x)$ with the averaging radius h

$$v_{h,\gamma}(x) = \frac{1}{h^n} \int\limits_{I_h(x)} \omega\left(\frac{x-y}{h}\right)\theta_\gamma(y)dy.$$

We define the function $\varphi_m(x)$ by the identity $\varphi_m(x) = v_{h,\gamma}(x)$, where

$$\gamma = \frac{2m+1}{2m(m+1)}, \quad h = \frac{1}{2m(m+1)},$$

that is,

$$\varphi_m(x) = 2^n m^n (m+1)^n \int\limits_{2m(m+1)|x-y|<1} \omega\left(2m(m+1)(x-y)\right)\chi_m(y)dy,$$

where $\chi_m(y)$ is the characteristic function of the ball of radius $\dfrac{2m+1}{2m(m+1)}$ centered at the origin. It is easy to make sure that this function possesses all properties given in Lemma 3.1. The proof is complete. □

Lemma 3.2. *Let m be a natural number and $\chi_{2,m}(x,y)$ be a characteristic function of ball*

$$J_{2,m}(y) = \left\{x \in \mathbb{R}^n : |x-y| < \frac{1}{m(m+1)\varkappa}\rho(y)\right\}. \tag{3.1}$$

Then

$$\frac{1}{(m^2+1)^n} \leqslant \rho^{-n}(x)\varkappa^n\omega_n^{-1}\left(1+\frac{1}{m}\right)^n \int\limits_\Omega \chi_{2,m}(x,y)dy \leqslant \frac{1}{(m^2-1)^n}, \tag{3.2}$$

where ω_n is the area of unit sphere in $\mathbb{R}^n$.

Proof. Let $x, y \in \Omega$. Using the identity

$$\rho(x) - \rho(y) = \int\limits_0^1 \frac{d}{dt}\rho(y+t(x-y))dt, \tag{3.3}$$

by the inequality (see (2.2)) $|\nabla\rho(x)| \leqslant \varkappa$ we have

$$|\rho(x)-\rho(y)| \leqslant \varkappa|x-y|.$$

If (see (2.7)) $x \in J_{1,m}(y)$, then

$$|\rho(x)-\rho(y)| \leqslant \frac{1}{m^2}\rho(y).$$

This is why

$$\left(1-\frac{1}{m^2}\right)\rho(y) < \rho(x) < \left(1+\frac{1}{m^2}\right)\rho(y) \quad \text{for} \quad y \in \Omega, x \in J_{1,m}(y). \tag{3.4}$$

Let

$$G_{2,m}(x) = \left\{y \in \mathbb{R}^n : |x-y| < \frac{\rho(y)}{m(m+1)\varkappa}\right\}.$$

Then

$$\int\limits_\Omega \chi_{2,m}(x,y)dy = \int\limits_{G_{2,m}(x)} 1 \cdot dy = |G_{2,m}(x)|. \tag{3.5}$$

Since $J_{2,m}(y) \subset J_{1,m}(y)$, by using (3.4) for all $y \in G_{2,m}(x)$, we find

$$|x-y| < \frac{\rho(y)}{m(m+1)\varkappa} < \frac{m\rho(x)}{\varkappa(m+1)(m^2-1)}.$$

Hence, the set $G_{2,m}(x)$ is contained in the ball of radius $R = \dfrac{m\rho(x)}{\varkappa(m+1)(m^2-1)}$. This is why

$$|G_{2,m}(x)| \leqslant \omega_n \left(\frac{m\rho(x)}{\varkappa(m+1)(m^2-1)}\right)^n,$$

that is,

$$\rho^{-n}(x)\varkappa^n\omega_n^{-1}\left(1+\frac{1}{m}\right)^n |G_{2,m}(x)| \leqslant \left(\frac{1}{m^2-1}\right)^n.$$

By (3.5) this implies the right inequality in (3.2).

We consider the set

$$Q_{2,m}(x) = \left\{y \in \mathbb{R}^n : |x-y| < \frac{\varepsilon_m\rho(x)}{\varkappa}\right\},$$

where $\varepsilon_m = \dfrac{m}{(m+1)(m^2+1)}$. By (3.3) in view of the inequality $|\nabla\rho(x)| \leqslant \varkappa$ (see (2.2)) for all $y \in Q_{2,m}(x)$ we have

$$|\rho(x) - \rho(y)| \leqslant \varkappa|x-y| < \varepsilon_m\rho(x).$$

Therefore,

$$(1-\varepsilon_m)\rho(x) \leqslant \rho(y) \leqslant (1+\varepsilon_m)\rho(x), \qquad y \in Q_{2,m}(x).$$

Using this inequality for $y \in Q_{2,m}(x)$, we have

$$|x-y| < \varepsilon_m\rho(x)\varkappa^{-1} \leqslant \varepsilon_m\varkappa^{-1}(1-\varepsilon_m)^{-1}\rho(y).$$

Since $\varepsilon_m(1-\varepsilon_m)^{-1} < [m(m+1)]^{-1}$, this implies that

$$|x-y| \leqslant \frac{\rho(y)}{m(m+1)\varkappa},$$

that is, $y \in G_{2,m}(x)$. This is why $|Q_{2,m}(x)| \leqslant |G_{2,m}(x)|$. Then by the identity

$$|Q_{2,m}(x)| = \omega_n\left(\frac{\varepsilon_m\rho(x)}{\varkappa}\right)^n = \omega_n\left(\frac{m}{\varkappa(m+1)(m^2+1)}\right)^n \rho^n(x)$$

we have

$$\omega_n\left(\frac{m}{\varkappa(m+1)(m^2+1)}\right)^n \rho^n(x) \leqslant |G_{2,m}(x)|,$$

that is,

$$\left(\frac{1}{m^2+1}\right)^n \leqslant \omega_n^{-n}\varkappa^n\left(1+\frac{1}{m}\right)^n \rho^{-n}(x)\,|G_{2,m}(x)|.$$

Hence, taking into consideration identity (3.5), we obtain the left inequality in (3.2). The proof is complete. □

Similarly to Lemma 3.2, we prove the next statement.

Lemma 3.3. *Let m be a natural number and $\chi_{1,m}(x,y)$ be a characteristic function of ball $J_{1,m}(y)$. Then*

$$\frac{1}{(m^2+1)^n} \leqslant \rho^{-n}(x)\varkappa^n\omega_n^{-1}\int\limits_\Omega \chi_{1,m}(x,y)dy \leqslant \frac{1}{(m^2-1)^n}.$$

Lemma 3.4. *Let m be a natural number and $\chi^{(m)}(z,y)$ be a characteristic function of set*

$$J^{(m)}(y) = \left\{z \in \mathbb{R}^n : \frac{1}{m(m+1)\varkappa}\rho(y) \leqslant |z-y| < \frac{1}{m^2\varkappa}\rho(y)\right\}. \tag{3.6}$$

Then

$$\int_\Omega \chi^{(m)}(z,y)dy \leqslant \frac{\sigma_1(m)}{(m^2-1)^n}\rho^n(z), \tag{3.7}$$

where

$$\sigma_1(m) = \omega_n \varkappa^{-n}\left[1-\left(\frac{m^2-m}{m^2+1}\right)^n\right]. \tag{3.8}$$

Proof. We first mention that $\sigma_1(m) > 0$ for $m \geqslant 2$ and

$$\lim_{m\to\infty} \sigma_1(m) = 0. \tag{3.9}$$

For the convenience of reading we introduce the notation

$$F_m(z) = \int_\Omega \chi^{(m)}(z,y)\,dy. \tag{3.10}$$

For all $y \in \Omega$ and all $z \in J_{1,m}(y)$, by using inequality (3.4), we have

$$\rho(y) \leqslant \left(1-\frac{1}{m^2}\right)^{-1}\rho(z), \qquad \left(1+\frac{1}{m^2}\right)^{-1}\rho(z) \leqslant \rho(y) \qquad (z \in J_{1,m}(y), y \in \Omega)\,. \tag{3.11}$$

Let $z \in J^{(m)}(y)$. Then

$$\frac{1}{m(m+1)\varkappa}\rho(y) \leqslant |z-y| < \frac{1}{m^2\varkappa}\rho(y).$$

Now, using inequalities (3.11), we replace $\rho(y)$ by $\rho(z)$. As the result we get

$$\frac{m}{(m+1)(m^2+1)\varkappa}\rho(z) \leqslant |z-y| < \frac{1}{\varkappa(m^2-1)}\rho(z).$$

By these inequalities it follows from (3.10) that

$$F_m(z) \leqslant \int\limits_{\frac{m}{(m+1)(m^2+1)\varkappa}\rho(z)\leqslant|z-y|<\frac{1}{\varkappa(m^2-1)}\rho(z)} dy \eqqcolon F_m^0(z). \tag{3.12}$$

The right hand side of this inequality is equal to the volume of spherical layer with radii

$$r_1 = \frac{m}{(m+1)(m^2+1)\varkappa}\rho(z), \qquad r_2 = \frac{1}{\varkappa(m^2-1)}\rho(z).$$

This is why it is equal to

$$\begin{aligned} F_m^0(z) &= \omega_n\left(r_2^n - r_1^n\right) = \omega_n\left(\frac{1}{\varkappa(m^2-1)}\rho(z)\right)^n - \omega_n\left(\frac{m}{(m+1)(m^2+1)\varkappa}\rho(z)\right)^n \\ &= \omega_n\frac{1}{(m^2-1)^n\varkappa^n}\rho^n(z)\left[1-\left(\frac{m^2-m}{m^2+1}\right)^n\right]. \end{aligned} \tag{3.13}$$

By (3.12), (3.13), identity (3.10) implies estimate (3.7). The proof is complete. □

Lemma 3.5. *For each real number θ and all $y \in \Omega$, $z \in J_{1,m}(y)$ the inequality holds*

$$\left(\frac{m^2}{m^2+\operatorname{sgn}\theta}\right)^\theta \rho^\theta(z) \leqslant \rho^\theta(y) \leqslant \left(\frac{m^2}{m^2-\operatorname{sgn}\theta}\right)^\theta \rho^\theta(z). \tag{3.14}$$

Proof. Using inequality (3.4), we find

$$\frac{m^2}{m^2+1}\rho(z) \leqslant \rho(y) \leqslant \frac{m^2}{m^2-1}\rho(z) \tag{3.15}$$

for all $y \in \Omega$, $z \in J_{1,m}(y)$. This implies that

$$\left(\frac{m^2-1}{m^2}\right)\rho^{-1}(z) \leqslant \rho^{-1}(y) \leqslant \left(\frac{m^2+1}{m^2}\right)\rho^{-1}(z). \tag{3.16}$$

Now, taking the powers $\theta > 0$ of inequalities (3.15), (3.16), we obtain

$$\left(\frac{m^2}{m^2+1}\right)^{\theta}\rho^{\theta}(z) \leqslant \rho^{\theta}(y) \leqslant \left(\frac{m^2}{m^2-1}\right)^{\theta}\rho^{\theta}(z),$$
$$\left(\frac{m^2}{m^2-1}\right)^{-\theta}\rho^{-\theta}(z) \leqslant \rho^{-\theta}(y) \leqslant \left(\frac{m^2}{m^2+1}\right)^{-\theta}\rho^{-\theta}(z).$$

These two inequalities imply inequality (3.14). The proof is complete. □

Lemma 3.6. *Let m be a natural number and $y \in \Omega$. Then for each real number θ and each $z \in J_{2,m}(y)$ the inequality holds*

$$\left|\rho^{\theta}(y) - \rho^{\theta}(z)\right| \leqslant \frac{1}{m^2}N_1(m,\theta)\rho^{\theta}(z), \tag{3.17}$$

where

$$N_1(m,\theta) = n|\theta|\left(1+\frac{1}{m^2}\right)^{2|\theta|+1}.$$

Proof. Since (see (2.2)) $|\nabla\rho(x)| \leqslant \varkappa$, $x \in \Omega$, by using the identity

$$\rho^{\theta}(y) - \rho^{\theta}(z) = \int\limits_0^1 \frac{d}{dt}\left\{\rho^{\theta}(y+t(z-y))\right\}dt,$$

we obtain

$$\left|\rho^{\theta}(y) - \rho^{\theta}(z)\right| \leqslant n\varkappa|\theta|\int\limits_0^1 \rho^{\theta-1}(y+t(z-y))dt \leqslant n\varkappa|\theta||z-y|\max_{0\leqslant t\leqslant 1}\rho^{\theta-1}(y+t(z-y)). \tag{3.18}$$

We then note that $z \in J_{2,m}(y)$ and this is why for $\xi = y + t(z-y)$, $0 < t < 1$, we have

$$|\xi - y| = t|z-y| \leqslant |z-y| < \frac{1}{m(m+1)\varkappa}\rho(y) < \frac{1}{m^2\varkappa}\rho(y).$$

By these inequality it follows from (3.15) that

$$\left(1+\frac{1}{m^2}\right)^{-1}\rho(\xi) \leqslant \rho(y) \leqslant \left(1-\frac{1}{m^2}\right)^{-1}\rho(\xi).$$

Then by this inequality we find

$$\left(1-\frac{1}{m^2}\operatorname{sgn}(\theta-1)\right)^{\theta-1}\rho^{\theta-1}(y) \leqslant \rho^{\theta-1}(\xi) \leqslant \left(1+\frac{1}{m^2}\operatorname{sgn}(\theta-1)\right)^{\theta-1}\rho^{\theta-1}(y).$$

By (3.18) this implies

$$\left|\rho^{\theta}(y) - \rho^{\theta}(z)\right| \leqslant n|\theta|\frac{1}{m^2}\left(1+\frac{1}{m^2}\operatorname{sgn}(\theta-1)\right)^{\theta-1}\rho^{\theta}(y).$$

Then by applying inequality (3.14), we arrive at the inequality

$$\left|\rho^\theta(y)-\rho^\theta(z)\right| \leqslant n|\theta|\frac{1}{m^2}\left(1+\frac{1}{m^2}\operatorname{sgn}(\theta-1)\right)^{\theta-1}\left(1-\frac{1}{m^2}\operatorname{sgn}\theta\right)^{-\theta}\rho^\theta(z).$$

After simple transformation this implies inequality (3.17). The proof is complete. □

Lemma 3.7. *For each real β and each $m \geqslant 2$ the inequality holds*

$$\begin{aligned}\frac{1}{m^{2n}}M_{11}(\beta)\,\|f;L_{2,\beta}(\Omega)\|^2 &\leqslant \int\limits_\Omega \rho^{2\beta-n}(y)\Bigg(\int\limits_{J_{2,m}(y)}|f(z)|^2\,dz\Bigg)dy \\ &\leqslant \frac{1}{m^{2n}}M_{12}(\beta)\,\|f;L_{2,\beta}(\Omega)\|^2,\end{aligned}\tag{3.19}$$

where

$$M_{11}(\beta)=\omega_n\varkappa^{-n}2^{-|\beta|}3^{-n}, \qquad M_{12}(\beta)=\omega_n\varkappa^{-n}2^{|\beta|}2^n.\tag{3.20}$$

Proof. Using inequality (3.14) and Lemma 3.2, we obtain

$$\begin{aligned}\int\limits_\Omega \rho^{2\beta-n}(y)\Bigg(\int\limits_{J_{2,m}(y)}|f(z)|^2\,dz\Bigg)dy &= \int\limits_\Omega \rho^{2\beta-n}(y)\Bigg(\int\limits_\Omega \chi_{2,m}(z,y)\,|f(z)|^2\,dz\Bigg)dy \\ &\geqslant \left(\frac{m^2}{m^2-1}\right)^{-n}\left(\frac{m^2}{m^2+\operatorname{sgn}\beta}\right)^{2\beta}\int\limits_\Omega \rho^{2\beta-n}(z)\Bigg(\int\limits_\Omega \chi_{2,m}(z,y)dy\Bigg)|f(z)|^2\,dz \\ &\geqslant \omega_n\left(\frac{m^2}{m^2+\operatorname{sgn}\beta}\right)^{2\beta}\left(\frac{m-1}{\varkappa m(m^2+1)}\right)^n\int\limits_\Omega \rho^{2\beta}(z)\,|f(z)|^2\,dz.\end{aligned}\tag{3.21}$$

We then note that

$$\omega_n\left(\frac{m^2}{m^2+\operatorname{sgn}\beta}\right)^{2\beta}\left(\frac{m-1}{\varkappa m(m^2+1)}\right)^n \geqslant \frac{1}{m^{2n}}M_{11}(\beta)$$

for $m \geqslant 2$ and each real β. This is why relations (3.21) imply the left inequality in (3.19).

We proceed to proving the right inequality in (3.19). As above, by using inequality (3.14) and Lemma 3.2, we obtain

$$\int\limits_\Omega \rho^{2\beta-n}(y)\Bigg(\int\limits_{J_{2,m}(y)}|f(z)|^2\,dz\Bigg)dy \leqslant \frac{1}{m^{2n}}M_{12}(\beta,m)\,\|v;L_{2,\beta}(\Omega)\|^2,\tag{3.22}$$

where

$$M_{12}(\beta,m)=\frac{\omega_n}{\varkappa^n}\left(\frac{m^2}{m^2-\operatorname{sgn}\beta}\right)^{2\beta}\left(\frac{m^2+1}{m^2-1}\right)^n.$$

By direct calculation we can make sure that $M_{12}(\beta,m) \leqslant M_{12}(\beta)$ for $m \geqslant 2$ and each real β. This is why inequality (3.22) implies the right inequality in (3.19). The proof is complete. □

Lemma 3.8. *The inequality*

$$\int\limits_\Omega \rho^{2\beta-n}(y)\Bigg(\int\limits_{J^{(m)}(y)}|f(z)|^2\,dz\Bigg)dy \leqslant \frac{1}{m^{2n}}M_2(m,\beta)\,\|f;L_{2,\beta}(\Omega)\|^2\tag{3.23}$$

holds, where

$$M_2(m,\beta)=2^n4^{|\beta|}\sigma_1(m),\tag{3.24}$$

where the quantity $\sigma_1(m)$ is determined by identity (3.8).

Proof. Using inequality (3.14) and Lemma 3.4, we obtain

$$\int_{\Omega} \rho^{2\beta-n}(y)\left(\int_{J^{(m)}(y)} |f(z)|^2\,dz\right)dy = \int_{\Omega} \rho^{2\beta-n}(y)\left(\int_{\Omega} \chi^{(m)}(z,y)\,|f(z)|^2\,dz\right)dy$$
$$\leqslant \frac{1}{m^{2n}}\left(\frac{m^2+1}{m^2-1}\right)^n\left(\frac{m^2}{m^2-\operatorname{sgn}\beta}\right)^{2\beta}\sigma_1(m)\int_{\Omega}\rho^{2\beta}(z)\,|f(z)|^2\,dz.$$

Hence, for $m \geqslant 2$ we get

$$\int_{\Omega} \rho^{2\beta-n}(y)\left(\int_{J^{(m)}(y)} |f(z)|^2\,dz\right)dy \leqslant \frac{1}{m^{2n}}2^n 4^{|\beta|}\sigma_1(m)\int_{\Omega}\rho^{2\beta}(z)\,|f(z)|^2\,dz.$$

Introducing notation (3.24), we obtain inequality (3.23). The proof is complete. □

We note that $M_2(m,\beta) > 0$ for $m \geqslant 2$ and

$$\lim_{m\to\infty} M_2(m,\beta) = 0 \tag{3.25}$$

by identity (3.9).

Lemma 3.9. *The inequality*

$$\int_{\Omega} \rho^{2\beta-n}(y)\left(\int_{J_{1,m}(y)} |f(z)|^2\,dz\right)dy \leqslant \frac{1}{m^{2n}}M_3(\beta)\,\|f; L_{2,\beta}(\Omega)\|^2 \tag{3.26}$$

holds, where

$$M_3(\beta) = \omega_n \varkappa^{-n} 4^{|\beta|} 2^n. \tag{3.27}$$

Proof. Using inequality (3.14) and Lemma 3.3, we obtain

$$\int_{\Omega} \rho^{2\beta-n}(y)\left(\int_{J_{1,m}(y)} |f(z)|^2\,dz\right)dy$$
$$\leqslant \left(\frac{m^2}{m^2+1}\right)^{-n}\left(\frac{m^2}{m^2-\operatorname{sgn}\beta}\right)^{2\beta}\int_{\Omega}\rho^{2\beta-n}(z)\left(\int_{\Omega}\chi_{1,m}(z,y)dy\right)|f(z)|^2\,dz$$
$$\leqslant \omega_n\varkappa^{-n}\frac{1}{m^{2m}}\left(\frac{m^2+1}{m^2-1}\right)^n\left(\frac{m^2}{m^2-\operatorname{sgn}\beta}\right)^{2\beta}\int_{\Omega}\rho^{2\beta}(z)\,|f(z)|^2\,dz.$$

This implies inequality (3.26) once we observe that

$$\left(\frac{m^2+1}{m^2-1}\right)^n \leqslant 2^n \quad\text{and}\quad \left(\frac{m^2}{m^2-\operatorname{sgn}\beta}\right)^{2\beta} \leqslant 4^{|\beta|}.$$

The proof is complete. □

Lemma 3.10. *Let a real–valued function $\Phi(z)$ belongs to the space $L_1(\Omega)$. Then for natural $m \geqslant 3$ the inequality*

$$\int_{\Omega}\left(\int_{J_{2,m}(y)} \rho^{-n}(z)\Phi(z)dz\right)dy \leqslant \varkappa^{-n}\omega_n\left(\frac{m}{m+1}\right)^n\frac{1}{(m^2-1)^n}\int_{\Omega}\Phi(z)dz$$
$$+\varkappa^{-n}\omega_n\left[\frac{1}{(m^2-1)^n}-\frac{1}{(m^2+1)^n}\right]\int_{\Omega}\Phi^-(z)dz \tag{3.28}$$

holds, where $\Phi^-(z) = \dfrac{|\Phi(z)| - \Phi(z)}{2}$.

Proof. We introduce the auxiliary function

$$\Phi^+(z) = \frac{|\Phi(z)| + \Phi(z)}{2}.$$

We note that the functions $\Phi^+(z)$, $\Phi^-(z)$ are non–negative and $\Phi(z) = \Phi^+(z) - \Phi^-(z)$. Taking this into consideration and applying Lemma 3.2, we obtain

$$\begin{aligned}
\int\limits_\Omega \Bigg(\int\limits_{J_{2,m}(y)} \rho^{-n}(z)\Phi^+(z)dz \Bigg) dy &= \int\limits_\Omega \Bigg(\int\limits_\Omega \chi_{2,m}(z,y)\rho^{-n}(z)\Phi^+(z)dz \Bigg) dy \\
&= \int\limits_\Omega \Bigg(\int\limits_\Omega \chi_{2,m}(z,y)dy \Bigg) \rho^{-n}(z)\Phi^+(z)\,dz \\
&\leqslant \varkappa^{-n}\omega_n \frac{1}{(m^2-1)^n} \left(\frac{m}{m+1} \right)^n \int\limits_\Omega \Phi^+(z)\,dz,
\end{aligned}$$

$$\int\limits_\Omega \Bigg(\int\limits_{J_{2,m}(y)} \rho^{-n}(z)\Phi^-(z)dz \Bigg) dy = \int\limits_\Omega \Bigg(\int\limits_\Omega \chi_{2,m}(z,y)\rho^{-n}(z)\Phi^-(z)dz \Bigg) dy,$$

$$\int\limits_\Omega \Bigg(\int\limits_\Omega \chi_{2,m}(z,y)dy \Bigg) \rho^{-n}(z)\Phi^-(z)\,dz \geqslant \varkappa^{-n}\omega_n \frac{1}{(m^2+1)^n} \left(\frac{m}{m+1} \right)^n \int\limits_\Omega \Phi^-(z)\,dz.$$

By the identity $\Phi(z) = \Phi^+(z) - \Phi^-(z)$ these inequalities imply

$$\begin{aligned}
\int\limits_\Omega \Bigg(\int\limits_{J_{2,m}(y)} \rho^{-n}(z)\Phi(z)dz \Bigg) dy \leqslant & \varkappa^{-n}\omega_n \left(\frac{m}{m+1} \right)^n \frac{1}{(m^2-1)^n} \int\limits_\Omega \Phi^+(z)dz \\
& - \varkappa^{-n}\omega_n \left(\frac{m}{m+1} \right)^n \frac{1}{(m^2+1)^n} \int\limits_\Omega \Phi^-(z)dz.
\end{aligned}$$

Substituting here $\Phi^+(z) = \Phi(z) + \Phi^-(z)$, we obtain inequality ((3.28)). The proof is complete. □

Lemma 3.11 ([15, Sect. 4.4, Thm. 7])**.** *Let G be a bounded domain in $\mathbb{R}^n$ obeying the cone condition, an integer j such that $0 < j < 2r$ and let $\mu_0 > 0$. Then for each $f \in W_2^{2r}(G)$ and each $\mu \in (0, \mu_0]$ the inequality holds*

$$\left\| f; L_2^j(G) \right\|^2 \leqslant \mu \left\| f; L_2^{2r}(G) \right\|^2 + K_1 \mu^{-\frac{j}{2r-j}} \left\| f; L_2(G) \right\|^2, \tag{3.29}$$

where the number $K_1 > 0$ is independent of f and μ.

Lemma 3.12 ([6, Lm. 2.2])**.** *Let $\mu_0 > 0$ and an integer j such that $0 < j < 2r$. Then for each $\mu \in (0, \mu_0]$ and all $v \in C_0^\infty(\Omega)$ the inequality*

$$\left\| v; L^j_{2;\alpha-2r+j}(\Omega) \right\| \leqslant \mu \left\| v; L^{2r}_{2;\alpha}(\Omega) \right\| + K_2 \mu^{-\frac{j}{2r-j}} \left\| v; L_{2;\alpha-2r}(\Omega) \right\| \tag{3.30}$$

holds, where the number $K_2 > 0$ is independent of v and μ.

4. Proof of Theorem 1

We first prove Theorem 2.1 in the case, when operator (2.4) has zero lower order coefficients, that is, we consider operator (2.11).

Let y be an arbitrary fixed point in the domain Ω. Freezing the coefficients of operator (2.11) at the point y, we consider the operator

$$L_{0,y}[u(x)] = \sum_{|k|=2r} b_k(y)u^{(k)}(x), \qquad u \in C_0^\infty(\mathbb{R}^n). \tag{4.1}$$

Since this operator an elliptic one with constant coefficients (see (2.5)), there exists a number $c_1 > 0$ (see, for instance, [4]) such that

$$c_1 \left\| u; W_2^{2r}(g) \right\| \leqslant \| L_{0,y} u; L_2(g) \| , \qquad u \in C_0^\infty(g), \tag{4.2}$$

if $\operatorname{diam} g$ is small enough. We then suppose that the number m_0 is large enough so that for $g = B_{m_0}(0)$ inequality (4.2) holds. Since $B_m(0) \subset B_{m_0}(0)$ for $m \geqslant m_0$, inequality (4.2) holds with $g = B_m(0)$ for all $m \geqslant m_0$.

We consider an arbitrary function $u(x) \in C^\infty(B_m(0))$. Since $\varphi_m \in C_0^\infty(B_m(0))$, we obtain $v_m(x) = \varphi_m(x)u(x) \in C_0^\infty(B_m(0))$. Using now inequality (4.2) for the function $v_m(x)$, by the identity $v_m(x) = u(x)$, $x \in B_{m+1}(0)$, we have

$$\sum_{|k|=2r} \int\limits_{B_{m+1}(0)} \left| u^{(k)}(x) \right|^2 dx + \int\limits_{B_{m+1}(0)} |u(x)|^2 dx \leqslant C_5 \left\{ \int\limits_{B_m(0)} |L_{0,y} v_m(x)|^2 dx \right\} \tag{4.3}$$

Using the Leibnitz rule of differentiating a product of functions, we represent the expression $L_{0,y} v_m(x)$ as

$$L_{0,y}[v_m](x) = L_y^{(1)}[v_m](x) + L_y^{(2)}[v_m](x), \tag{4.4}$$

where

$$L_y^{(1)}[v_m](x) = \sum_{|k|=2r} b_k(y)\varphi_m(x)u^{(k)}(x), \tag{4.5}$$

$$L_y^{(2)}[v_m](x) = \sum_{|k|=2r} \sum_{0 \neq \nu \leqslant k} C_{k,\nu} b_k(y) u^{(k-\nu)}(x) \varphi_m^{(\nu)}(x), \tag{4.6}$$

$C_{k,\nu}$ are some constants numbers. We note that in identity (4.6) the multi–indices k, ν satisfies the condition $0 \leqslant |k - \nu| \leqslant 2r - 1$.

Since $\varphi_m(x) = 1$ for all $x \in B_{m+1}(0)$, we obtain

$$\int\limits_{B_m(0)} \left| L_y^{(1)}[v_m](x) \right|^2 dx = \int\limits_{B_{m+1}(0)} |L_{0,y}[u](x)|^2 dx + \int\limits_{B^{(m)}(0)} |\varphi_m(x) L_{0,y}[u](x)|^2 dx, \tag{4.7}$$

where

$$B^{(m)}(0) = B_m(0) \setminus B_{m+1}(0) = \left\{ x \in \mathbb{R}^n : \frac{1}{m+1} \leqslant |x| < \frac{1}{m} \right\}. \tag{4.8}$$

It follows from the boundedness of coefficients $b_k(x)$ and (4.1) that

$$\int\limits_{B^{(m)}(0)} |\varphi_m(x) L_{0,y}[u](x)|^2 dx \leqslant C_6 \sum_{|k|=2r} \int\limits_{B^{(m)}(0)} \left| u^{(k)}(x) \right|^2 dx. \tag{4.9}$$

Applying inequalities (4.7), (4.9), we find

$$\int\limits_{B_m(0)} \left| L_y^{(1)}[v_m](x) \right|^2 dx \leqslant \int\limits_{B_{m+1}(0)} |L_{0,y}[u](x)|^2 dx + C_6 \sum_{|k|=2r} \int\limits_{B^{(m)}(0)} \left| u^{(k)}(x) \right|^2 dx. \tag{4.10}$$

Using identity (4.6), by the boundedness of coefficients $b_k(x)$ and inequality 5 in Lemma 3.1, we obtain

$$\begin{aligned}\int\limits_{B_m(0)} \left|L_y^{(2)}[v_m](x)\right|^2 dx &\leqslant C_7 \sum_{|k|=2r} \sum_{0\neq\nu\leqslant k} \int\limits_{B_m(0)} \left|u^{(k-\nu)}(x)\right|^2 \left|\varphi^{(\nu)}(x)\right|^2 dx \\ &\leqslant C_8 \sum_{|k|=2r} \sum_{0\neq\nu\leqslant k} [2m(m+1)]^{2|\nu|} \int\limits_{B_m(0)} \left|u^{(k-\nu)}(x)\right|^2 dx.\end{aligned} \tag{4.11}$$

Now, to estimate the integrals in the right hand side, we apply Lemma 3.11. Using inequality (3.29) in this lemma for $G = B_m(0)$, $0 \neq |\nu| \leqslant 2r$, $j = |k-\nu|$, $|k| = 2r$, we obtain

$$\int\limits_{B_m(0)} \left|u^{(k-\nu)}(x)\right|^2 dx \leqslant \mu \sum_{|l|=2r} \int\limits_{B_m(0)} \left|u^{(l)}(x)\right|^2 dx + c_2 \mu^{-\frac{|k-\nu|}{2r-|k-\nu|}} \int\limits_{B_m(0)} |u(x)|^2 dx. \tag{4.12}$$

By this inequality, it follows from (4.11) that

$$\begin{aligned}\int\limits_{B_m(0)} \left|L_y^{(2)}[v_m](x)\right|^2 dx \leqslant & C_8 \sum_{|k|=2r} \sum_{1\leqslant|k-\nu|\leqslant 2r-1} [2m(m+1)]^{2|\nu|} \int\limits_{B_m(0)} \left|u^{(k-\nu)}(x)\right|^2 dx \\ & + C_8[2m(m+1)]^{4r} \int\limits_{B_m(0)} |u(x)|^2 dx \\ \leqslant & \mu\left[C_9 \sum_{|k|=2r} \sum_{1\leqslant|k-\nu|\leqslant 2r-1} [2m(m+1)]^{2|\nu|}\right] \left\|u; L_2^{2r}(B_m(0))\right\|^2 \\ & + \left[C_{10} \sum_{|k|=2r} \sum_{1\leqslant|k-\nu|\leqslant 2r-1} [2m(m+1)]^{2|\nu|} \mu^{-\frac{|k-\nu|}{2r-|k-\nu|}} \right. \\ & \left. + C_8[2m(m+1)]^{4r}\right] \|u; L_2(B_m(0))\|^2 .\end{aligned}$$

Since $m \geqslant m_0$ and μ is a sufficiently small positive number, supposing that $m \geqslant 3$ and $0 < \mu < \frac{1}{2}$, we hence get

$$\int\limits_{B_m(0)} \left|L_y^{(2)}[v_m](x)\right|^2 dx \leqslant \mu\delta_1(m) \left\|u; L_2^{2r}(B_m(0))\right\|^2 + K_1(m,\mu) \|u; L_2(B_m(0))\|^2 , \tag{4.13}$$

where $\delta_1(m) = C_{13} m^{8r}$, $K_1(m,\mu) = C_{14} m^{8r} \mu^{-(2r-1)}$.

Then we choose a number $\mu \in (0, \frac{1}{2})$ so that, starting from some number m^*, the inequality $\mu\delta_1(m) \leqslant \frac{1}{m}$ holds for all $m \geqslant m^*$. Then it follows from (4.13) that

$$\int\limits_{B_m(0)} \left|L_y^{(2)}[v_m](x)\right|^2 dx \leqslant \frac{1}{m} \left\|u; L_2^{2r}(B_m(0))\right\|^2 + K_2(m) \|u; L_2(B_m(0))\|^2 , \tag{4.14}$$

where

$$K_2(m) = C_{15} m^{16r^2+2r-1}. \tag{4.15}$$

Now, applying obtained inequalities (4.10), (4.14), by (4.3), (4.4) we find

$$\sum_{|k|=2r}\int\limits_{B_{m+1}(0)}\left|u^{(k)}(x)\right|^2dx+\int\limits_{B_{m+1}(0)}|u(x)|^2dx\leqslant C_5\int\limits_{B_{m+1}(0)}|L_{0,y}[u](x)|^2dx$$
$$+C_6\sum_{|k|=2r}\int\limits_{B^{(m)}(0)}\left|u^{(k)}(x)\right|^2dx+\frac{1}{m}\sum_{|l|=2r}\int\limits_{B_m(0)}\left|u^{(l)}(x)\right|^2dx \tag{4.16}$$
$$+K_2(m)\int\limits_{B_m(0)}|u(x)|^2dx.$$

We consider the mapping $z\to x$ defined by means of the identity

$$x=(z-y)m\varkappa\rho(y), \tag{4.17}$$

where $\varkappa$ is the constant in condition (2.1), and y is an arbitrary fixed point in the domain Ω. It maps the sets (see (2.7), (3.1), (3.6)) $J_{1,m}(y)$, $J_{2,m}(y)$, $J^{(m)}(y)$ into the sets $B_m(0)$, $B_{m+1}(0)$, $B^{(m)}(0)$, respectively.

We choose an arbitrary function v in the class $C_0^\infty(\Omega)$. It is easy to verify that $J_{1,m}(y)\subset\Omega$ for all $y\in\Omega$, $m\geqslant 2$. This is why the function $\widehat{v}_y(x)=v\left(\dfrac{x\rho(y)}{m\varkappa}+y\right)$ is defined for all $x\in B_m(0)$ and belongs to the class $C^\infty(B_m(0))$.

We note that if $u(x)=\widehat{v}_y(x)$, then

$$u^{(k)}(x)=(m\varkappa)^{-|k|}\rho^{|k|}(y)\widehat{v}_y^{(k)}(x). \tag{4.18}$$

Using this identity, by (4.1) we have

$$L_{0,y}[u(x)]=(m\varkappa)^{-2r}\rho^{2r}(y)\sum_{|k|=2r}b_k(y)\widehat{v}_y^{(k)}(x)=(m\varkappa)^{-2r}\rho^{2r}(y)L_{0,y}\left[\widehat{v}_y(x)\right], \tag{4.19}$$

Taking into consideration identities (4.18), (4.19) and applying inequality (4.16) for the function $u(x)=\widehat{v}_y(x)$, we obtain an inequality, which after the multiplication by $\rho^{2\alpha}(y)$ becomes

$$(m\varkappa)^{-4r}\rho^{4r+2\alpha}(y)\sum_{|k|=2r}\int\limits_{B_{m+1}(0)}\left|\widehat{v}_y^{(k)}(x)\right|^2dx+\rho^{2\alpha}(y)\int\limits_{B_{m+1}(0)}|\widehat{v}_y(x)|^2dx$$
$$\leqslant C_5(m\varkappa)^{-4r}\rho^{4r+2\alpha}(y)\int\limits_{B_{m+1}(0)}\left|L_{0,y}\left[\widehat{v}_y(x)\right]\right|^2dx$$
$$+C_{10}(m\varkappa)^{-4r}\rho^{4r+2\alpha}(y)\sum_{|k|=2r}\int\limits_{B^{(m)}(0)}\left|\widehat{v}^{(k)}(x)\right|^2dx \tag{4.20}$$
$$+\frac{1}{m}(m\varkappa)^{-4r}\rho^{4r+2\alpha}(y)\sum_{|k|=2r}\int\limits_{B_m(0)}\left|\widehat{v}_y^{(k)}(x)\right|^2dx$$
$$+K_2(m)\rho^{2\alpha}(y)\int\limits_{B_m(0)}|\widehat{v}_y(x)|^2dx.$$

We make the change of variable in the integrals of this inequality $z=y+\frac{x\rho(y)}{m\varkappa}$. At the same time we employ the identities

$$x=\frac{(z-y)m\varkappa}{\rho(y)},\qquad dx=\left(\frac{m\varkappa}{\rho(y)}\right)^n dz,$$

$$\begin{aligned}
\widehat{v}_y(x) &= v\left(\frac{x\rho(y)}{m\varkappa}+y\right) = v(z),\\
\widehat{v}_y^{(k)}(x) &= v^{(k)}\left(\frac{x\rho(y)}{m\varkappa}+y\right) = v^{(k)}(z),\\
L_{0,y}\left[\widehat{v}_y(x)\right] &= \sum_{|k|=2r} b_k(y)\widehat{v}_y^{(k)}(x) = \sum_{|k|=2r} b_k(y)v^{(k)}\left(\frac{x\rho(y)}{m\varkappa}+y\right)\\
&= \sum_{|k|=2r} b_k(y)v^{(k)}(z) = L_{0,y}\left[v(z)\right].
\end{aligned}$$

The integrals become

$$\int\limits_{B_m(0)} \left|\widehat{v}_y(x)\right|^2 dx = \left(\frac{m\varkappa}{\rho(y)}\right)^n \int\limits_{J_{1,m}(y)} |v(z)|^2 dz, \tag{4.21}$$

$$\int\limits_{B_{m+1}(0)} \left|\widehat{v}_y^{(k)}(x)\right|^2 dx = \left(\frac{m\varkappa}{\rho(y)}\right)^n \int\limits_{J_{2,m}(y)} \left|v^{(k)}(z)\right|^2 dz, \tag{4.22}$$

$$\int\limits_{B_m(0)} \left|\widehat{v}_y^{(k)}(x)\right|^2 dx = \left(\frac{m\varkappa}{\rho(y)}\right)^n \int\limits_{J_{1,m}(y)} \left|v^{(k)}(z)\right|^2 dz, \tag{4.23}$$

$$\int\limits_{B_{m+1}(0)} \left|L_{0,y}\left[\widehat{v}_y(x)\right]\right|^2 dx = \left(\frac{m\varkappa}{\rho(y)}\right)^n \int\limits_{J_{2,m}(y)} \left|L_{0,y}\left[v(z)\right]\right|^2 dz, \tag{4.24}$$

$$\int\limits_{B^{(m)}(0)} \left|\widehat{v}^{(k)}(x)\right|^2 dx = \left(\frac{m\varkappa}{\rho(y)}\right)^n \int\limits_{J^{(m)}(y)} \left|v^{(k)}(z)\right|^2 dz. \tag{4.25}$$

We replace the integrals in (4.20) by (4.21)–(4.25):

$$\begin{aligned}
(m\varkappa)^{-4r}\rho^{4r+2\alpha}(y)\left(\frac{m\varkappa}{\rho(y)}\right)^n \sum_{|k|=2r}\int\limits_{J_{2,m}(y)} \left|v^{(k)}(z)\right|^2 dz + \rho^{2\alpha}(y)\left(\frac{m\varkappa}{\rho(y)}\right)^n \int\limits_{J_{2,m}(y)} |v(z)|^2 dz&\\
\leqslant C_5\,(m\varkappa)^{-4r}\rho^{4r+2\alpha}(y)\left(\frac{m\varkappa}{\rho(y)}\right)^n \int\limits_{J_{2,m}(y)} \left|L_{0,y}\left[v(z)\right]\right|^2 dz&\\
+ C_{10}\,(m\varkappa)^{-4r}\rho^{4r+2\alpha}(y)\left(\frac{m\varkappa}{\rho(y)}\right)^n \sum_{|k|=2r}\int\limits_{J^{(m)}(y)} \left|v^{(k)}(z)\right|^2 dz&\\
+ \frac{1}{m}(m\varkappa)^{-4r}\rho^{4r+2\alpha}(y)\left(\frac{m\varkappa}{\rho(y)}\right)^n \sum_{|k|=2r}\int\limits_{J_{1,m}(y)} \left|v^{(k)}(z)\right|^2 dz&\\
+ K_2(m)\rho^{2\alpha}(y)\left(\frac{m\varkappa}{\rho(y)}\right)^n \int\limits_{J_{1,m}(y)} |v(z)|^2 dz.&
\end{aligned}$$

Therefore,

$$(m\varkappa)^{-4r+n}\rho^{4r+2\alpha-n}(y)\sum_{|k|=2r}\int\limits_{J_{2,m}(y)} \left|v^{(k)}(z)\right|^2 dz + (m\varkappa)^n \rho^{2\alpha-n}(y)\int\limits_{J_{2,m}(y)} |v(z)|^2 dz$$

$$\leqslant C_5 (m\varkappa)^{-4r+n} \rho^{4r+2\alpha-n}(y) \int\limits_{J_{2,m}(y)} |L_{0,y}[v(z)]|^2 dz$$
$$+ C_{10} (m\varkappa)^{-4r+n} \rho^{4r+2\alpha-n}(y) \sum_{|k|=2r} \int\limits_{J^{(m)}(y)} \left|v^{(k)}(z)\right|^2 dz$$
$$+ \frac{1}{m} (m\varkappa)^{-4r+n} \rho^{4r+2\alpha-n}(y) \sum_{|k|=2r} \int\limits_{J_{1,m}(y)} \left|v^{(k)}(z)\right|^2 dz$$
$$+ K_1(m)\rho^{2\alpha-n}(y) (m\varkappa)^n \int\limits_{J_{1,m}(y)} |v(z)|^2 dz.$$

We multiply this inequality by $(m\varkappa)^{4r-n} \rho^{-4r}(y)$ and integrate the result in $y \in \Omega$

$$\begin{aligned}
&\sum_{|k|=2r} \int\limits_{\Omega} \rho^{2\alpha-n}(y) \left(\int\limits_{J_{2,m}(y)} \left|v^{(k)}(z)\right|^2 dz \right) dy \\
&\quad + (m\varkappa)^{4r} \int\limits_{\Omega} \rho^{2\alpha-4r-n}(y) \left(\int\limits_{J_{2,m}(y)} |v(z)|^2 dz \right) dy \\
&\leqslant C_5 \int\limits_{\Omega} \rho^{2\alpha-n}(y) \left(\int\limits_{J_{2,m}(y)} |L_{0,y}[v(z)]|^2 dz \right) dy \\
&\quad + C_{10} \sum_{|k|=2r} \int\limits_{\Omega} \rho^{2\alpha-n}(y) \left(\int\limits_{J^{(m)}(y)} \left|v^{(k)}(z)\right|^2 dz \right) dy \\
&\quad + \frac{1}{m} \sum_{|k|=2r} \int\limits_{\Omega} \rho^{2\alpha-n}(y) \left(\int\limits_{J_{1,m}(y)} \left|v^{(k)}(z)\right|^2 dz \right) dy \\
&\quad + K_2(m) (m\varkappa)^{4r} \int\limits_{\Omega} \rho^{2\alpha-4r-n}(y) \left(\int\limits_{J_{1,m}(y)} |v(z)|^2 dz \right) dy.
\end{aligned} \tag{4.26}$$

We proceed to estimating the integrals in inequality (4.26). First, by using Lemma 3.7, we estimate from below the integrals in the left hand side of inequality (4.26)

$$\sum_{|k|=2r} \int\limits_{\Omega} \rho^{2\alpha-n}(y) \left(\int\limits_{J_{2,m}(y)} \left|v^{(k)}(z)\right|^2 dz \right) dy \geqslant \frac{1}{m^{2n}} M_{11}(\alpha) \left\|v; L^{2r}_{2,\alpha}(\Omega)\right\|^2, \tag{4.27}$$

$$\int\limits_{\Omega} \rho^{2\alpha-4r-n}(y) \left(\int\limits_{J_{2,m}(y)} |v(z)|^2 dz \right) dy \geqslant \frac{1}{m^{2n}} M_{11}(\alpha-2r) \left\|v; L_{2,\alpha-2r}(\Omega)\right\|^2, \tag{4.28}$$

where

$$M_{11}(\alpha) = \omega_n \varkappa^{-n} 2^{-|\alpha|} 3^{-n}, \qquad M_{11}(\alpha-2r) = \omega_n \varkappa^{-n} 2^{-|\alpha-2r|} 3^{-n}.$$

Then we apply Lemma 3.8 to estimate from above the second integral in the right hand side of inequality (4.26), and as the result we get

$$\sum_{|k|=2r} \int\limits_{\Omega} \rho^{2\alpha-n}(y) \Bigg(\int\limits_{J^{(m)}(y)} \left|v^{(k)}(z)\right|^2 dz \Bigg) dy \leqslant \frac{1}{m^{2n}} M_2(m,\alpha) \left\|v; L^{2r}_{2,\alpha}(\Omega)\right\|^2, \tag{4.29}$$

where $M_2(m,\alpha)$ is determined by identity (3.24) and satisfies (3.25).

Now, using Lemma 3.9, we estimate from above the last integrals in the right hand side of inequality (4.26). As the result we obtain the inequalities

$$\sum_{|k|=2r} \int\limits_{\Omega} \rho^{2\alpha-n}(y) \Bigg(\int\limits_{J_{1,m}(y)} \left|v^{(k)}(z)\right|^2 dz \Bigg) dy \leqslant \frac{1}{m^{2n}} M_3(\alpha) \left\|v; L^{2r}_{2,\alpha}(\Omega)\right\|^2, \tag{4.30}$$

$$\int\limits_{\Omega} \rho^{2\alpha-4r-n}(y) \Bigg(\int\limits_{J_{1,m}(y)} |v(z)|^p dz \Bigg) dy \leqslant \frac{1}{m^{2n}} M_3(\alpha-2r) \left\|v; L_{2,\alpha-2r}(\Omega)\right\|^2, \tag{4.31}$$

where

$$M_3(\alpha) = \omega_n \varkappa^{-n} 4^{|\alpha|} 2^n, \qquad M_3(\alpha-2r) = \omega_n \varkappa^{-n} 4^{|\alpha-2r|} 2^n. \tag{4.32}$$

By the above inequalities (4.27)–(4.31) and (4.26) we find

$$\begin{aligned} \frac{1}{m^{2n}} M_{11}(\alpha) \left\|v; L^{2r}_{2,\alpha}(\Omega)\right\|^2 &+ (m\varkappa)^{4r} \frac{1}{m^{2n}} M_{11}(\alpha-2r) \left\|v; L_{2,\alpha-2r}(\Omega)\right\|^2 \\ &\leqslant C_5 \mathcal{L}^{(1)}_{\varepsilon,m}(v) + C_{10} \frac{1}{m^{2n}} M_2(m,\alpha) \left\|v; L^{2r}_{2,\alpha}(\Omega)\right\|^2 \\ &\quad + \frac{1}{m} \frac{1}{m^{2n}} M_3(\alpha) \left\|v; L^{2r}_{2,\alpha}(\Omega)\right\|^2 \\ &\quad + K_2(m)(m\varkappa)^{4r} \frac{1}{m^{2n}} M_3(\alpha-2r) \left\|v; L_{2,\alpha-2r}(\Omega)\right\|^2, \end{aligned}$$

where

$$\mathcal{L}_{1,m}(v) = \int\limits_{\Omega} \rho^{2\alpha-n}(y) \Bigg(\int\limits_{J_{2,m}(y)} \left|L_{0,y}\left[v(z)\right]\right|^2 dz \Bigg) dy. \tag{4.33}$$

Multiplying both sides of this inequality by m^{2n} and collecting the like terms, we obtain

$$M_5(m,\alpha) \left\|v; L^{2r}_{2,\alpha}(\Omega)\right\|^2 - M_6(m,\alpha) \left\|v; L_{2,\alpha-2r}(\Omega)\right\|^2 \leqslant C_5 m^{2n} \mathcal{L}_{1,m}(v), \tag{4.34}$$

where

$$M_5(m,\alpha) = M_{11}(\alpha) - C_{10} M_2(m,\alpha) - \frac{1}{m} M_3(\alpha), \tag{4.35}$$

$$M_6(m,\alpha) = K_2(m)(m\varkappa)^{4r} M_3(\alpha-2r) - (m\varkappa)^{4r} M_{11}(\alpha-2r). \tag{4.36}$$

We are going to estimate from above the functional $\mathcal{L}_{1,m}(v)$. According to Lemma 3.6, the inequality holds

$$\left|\rho^{2\alpha-n}(y) - \rho^{2\alpha-n}(z)\right| \leqslant \frac{1}{m^2} N_2(\alpha) \rho^{2\alpha-n}(z), \quad z \in J_{2,m}(y), \tag{4.37}$$

where

$$N_2(\alpha) = n(2|\alpha|+n) 2^{4|\alpha|+2n+1}. \tag{4.38}$$

We introduce the notation

$$\mathcal{L}_{2,m}(v) = \int\limits_{\Omega} \Bigg(\int\limits_{\Omega} \chi_{2,m}(z,y) \rho^{2\alpha-n}(z) \Bigg| \sum_{|k|=2r} b_k(y) v^{(k)}(z) \Bigg|^2 dz \Bigg) dy.$$

We note that (see (4.33))

$$\mathcal{L}_{1,m}(v) = \int\limits_{\Omega} \rho^{2\alpha-n}(y)\Bigg(\int\limits_{\Omega} \chi_{2,m}(z,y)\Bigg|\sum_{|k|=2r} b_k(y)v^{(k)}(z)\Bigg|^2 dz\Bigg)dy.$$

In view of this identity, we apply inequality (4.37) and we get

$$\begin{aligned}|\mathcal{L}_{1,m}(v) - \mathcal{L}_{2,m}(v)| &\leqslant \int\limits_{\Omega}\Bigg(\int\limits_{\Omega} \chi_{2,m}(z,y)\left|\rho^{2\alpha-n}(y) - \rho^{2\alpha-n}(z)\right|\Bigg|\sum_{|k|=2r} b_k(y)v^{(k)}(z)\Bigg|^2 dz\Bigg)dy \\ &\leqslant \frac{1}{m^2}N_2(\alpha)\int\limits_{\Omega}\Bigg(\int\limits_{\Omega} \chi_{2,m}(z,y)\rho^{2\alpha-n}(z)\Bigg|\sum_{|k|=2r} b_k(y)v^{(k)}(z)\Bigg|^2 dz\Bigg)dy.\end{aligned}$$

Since the coefficients $b_k(x)$ are bounded, this implies

$$|\mathcal{L}_{1,m}(v) - \mathcal{L}_{2,m}(v)| \leqslant C_3\frac{1}{m^2}N_2(\alpha)\sum_{|k|=2r}\int\limits_{\Omega}\Bigg(\int\limits_{\Omega} \chi_{2,m}(z,y)\rho^{2\alpha-n}(z)\left|v^{(k)}(z)\right|^2 dz\Bigg)dy, \tag{4.39}$$

where C_3 is some constant. Now, using Lemma 3.2, we estimate the integral in the right hand side of inequality (4.39)

$$\begin{aligned}&\int\limits_{\Omega}\Bigg(\int\limits_{\Omega} \chi_{2,m}(z,y)\rho^{2\alpha-n}(z)\left|v^{(k)}(z)\right|^2 dz\Bigg)dy \\ &\qquad = \int\limits_{\Omega} \rho^{2\alpha-n}(z)\left|v^{(k)}(z)\right|^2\Bigg(\int\limits_{\Omega} \chi_{2,m}(z,y)dy\Bigg)dz \\ &\qquad \leqslant \frac{1}{(m^2-1)^n}\varkappa^{-n}\omega_n\left(\frac{m}{m+1}\right)^n\int\limits_{\Omega} \rho^{2\alpha}(z)\left|v^{(k)}(z)\right|^2 dz \\ &\qquad \leqslant \frac{1}{m^{2n}}\varkappa^{-n}\omega_n 2^n\int\limits_{\Omega} \rho^{2\alpha}(z)\left|v^{(k)}(z)\right|^2 dz.\end{aligned} \tag{4.40}$$

By (4.39) this implies

$$|\mathcal{L}_{1,m}(v) - \mathcal{L}_{2,m}(v)| \leqslant \frac{1}{m^{2n+2}}N_3(\alpha)\left\|v; L^{2r}_{2,\alpha}(\Omega)\right\|^2,$$

where $N_3(\alpha) = C_3N_2(\alpha)\omega_n\varkappa^{-n}2^n$. By identity (4.38) the inequality

$$N_3(\alpha) = C_3n(2|\alpha|+n)2^{4|\alpha|+2n+1}\omega_n\varkappa^{-n}2^n \leqslant C_3(2|\alpha|+n)2^{4|\alpha|+3n+1}$$

holds. This is why

$$|\mathcal{L}_{1,m}(v) - \mathcal{L}_{2,m}(v)| \leqslant \frac{1}{m^{2n+2}}N_4(\alpha)\left\|v; L^{2r}_{2,\alpha}(\Omega)\right\|^2, \tag{4.41}$$

where $N_4(\alpha) = C_3(2|\alpha|+n)2^{4|\alpha|+3n+1}$.

We introduce the notation

$$\mathcal{L}_{3,m}(v) = \int\limits_{\Omega}\Bigg(\int\limits_{\Omega} \chi_{2,m}(z,y)\rho^{2\alpha-n}(z)\Bigg|\sum_{|k|=2r} b_k(z)v^{(k)}(z)\Bigg|^2 dz\Bigg)dy. \tag{4.42}$$

We note that

$$|\mathcal{L}_{2,m}(v) - \mathcal{L}_{3,m}(v)| \leqslant C_4 \sum_{|k|=2r} \int_\Omega \left(\int_\Omega \chi_{2,m}(z,y) \rho^{2\alpha-n}(z) \left| b_k(y) - b_k(z) \right| \left| v^{(k)}(z) \right|^2 dz \right) dy, \tag{4.43}$$

where C_4 is some positive constant. Now, by Condition II) of Theorem 2.1 (see (2.6), (2.7)) and above proven inequality (4.39), it follows from (4.43) that

$$|\mathcal{L}_{2,m}(v) - \mathcal{L}_{3,m}(v)| \leqslant \nu C_4 \sum_{|k|=2r} \int_\Omega \left(\int_\Omega \chi_{2,m}(z,y) \rho^{2\alpha-n}(z) \left| v^{(k)}(z) \right|^2 dz \right) dy$$
$$\leqslant \nu \frac{1}{m^{2n}} C_4 \varkappa^{-n} \omega_n 2^n \sum_{|k|=2r} \int_\Omega \rho^{2\alpha}(z) \left| v^{(k)}(z) \right|^2 dz.$$

Thus, we have proved that

$$|\mathcal{L}_{2,m}(v) - \mathcal{L}_{3,m}(v)| \leqslant \nu \frac{1}{m^{2n}} N_5 \left\| v; L^{2r}_{2,\alpha}(\Omega) \right\|^2, \qquad m \in \mathbb{N}, \quad m \geqslant m_\nu, \tag{4.44}$$

where $N_5 = C_4 \varkappa^{-n} \omega_n 2^n$.

Applying Lemma 3.2, we estimate functional (4.42)

$$\mathcal{L}_{3,m}(v) = \int_\Omega \left(\int_\Omega \chi_{2,m}(z,y) \rho^{-n}(z) \left| L_0[v(z)] \right|^2 dz \right) dy$$
$$= \int_\Omega \left(\int_\Omega \chi_{2,m}(z,y) dy \right) \rho^{-n}(z) \left| L_0[v(z)] \right|^2 dz$$
$$\leqslant \frac{1}{m^{2n}} \frac{\omega_n}{\varkappa^n} \left(\frac{m^2}{m^2-1} \right)^n \int_\Omega \left| L_0[v(z)] \right|^2 dz \leqslant \frac{1}{m^{2n}} \frac{\omega_n}{\varkappa^n} 2^n \int_\Omega \left| L_0[v(z)] \right|^2 dz.$$

Therefore,

$$\mathcal{L}_{3,m}(v) \leqslant \frac{1}{m^{2n}} N_6 \int_\Omega \left| L_0[v(z)] \right|^2 dz, \tag{4.45}$$

where $N_6 = \omega_n \varkappa^{-n} 2^n$. Employing above proven inequalities (4.41), (4.44), (4.45), we estimate functional (4.33)

$$\mathcal{L}_{1,m}(v) \leqslant |\mathcal{L}_{1,m}(v) - \mathcal{L}_{2,m}(v)| + |\mathcal{L}_{2,m}(v) - \mathcal{L}_{3,m}(v)| + \mathcal{L}_{3,m}(v)$$
$$\leqslant \frac{1}{m^{2n+2}} N_4(m,\alpha) \left\| v; L^{2r}_{2,\alpha}(\Omega) \right\|^2 + \nu \frac{1}{m^{2n}} N_5 \left\| v; L^{2r}_{2,\alpha}(\Omega) \right\|^2 + \frac{1}{m^{2n}} N_6 \int_\Omega \left| L_0[v(z)] \right|^2 dz.$$

By this inequality it follows from (4.34) that

$$M_5(m,\alpha) \left\| v; L^{2r}_{2,\alpha}(\Omega) \right\|^2 - M_6(m,\alpha) \left\| v; L_{2,\alpha-2r}(\Omega) \right\|^2 \leqslant C_5 m^{2n} \mathcal{L}_{1,m}(v)$$
$$\leqslant C_5 \left(\frac{1}{m^2} N_4(\alpha) + \nu N_5 \right) \left\| v; L^{2r}_{2,\alpha}(\Omega) \right\|^2 + C_5 N_6 \int_\Omega \left| L_0[v(z)] \right|^2 dz,$$

where

$$M_5(m,\alpha) = M_{11}(\alpha) - C_{10} M_2(m,\alpha) - \frac{1}{m} M_3(\alpha), \tag{4.46}$$
$$M_6(m,\alpha) = K_2(m)(m\varkappa)^{4r} M_3(\alpha - 2r) - (m\varkappa)^{4r} M_{11}(\alpha - 2r). \tag{4.47}$$

Thus, we have proved the inequality

$$C_0(m,\alpha,\nu)\left\|v;L^{2r}_{2,\alpha}(\Omega)\right\|^2-C_1(m,\alpha)\left\|v;L_{2,\alpha-2r}(\Omega)\right\|^2 \leqslant \int\limits_\Omega |L_0[v(z)]|^2\,dz,\quad m\geqslant m(\nu), \tag{4.48}$$

where

$$C_0(m,\alpha,\nu)=\frac{1}{C_5N_6}\left[M_{5,\alpha}(m)-C_5\left(\frac{1}{m^2}N_4(\alpha)+\nu N_5\right)\right], \tag{4.49}$$

$$C_1(m,\alpha)=\frac{M_6(m,\alpha)}{C_5N_6}. \tag{4.50}$$

By (3.20) and (3.24) it follows from (4.46) that for some sufficiently large m_1 for all $m\geqslant m_1$ the inequality $M_5(m,\alpha)\geqslant\frac{M_{11}(\alpha)}{2}>0$ holds. By (4.49) this implies that for sufficiently large $m_2>m_1$ and some sufficiently large $\nu_1>0$ for all $m\geqslant m_2$ and all $\nu\in(0,\nu_1)$ the inequality holds

$$C_0(m,\nu)\geqslant\frac{1}{4}M_{11}(\alpha)>0. \tag{4.51}$$

On the other hand, by (4.15) it follows from (4.47) that

$$M_6(m,\alpha)=K_2(m)(m\varkappa)^{4r}M_3(\alpha-2r)-(m\varkappa)^{4r}M_{11}(\alpha-2r)\geqslant\frac{1}{2}M_3(\alpha-2r)$$

for all $m\geqslant m_3$, where m_3 is some number greater than m_2. This is why by (4.50) we have

$$C_1(m,\alpha)=\frac{M_6(m,\alpha)}{C_5N_6}\geqslant\frac{1}{2}M_3(\alpha-2r)\frac{1}{C_5N_6}>0\quad(m\geqslant m_3). \tag{4.52}$$

Taking into consideration inequalities (4.51), (4.52) and choosing the number m large enough and the number ν small enough, by (4.48) we obtain the inequality

$$c\left\|v;L^{2r}_{2,\alpha}(\Omega)\right\|^2\leqslant\int\limits_\Omega|L_0[v(z)]|^2\,dz+K\left\|v;L_{2,\alpha-2r}(\Omega)\right\|^2\quad\text{for all}\quad v\in C_0^\infty(\Omega). \tag{4.53}$$

Thus, the proof of Theorem 2.1 for operator (2.11) is complete.

We proceed to the proof of Theorem 2.1 in the case of operators with non–zero lower order coefficients. We represent operator (2.4) in the form

$$L[u(x)]=L_0[u(x)]+L_1[u(x)], \tag{4.54}$$

where $L_0[u(x)]$ is defined by identity (2.11) and

$$L_1[u(x)]=\sum_{|k|\leqslant 2r-1}\rho^{\alpha_{|k|}}(x)b_k(x)u^{(k)}(x). \tag{4.55}$$

Since the coefficients $b_k(x)$ are bounded and $\alpha_j\geqslant\alpha-2r+j$ for all $j\leqslant 2r-1$, we have

$$\left(\int\limits_\Omega|L_1[v(x)]|^2\,dx\right)^{\frac12}=\|L_1[v];L_2(\Omega)\|\leqslant\sum_{|k|=j\leqslant 2r-1}\left\|\rho^{\alpha_j}b_kv^{(k)};L_2(\Omega)\right\| \leqslant M_1\sum_{j=0}^{2r-1}\left\|v;L^j_{2,\alpha_j}(\Omega)\right\|\leqslant M_2\sum_{j=0}^{2r-1}\left\|v;L^j_{2,\alpha-2r+j}(\Omega)\right\|. \tag{4.56}$$

Applying inequality (3.30) in Lemma 3.12, we hence obtain

$$\left(\int\limits_\Omega|L_1[v(x)]|^2\,dx\right)^{\frac12}\leqslant\mu\|v;L^{2r}_{2;\alpha}(\Omega)\|+C(\mu)\|v;L_{2;\alpha-2r}(\Omega)\|,\quad v\in C_0^\infty(\Omega), \tag{4.57}$$

where μ is a sufficiently small positive number. Using inequalities (4.53), (4.57), we find

$$\left(\int\limits_{\Omega} |L[v(x)]|^2\, dx\right)^{\frac{1}{2}} = \|L[v]; L_2(\Omega)\| \geqslant \|L_0[v]; L_2(\Omega)\| - \|L_1[v]; L_2(\Omega)\|$$
$$\geqslant (c-\mu)\left\|v; L^{2r}_{2,\alpha}(\Omega)\right\| - C(\mu)\left\|v; L_{2;\alpha-2r}(\Omega)\right\|.$$

We fix an appropriate value of the parameter μ in this inequality and get inequality (2.9). The proof of Theorem 2.1 is complete.

5. Proof of Theorem 2.2

We consider operator (2.11), which satisfies the assumptions of Theorem 2.2. Let y be an arbitrary fixed point in the domain Ω. We consider operator (4.1) with coefficients frozen at the point y. Since the coefficients of such an operator are constant, by condition (2.10) (see, for instance, [4]), we have

$$\mathrm{Re}(L_{0,y}[u], u)_0 = \mathrm{Re}\int\limits_{\mathbb{R}^n} L_{0,y}[u(x)]\cdot\overline{u(x)}dx \geqslant 0, \quad u(x)\in C_0^\infty(\mathbb{R}^n). \tag{5.1}$$

We consider an arbitrary function $u(x) \in C^\infty(B_m(0))$. Since $v_m(x) = \varphi_m(x)u(x) \in C_0^\infty(B_m(0))$, by using (5.1) for the function $v_m(x)$, we get

$$\mathrm{Re}(L_{0,y}[v_m], v_m)_0 = \sum_{|k|=2r}\mathrm{Re}\int\limits_{\mathbb{R}^n} b_k(y)v_m^{(k)}(x)\overline{v_m(x)}dx \geqslant 0. \tag{5.2}$$

We represent the expression $L_{0,y}v_m(x)$ in form (4.4) and we obtain the identity

$$(L_{0,y}[v_m], v_m)_0 = (L_y^{(1)}[v_m], v_m)_0 + (L_y^{(2)}[v_m], v_m)_0, \tag{5.3}$$

where $L_y^{(2)}[v_m], L_y^{(1)}[v_m]$ are determined by identities (4.5), (4.6), respectively.

Since $\varphi_m \in C_0^\infty(B_m(0))$ and $\varphi_m(x) = 1$ for all $x \in B_{m+1}(0)$, we have

$$\begin{aligned}(L_y^{(1)}[v_m], v_m)_0 = \int\limits_{B_m(0)} L_y^{(1)}[v_m](x)\overline{v_m(x)}dx = \int\limits_{B_{m+1}(0)} L_{0,y}[u](x)\overline{u(x)}dx \\ + \int\limits_{B^{(m)}(0)} \varphi_m^2(x)L_{0,y}[u](x)\overline{u(x)}dx,\end{aligned} \tag{5.4}$$

where $B^{(m)}(0)$ is defined by identity (4.8). In view of the boundedness of coefficients $b_k(x)$ we obtain

$$\begin{aligned}\left|\int\limits_{B^{(m)}(0)} \varphi_m^2(x)L_{0,y}[u(x)]\overline{u(x)}dx\right| \leqslant & M_0\left(\sum_{|k|=2r}\int\limits_{B^{(m)}(0)} \left|u^{(k)}(x)\right|^2 dx\right)^{\frac{1}{2}}\left(\int\limits_{B^{(m)}(0)} |u(x)|^2\,dx\right)^{\frac{1}{2}} \\ =& M_0\left\|u; L_2^{2r}(B^{(m)}(0))\right\|\left\|u; L_2(B^{(m)}(0))\right\|.\end{aligned} \tag{5.5}$$

We then observe that there exists some natural number m^* such that for all $m \geqslant m^*$ inequality (4.14) holds. Using this inequality, we obtain

$$\begin{aligned}\left|\int\limits_{B_m(0)} L_y^{(2)}[v_m(x)]\overline{v_m(x)}dx\right| &\leqslant \left\|L_y^{(2)}[v_m]; L_2(B_m(0))\right\| \|u; L_2(B_m(0))\| \\ &\leqslant \frac{1}{m}\left\|u; L_2^{2r}(B_m(0))\right\| \|u; L_2(B_m(0))\| \\ &\quad + K_2(m)\|u; L_2(B_m(0))\| \|u; L_2(B_m(0))\|.\end{aligned} \tag{5.6}$$

Using above obtained relations (5.2)–(5.6), we find

$$\begin{aligned}0 \leqslant \operatorname{Re}(L_{0,y}[v_m], v_m)_0 &= \operatorname{Re}(L_y^{(1)}[v_m], v_m)_0 + \operatorname{Re}(L_y^{(2)}[v_m], v_m)_0 \\ &= \operatorname{Re}\int\limits_{B_{m+1}(0)} L_{0,y}[u(x)]\overline{u(x)}dx \\ &\quad + \operatorname{Re}\int\limits_{B^{(m)}(0)} \varphi_m^2(x)L_{0,y}[u(x)]\overline{u(x)}dx + \operatorname{Re}(L_y^{(2)}[v_m], v_m)_0 \\ &\leqslant \operatorname{Re}\int\limits_{B_{m+1}(0)} L_{0,y}[u(x)]\overline{u(x)}dx \\ &\quad + M_0\left\|u; L_2^{2r}(B^{(m)}(0))\right\| \left\|u; L_2(B^{(m)}(0))\right\| \\ &\quad + \frac{1}{m}\left\|u; L_2^{2r}(B_m(0))\right\| \|u; L_2(B_m(0))\| \\ &\quad + K_2(m)\|u; L_2(B_m(0))\|^2.\end{aligned} \tag{5.7}$$

As in the proof of Theorem 2.1, we consider the mapping $z \to x$, defined by means of the identity $x = (z-y)m\varkappa\rho(y)$, where y is an arbitrary fixed point in the domain Ω (see (4.17)). It maps the sets (see (2.7), (3.1), (3.6)) $J_{1,m}(y)$, $J_{2,m}(y)$, $J^{(m)}(y)$ respectively into the sets $B_m(0)$, $B_{m+1}(0)$, $B^{(m)}(0)$.

We choose an arbitrary function v in the class $C_0^\infty(\Omega)$. The function $\widehat{v}_y(x) = v\left(\frac{x\rho(y)}{m\varkappa} + y\right)$ is defined for all points $x \in B_m(0)$, and the function $u(x) = \widehat{v}_y(x)$ satisfies identities (4.18), (4.19). In view of these identities and by (5.7), for the function $u(x) = \widehat{v}_y(x)$ we obtain an inequality, which after the multiplication by $\rho^\alpha(y)$ becomes

$$\begin{aligned}&\operatorname{Re}(m\varkappa)^{-2r}\rho^{2r+\alpha}(y)\int\limits_{B_{m+1}(0)} L_{0,y}\left[\widehat{v}_y(x)\right]\cdot\overline{\widehat{v}_y(x)}dx \\ &\geqslant -M_0(m\varkappa)^{-2r}\rho^{2r+\alpha}(y)\left(\sum_{|k|=2r}\int\limits_{B^{(m)}(0)} \left|\widehat{v}_y^{(k)}(x)\right|^2 dx\right)^{\frac{1}{2}}\left(\int\limits_{B^{(m)}(0)} |\widehat{v}_y(x)|^2 dx\right)^{\frac{1}{2}} \\ &\quad - \frac{1}{m}(m\varkappa)^{-2r}\rho^{2r+\alpha}(y)\left(\sum_{|k|=2r}\int\limits_{B_m(0)} \left|\widehat{v}_y^{(k)}(x)\right|^2 dx\right)^{\frac{1}{2}}\left(\int\limits_{B_m(0)} |\widehat{v}_y(x)|^2 dx\right)^{\frac{1}{2}} \\ &\quad - K_2(m)\rho^\alpha(y)\int\limits_{B_m(0)} |\widehat{v}_y(x)|^2 dx.\end{aligned} \tag{5.8}$$

Then in the integrals of this inequality we change the variables $z = y + x\rho(y)(m\varkappa)^{-1}$ and observe that the integrals are transformed in accordance with (4.21) – (4.25). Now, using these identities, by (5.8) we obtain an inequality, which is after multiplication by $(m\varkappa)^{2r-n}\rho^{-2r}(y)$, becomes

$$\rho^{\alpha-n}(y)\,\mathrm{Re}\int\limits_{J_{2,m}(y)} L_{0,y}\left[v(z)\right]\cdot\overline{v(z)}dz$$
$$\geqslant -M_0\Bigg(\rho^{2\alpha-n}(y)\sum_{|k|=2r}\int\limits_{J^{(m)}(y)}\left|v^{(k)}(z)\right|^2dz\Bigg)^{\frac{1}{2}}\Bigg(\rho^{-n}(y)\int\limits_{J^{(m)}(y)}|v(z)|^2dz\Bigg)^{\frac{1}{2}}$$
$$-\frac{1}{m}\Bigg(\rho^{2\alpha-n}(y)\sum_{|k|=2r}\int\limits_{J_{1,m}(y)}\left|v^{(k)}(z)\right|^2dz\Bigg)^{\frac{1}{2}}\Bigg(\rho^{-n}(y)\int\limits_{J_{1,m}(y)}|v(z)|^2dz\Bigg)^{\frac{1}{2}}$$
$$-K_2(m)\rho^{-2r+\alpha-n}(y)\int\limits_{J_{1,m}(y)}|v(z)|^2dz.$$

Now we integrate this inequality in $y\in\Omega$ and we estimate the right hand side by means of the Cauchy — Bunyakovsky inequality

$$-\int\limits_{\Omega}\{\mathbb{U}(y)\}^{\frac{1}{2}}\left(\mathbb{V}(y)\right)^{\frac{1}{2}}dy\geqslant-\Bigg(\int\limits_{\Omega}\mathbb{U}(y)dy\Bigg)^{\frac{1}{2}}\cdot\Bigg(\int\limits_{\Omega}\mathbb{V}(y)dy\Bigg)^{\frac{1}{2}}$$

and as the result we arrive at the inequality

$$\begin{aligned}
&\mathrm{Re}\int\limits_{\Omega}\rho^{\alpha-n}(y)\Bigg(\int\limits_{J_{2,m}(y)}L_{0,y}\left[v(z)\right]\cdot\overline{v(z)}dz\Bigg)dy\\
&\qquad\geqslant -M_0\Bigg(\sum_{|k|=2r}\int\limits_{\Omega}\rho^{2\alpha-n}(y)\Bigg(\int\limits_{J^{(m)}(y)}\left|v^{(k)}(z)\right|^2dz\Bigg)dy\Bigg)^{\frac{1}{2}}\\
&\qquad\quad\cdot\Bigg(\int\limits_{\Omega}\rho^{-n}(y)\Bigg(\int\limits_{J^{(m)}(y)}|v(z)|^2dz\Bigg)dy\Bigg)^{\frac{1}{2}}\\
&\qquad\quad-\frac{1}{m}\Bigg(\sum_{|k|=2r}\int\limits_{\Omega}\rho^{2\alpha-n}(y)\Bigg(\int\limits_{J_{1,m}(y)}\left|v^{(k)}(z)\right|^2dz\Bigg)dy\Bigg)^{\frac{1}{2}}\\
&\qquad\quad\cdot\Bigg(\int\limits_{\Omega}\rho^{-n}(y)\Bigg(\int\limits_{J_{1,m}(y)}|v(z)|^2dz\Bigg)dy\Bigg)^{\frac{1}{2}}\\
&\qquad\quad-K_2(m)\int\limits_{\Omega}\rho^{-2r+\alpha-n}(y)\Bigg(\int\limits_{J_{1,m}(y)}|v(z)|^2dz\Bigg)dy.
\end{aligned}\tag{5.9}$$

We proceed to estimating the integrals in this inequality. We observe that for the first integral in the right hand side of (5.9) the inequality (4.29) holds. Similarly to this inequality, by means of Lemma 3.8 we prove that

$$\int\limits_{\Omega} \rho^{-n}(y)\Bigg(\int\limits_{J^{(m)}(y)} |v(z)|^2\,dz\Bigg)dy \leqslant \frac{1}{m^{2n}} M_2(m,0)\,\|v;L_2(\Omega)\|^2\,, \tag{5.10}$$

where $M_2(m,0) = 2^n\sigma_1(m)$.

To estimate the remaining integrals in the right hand side of (5.9) we apply Lemma 3.9. As the result we obtain the inequalities

$$\sum_{|k|=2r}\int\limits_{\Omega} \rho^{2\alpha-n}(y)\Bigg(\int\limits_{J_{1,m}(y)} \left|v^{(k)}(z)\right|^2 dz\Bigg)dy \leqslant \frac{1}{m^{2n}} M_3(\alpha)\,\left\|v;L^{2r}_{2,\alpha}(\Omega)\right\|^2, \tag{5.11}$$

$$\int\limits_{\Omega} \rho^{-n}(y)\Bigg(\int\limits_{J_{1,m}(y)} |v(z)|^2dz\Bigg)dy \leqslant \frac{1}{m^{2n}} M_3(0)\,\|v;L_2(\Omega)\|^2\,, \tag{5.12}$$

$$\int\limits_{\Omega} \rho^{-2r+\alpha-n}(y)\Bigg(\int\limits_{J_{1,m}(y)} |v(z)|^2dz\Bigg)dy \leqslant \frac{1}{m^{2n}} M_3(\alpha-2r)\,\|f;L_{2,\alpha-2r}(\Omega)\|^2\,, \tag{5.13}$$

where the constants $M_3(\alpha)$, $M_3(0)$, $M_3(\alpha-2r)$ are defined by (4.32). Applying now inequalities (4.39), (5.10)–(5.13), by (5.9) we obtain

$$\begin{aligned}
\mathrm{Re}\int\limits_{\Omega} \rho^{\alpha-n}(y)\Bigg(\int\limits_{J_{2,m}(y)} L_{0,y}\,[v(z)]\cdot\overline{v(z)}dz\Bigg)dy& \\
\geqslant -\frac{1}{m^{2n}} M_0\sqrt{M_2(m,\alpha)M_2(m,0)}\,&\left\|v;L^{2r}_{2,\alpha}(\Omega)\right\|\,\|v;L_2(\Omega)\| \\
-\frac{1}{m}\frac{1}{m^{2n}}\sqrt{M_3(\alpha)M_3(0)}\,&\left\|v;L^{2r}_{2,\alpha}(\Omega)\right\|\,\|v;L_2(\Omega)\| \\
-\frac{1}{m^{2n}} K_2(m)M_3(\alpha-2r)\,&\|f;L_{2,\alpha-2r}(\Omega)\|^2\,.
\end{aligned}$$

Introducing the notation

$$M_6(m,\alpha) = M_0\sqrt{M_2(m,\alpha)M_2(m,0)} + \frac{1}{m}\sqrt{M_3(\alpha)M_3(0)}, \tag{5.14}$$

$$M_7(m,\alpha) = K_2(m)M_3(\alpha-2r), \tag{5.15}$$

we arrive at the inequality

$$\begin{aligned}
\mathrm{Re}\int\limits_{\Omega} \rho^{\alpha-n}(y)\Bigg(\int\limits_{J_{2,m}(y)} L_{0,y}\,[v(z)]\cdot\overline{v(z)}dz\Bigg)dy& \\
\geqslant -\frac{1}{m^{2n}} M_6(m,\alpha)\,&\left\|v;L^{2r}_{2,\alpha}(\Omega)\right\|\,\|v;L_2(\Omega)\| \\
-\frac{1}{m^{2n}} M_7(m,\alpha)\,&\|v;L_{2,\alpha-2r}(\Omega)\|^2\,.
\end{aligned} \tag{5.16}$$

It follows from (3.24), (4.32), (5.14) that

$$\begin{aligned}
M_6(m,\alpha) &= M_0\sqrt{2^n4^{|\alpha|}\sigma_1(m)2^n\sigma_1(m)} + \frac{1}{m}\sqrt{\omega_n\varkappa^{-n}4^{|\alpha|}2^n\omega_n\varkappa^{-n}2^n} \\
&= M_02^n2^{|\alpha|}\sigma_1(m) + \frac{1}{m}\omega_n\varkappa^{-n}2^{|\alpha|}2^n = 2^{|\alpha|+n}\left[M_0\sigma_1(m) + \frac{1}{m}\omega_n\varkappa^{-n}\right],
\end{aligned} \tag{5.17}$$

Since (see (3.9)) $\lim\limits_{m\to\infty}\sigma_1(m)=0$, this implies that

$$\lim_{m\to\infty} M_6(m,\alpha)=0. \tag{5.18}$$

Using the identity

$$\begin{aligned}\operatorname{Re}\int\limits_{\Omega}\Bigg(\int\limits_{J_{2,m}(y)}\rho^{-n}(z)L_0\left[v(z)\right]\cdot\overline{v(z)}dz\Bigg)dy &= \operatorname{Re}\int\limits_{\Omega}\rho^{\alpha-n}(y)\Bigg(\int\limits_{J_{2,m}(y)}L_{0,y}\left[v(z)\right]\cdot\overline{v(z)}dz\Bigg)dy\\ &+\operatorname{Re}\int\limits_{\Omega}\Bigg(\int\limits_{J_{2,m}(y)}\rho^{-n}(z)L_0\left[v(z)\right]\cdot\overline{v(z)}dz\Bigg)dy\\ &-\operatorname{Re}\int\limits_{\Omega}\rho^{\alpha-n}(y)\Bigg(\int\limits_{J_{2,m}(y)}L_{0,y}\left[v(z)\right]\cdot\overline{v(z)}dz\Bigg)dy,\end{aligned}$$

we obtain

$$\begin{aligned}&\operatorname{Re}\int\limits_{\Omega}\Bigg(\int\limits_{J_{2,m}(y)}\rho^{-n}(z)L_0\left[v(z)\right]\cdot\overline{v(z)}dz\Bigg)dy\\ &\qquad\geqslant-\frac{1}{m^{2n}}M_6(m,\alpha)\left\|v;L^{2r}_{2,\alpha}(\Omega)\right\|\left\|v;L_2(\Omega)\right\|\\ &\qquad\quad-\frac{1}{m^{2n}}M_7(m,\alpha)\left\|v;L_{2,\alpha-2r}(\Omega)\right\|^2-\mathcal{M}_m[v],\end{aligned}\tag{5.19}$$

where

$$\mathcal{M}_m[v]=\left|\int\limits_{\Omega}\Bigg(\int\limits_{J_{2,m}(y)}\rho^{-n}(z)L_0\left[v(z)\right]\cdot\overline{v(z)}dz\Bigg)dy-\int\limits_{\Omega}\rho^{\alpha-n}(y)\Bigg(\int\limits_{J_{2,m}(y)}L_{0,y}\left[v(z)\right]\cdot\overline{v(z)}dz\Bigg)dy\right|.$$

We introduce the following auxiliary functionals

$$\begin{aligned}\mathcal{M}^{(1)}_m[v]=&\Bigg|\int\limits_{\Omega}\Bigg(\int\limits_{J_{2,m}(y)}\rho^{-n}(z)L_0\left[v(z)\right]\cdot\overline{v(z)}dz\Bigg)dy\\ &-\int\limits_{\Omega}\Bigg(\int\limits_{J_{2,m}(y)}\rho^{\alpha-n}(z)L_{0,y}\left[v(z)\right]\cdot\overline{v(z)}dz\Bigg)dy\Bigg|,\end{aligned}\tag{5.20}$$

$$\begin{aligned}\mathcal{M}^{(2)}_m[v]=&\Bigg|\int\limits_{\Omega}\Bigg(\int\limits_{J_{2,m}(y)}\rho^{\alpha-n}(z)L_{0,y}\left[v(z)\right]\cdot\overline{v(z)}dz\Bigg)dy\\ &-\int\limits_{\Omega}\rho^{\alpha-n}(y)\Bigg(\int\limits_{J_{2,m}(y)}L_{0,y}\left[v(z)\right]\cdot\overline{v(z)}dz\Bigg)dy\Bigg|,\end{aligned}\tag{5.21}$$

and observe that

$$\mathcal{M}_m[v]\leqslant\mathcal{M}^{(1)}_m[v]+\mathcal{M}^{(2)}_m[v],\qquad v\in C^{\infty}_0(\mathbb{R}^n).\tag{5.22}$$

Using Condition II) of Theorem 2.1 and applying Lemma 3.2, for functional (5.20) we get

$$\mathcal{M}^{(1)}_m[v]=\left|\sum_{|k|=2r}\int\limits_{\Omega}\rho^{\alpha-n}(z)\Bigg(\int\limits_{J_{2,m}(y)}(b_k(z)-b_k(y))dy\Bigg)v^{(k)}(z)\overline{v(z)}dz\right|$$

$$\leqslant \sum_{|k|=2r} \int_\Omega \rho^{\alpha-n}(z) \left(\int_\Omega \chi_{2,m}(z,y)|b_k(z) - b_k(y)|dy \right) \left|v^{(k)}(z)\right| |v(z)|dz$$
$$\leqslant \nu \sum_{|k|=2r} \int_\Omega \rho^{\alpha-n}(z) \left(\int_\Omega \chi_{2,m}(z,y)dy \right) \left|v^{(k)}(z)\right| |v(z)|dz$$
$$\leqslant \nu \varkappa^{-n}\omega_n \left(\frac{m}{m+1}\right)^n \frac{1}{(m^2-1)^n} \sum_{|k|=2r} \int_\Omega \rho^{\alpha}(z) \left|v^{(k)}(z)\right| |v(z)|dz$$
$$\leqslant \nu \varkappa^{-n}\omega_n \frac{1}{m^{2n}} \left\|v; L^{2r}_{2,\alpha}(\Omega)\right\| \cdot \|v; L_2(\Omega)\| .$$

Thus, we have proved the inequality

$$\mathcal{M}^{(1)}_m[v] \leqslant \nu \frac{M_1}{m^{2n}} \left\|v; L^{2r}_{2,\alpha}(\Omega)\right\| \cdot \|v; L_2(\Omega)\| , \qquad M_1 = \varkappa^{-n}\omega_n. \tag{5.23}$$

We proceed to estimating functional (5.21). Taking into consideration the boundedness of coefficients $b_k(z)$ and applying Lemmas 3.2, 3.6, we obtain

$$\mathcal{M}^{(2)}_m[v] = \left| \int_\Omega \left(\int_{J_{2,m}(y)} \left(\rho^{\alpha-n}(z) - \rho^{\alpha-n}(y)\right) L_{0,y}\left[v(z)\right] \cdot \overline{v(z)}dz \right) dy \right|$$
$$\leqslant \sum_{|k|=2r} \int_\Omega \left(\int_\Omega \chi_{2,m}(z,y) \left|\rho^{\alpha-n}(z) - \rho^{\alpha-n}(y)\right| |b_k(y)|\, dy \right) |v^{(k)}(z)||v(z)|dz$$
$$\leqslant \frac{1}{m^2} N_1(m, \alpha-n) M_0 \sum_{|k|=2r} \int_\Omega \left(\int_\Omega \chi_{2,m}(z,y)\, dy \right) \rho^{\alpha-n}(z)|v^{(k)}(z)||v(z)|\, dz$$
$$\leqslant \frac{1}{m^2} N_1(m, \alpha-n) M_0 \varkappa^{-n}\omega_n \left(\frac{m}{m+1}\right)^n \frac{1}{(m^2-1)^n} \sum_{|k|=2r} \int_\Omega \rho^{\alpha}(z)|v^{(k)}(z)||v(z)|\, dz$$
$$\leqslant \frac{1}{m^{2n+2}} N_1(m, \alpha-n) M_0 \varkappa^{-n}\omega_n \left\|v; L^{2r}_{2,\alpha}(\Omega)\right\| \cdot \|v; L_2(\Omega)\| .$$

Since

$$N_1(m, \alpha-n) \leqslant n(|\alpha|+n) \left(1 + \frac{1}{m^2}\right)^{2|\alpha|+2n+1} \leqslant n(|\alpha|+n)4^{|\alpha|+n+1},$$

it follows from the above obtained inequality that

$$\mathcal{M}^{(2)}_m[v] \leqslant \frac{1}{m^{2n+2}} M^*_7(\alpha) \left\|v; L^{2r}_{2,\alpha}(\Omega)\right\| \cdot \|v; L_2(\Omega)\| , \tag{5.24}$$

where

$$M^*_7(\alpha) = M_0 \varkappa^{-n}\omega_n n(|\alpha|+n)4^{|\alpha|+n+1}. \tag{5.25}$$

By inequalities (5.23) and (5.24), in view of (5.22) we obtain

$$\mathcal{M}_m[v] \leqslant \mathcal{M}^{(1)}_m[v] + \mathcal{M}^{(2)}_m[v]$$
$$\leqslant \nu \frac{M_1}{m^{2n}} \left\|v; L^{2r}_{2,\alpha}(\Omega)\right\| \cdot \|v; L_2(\Omega)\| + \frac{1}{m^{2n+2}} M_7(\alpha) \left\|v; L^{2r}_{2,\alpha}(\Omega)\right\| \cdot \|v; L_2(\Omega)\|$$
$$\leqslant \frac{1}{m^{2n}} \left[\nu \varkappa^{-n}\omega_n + \frac{1}{m^2} M^*_7(\alpha)\right] \left\|v; L^{2r}_{2,\alpha}(\Omega)\right\| \cdot \|v; L_2(\Omega)\| , \qquad v \in C^\infty_0(\mathbb{R}^n).$$

Applying this inequality, by (5.19) we find

$$
\begin{aligned}
&\operatorname{Re}\int\limits_{\Omega}\Bigg(\int\limits_{J_{2,m}(y)}\rho^{-n}(z)L_0\left[v(z)\right]\cdot\overline{v(z)}dz\Bigg)dy\\
&\geqslant -\frac{1}{m^{2n}}M_6(m,\alpha)\left\|v;L^{2r}_{2,\alpha}(\Omega)\right\|\left\|v;L_2(\Omega)\right\|\\
&\quad-\frac{1}{m^{2n}}M_7(m,\alpha)\left\|v;L_{2,\alpha-2r}(\Omega)\right\|^2\\
&\quad-\frac{1}{m^{2n}}\left[\nu M_1+\frac{1}{m^2}M^*_7(\alpha)\right]\left\|v;L^{2r}_{2,\alpha}(\Omega)\right\|\cdot\left\|v;L_2(\Omega)\right\|.
\end{aligned}
\tag{5.26}
$$

Applying Lemma 3.10, for the function $\Phi(z)=\operatorname{Re}L_0\left[v(z)\right]\cdot\overline{v(z)}$ we obtain

$$
\begin{aligned}
&\operatorname{Re}\int\limits_{\Omega}\Bigg(\int\limits_{J_{2,m}(y)}\rho^{-n}(z)L_0\left[v(z)\right]\cdot\overline{v(z)}dz\Bigg)dy\\
&\leqslant\varkappa^{-n}\omega_n\left(\frac{m}{m+1}\right)^n\frac{1}{(m^2-1)^n}\operatorname{Re}\int\limits_{\Omega}L_0\left[v(z)\right]\cdot\overline{v(z)}dz\\
&\quad+\varkappa^{-n}\omega_n\left[\frac{1}{(m^2-1)^n}-\frac{1}{(m^2+1)^n}\right]\int\limits_{\Omega}\left|L_0\left[v(z)\right]\right||v(z)|dz.
\end{aligned}
$$

Since

$$
\int\limits_{\Omega}\left|L_0\left[v(z)\right]\right||v(z)|dz\leqslant M_2\sum_{|k|=2r}\int\limits_{\Omega}\rho^{\alpha}(z)\left|v^{(k)}(z)\right||v(z)|dz\leqslant M_2\left\|v;L^{2r}_{2,\alpha}(\Omega)\right\|\left\|v;L_2(\Omega)\right\|,
$$

this implies that

$$
\begin{aligned}
&\operatorname{Re}\int\limits_{\Omega}\Bigg(\int\limits_{J_{2,m}(y)}\rho^{-n}(z)L_0\left[v(z)\right]\cdot\overline{v(z)}dz\Bigg)dy\\
&\leqslant\varkappa^{-n}\omega_n\left(\frac{m}{m+1}\right)^n\frac{1}{(m^2-1)^n}\operatorname{Re}\int\limits_{\Omega}L_0\left[v(z)\right]\cdot\overline{v(z)}dz\\
&\quad+\varkappa^{-n}\omega_n\left[\frac{1}{(m^2-1)^n}-\frac{1}{(m^2+1)^n}\right]M_2\left\|v;L^{2r}_{2,\alpha}(\Omega)\right\|\left\|v;L_2(\Omega)\right\|.
\end{aligned}
$$

Applying this inequality and using (5.26), after simple transformations we obtain

$$
\begin{aligned}
\operatorname{Re}\int\limits_{\Omega}L_0\left[v(z)\right]\cdot\overline{v(z)}dz\geqslant&-M^*_8(m,\alpha)\left\|v;L^{2r}_{2,\alpha}(\Omega)\right\|\left\|v;L_2(\Omega)\right\|\\
&-M^*_9(m,\alpha,\nu)\left\|v;L_{2,\alpha-2r}(\Omega)\right\|\left\|v;L_2(\Omega)\right\|\\
&-M^*_{10}(m,\alpha)\left\|v;L_{2,\alpha-2r}(\Omega)\right\|^2,
\end{aligned}
\tag{5.27}
$$

where

$$
M^*_8(m,\alpha)=\varkappa^n\omega_n^{-1}\left(\frac{m+1}{m}\right)^n(m^2-1)^n\frac{1}{m^{2n}}M_6(m,\alpha),
\tag{5.28}
$$

$$M_9^*(m,\alpha,\nu) = \varkappa^n\omega_n^{-1}\left(\frac{m+1}{m}\right)^n (m^2-1)^n\frac{1}{m^{2n}} \cdot\left[\nu M_1+\frac{1}{m^2}M_7^*(\alpha)+\frac{\omega_n}{\varkappa^n}\left(\frac{m^{2n}}{(m^2-1)^n}-\frac{m^{2n}}{(m^2+1)^n}\right)\right], \tag{5.29}$$

$$M_{10}^*(m,\alpha) = \varkappa^n\omega_n^{-1}\left(\frac{m+1}{m}\right)^n (m^2-1)^n\frac{1}{m^{2n}}M_7(m,\alpha). \tag{5.30}$$

We estimate $M_8^*(m,\alpha)$, $M_9^*(m,\alpha,\nu)$, $M_{10}^*(m,\alpha)$ from above. By (5.28) we have

$$M_8^*(m,\alpha) = \varkappa^n\omega_n^{-1}\left(\frac{m+1}{m}\right)^n (m^2-1)^n\frac{1}{m^{2n}}M_6(m,\alpha) \leqslant \varkappa^n\omega_n^{-1}2^n\left(\frac{m^2-1}{m^2}\right)^n M_6(m,\alpha) \leqslant \varkappa^n\omega_n^{-1}2^n M_6(m,\alpha).$$

Thus, we have proved that

$$M_8^*(m,\alpha)\leqslant M_8(m,\alpha),\qquad M_8(m,\alpha)=\varkappa^n\omega_n^{-1}2^nM_6(m,\alpha). \tag{5.31}$$

We note that (see (5.18)) $M_6(m,\alpha)\to 0+$ as $m\to+\infty$ and this is why

$$\lim_{m\to+\infty} M_8(m,\alpha) = 0+\,. \tag{5.32}$$

Using (5.29), we find

$$\begin{aligned} M_9^*(m,\alpha,\nu) =&\varkappa^n\omega_n^{-1}\left(\frac{m+1}{m}\right)^n\left(\frac{m^2-1}{m^2}\right)^n \\ &\cdot\left[\nu M_1+\frac{1}{m^2}M_7(\alpha)+\frac{\omega_n}{\varkappa^n}\left(\frac{m^{2n}}{(m^2-1)^n}-\frac{m^{2n}}{(m^2+1)^n}\right)\right] \\ \leqslant&\varkappa^n\omega_n^{-1}2^n\left[\nu M_1+\frac{1}{m^2}M_7^*(\alpha)+\frac{\omega_n}{\varkappa^n}\left(\left(\frac{m^2}{m^2-1}\right)^n-\left(\frac{m^2}{m^2+1}\right)^n\right)\right]. \end{aligned}$$

Therefore,

$$M_9^*(m,\alpha,\nu)\leqslant M_9(m,\alpha,\nu), \tag{5.33}$$

where

$$M_9(m,\alpha,\nu) = \varkappa^n\omega_n^{-1}2^n\left[\nu M_1+\frac{1}{m^2}M_7^*(\alpha)+\frac{\omega_n}{\varkappa^n}\left(\left(\frac{m^2}{m^2-1}\right)^n-\left(\frac{m^2}{m^2+1}\right)^n\right)\right]. \tag{5.34}$$

We observe that

$$\lim_{m\to+\infty}\lim_{\nu\to+0} M_9(m,\alpha,\nu) = 0. \tag{5.35}$$

We proceed to estimating $M_{10}^*(m,\alpha)$. It follows from (5.30) that

$$M_{10}^*(m,\alpha)\leqslant \varkappa^n\omega_n^{-1}2^n\left(\frac{m^2-1}{m^2}\right)^n M_7(m,\alpha)\leqslant \varkappa^n\omega_n^{-1}2^nM_7(m,\alpha).$$

Using this inequality and (4.15), (4.32), (5.15), we obtain

$$M_{10}^*(m,\alpha)\leqslant M_{10}(m,\alpha), \tag{5.36}$$

where

$$M_{10}(m,\alpha) = \varkappa^n\omega_n^{-1}2^nM_7(m,\alpha) = \varkappa^n\omega_n^{-1}2^nC_{15}m^{16r^2+2r-1}M_3(\alpha-2r). \tag{5.37}$$

Therefore, $\lim\limits_{m\to+\infty} M_{10}(m,\alpha) = +\infty$.

In view of (5.31), (5.33), (5.36), it follows from (5.27) that

$$\operatorname{Re}\int\limits_{\Omega} L_0\left[v(z)\right]\cdot\overline{v(z)}dz \geqslant -\, M_8(m,\alpha)\left\|v; L^{2r}_{2,\alpha}(\Omega)\right\|\left\|v; L_2(\Omega)\right\| - M_9(m,\alpha,\nu)\left\|v; L_{2,\alpha-2r}(\Omega)\right\|\left\|v; L_2(\Omega)\right\| - M_{10}(m,\alpha)\left\|v; L_{2,\alpha-2r}(\Omega)\right\|^2, \tag{5.38}$$

where the numbers $M_8(m,\alpha)$, $M_9(m,\alpha,\nu)$, $M_{10}(m,\alpha)$ are determined by identities (5.31), (5.34), (5.37), respectively.

Identities (5.32), (5.35) allow us to make the coefficients $M_8(m,\alpha)$, $M_9(m,\alpha,\nu)$ arbitrarily small. This is why, choosing appropriate values of parameters m and ν, by (5.38) we obtain inequality (2.12) of Theorem 2.2. The proof of Theorem 2.2 is complete.

6. Elliptic operators with strong degeneration

In this section we provide the proofs of Theorems 2.3, 2.4. We first prove Theorem 2.3 for operators only with higher order coefficients, that is, we consider operator (2.11).

We suppose that the coefficients of this operator satisfy the assumptions of Theorem 2.3. Since in this case the assumptions of Theorems 2.1 and 2.2 are satisfied, in accordance with these theorems inequalities (2.9), (2.12) hold.

Under the assumptions of Theorem 2.3 operator (2.11) has a strong degeneration, that is, the inequality $\alpha - 2r \geqslant 0$ holds. Therefore, $\rho^{\alpha-2r}(x) \leqslant \text{const}$, $x \in \Omega$, and this is why

$$\left\|v; L_{2,\alpha-2r}(\Omega)\right\|^2 = \int\limits_{\Omega}\left(\rho^{\alpha-2r}(x)|v(x)|\right)^2 dx \leqslant \text{const}\left\|v; L_2(\Omega)\right\|^2. \tag{6.1}$$

In view of this inequality by (2.9), (2.12) we have

$$c\left\|v; L^{2r}_{2,\alpha}(\Omega)\right\|^2 \leqslant \left\|L_0[v(z)]; L_2(\Omega)\right\|^2 + K\left\|v; L_2(\Omega)\right\|^2, \tag{6.2}$$

$$\operatorname{Re}\left(L_0[v], v\right)_0 \geqslant -\varepsilon\left\|v; L^{2r}_{2,\alpha}(\Omega)\right\|\left\|v; L_2(\Omega)\right\| - \delta\left\|v; L_2(\Omega)\right\|^2 - K_1(\varepsilon,\delta)\left\|v; L_2(\Omega)\right\|^2,$$

for all $v \in C^\infty_0(\Omega)$. By the last inequality for $\delta = \varepsilon$ we obtain

$$\operatorname{Re}\left(L_0[v], v\right)_0 \geqslant -\varepsilon\left\|v; L^{2r}_{2,\alpha}(\Omega)\right\|\left\|v; L_2(\Omega)\right\| - K_1(\varepsilon)\left\|v; L_2(\Omega)\right\|^2, \tag{6.3}$$

where $K_1(\varepsilon) = K_1(\varepsilon,\varepsilon) + \varepsilon$.

In what follows we suppose that λ is a non–negative parameter. Using the inequality $2A\cdot B \leqslant A^2 + B^2$, $A \geqslant 0$, $B \geqslant 0$, for $A = \left\|v; L^{2r}_{2,\alpha}(\Omega)\right\|$, $B = \lambda\left\|v; L_2(\Omega)\right\|$ we have

$$2\lambda\left\|v; L^{2r}_{2,\alpha}(\Omega)\right\|\left\|v; L_2(\Omega)\right\| \leqslant \left\|v; L^{2r}_{2,\alpha}(\Omega)\right\|^2 + \lambda^2\left\|v; L_2(\Omega)\right\|^2.$$

By this inequality it follows from (6.3) that

$$2\lambda\operatorname{Re}\left(L_0[v], v\right)_0 \geqslant -\varepsilon\left\|v; L^{2r}_{2,\alpha}(\Omega)\right\|^2 - \left(\varepsilon\lambda^2 + 2\lambda K_1(\varepsilon)\right)\left\|v; L_2(\Omega)\right\|^2. \tag{6.4}$$

Since $\lambda \geqslant 0$, we find

$$\begin{aligned}\left\|L_0[v] + \lambda v; L_2(\Omega)\right\|^2 &= \left(L_0[v] + \lambda v, L_0[v] + \lambda v\right)_0 \\ &= \left\|L_0[v]; L_2(\Omega)\right\|^2 + \lambda^2\left\|v; L_2(\Omega)\right\|^2 + 2\lambda\operatorname{Re}\left(L_0[v], v\right)_0.\end{aligned}$$

Using inequalities (6.2), (6.4), we get

$$\begin{aligned}\left\|L_0[v] + \lambda v; L_2(\Omega)\right\|^2 &= \left\|L_0[v]; L_2(\Omega)\right\|^2 + \lambda^2\left\|v; L_2(\Omega)\right\|^2 + 2\lambda\operatorname{Re}\left(L_0[v], v\right)_0 \\ &\geqslant (c-\varepsilon)\left\|v; L^{2r}_{2,\alpha}(\Omega)\right\|^2 + \left[\lambda^2 - K - \varepsilon\lambda^2 - 2\lambda K_1(\varepsilon)\right]\left\|v; L_2(\Omega)\right\|^2.\end{aligned}$$

Fixing some value of the parameter $\varepsilon > 0$, we hence get

$$\|L_0[v] + \lambda v; L_2(\Omega)\|^2 \geqslant c_1 \left\|v; L^{2r}_{2,\alpha}(\Omega)\right\|^2 + \Lambda(\lambda)\|v; L_2(\Omega)\|^2,$$

where $c_1 = c - \varepsilon > 0, \Lambda(\lambda) = (1-\varepsilon)\lambda^2 - K - 2\lambda K_1(\varepsilon)$. Therefore, there exists a number $\lambda_0 > 0$ such that for $\lambda \geqslant \lambda_0$ the inequality holds

$$\|L_0[v] + \lambda v; L_2(\Omega)\|^2 \geqslant c_1 \left\|v; W^{2r}_{2,\alpha}(\Omega)\right\|^2, \quad v \in C_0^\infty(\Omega).$$

This is the main inequality in Theorem 2.3 for operator (2.11), and hence, the proof of Theorem 2.3 for operator (2.11) is complete.

We proceed to proving Theorem 2.3 in the general case. We consider operator (2.4), which has non–zero lower order coefficients.

In what follows we shall employ the inequality

$$2A \cdot B \leqslant \frac{1}{q}A^2 + qB^2, \quad A \geqslant 0, \quad B \geqslant 0, \quad q > 0. \tag{6.5}$$

We introduce the notation $\alpha = \alpha_{2r}$ and represent operator (2.4) as (see (4.54)) $L[u](x) = L_0[u](x) + L_1[u](x)$, where the operators L_0, L_1 are defined by identities (2.11), (4.55), respectively. Since all coefficients $b_k(x)$ of operator (2.4) are bounded and the numbers α, α_j, $j \leqslant 2r-1$, satisfy the condition $\alpha_j \geqslant \alpha - 2r + j$, by inequality (3.30) of Lemma 3.12, for operator (4.55) we find

$$\begin{aligned}\left(\int\limits_\Omega |L_1[v(x)]|^2\,dx\right)^{\frac12} &\leqslant \sum_{|k|=j\leqslant 2r-1} \left\|\rho^{\alpha_j} b_k v^{(k)}; L_2(\Omega)\right\| \leqslant M_1 \sum_{j=0}^{2r-1} \left\|v; L^j_{2,\alpha_j}(\Omega)\right\| \\ &\leqslant M_2 \sum_{j=0}^{2r-1} \left\|v; L^j_{2,\alpha-2r+j}(\Omega)\right\| \leqslant \mu\|v; L^{2r}_{2;\alpha}(\Omega)\| + K_2(\mu)\|v; L_{2;\alpha-2r}(\Omega)\|,\end{aligned}$$

where μ is a sufficiently small positive number. By (6.1) this implies

$$\|L_1[v]; L_2(\Omega)\| \leqslant \mu \left\|v; L^{2r}_{2,\alpha}(\Omega)\right\| + K_2(\mu)\|v; L_2(\Omega)\|, \quad v \in C_0^\infty(\Omega). \tag{6.6}$$

In what follows instead of the expression $\mu\cdot\mathrm{const}$, $K_2(\mu)\cdot\mathrm{const}$ we shall again write μ, $K_2(\mu)$, respectively. Using (6.5), (6.6), we find

$$\begin{aligned}2\lambda\left|\mathrm{Re}\,(L_1[v], v)_0\right| &\leqslant 2\lambda \int\limits_\Omega |L_1[v(z)]|\,|v(z)|dz \leqslant 2\lambda\|L_1[v]; L_2(\Omega)\| \cdot \|v; L_2(\Omega)\| \\ &\leqslant \frac{\lambda^2}{q}\|v; L_2(\Omega)\|^2 + q\|L_1[v]; L_2(\Omega)\|^2 \leqslant q\mu^2\left\|v; L^{2r}_{2,\alpha}(\Omega)\right\|^2 \\ &\quad + qK_2(\mu)^2\|v; L_2(\Omega)\|^2 + \frac{\lambda^2}{q}\|v; L_2(\Omega)\|^2.\end{aligned} \tag{6.7}$$

Taking into consideration the identity $L = L_0 + L_1$ and applying inequalities (6.4), (6.7), we find

$$\begin{aligned}2\lambda\,\mathrm{Re}\,(L[v], v)_0 &= 2\lambda\,\mathrm{Re}\,(L_0[v], v)_0 + 2\lambda\,\mathrm{Re}\,(L_1[v], v)_0 \\ &\geqslant -\left(\varepsilon + q\mu^2\right)\left\|v; L^{2r}_{2,\alpha}(\Omega)\right\|^2 - \left(qK_2(\mu)^2 + \frac{\lambda^2}{q} + \varepsilon\lambda^2\right)\|v; L_2(\Omega)\|^2.\end{aligned}$$

Since $\left\|v; L^{2r}_{2,\alpha}(\Omega)\right\| \leqslant \left\|v; W^{2r}_{2,\alpha}(\Omega)\right\|$, this yields

$$2\lambda\,\mathrm{Re}\,(L[v], v)_0 \geqslant -\left(\varepsilon + q\mu^2\right)\left\|v; W^{2r}_{2,\alpha}(\Omega)\right\|^2 - \left(qK_2(\mu)^2 + \frac{\lambda^2}{q} + \varepsilon\lambda^2\right)\|v; L_2(\Omega)\|^2 \tag{6.8}$$

for all $v \in C_0^\infty(\Omega)$. Here ε, μ are arbitrarily small positive numbers.

In the case of strong degeneration inequality (2.9) of Theorem 2.1 implies

$$\varkappa_1 \left\| v; W^{2r}_{2,\alpha}(\Omega) \right\|^2 \leqslant \| L[v]; L_2(\Omega) \|^2 + K_4 \| v; L_2(\Omega) \|^2 . \tag{6.9}$$

We note that $\lambda \geqslant 0$ and hence

$$\| \mathbb{L}v + \lambda v; L_2(\Omega) \|^2 = \| L[v]; L_2(\Omega) \|^2 + \lambda^2 \| v; L_2(\Omega) \|^2 + 2\lambda \operatorname{Re} (L[v], v)_0 .$$

By inequalities (6.8), (6.9) this implies

$$\begin{aligned} \| \mathbb{L}v + \lambda v; L_2(\Omega) \|^2 \geqslant (\varkappa_1 - \varepsilon - q\mu^2) \left\| v; W^{2r}_{2,\alpha}(\Omega) \right\|^2 \\ + \left(\lambda^2 - \varepsilon\lambda^2 - \frac{\lambda^2}{q} - qK(\mu)^2 - K_3^2 \right) \| v; L_2(\Omega) \|^2 . \end{aligned} \tag{6.10}$$

Without loss of generality we can suppose that the number $\varkappa_1$ in inequality (6.9) is such that $0 < 2\varkappa_1 < 1$. Then for $\varepsilon = \frac{\varkappa_1}{8}$, $\mu = \sqrt{\frac{\varkappa_1}{8}}$, $q = 3$, by inequality (6.10) we find

$$\| \mathbb{L}v + \lambda v; L_2(\Omega) \|^2 \geqslant \frac{\varkappa_1}{2} \left\| v; W^{2r}_{2,\alpha}(\Omega) \right\|^2 + \left(\frac{\lambda^2}{2} - M_* \right) \| v; L_2(\Omega) \|^2 ,$$

where $M_* = 3K(\mu)^2 + K_3^2 > 0$. Thus, we have proved, that under the above assumptions there exist positive numbers $\varkappa_0$, λ_0 such that for $\lambda > \lambda_0$ inequality (2.13) of Theorem 2.3 holds. This completes the proof of Theorem 2.3.

Now we proceed to considering adjoint operators, namely, we are going to prove Theorem 2.4. It is convenient to represent operator (2.4) as

$$L[u](x) = \sum_{j=0}^{2r} \mathcal{L}_j[u](x), \tag{6.11}$$

$$\mathcal{L}_j[u](x) = \sum_{|k|=j} \rho^{\alpha|k|}(x) b_k(x) u^{(k)}(x), \quad u \in C_0^\infty(\Omega).$$

We consider the operator $\mathcal{L}_{2r}[u]$. The adjoint operator is defined by the identity

$$\int\limits_\Omega \mathcal{L}_{2r}[u](x)\overline{v(x)}dx = \int\limits_\Omega u(x)\overline{\mathcal{L}^*_{2r}[v](x)}dx. \tag{6.12}$$

Integrating by parts, we find the expressions for the adjoint operator

$$\mathcal{L}^*_{2r}[v](x) = \sum_{|k|=2r} \left(\rho^\alpha(x)\overline{b_k(x)}v(x) \right)^{(k)} = \sum_{|k'+k''|=2r} c_{k'k''} \left(\rho^\alpha(x)\overline{b_{k'+k''}(x)} \right)^{(k')} v^{(k'')}(x) \tag{6.13}$$

for all $v \in C_0^\infty(\Omega)$. Denoting the multi–index k' by l, and k'' by k, we obtain

$$\mathcal{L}^*_{2r}[v](x) = \sum_{|k|\leqslant 2r} \widehat{a}_{2r,k}(x) v^{(k)}(x), \tag{6.14}$$

where

$$\widehat{a}_{2r,k}(x) = \sum_{|l|=2r-|k|} c_{lk} \left(\rho^\alpha(x)\overline{b_{l+k}(x)} \right)^{(l)}, \quad |k| \leqslant 2r.$$

We then have

$$\widehat{a}_{2r,k}(x) = \sum_{|l|=2r-|k|} c_{lk} \left(\rho^\alpha(x)\overline{b_{l+k}(x)} \right)^{(l)} = \sum_{|l|=2r-|k|} \sum_{l=l'+l''} c_{lk} c_{l'l''} \left(\rho^\alpha(x) \right)^{(l')} \left(\overline{b_{l+k}(x)} \right)^{(l'')} . \tag{6.15}$$

We note that the function $\rho(x)$ possesses the property

$$\left| (\rho^\alpha(x))^{(l')} \right| \leqslant C\rho^{\alpha-|l'|}(x) \tag{6.16}$$

and the coefficients of operator (6.11) satisfies condition (2.14). This is why

$$|\widehat{a}_{2r,k}(x)| \leqslant \sum_{|l|=2r-|k|} \sum_{l=l'+l''} c_{lk}c_{l'l''}CM_{l''}\rho^{\alpha-|l'|}(x)\rho^{-|l''|}(x)$$
$$= \sum_{|l|=2r-|k|} \sum_{l=l'+l''} c_{lk}c_{l'l''}CM_{l''}\rho^{\alpha-|l'|-|l''|}(x)$$
$$\leqslant \sum_{|l|=2r-|k|} C_l\rho^{\alpha-|l|}(x) \leqslant M_k\rho^{\alpha-2r+|k|}(x).$$

This inequality allows us to represent operator (6.14) as

$$\mathcal{L}^*_{2r}[v](x) = \sum_{|k|\leqslant 2r} \rho^{\alpha-2r+|k|}(x)a_{2r,k}(x)v^{(k)}(x), \tag{6.17}$$

where all coefficients $a_{2r,k}(x)$ are bounded and are defined by the identity

$$a_{2r,k}(x) = \frac{\widehat{a}_{2r,k}(x)}{\rho^{\alpha-2r+|k|}(x)}. \tag{6.18}$$

For $|k| = 2r$ it follows from (6.15) that $\widehat{a}_{2r,k}(x) = \rho^{\alpha}(x)\overline{b_k(x)}$. By (6.18) this implies $a_{2r,k}(x) = \overline{b_k(x)}$, $|k| = 2r$. By this identity assumptions I) – III) of Theorem 2.1 (see (2.5)–(2.7)) are satisfied for the coefficients $a_{2r,k}(x)$, $|k| = 2r$, of operator $\mathcal{L}^*_{2r}$.

Similarly to the case $\mathcal{L}^*_{2r}$ one can study the operators $\mathcal{L}_j[u]$, $j \leqslant 2r-1$, and prove that these operators admit the representation

$$\mathcal{L}^*_j[v](x) = \sum_{|k|\leqslant j} \rho^{\alpha_j-j+|k|}(x)a_{j,k}(x)v^{(k)}(x), \tag{6.19}$$

where all coefficients $a_{j,k}(x)$ are bounded.

Now using (6.11), (6.17), (6.19), we represent the adjoint operator $L^*[v]$ as

$$L^*[v](x) = \sum_{j=0}^{2r} \mathcal{L}^*_j[v](x) = \sum_{j=0}^{2r}\sum_{|k|\leqslant j} \rho^{\alpha_j-j+|k|}(x)a_{j,k}(x)v^{(k)}(x), \tag{6.20}$$

where all coefficients $a_{j,k}(x)$ are bounded and higher order coefficients $a_{2r,k}(x)$, $|k| = 2r$, satisfy assumptions I) – II) of Theorem 2.1 and the weak positivity condition in Theorem 2.2. This is can apply Theorem 2.3 to operator (6.20) if $\alpha_j - j + |k| \geqslant \alpha_{2r} - 2r + |k|$ for all $0 \leqslant j \leqslant 2r-1$ and for all $|k| \leqslant j$. We note that this condition is equivalent to the condition $\alpha_j \geqslant \alpha_{2r} + j - 2r$, which holds under the assumptions of Theorem 2.1, see condition (2.8).

Thus, we have show that operator (6.20) satisfies all assumption of Theorem 2.3. Applying this theorem, we obtain inequality (2.15) for adjoint operator. The proof of Theorem 2.4 is complete.

Concerning the proof of Corollary 2.1, we note that by means of inequality (2.13) (see Theorem 2.3) we can prove that under the assumptions of Theorem 2.4 the domain of operator $\mathbb{L} + \lambda I$ is closed, while inequality (2.15) (see Theorem 2.4) implies that the kernel of adjoint operator $\mathbb{L}^* + \lambda I$ is empty, that is, $\overline{R(\mathbb{L} + \lambda I)} = R(\mathbb{L} + \lambda I)$ and $N(\mathbb{L}^* + \lambda \mathbb{I}) = \emptyset$. This is why it follows from identity $L_2(\Omega) = \overline{R(\mathbb{L} + \lambda I)} \oplus N(\mathbb{L}^* + \lambda I)$ that $R(\mathbb{L} + \lambda I) = L_2(\Omega)$.

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