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STABILITY DEGREE OF MAXIMAL TERM OF DIRICHLET SERIES

N.N. AITKUZHINA, R.A. GAISIN

Abstract. We study the stability of a maximal term of a Dirichlet series with positive exponents, the sum of which is an entire function. For a class of entire Dirichlet series defined by a certain convex growth majorant, we prove a theorem on the quantitative estimate of the equivalence degree (outside of some exceptional c_q-set) of the logarithms of maximal terms in the original series and the modified Dirichlet series. A similar problem for entire Dirichlet series of an arbitrary rapid growth, but with no quantitative estimate of the stability degree for the maximal term, was first studied by A.M. Gaisin in the late 1990s and early 2000s. He then obtained a stability criterion, which was the equivalence of the logarithms of the maximal terms of the original and modified series on the asymptotic set. This result, as well as the corresponding stability statements for Dirichlet series converging only in a certain half–plane obtained by A.M. Gaisin and T.I. Belous, found useful applications in the theory of asymptotic properties of Dirichlet series, specifically in proving Pólya type identities. The formulation of the stability problem considered in this paper is relevant for its applications to the minimum modulus problem, as well as to other related problems in analysis and complex dynamics.

Keywords: Dirichlet series, convex growth majorant, maximal term, stability degree.

Mathematics Subject Classification: 30D10

1. Introduction

The study of the stability of maximal term of a Dirichlet series with positive exponents, the sum of which is an entire function, is relevant in connection with theorems on the asymptotics of entire Dirichlet series on various continua extending to infinity (e.g., on curves, strips), in which a key role is played by Leontiev formulas for the coefficients calculated in terms of a biorthogonal system of functions (see [4]). The functions of this system contain a factor depending on the reciprocal of the derivative of the characteristic function, which is an even Weierstrass product, the zero set of which coincides with the sequence of exponents of the Dirichlet series (see [4]).

The stability of maximal term of Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \quad s = \sigma + it, \tag{1.1}$$

with positive exponents, which converges absolutely in the entire plane, was first studied in [1]. This concept proved to be very useful in studying the asymptotic behavior of the sum of a Dirichlet series on curves extending to infinity, namely, in proving the well–known Pólya conjecture. Similar studies were later made for Dirichlet series of a given growth, in particular, of finite Ritt order (see [2], [3], [6]). A key role in such problems is played by lemmas of the

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Borel — Nevanlinna type. Note that in such problems λ_n are necessarily zeros of some entire function of exponential type. However, the study of the stability of maximal term is also of an independent interest. Under this approach, the exponents of series admit an optimal choice [5].

In [2], [6], Dirichlet series in the classes $D(\Phi)$ and $\underline{D}(\Phi)$, defined by some convex majorant Φ, were studied. Criteria for the stability of maximal term of Dirichlet series (1.1) were obtained in terms of the function φ inverse to Φ. The essence of the main results in [2], [6] are theorems on relations of type

$$\ln \mu(\sigma) = (1 + o(1)) \ln \mu^*(\sigma), \tag{1.2}$$

which hold as $\sigma \to \infty$ outside some exceptional sets $E \subset \mathbb{R}_+ = [0, \infty)$ of zero lower density. Here $\mu(\sigma)$ is the maximal term of series (1.1), and $\mu^*(\sigma)$ is the maximal term of the Dirichlet series with the same exponents but with the modified coefficients of form $a_n b_n$, $n = 1, 2, \ldots$. However, in these papers the behavior of the infinitesimal quantity $o(1)$ in identity (1.2) was not discussed at all; this identity means the stability of $\mu(\sigma)$. The aim of this paper is to obtain an estimate of form

$$\left|1 - \frac{\ln \mu^*(\sigma)}{\ln \mu(\sigma)}\right| < \frac{\text{const}}{\sigma^{\gamma+\mu}}, \qquad 0 < \mu < \beta,$$

in the class $D^p(\Phi)$, which a certain analogue of $D(\Phi)$ depending on the parameter $p > 0$. The estimate should hold as $\sigma \to \infty$ outside some c_q–set $E^p_{\alpha\beta\gamma} \subset \mathbb{R}_+$,

$$\overline{\text{dens}}\, E^p_{\alpha\beta\gamma} \leqslant q < 1,$$

where α, β, γ are given parameters, $q = q(\alpha) = O(\alpha)$ as $\alpha \to 0$, $\beta \in (0, 1)$, $\gamma \in (0, 1)$, $\beta + \gamma < 1$, $0 < \alpha \leqslant \alpha_0 < 1$.

2. Necessary information and main result

Let $\Lambda = \{\lambda_n\}$, $0 < \lambda_n \uparrow \infty$, be a sequence, which obeys the condition

$$\lim_{n\to\infty} \frac{\ln n}{\lambda_n} = 0. \tag{2.1}$$

We denote by $D(\Lambda)$ the class of all functions F that can be represented by Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \qquad s = \sigma + it, \tag{2.2}$$

in the entire plane.

It follows from condition (2.1) that if series (2.2) converges in the entire plane, then it converges absolutely, and its sum F is an entire function [4]. By L we denote the class of all continuous and infinitely increasing positive functions on $\mathbb{R}_+$. Let Φ be a convex function in L,

$$D_m(\Phi) = \{F \in D(\Lambda) : \ln M_F(\sigma) \leqslant \Phi(m\sigma)\}, \qquad m \geqslant 1,$$

where $M_F(\sigma) = \sup\limits_{|t|<\infty} |F(\sigma + it)|$. We let

$$D(\Phi) = \bigcup_{m=1}^{\infty} D_m(\Phi).$$

Together with series (2.2) we introduce the series

$$F_b^*(s) = \sum_{n=1}^{\infty} a_n b_n e^{\lambda_n s}, \tag{2.3}$$

where the sequence $b=\{b_n\}$ of complex numbers b_n ($b_n\neq 0$ for $n\geqslant N$) satisfies the condition

$$\varlimsup_{n\to\infty}\frac{|\ln|b_n||}{\lambda_n}<\infty. \tag{2.4}$$

Let $E\subset[0,\infty)$ be a Lebesgue measurable set. The upper DE and lower dE densities of the set E are the quantities

$$DE=\overline{\text{dens}}\,E=\varlimsup_{\sigma\to\infty}\frac{\text{mes}(E\cap[0,\sigma])}{\sigma},\qquad dE=\underline{\text{dens}}\,E=\varliminf_{\sigma\to\infty}\frac{\text{mes}(E\cap[0,\sigma])}{\sigma}.$$

In what follows we suppose that all exceptional sets $E\subset[0,\infty)$, outside of which asymptotic estimates will be obtained, are unions of segments of the form $[a_n,a'_n]$, where (see [1])

$$0<a_1<a'_1\leqslant a_2<a'_2\leqslant\ldots\leqslant a_n<a'_n\leqslant\ldots.$$

Let $0\leqslant q<1$. The set $E\subset\mathbb{R}_+$ is called the C_q–set if $DE\leqslant q$, and the c_q–set if $dE\leqslant q$.
Let

$$d_w=\inf_{x\geqslant e}\frac{\ln w(x)}{\ln x},$$

and φ be the inverse function to Φ such that

$$\varlimsup_{x\to\infty}\frac{\varphi(x^2)}{\varphi(x)}<\infty. \tag{2.5}$$

It follows from (2.5) that for some $c>1$

$$\varphi(x)\leqslant c\varphi(x^{\frac{1}{2}})\leqslant\ldots\leqslant c^n\varphi(x^{\frac{1}{2^n}}).$$

We let $n=\left[\dfrac{\ln\ln x}{\ln 2}\right]$, where $[a]$ is the integer part of a. Then $\ln\varphi(x)=O(\ln\ln x)$, $x\to\infty$.

We introduce the classes of functions

$$\underline{W}(\varphi)=\left\{w\in L: d_w>0,\quad \lim_{x\to\infty}\frac{w(x)}{x\varphi(x)}=0,\quad \varliminf_{x\to\infty}\frac{1}{\varphi(x)}\int\limits_1^x\frac{w(t)}{t^2}\,dt=0\right\},$$

$$W(\varphi)=\left\{w\in L: d_w>0,\quad \lim_{x\to\infty}\frac{1}{\varphi(x)}\int\limits_1^x\frac{w(t)}{t^2}\,dt=0\right\}.$$

We say that a sequence $\{b_n\}$ ($b_n\neq 0$ as $n\geqslant N$) is $W(\varphi)$–normal if there exists a function $\theta\in L$ such that

$$\lim_{x\to\infty}\frac{1}{\varphi(x)}\int\limits_1^x\frac{\theta(t)}{t^2}\,dt=0, \tag{2.6}$$

and

$$\ln\frac{1}{|b_n|}\leqslant\theta(\lambda_n),\qquad n\geqslant N.$$

Let $n(t)=\sum\limits_{\lambda_n\leqslant t}1$ be the counting function of sequence Λ, and $n_l(t)$ be the smallest concave majorant of $\ln n(t)$. By condition (2.1) it is well defined and $n_l(t)=o(t)$ for $t\to\infty$.

By $\mu(\sigma)$ and $\mu^*_b(\sigma)$ we denote the maximal terms of series (2.2) and (2.3), respectively, i.e.

$$\mu(\sigma)=\max_{n\geqslant 1}\left\{|a_n|\,e^{\lambda_n\sigma}\right\},\qquad \mu^*_b(\sigma)=\max_{n\geqslant 1}\left\{|a_n|\,|b_n|\,e^{\lambda_n\sigma}\right\}.$$

In [2] the following theorem was proved.

Theorem 2.1. *Let $\{b_n\}$ be a sequence of complex numbers ($b_n \neq 0, n \geqslant N$) satisfying condition (2.4), and let Φ be a convex function in the class L. Assume that the function φ, inverse to Φ, satisfies condition (2.5), and that the majorant $n_l(t)$ satisfies integral condition (2.6) of $W(\varphi)$–normality* [1].

If there exists a function $w \in \underline{W}(\varphi)$ such that

$$|\ln |b_n|| \leqslant w(\lambda_n), \qquad n \geqslant N, \tag{2.7}$$

then each function $F \in D(\Phi)$ satisfies the asymptotic identity

$$\ln \mu(\sigma) = (1 + o(1)) \ln \mu_b^*(\sigma), \tag{2.8}$$

as $\sigma \to \infty$ outside some set $E \subset [0, \infty)$ of zero lower density. For $W(\varphi)$–normal sequence $\{b_n\}$ condition (2.7) is necessary for (2.8).

We note that a similar result for Dirichlet series of arbitrary growth was proved in [1]. Let p be some positive number,

$$\underline{W}^p(\varphi) = \left\{ w \in L : d_w > 0, \quad d_p(w) = \overline{\lim_{x\to\infty}} \frac{w(x)}{x\varphi^p(x)} < \infty, \ \underline{\lim}_{x\to\infty} \frac{1}{\varphi^p(x)} \int\limits_1^x \frac{w(t)}{t^2}\, dt < \infty \right\}.$$

In the same way we define the class $W^p(\varphi)$ by replacing $\underline{\lim}$ by $\overline{\lim}$ and the notion of $W^p(\varphi)$–normal sequence. It is obvious that if $w \in W^p(\varphi)$, then $d_p(w) < \infty$.

Our main result is as follows; an exact formulation is provided in Theorem 3.2. Let Φ be an increasing convex on $\mathbb{R}_+$ function, $p > 0$,

$$D^p(\Phi) = \bigcup_{m=1}^{\infty} \left\{ F \in D(\Lambda) : \ \ln M_F(\sigma) \leqslant \Phi(m\sigma^{\frac{1}{p}}) \right\}.$$

We establish for given β, $\gamma \in (0, 1)$, $\nu = 1 - \beta - \gamma > 0$ such that for $n \geqslant N$

$$|\ln |b_n|| \leqslant w(\lambda_n), \quad w \in \underline{W}^{p\nu}(\varphi)$$

(φ is the inverse function for Φ, $\varphi(x^2) = O(\varphi(x))$, $x \to \infty$) for each function $F \in D^p(\Phi)$ as $\sigma \to \infty$ outside some c_q–set the estimate

$$\left| 1 - \frac{\ln \mu_b^*(\sigma)}{\ln \mu(\sigma)} \right| \leqslant \frac{2}{\sigma^{\gamma+\mu}}$$

holds, where μ is arbitrary number $(0, \beta)$, which can be arbitrarily close to β.

3. Proof of main results

The proof of Theorem 3.2 is based on the following Borel — Nevanlinna type theorem.

Theorem 3.1. *Let $\Phi \in L$ and the function φ inverse to Φ satisfy condition (2.5). Let $u(\sigma)$ be a non–decreasing positive continuous on $[r_0, \infty)$ function and*

$$\lim_{\sigma\to\infty} u(\sigma) = \infty, \qquad \overline{\lim_{\sigma\to\infty}} \frac{u(\sigma)}{\ln \Phi(\sigma^{\frac{1}{\sigma}})} < \infty, \quad p > 0.$$

Let $\beta \in (0, 1)$, $\gamma \in (0, 1)$, $\nu = 1 - \beta - \gamma$, $\nu \in (0, 1)$. Suppose that $w \in \underline{W}^{p\nu}(\varphi)$ and $\{x_n\}$ be a sequence chosen so that

$$T_{p\nu}^- = \lim_{x_n\to\infty} \frac{1}{\varphi^{p\nu}(x_n)} \int\limits_1^{x_n} \frac{w(t)}{t^2}\, dt < \infty. \tag{3.1}$$

[1] Without loss of generality, we can assume that $d_{n_l} > 0$, i.e. $n_l(t)$ belongs to class $W(\varphi)$ (see [2]).

If τ_n is such that $v(\tau_n) = x_n$, $n \geqslant 1$, where $v = v(\sigma)$ is a solution to equation

$$w(v) = e^{u(\sigma)}, \tag{3.2}$$

then as $\sigma \to \infty$ outside some c_q-set $E^p_{\alpha\beta\gamma} \subset [0, \infty)$,

$$\overline{\lim_{\tau_n \to \infty}} \frac{\operatorname{mes}(E \cap [0, \tau_n])}{\tau_n} \leqslant q < 1,$$

the estimate

$$u\left(\sigma + \sigma^\gamma \frac{w(v(\sigma))}{v(\sigma)}\right) < u(\sigma) + \frac{1}{\alpha\sigma^\beta}$$

holds, where $0 < \alpha \leqslant \alpha_0 < 1$.

Proof. Let $w = w(x)$ be a function in the class $\underline{W}^p(\varphi)$, $\alpha \in (0, 1)$, while the parameters β and γ are chosen in the formulation of theorem and fixed.

We are going to show that outside some set $E \subset [0, \infty)$, $E = E^p_{\alpha\beta\gamma}$ of lower density $dE^p_{\alpha\beta\gamma} \leqslant q < 1$ the estimate

$$u\left(\sigma + \sigma^\gamma \frac{w(v(\sigma))}{v(\sigma)}\right) < u(\sigma) + \frac{1}{\alpha\sigma^\beta}$$

holds. Indeed, let $E^p_{\alpha\beta\gamma} \subset [0, \infty)$ be a set, on which

$$u\left(\sigma + \sigma^\gamma \frac{w(v(\sigma))}{v(\sigma)}\right) \geqslant u(\sigma) + \frac{1}{\alpha\sigma^\beta}, \tag{3.3}$$

where $\beta \in (0, 1)$, $\gamma \in (0, 1)$, $\nu = 1 - \beta - \gamma$, $\nu \in (0, 1)$.

Let $E(\sigma) = E^p_{\alpha\beta\gamma} \cap [\sigma, \infty)$. If $E(\sigma) = \emptyset$ for some σ, then the proof is complete. Otherwise, by σ_1 we denote the smallest number such that $\sigma_1 \geqslant 0$, $\sigma_1 \in E^p_{\alpha\beta\gamma}$, and σ_1' is the smallest σ, for which

$$u(\sigma) = u(\sigma_1) + \frac{1}{\alpha\sigma_1^\beta}.$$

Then by (3.3) we have

$$0 < \sigma_1' - \sigma_1 \leqslant \sigma_1^\gamma \frac{w(v(\sigma_1))}{v(\sigma_1)}.$$

Let $\sigma_2 = \inf\{\sigma : \sigma \in E(\sigma_1')\}$, and σ_2' be the smallest of σ, for which

$$u(\sigma) = u(\sigma_2) + \frac{1}{\alpha\sigma_2^\beta}.$$

It is clear that

$$0 < \sigma_2' - \sigma_2 \leqslant \sigma_2^\gamma \frac{w(v(\sigma_2))}{v(\sigma_2)}, \qquad u(\sigma_2) - u(\sigma_1) \geqslant \frac{1}{\alpha\sigma_1^\beta}.$$

Proceeding in the same way, we find sequences $\{\sigma_n\}$, $\{\sigma_n'\}$, such that

$$0 < \sigma_n' - \sigma_n \leqslant \sigma_n^\gamma \frac{w(v(\sigma_n))}{v(\sigma_n)}, \qquad u(\sigma_n) - u(\sigma_{n-1}) \geqslant \frac{1}{\alpha\sigma_{n-1}^\beta}. \tag{3.4}$$

This construction shows that

$$E^p_{\alpha\beta\gamma} \subset \bigcup_{n=1}^{\infty} [\sigma_n, \sigma_n'] .$$

We denote $v_n = v(\sigma_n)$, $\delta_n = \dfrac{w(v_n)}{v_n}$, $n \geqslant 1$. Let $\{x_n\}$, $0 < x_n \uparrow \infty$, be the sequence in condition (3.1).

Let $\{\tau_j\}$ be a sequence, where τ_j is a solution to the equation $v(\tau) = x_j$, $j \geqslant 1$. It is clear that $0 < \tau_j \uparrow \infty$, and for each $j \geqslant 0$ there exists $k \geqslant 1$ such that $\sigma_{k-1} \leqslant \tau_j < \sigma_k$. Therefore, we have

$$\operatorname{mes}(E^p_{\alpha\beta\gamma} \cap [0, \tau_j]) \leqslant \sum_{n=1}^{k-1} \sigma_n^\gamma \delta_n \leqslant \sigma_{k-1}^\gamma \delta_{k-1} + \tau_j^\gamma \sum_{n=1}^{k-2} \delta_n. \tag{3.5}$$

If $2v_n \leqslant v_{n+1}$, then

$$2\delta_n \leqslant w(v_n) \int_{v_n}^{v_{n+1}} \frac{dt}{t^2} \leqslant 2 \int_{v_n}^{v_{n+1}} \frac{w(t)}{t^2}\, dt. \tag{3.6}$$

If $2v_n > v_{n+1}$, in view of equation (3.2), (3.4) and the monotonicity of function $w = w(t)$, we have

$$\begin{aligned} \delta_n &\leqslant \alpha\sigma_n^\gamma \frac{w(v_n)}{v_n} (u(\sigma_{n+1}) - u(\sigma_n)) \leqslant 2\alpha\sigma_n^\gamma \int_{v_n}^{v_{n+1}} \frac{w(t)}{t}\, d\ln w(t) \\ &= 2\alpha\sigma_n^\gamma \left(\frac{w(v_{n+1})}{v_{n+1}} - \frac{w(v_n)}{v_n} + \int_{v_n}^{v_{n+1}} \frac{w(t)}{t^2}\, dt \right). \end{aligned} \tag{3.7}$$

Since it is obvious that

$$\int_{v_n}^{v_{n+1}} \frac{dw(t)}{t} \geqslant 0,$$

we find

$$\int_{v_n}^{v_{n+1}} \frac{w(t)\, dt}{t^2} \leqslant \frac{w(v_{n+1})}{v_{n+1}} - \frac{w(v_n)}{v_n} + 2 \int_{v_n}^{v_{n+1}} \frac{w(t)}{t^2}\, dt.$$

By (3.6), (3.7) we hence conclude that

$$\delta_n \leqslant 2 \max(1, \alpha\sigma_n^\beta) \left(\frac{w(v_{n+1})}{v_{n+1}} - \frac{w(v_n)}{v_n} + 2 \int_{v_n}^{v_{n+1}} \frac{w(t)}{t^2}\, dt \right). \tag{3.8}$$

Thus, if $\sigma_{k-1} \leqslant \tau_j < \sigma_k$, then by (3.5), (3.8) we have

$$\frac{\operatorname{mes}\left(E^p_{\alpha\beta\gamma} \cap [0, \tau_j]\right)}{\tau_j} \leqslant \frac{w(v_{k-1})}{\tau_j^{1-\gamma} v_{k-1}} + \frac{2\alpha}{\tau_j^\nu} \left(\frac{w(v_{k-1}}{v_{k-1}} + 2 \int_{v_1}^{v_{k-1}} \frac{w(t)}{t^2}\, dt \right), \quad \nu = 1 - \gamma - \beta. \tag{3.9}$$

By the assumptions of the theorem, there exists $c > 0$ such that $u(\sigma) \leqslant c \ln \Phi(\sigma^{\frac{1}{p}})$. Then, in view of the inequality $d_w > 0$ and equation (3.2), we obtain that for some $m \in \mathbb{N}$

$$(v(\sigma))^{\frac{1}{m}} \leqslant w(v(\sigma)) \leqslant \Phi^c(\sigma^{\frac{1}{p}}),$$

i.e.

$$v(\sigma) \leqslant \Phi^{mc}(\sigma^{\frac{1}{p}}).$$

Taking into consideration property (2.5) of the function φ, we hence obtain that

$$\frac{1}{\sigma} \leqslant c_1^{-1}(p) \frac{1}{\varphi^p(v)}, \qquad v = v(\sigma), \qquad \sigma \geqslant x_1. \tag{3.10}$$

Therefore, in view of (3.10), by (3.9) with $j \geqslant j_0$, $\sigma_{k-1} \leqslant \tau_j < \sigma_k$, we have

$$\begin{aligned} \varlimsup_{\tau_j\to\infty} \frac{\operatorname{mes}\left(E^p_{\alpha\beta\gamma}\cap[0,\tau_j]\right)}{\tau_j} \leqslant & c_1^{-(1-\gamma)} \varlimsup_{k\to\infty} \frac{w(v_{k-1})}{\varphi^{(\nu+\beta)p}(v_{k-1})v_{k-1}} \\ & + 2\alpha c_1^{-\nu} \varlimsup_{k\to\infty} \left(\frac{w(v_{k-1})}{v_{k-1}\varphi^{p\nu}(v_{k-1})} + \frac{2}{\varphi^{p\nu}(v_j)} \int_{v_1}^{v_j} \frac{w(t)}{t^2}\,dt \right) \\ \leqslant & c_1^{-(1-\gamma)} \varlimsup_{k\to\infty} \frac{w(v_{k-1})}{\varphi^{(\nu+\beta)p}(v_{k-1})v_{k-1}} + 2\alpha c_1^{-\nu}(d_{p\nu}(w) + 2T^-_{p\nu}), \end{aligned} \tag{3.11}$$

where $v_j = v(\tau_j)$, $p > 0$, $\nu = 1-\gamma-\beta$, $\nu \in (0,1)$, $0 < \alpha < 1$. Since $w \in \underline{W}^{p\nu}(\varphi)$, we have $d_{p\nu}(w) < \infty$, $T^-_{p\nu}(w) < \infty$, and the first term on the right hand side of the second inequality in (3.11) is zero. Since $x_j = v(\tau_j) = v_j$, by (3.11) we finally obtain

$$\varlimsup_{\tau_j\to\infty} \frac{\operatorname{mes}\left(E^p_{\alpha\beta\gamma}\cap[0,\tau_j]\right)}{\tau_j} \leqslant 2\alpha c_1^{-\nu}(d_p(w) + 2T^-_{p\nu}) \leqslant q < 1$$

for $0 < \alpha \leqslant \alpha_0$. The proof is complete. □

Let Φ be a convex function in L, $p > 0$,

$$D^p_m(\Phi) = \left\{ F \in D(\Lambda) :\ \ln M_F(\sigma) \leqslant \Phi(m\sigma^{\frac{1}{p}}) \right\}, \quad m \geqslant 1,$$

$$D^p(\Phi) = \bigcup_{m=1}^{\infty} D^p_m(\Phi).$$

If estimate $\ln M_F(\sigma) \leqslant \Phi(m\sigma^{\frac{1}{p}})$ holds only for some sequence $\{x_n\}$, $0 < x_n \uparrow \infty$, then instead of $D^p_m(\Phi)$ we arrive at the definition of the class $\underline{D}^p_m(\Phi)$. Then

$$\underline{D}^p(\Phi) = \bigcup_{m=1}^{\infty} \underline{D}^p_m(\Phi).$$

We are in position to formulate our main theorem.

Theorem 3.2. *Let $\{b_n\}$ be a sequence of complex numbers ($b_n \neq 0$, $n \geqslant \mathbb{N}$) satisfying condition (2.4), Φ be a convex function in the class L, and let parameters α, β, γ be such that $\alpha \in (0,1)$, $\beta \in (0,1)$, $\gamma \in (0,1)$, $\nu \in (0,1)$, where $\nu = 1-\beta-\gamma$. Suppose that condition (2.5) holds for the function φ, which is inverse to Φ, and the function $n_l(t)$ belongs to the class $W^{p\nu}(\varphi)$.*

If there exists a function $w \in \underline{W}^{p\nu}(\varphi)$ such that

$$|\ln|b_n|| \leqslant w(\lambda_n), \quad n \geqslant N, \tag{3.12}$$

then for each function $F \in D^p(\Phi)$ with $\sigma \to \infty$ outside some exceptional c_q-set $E^p_{\alpha\beta\gamma} \subset \mathbb{R}_+$, $q = q(\alpha) = O(\alpha)$ with $\alpha \to 0$ the estimate

$$\left| 1 - \frac{\ln \mu^*_b(\sigma)}{\ln \mu(\sigma)} \right| \leqslant \frac{2}{\sigma^{\gamma+\mu}}$$

holds. Here μ is arbitrary number $(0,\beta)$, which can be arbitrarily close to β.

In what follows, without loss of generality, we can assume that $n_l(t) \leqslant w(t)$, $t > 0$. Otherwise, we consider the function $w(t)+n_l(t)$, which obviously belongs to $\underline{W}^{p\nu}(\varphi)$ since $n_l(t) \in W^{p\nu}(\varphi)$, $w(t) \in \underline{W}^{p\nu}(\varphi)$.

Proof. Sufficiency. Let condition (2.7) hold, where $w \in \underline{W}^{p\nu}(\varphi)$, and $v = v(\sigma)$ is a solution of the equation

$$w(v) = \sigma^{-\gamma-\mu} \ln \mu(\sigma), \qquad 0 < \gamma < 1, \qquad \mu \in (0, \beta), \qquad \gamma + \beta < 1.$$

The function $\ln \mu(\sigma)$ is convex, therefore the function $\sigma^{-\gamma-\mu} \ln \mu(\sigma)$ increases unboundedly as $\sigma \uparrow \infty$. We let

$$R_v = \sum_{\lambda_j > v} |a_j| \, e^{\lambda_j \sigma}, \qquad h = \frac{3\sigma^\gamma w(v)}{v}, \qquad v = v(\sigma).$$

According to the assumptions of theorem,

$$\ln n = \ln n(\lambda_n) \leqslant n_l(\lambda_n), \qquad n \geqslant 0.$$

Since the function $n_l(t)$ is concave, we also have the inequality

$$n_l(\lambda_n) \leqslant \frac{w(v)}{v} \lambda_n, \qquad \lambda_n \geqslant v.$$

Therefore,

$$R_v \leqslant \mu(\sigma + h) \sum_{\lambda_n > v} e^{-\lambda_n h} \leqslant \mu(\sigma + h) c_0 \exp(\max_{t \geqslant v} \psi(t)), \qquad h = 3\sigma^\gamma \frac{w(v)}{v}, \qquad v = v(\sigma),$$

where

$$\psi(t) = 2n_l(t) - ht, \qquad c_0 = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Hence, in view of the above estimates, as $\sigma \to \infty$,

$$\max_{t \geqslant v}(\psi(t)) \leqslant 2\frac{w(v)}{v} t - h(t) \leqslant -3\sigma^\gamma w(v)(1 + o(1)) < -2\sigma^\gamma w(v), \qquad v = v(\sigma).$$

Thus,

$$R_v \leqslant c_0 \mu(\sigma + h) \exp[-2\sigma^\gamma w(v)], \qquad v = v(\sigma), \qquad \sigma \geqslant \sigma_0. \tag{3.13}$$

We let

$$u(\sigma) = \ln \frac{\ln \mu(\sigma)}{\sigma^{\gamma+\mu}}, \qquad \mu \in (0, \beta), \qquad \gamma + \beta < 1.$$

Since $\mu(\sigma) \leqslant M_F(\sigma)$ and $F \in D^p(\Phi)$, there exists $k > 1$ such that the estimate

$$u(\sigma) \leqslant \ln \Phi(k\sigma^{\frac{1}{p}}), \qquad \sigma \geqslant \sigma_1, \tag{3.14}$$

holds. It is clear $v = v(\sigma)$ is a solution of the equation

$$w(v) = e^{u(\sigma)}.$$

Then in view of (3.14) we have

$$w(v(\sigma)) = e^{u(\sigma)} \leqslant \Phi(k\sigma^{\frac{1}{p}}), \qquad k > 1.$$

This yields

$$\varphi(w(v(\sigma))) \leqslant k\sigma^{\frac{1}{p}}.$$

Hence,

$$\frac{1}{\sigma} \leqslant \frac{c_1}{[\varphi(w(v))]^p}, \qquad \sigma \geqslant \sigma_1, \qquad c_1 = k^p, \qquad v = v(\sigma). \tag{3.15}$$

In view of condition (2.5) and the inequality $\sqrt{x} \leqslant w(x)$ for $x \geqslant x_0$, we find

$$\varphi(x) \leqslant c_2 \varphi(w(x)), \qquad x \geqslant x_0, \qquad 0 < c_2 < \infty. \tag{3.16}$$

Finally, by (3.15), (3.16) we obtain the estimate

$$\frac{1}{\sigma} \leqslant \frac{c_3}{(\varphi(v))^p}, \qquad v = v(\sigma), \qquad \sigma \geqslant \sigma_2, \qquad 0 < c_3 < \infty. \tag{3.17}$$

Since $w \in \underline{W}^{p\nu}(\varphi)$, we obtain

$$d_{p\nu}(w) = \varlimsup_{x\to\infty} \frac{w(x)}{x\varphi^{p\nu}(x)} < \infty, \tag{3.18}$$

and for the sequence $\{x_n\}$ we also have

$$T^{-}_{p\nu}(w) = \lim_{x_n\to\infty} \frac{1}{\varphi^{p\nu}(x_n)} \int\limits_1^{x_n} \frac{w(t)}{t^2}\, dt < \infty. \tag{3.19}$$

It is obvious that under the replacement of the condition $u(\sigma) \leqslant c\Phi(\sigma^{\frac{1}{p}})$ by $u(\sigma) \leqslant \Phi(k\sigma^{\frac{1}{p}})$, $k > 1$, the statement of Theorem 3.1 remains true provided all other assumptions remain same. This is why, applying Theorem 3.1 for the functions u, w and taking into consideration (3.12)–(3.19) as well as the fact that τ_j is a root of the equation $v(\tau) = x_j$, $j \geqslant 1$, outside some set $E_1 \subset [0,\infty)$, $E_1 = E_1^p(\alpha,\beta,\gamma)$, $0 < \alpha < \alpha_1 < 1$,

$$\varlimsup_{\tau_j\to\infty} \frac{\operatorname{mes}(E_1 \cap [0,\tau_j])}{\tau_j} \leqslant q_1 < \frac{1}{2}, \tag{3.20}$$

by (3.13) we obtain

$$u(\sigma+h) < u(\sigma) + \frac{1}{\alpha\sigma^\beta},$$

that is,

$$\ln\ln\mu(\sigma+h) - (\gamma+\mu)\ln(\sigma+h) < \ln\ln\mu(\sigma) - (\gamma+\mu)\ln\sigma + \frac{1}{\alpha\sigma^\beta}.$$

This implies that as $\sigma\to\infty$, outside E_1 we have

$$\begin{aligned}\ln\ln\mu(\sigma+h) &< \ln\ln\mu(\sigma) + \frac{1}{\alpha\sigma^\beta} + (\gamma+\mu)\ln\left(1+\frac{h}{\sigma}\right)\\ &\leqslant \ln\ln\mu(\sigma) + \frac{1}{\alpha\sigma^\beta}\left(1+3(\gamma+\beta)\alpha d_{p\nu}(w)\right).\end{aligned}$$

Since $0<\mu<\beta$, taking into consideration the elementary inequality $e^\varepsilon < 1+2\varepsilon$, $\varepsilon\in(0,\ln 2)$, the identity $w(v) = \sigma^{-\gamma-\mu}\ln\mu(\sigma)$, by (3.13) we obtain

$$R_v \leqslant c_0\mu^{1+\frac{a}{\alpha\sigma^\beta}}\exp\left(-2\sigma^\gamma w(v)\right) = c_0\mu(\sigma)^{1-2(1+o(1))\sigma^{-\mu}} = o(1)\mu(\sigma), \quad v = v(\sigma), \quad \sigma\geqslant\sigma_3,$$

where $a = 1+3(\gamma+\beta)\alpha d_{p\nu}(w)$, $0<\mu<\beta$. Thus, for $\sigma\geqslant\sigma_4$ and $\sigma\notin E_1$ we obtain $\lambda_{\nu(\sigma)} \leqslant v(\sigma)$, where $\nu=\nu(\sigma)$ is the central index of series (2.2). In view of (2.7), for $\sigma\to\infty$ outside E_1 we then have

$$\begin{aligned}&\mu(\sigma) = |a_\nu|\, e^{\lambda_\nu\sigma} = |a_\nu b_\nu|\, e^{\lambda_\nu\sigma}|b_\nu|^{-1} \leqslant \mu_b^*(\sigma)e^{w(\lambda_\nu)} \leqslant \mu_b^*(\sigma)e^{w(v)} = \mu_b^*(\sigma)\mu(\sigma)^{\sigma^{\gamma+\mu}},\\ &0<\gamma<1, \quad 0<\mu<\beta, \quad \gamma+\beta<1.\end{aligned}$$

This means that for $\sigma\to\infty$ outside $E_1 = E_1^p(\alpha,\beta,\gamma)$ we have

$$\left(1-\frac{1}{\sigma^{\gamma+\mu}}\right)\ln\mu(\sigma) \leqslant \ln\mu_b^*(\sigma). \tag{3.21}$$

Since $|b_n| \leqslant e^{w(\lambda_n)}$, $n\geqslant N$, we find

$$\mu_b^*(\sigma) = |a_n b_n|\, e^{\lambda_k\sigma} \leqslant \mu(\sigma)e^{w(\lambda_k)}, \qquad k\geqslant N, \tag{3.22}$$

where $k=k(\sigma)$ is the central index of series (2.3).

Let $p=p(\sigma)$ be a solution of the equation $w(p) = \sigma^{-\gamma-\mu}\ln\mu_b^*(\sigma)$, and

$$R_p^* = \sum_{\lambda_n>p} |a_n b_n|\, e^{\lambda_n\sigma}, \qquad p = p(\sigma).$$

We let

$$u^*(\sigma) = \ln \frac{\ln \mu(\sigma)}{\sigma^{\gamma+\mu}}.$$

Applying Theorem 3.1, by the same arguing as in derivation of the estimate for R_v we obtain[1]

$$R_p^* \leqslant c_0[\mu_b^*(\sigma)]^{1-2(1+o(1))\sigma^{-\mu}} = o(1)\mu_b^*(\sigma), \qquad c_0 = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

as $\sigma \to \infty$ outside some set $E_2 \subset [0, \infty)$, $E_2 = E_2^p(\alpha, \beta, \gamma)$, $0 < \alpha < \alpha_2 < 1$,

$$\varlimsup_{t_j \to \infty} \frac{\operatorname{mes}(E_2 \cap [0, t_j])}{t_j} \leqslant q_2 < \frac{1}{2}. \tag{3.23}$$

Here t_j is a solution to the equation $p(\sigma) = x_j$, and $\{x_j\}$ is the sequence involved in condition (3.19). It follows that $\lambda_{k(\sigma)} \leqslant p(\sigma)$ if $\sigma \geqslant \sigma_2$, $\sigma \notin E_2$. Hence, by (3.22) we obtain that as $\sigma \to \infty$ outside the set E_2

$$\mu_b^*(\sigma) \leqslant \mu(\sigma)e^{w(p(\sigma))} = \mu(\sigma)\mu_b^*(\sigma)^{\frac{1}{\sigma^{\gamma+\mu}}}, \quad \gamma \in (0,1), \quad \mu \in (0, \beta).$$

that is,

$$\left(1 - \frac{1}{\sigma^{\gamma+\nu}}\right) \ln \mu_b^*(\sigma) \leqslant \ln \mu(\sigma). \tag{3.24}$$

Let us verify that $dE < 1$, where $E = E_1 \cup E_2$. This fact does not implied immediately by (3.20) and (3.23). This is why we consider the sequence $\{\tau_j^*\}$, where $\{\tau_j^*\} = \min(\tau_j, t_j)$. Since $x_j = p(t_j) = v(\tau_j)$, and $F_b^* \in D^p(\Phi)$, then, as in (3.17), we obtain that $\varphi^p(p(\sigma)) \leqslant c_4\sigma$, $0 < c_4 < \infty$, and therefore,

$$\frac{1}{\tau_j^*} \leqslant \frac{A}{(\varphi(x_j))^p}, \quad j \geqslant 1.$$

Hence, if $E = E_1 \cup E_2$, then by estimates of type (3.11) for each of the sets E_1 and E_2 as $\tau_j^* \to \infty$ we obtain

$$\begin{aligned}\varlimsup_{\tau_j^* \to \infty} \frac{\operatorname{mes}(E \cap [0, \tau_j^*])}{\tau_j^*} &\leqslant \varlimsup_{x_j \to \infty} \frac{A}{(\varphi(x_j))^p} (\operatorname{mes}(E_1 \cap [0, \tau_j]) + \operatorname{mes}(E_2 \cap [0, t_j])) \\ &= A \varlimsup_{\tau_j \to \infty} \frac{\operatorname{mes}(E_1 \cap [0, \tau_j])}{\varphi^p(v_j)} + A \varlimsup_{t_j \to \infty} \frac{\operatorname{mes}(E_2 \cap [0, t_j])}{\varphi^p(p_j)} \leqslant q_1 + q_2 = q < 1,\end{aligned}$$

where $v_j = v(\tau_j)$, $p_j = p(t_j)$.

Thus, by (3.21), (3.24) we finally obtain that as $\sigma \to \infty$ outside E, $E = E^p(\alpha, \beta, \gamma)$, $dE \leqslant q < 1$,

$$\left|1 - \frac{\ln^* \mu_b(\sigma)}{\ln \mu(\sigma)}\right| \leqslant \frac{2}{\sigma^{\gamma+\mu}}, \quad \gamma \in (0,1), \quad \mu \in (0, \beta), \quad \beta \in (0,1), \quad \gamma + \beta < 1, \tag{3.25}$$

if $0 < \alpha < \min(\alpha_1, \alpha_2)$. The proof is complete. □

Open question: whether condition (3.12) is necessary for the validity of estimate (3.25) for each function $F \in D^{p\nu}(\Phi)$?

[1]Since $F \in D^p(\Phi)$, it is obvious that $F_b^* \in D^p(\Phi)$, where F_b^* is the sum of series (2.3).

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Narkes Nurmukhametovna Aitkuzhina,
Ufa University of Science and Technology,
Zaki Validi str. 32,
450076, Ufa, Russia
E-mail: `yusupovan@rambler.ru`

Rashit Akhtyarovich Gaisin,
Institute of Mathematics,
Ufa Federal Research Center, RAS
Chernyshevsky str. 112,
450008, Ufa Russia
E-mail: `rashit.gajsin@mail.ru`