ISSN 2304-0122 Ufa Mathematical Journal. Vol. 17. No 4 (2025). P. 95-103.

doi:10.13108/2025-17-4-95

CAUCHY PROBLEM FOR PARABOLIC EQUATIONS WITH MULTIPLE SPATIAL TRANSLATIONS AND SUMMABLE INITIAL FUNCTIONS

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Abstract. We consider the Cauchy problem for parabolic differential–difference equations with multiple spatial translations in lower order terms. The function in the initial condition is supposed to be summable. The solution to the problem is constructed as the convolution of the kernel of parabolic equation with the initial function. We study the behavior and smoothness of the solution and its derivatives for large time.

Keywords: differential–difference operator, parabolic equation, Cauchy problem.

Mathematics Subject Classification: 35R10, 35K15

1. Introduction

Elliptic and parabolic differential–difference equations with translations in spatial variables is a subject of detailed study by many authors for a long time, the related problems are widely elucidated in domestic and foreign scientific literature and have various applications [3]–[5], [7], [9], [15], [16], [19]. The study of these types of equations is closely related to the study of elliptic and parabolic differential equations with nonlocal boundary conditions, which have applications in plasma theory, multidimensional diffusion processes and nonlinear optics [1], [6], [10], [11], [13], [14], [18]. In its turn, the Cauchy problem for differential–difference equations was studied in details in [3], here the case of essentially bounded functions in the initial condition was systematically investigated, which led to a stabilization effect of the solution similar to one occurring in the heat equation or the Laplace equation in a half–space.

In [17], the Cauchy problem for parabolic difference–differential equations with summable initial functions was posed and studied. The existence of a solution to this problem was proved, and the solution itself was constructed as a convolution of the initial function with a Poisson–type kernel. The behavior of the solution and its derivatives at large time was also studied. The decay rate of the solution and its derivatives was demonstrated, and a sufficient condition for convergence was explicitly found. This paper is a continuation of the previous study; the key difference is that the equation now includes multiple spatial translations, namely, a superposition of lower order terms with translations is considered instead of a single term.

It should also be mentioned that boundary value problems for parabolic differential–difference equations with spatial translations were studied earlier and investigated by means of semigroup methods in [8], [12].

G.L. Rossovskii, Cauchy problem for parabolic equations with multiple spatial translations and summable initial functions.

The research by G.L. Rossovskii is financially supported by the Ministry of Science and Higher Education of Russian Federation in the framework of megagrant no. 075-15-2022-1115.

Submitted November 26, 2024.

2. Formulation of problem and integral representation of solution

We suppose that parameters a_k and h_k, $k = 1, \ldots, n$, are real. We consider the following parabolic differential–difference equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sum_{k=1}^{n} a_k u(x - h_k, t) \tag{2.1}$$

in the half–plane $x \in \mathbb{R}$, $t > 0$, with the initial condition

$$u(x, 0) = u_0(x), \qquad u_0 \in L_1(\mathbb{R}). \tag{2.2}$$

We define the function

$$\mathcal{E}(x,t) = \int\limits_0^{+\infty} e^{-t\left(\xi^2 - \sum\limits_{k=1}^{n} a_k \cos h_k \xi\right)} \cos\left(x\xi - t\sum_{k=1}^{n} a_k \sin h_k \xi\right) d\xi \tag{2.3}$$

in the aforementioned half–plane $x \in \mathbb{R}$, $t > 0$. The function (2.3) is called the Poisson kernel. It is known [3, Sec. 1.1] that this function satisfies Equation (2.1) in the classical sense in the entire half–plane. The solution of the posed problem can be written by means of the Poisson kernel.

It is obtained easily by the formal application of the Fourier transform in the variable x to the equation and initial condition. This leads us to the initial problem for the ordinary differential equation with a parameter ξ

$$\frac{\partial \tilde{u}}{\partial t} = \left(-\xi^2 + \sum_{k=1}^{n} a_k e^{-i\xi h_k}\right) \tilde{u}(\xi, t), \qquad \tilde{u}(\xi, 0) = \tilde{u}_0(\xi). \tag{2.4}$$

Solving (2.4) and applying the inverse Fourier transform, we obtain the solution of the initial problem (2.1), (2.2) as the convolution of the Poisson kernel (2.3) and the function in the initial condition (2.2):

$$u(x,t) = \frac{1}{\pi} \int\limits_{-\infty}^{+\infty} \mathcal{E}(x - \xi, t) u_0(\xi)\, d\xi. \tag{2.5}$$

As it was shown in detail in [3, Sec. 1.1] and [17, Sec. 2], the Poisson kernel (2.3) satisfies the estimate

$$\begin{aligned} \bigl|\mathcal{E}(x,t)\bigr| &\leqslant \int\limits_0^{+\infty} e^{-t\left(\xi^2 - \sum\limits_{k=1}^{n} a_k \cos h_k \xi\right)} d\xi = \int\limits_0^{+\infty} e^{-t\xi^2} e^{t\sum\limits_{k=1}^{n} a_k \cos h_k \xi} d\xi \\ &\leqslant e^{t\sum\limits_{k=1}^{n} |a_k|} \int\limits_0^{+\infty} e^{-t\xi^2}\, d\xi = \frac{e^{t\sum\limits_{k=1}^{n} |a_k|}}{\sqrt{t}} \int\limits_0^{+\infty} e^{-\xi^2}\, d\xi = e^{t\sum\limits_{k=1}^{n} |a_k|} \frac{\sqrt{\pi}}{2\sqrt{t}}. \end{aligned} \tag{2.6}$$

Since the function in the initial condition $u_0(x)$ is summable, for function $u(x,t)$ satisfies the estimate, which is similar to the above estimate for the kernel with the initial function taken into consideration

$$|u(x,t)| \leqslant \frac{e^{t\sum\limits_{k=1}^{n} |a_k|}}{2\sqrt{\pi t}} \int\limits_{-\infty}^{+\infty} |u_0(\xi)|\, d\xi \leqslant \frac{e^{t\sum\limits_{k=1}^{n} |a_k|}}{2\sqrt{\pi t}} \|u_0\|_{L_1(\mathbb{R})}. \tag{2.7}$$

Now we proceed to justifying the formal differentiation in Equation (2.1). We first formally differentiate the function $\mathcal{E}(x,t)$ twice in the variable x:

$$\frac{\partial^2\mathcal{E}}{\partial x^2} = -\int\limits_0^{+\infty} \xi^2 e^{-t\left(\xi^2-\sum\limits_{k=1}^{n} a_k\cos h_k\xi\right)}\cos\left(x\xi - t\sum_{k=1}^{n} a_k \sin h_k\xi\right)d\xi.$$

This implies the estimate

$$\begin{aligned}\left|\frac{\partial^2\mathcal{E}}{\partial x^2}\right| &\leqslant \int\limits_0^{+\infty} \xi^2 e^{-t\xi^2} e^{t\sum\limits_{k=1}^{n} a_k\cos h_k\xi}\,d\xi \leqslant e^{t\sum\limits_{k=1}^{n}|a_k|}\int\limits_0^{+\infty}\xi^2 e^{-t\xi^2}\,d\xi \\ &= \left[\begin{array}{c} t\xi^2 = y \\ d\xi = \frac{1}{2\sqrt{t}\sqrt{y}}\,dy\end{array}\right] = e^{t\sum\limits_{k=1}^{n}|a_k|}\frac{1}{2t^{\frac{3}{2}}}\int\limits_0^{+\infty}\sqrt{y}e^{-y}\,dy \\ &= e^{t\sum\limits_{k=1}^{n}|a_k|}\frac{1}{2t^{\frac{3}{2}}}\Gamma\left(\frac{1}{2}\right) = e^{t\sum\limits_{k=1}^{n}|a_k|}\frac{\sqrt{\pi}}{2t^{\frac{3}{2}}}.\end{aligned} \tag{2.8}$$

The obtained integral converges absolutely for all positive values of t. In its turn, this proves the validity of formal differentiation.

The formal differentiation of the function $\mathcal{E}(x,t)$ in the variable t is not needed since the above obtained estimates (2.6) and (2.8) and the fact that $\mathcal{E}(x,t)$ satisfies Equation (2.1) in the half–plane $x \in \mathbb{R}$, $t > 0$, ensure that

$$\left|\frac{\partial\mathcal{E}}{\partial t}\right| \leqslant \frac{\sqrt{\pi}}{2}e^{t\sum\limits_{k=1}^{n}|a_k|}\left(\frac{1}{t^{\frac{3}{2}}}+\frac{1}{t^{\frac{1}{2}}}\right). \tag{2.9}$$

The same is obviously true for the function $u(x,t)$ since the latter is constructed as the convolution of the just studied function $\mathcal{E}(x,t)$ and the summable initial function $u_0(x)$. Thus, the function $u(x,t)$ satisfies Equation (2.1) in the classical sense in the half–plane $x \in \mathbb{R}$, $t > 0$.

We reformulate the obtained fact as the next theorem.

Theorem 2.1. *The function (2.5) solves the Cauchy problem (2.1), (2.2) in distribution sense and satisfies Equation (2.1) in the half–plane $(-\infty,+\infty)\times(0,+\infty)$ in the classical sense.*

3. Properties of solution

3.1. Behavior of solution. The estimates obtained for the function $\mathcal{E}(x,t)$ and its derivatives in the variables x and t are rather simple and sufficient to prove the solvability of problem (2.1), (2.2) (the same is true for the function $u(x,t)$), but they are not so useful for studying the behavior of solution as $t \to \infty$.

This is why we are going to obtain sharper estimates for the function $\mathcal{E}(x,t)$. We consider the structure of the integrand in the integral representation for the function $\mathcal{E}(x,t)$ in more detail. Roughly speaking, for the (uniform) convergence of this integral, the exponent of exponential under the integral is to be negative. This requires the inequality

$$\xi^2 - \sum_{k=1}^{n} a_k\cos h_k\xi > 0. \tag{3.1}$$

It is obvious that

$$\xi^2 - \sum_{k=1}^{n} a_k \cos h_k\xi \geqslant \frac{1}{2}\xi^2$$

starting from some ξ_0. Since

$$\left|\sum_{k=1}^{n} a_k \cos h_k\xi\right| \leqslant \sum_{k=1}^{n} \left|a_k \cos h_k\xi\right| \leqslant \sum_{k=1}^{n} |a_k|,$$

we can take

$$\xi_0 = \sqrt{2\sum_{k=1}^{n} |a_k|}.$$

We return back to the integral in estimate for the function $\mathcal{E}(x,t)$. It is obvious that the inequality (3.1) supposes the existence of a constant $a_0 > 0$ such that $\xi^2 - \sum\limits_{k=1}^{n} a_k \cos h_k\xi \geqslant a_0$. Then

$$\begin{aligned} \big|\mathcal{E}(x,t)\big| \leqslant \int\limits_0^{+\infty} e^{-t\left(\xi^2 - \sum\limits_{k=1}^{n} a_k \cos h_k\xi\right)}\, d\xi = \int\limits_0^{\xi_0} e^{-t\left(\xi^2 - \sum\limits_{k=1}^{n} a_k \cos h_k\xi\right)}\, d\xi \\ + \int\limits_{\xi_0}^{+\infty} e^{-t\left(\xi^2 - \sum\limits_{k=1}^{n} a_k \cos h_k\xi\right)}\, d\xi \leqslant \int\limits_0^{\xi_0} e^{-a_0 t}\, d\xi + \int\limits_{\xi_0}^{+\infty} e^{-\frac{t\xi^2}{2}}\, d\xi \\ \leqslant e^{-a_0 t}\xi_0 + \int\limits_0^{+\infty} e^{-\frac{t\xi^2}{2}}\, d\xi = e^{-a_0 t}\xi_0 + \sqrt{\frac{\pi}{2t}} \\ = \sqrt{2\sum_{k=1}^{n} |a_k|}\, e^{-a_0 t} + \sqrt{\frac{\pi}{2t}}. \end{aligned} \tag{3.2}$$

Once we have studied the behavior of function $\mathcal{E}(x,t)$, we can formulate the following statements for the function $u(x,t)$.

Theorem 3.1. *Let the condition (3.1) hold. Then the solution (2.5) of Cauchy problem (2.1), (2.2) converges to zero as $t \to \infty$ uniformly in $x \in \mathbb{R}$. The convergence rate is given by the estimate*

$$|u(x,t)| \leqslant C\frac{\|u_0\|_{L_1(\mathbb{R})}}{\sqrt{t}},$$

where $C > 0$ is some constant.

Remark 3.1. *In the case of a single spatial translation, that is, as the equation involves only one lower order term, the condition (3.1) can be written in a more explicit form. Replacing $h\xi$ by η in*

$$\xi^2 - a\cos h\xi > 0,$$

where a, h are real and $a < 0$, we obtain

$$\frac{\eta^2}{|a|h^2} + \cos\eta > 0, \qquad \eta \in \mathbb{R}.$$

Therefore, we need to impose the following conditions for the parameters a, h

$$|a|h^2 < \frac{1}{\max\limits_{\eta\in\mathbb{R}} f(\eta)},$$

where

$$f(\eta) = -\frac{\cos\eta}{\eta^2}.$$

Remark 3.2. *It is also possible to write a rougher but more convenient form of the estimate (3.1), which relates only the coefficients a_k and h_k in Equation (2.1).*

As it was shown in [2] for the case of a hyperbolic equation with nonlocal potentials, we can obtain such condition under a more detailed study of the function

$$f(\xi) = \xi^2 - \sum_{k=1}^{n} a_k \cos h_k\xi \quad \text{for} \quad \xi > 0.$$

Differentiating this expression, we obtain

$$f'(\xi) = 2\xi + \sum_{k=1}^{n} a_k h_k \sin h_k\xi = 2\xi\left(1 + \sum_{k=1}^{n} \frac{a_k h_k^2}{2}\frac{\sin h_k\xi}{h_k\xi}\right) > 2\xi\left(1 - \sum_{k=1}^{n}\frac{|a_k|h_k^2}{2}\right).$$

The obtained derivative is positive for all $\xi \geqslant 0$ under the condition

$$\sum_{k=1}^{n} |a_k| h_k^2 \leqslant 2, \tag{3.3}$$

and this implies that the function $f(\xi)$ is non–decreasing for all $\xi \geqslant 0$, and $f(0) = -\sum\limits_{k=1}^{n} a_k$. Since the function $f(\xi)$ is even, the obtained value $f(0)$ remains minimal for all $\xi \in \mathbb{R}$.

3.2. Behavior of derivatives of solution. Using the same approach, we are going to study the behavior of derivatives of the obtained solution both in the spatial variable and time. We suppose that the condition (3.1) holds.

We first differentiate the function (2.3) in the variable x and apply the same technique as in the derivation of the estimate (3.2):

$$\begin{aligned}
\left|\frac{\partial\mathcal{E}}{\partial x}\right| &= \left|\int\limits_0^{+\infty} \xi e^{-t\left(\xi^2 - \sum\limits_{k=1}^{n} a_k\cos h_k\xi\right)} \sin\left(x\xi - t\sum_{k=1}^{n} a_k \sin h_k\xi\right) d\xi\right| \\
&\leqslant \int\limits_0^{+\infty} \xi e^{-t\left(\xi^2 - \sum\limits_{k=1}^{n} a_k\cos h_k\xi\right)} d\xi = \int\limits_0^{\xi_0} \xi e^{-t\left(\xi^2 - \sum\limits_{k=1}^{n} a_k\cos h_k\xi\right)} d\xi \\
&\quad + \int\limits_{\xi_0}^{+\infty} \xi e^{-t\left(\xi^2 - \sum\limits_{k=1}^{n} a_k\cos h_k\xi\right)} d\xi \leqslant \int\limits_0^{\xi_0} \xi e^{-a_0 t} d\xi + \int\limits_{\xi_0}^{+\infty} \xi e^{-\frac{t\xi^2}{2}} d\xi \\
&\leqslant \frac{\xi_0^2}{2} e^{-a_0 t} + \frac{1}{t}\int\limits_0^{+\infty} e^{-y^2} d\left(y^2\right) = e^{-a_0 t}\sum_{k=1}^{n} |a_k| + \frac{1}{t}.
\end{aligned}$$

Then we get the corresponding estimate for the function (2.5)

$$\left|\frac{\partial u}{\partial x}\right| \leqslant \frac{1}{\pi}\left|\frac{\partial\mathcal{E}}{\partial x}\right| \int\limits_{-\infty}^{+\infty} u_0 \; d\xi = \frac{\|u_0\|_{L_1(\mathbb{R})}}{\pi}\left[e^{-a_0 t}\sum_{k=1}^{n} |a_k| + \frac{1}{t}\right],$$

which means that

$$\lim_{t\to+\infty}\frac{\partial u(x,t)}{\partial x}=0,$$

that is, the first derivative of function $u(x,t)$ in the variable x converges to zero as $t\to\infty$ uniformly in $x\in\mathbb{R}$ with the rate t^{-1}.

We proceed to the higher derivatives in the spatial variable (starting from the second derivative):

$$\left|\frac{\partial^2\mathcal{E}}{\partial x^2}\right|\leqslant\int\limits_0^{+\infty}\xi^2 e^{-t\left(\xi^2-\sum\limits_{k=1}^{n}a_k\cos h_k\xi\right)}\,d\xi\leqslant\int\limits_0^{\xi_0}\xi^2 e^{-a_0t}\,d\xi+\int\limits_{\xi_0}^{+\infty}\xi^2 e^{-t\xi^2/2}\,d\xi$$
$$\leqslant\frac{\xi_0^3}{3}e^{-a_0t}+\left(\frac{2}{t}\right)^{\frac{3}{2}}\int\limits_0^{+\infty}y^2e^{-y^2}\,dy\leqslant\frac{e^{-a_0t}}{3}\left(2\sum_{k=1}^{n}|a_k|\right)^{\frac{3}{2}}+\frac{C_2}{t^{\frac{3}{2}}}.$$

where C_2 is some constant. Then for the function $u(x,t)$ we have

$$\left|\frac{\partial^2 u}{\partial x^2}\right|\leqslant\frac{\|u_0\|_{L_1(\mathbb{R})}}{\pi}\left[\frac{e^{-a_0t}}{3}\left(2\sum_{k=1}^{n}|a_k|\right)^{\frac{3}{2}}+\frac{C_2}{t^{\frac{3}{2}}}\right],$$

which implies

$$\lim_{t\to+\infty}\frac{\partial^2 u(x,t)}{\partial x^2}=0$$

uniformly in $x\in\mathbb{R}$ with the rate $t^{-\frac{3}{2}}$.

Now we look at the derivative of m order in the variable x:

$$\frac{\partial^m\mathcal{E}}{\partial x^m}=\int\limits_0^{+\infty}(-\xi)^m e^{-t\left(\xi^2-\sum\limits_{k=1}^{n}a_k\cos h_k\xi\right)}\cos\left(x\xi-t\sum_{k=1}^{n}a_k\sin h_k\xi+\frac{\pi m}{2}\right)\,d\xi$$

with the estimate

$$\left|\frac{\partial^m\mathcal{E}}{\partial x^m}\right|\leqslant\int\limits_0^{+\infty}\xi^m e^{-t\left(\xi^2-\sum\limits_{k=1}^{n}a_k\cos h_k\xi\right)}\,d\xi\leqslant\frac{\xi_0^{m+1}}{m+1}e^{-a_0t}+\left(\frac{2}{t}\right)^{\frac{m+1}{2}}\int\limits_0^{+\infty}y^me^{-y^2}\,dy$$
$$\leqslant\frac{e^{-a_0t}}{m+1}\left(2\sum_{k=1}^{n}|a_k|\right)^{\frac{m+1}{2}}+\frac{C_m}{t^{\frac{m+1}{2}}}.$$

Then for the function $u(x,t)$ we have

$$\left|\frac{\partial^m u}{\partial x^m}\right|\leqslant\frac{\|u_0\|_{L_1(\mathbb{R})}}{\pi}\left[\frac{e^{-a_0t}}{m+1}\left(2\sum_{k=1}^{n}|a_k|\right)^{\frac{m+1}{2}}+\frac{C_m}{t^{\frac{m+1}{2}}}\right],$$

which implies

$$\lim_{t\to+\infty}\frac{\partial^m u(x,t)}{\partial x^m}=0$$

uniformly in $x\in\mathbb{R}$ with the rate $t^{-\frac{m+1}{2}}$.

Now we additionally differentiate the function $\mathcal{E}(x,t)$ in the variable t, and then we estimate as above

$$\frac{\partial^{m+1}\mathcal{E}}{\partial x^m \partial t} = \int\limits_0^{+\infty} (-\xi)^m \sum_{k=1}^n a_k \sin h_k \xi e^{-t\left(\xi^2 - \sum\limits_{k=1}^n a_k \cos h_k \xi\right)} \sin\left(x\xi - t\sum_{k=1}^n a_k \sin h_k \xi + \frac{\pi m}{2}\right) d\xi$$
$$- \int\limits_0^{+\infty} (-\xi)^m \left(\xi^2 - \sum_{k=1}^n a_k \cos h_k \xi\right) e^{-t\left(\xi^2 - \sum\limits_{k=1}^n a_k \cos h_k \xi\right)} \cos\left(x\xi - t\sum_{k=1}^n a_k \sin h_k \xi + \frac{\pi m}{2}\right) d\xi$$
$$= \int\limits_0^{+\infty} (-1)^{m+1}(-\xi)^{m+2} e^{-t\left(\xi^2 - \sum\limits_{k=1}^n a_k \cos h_k \xi\right)} \cos\left(x\xi - t\sum_{k=1}^n a_k \sin h_k \xi + \frac{\pi m}{2}\right) d\xi$$
$$+ \int\limits_0^{+\infty} (-\xi)^m e^{-t\left(\xi^2 - \sum\limits_{k=1}^n a_k \cos h_k \xi\right)} \sum_{k=1}^n a_k \cos\left((x-h_k)\xi - t\sum_{k=1}^n a_k \sin h_k \xi + \frac{\pi m}{2}\right) d\xi,$$

hence,

$$\left|\frac{\partial^{m+1}\mathcal{E}}{\partial x^m \partial t}\right| \leqslant \int\limits_0^{+\infty} \xi^{m+2} e^{-t\left(\xi^2 - \sum\limits_{k=1}^n a_k \cos h_k \xi\right)} d\xi + \sum_{k=1}^n |a_k| \int\limits_0^{+\infty} \xi^m e^{-t\left(\xi^2 - \sum\limits_{k=1}^n a_k \cos h_k \xi\right)} d\xi$$
$$\leqslant \frac{\xi^{m+3}}{m+3} e^{-a_0 t} + C_1 \left(\frac{2}{t}\right)^{\frac{m+3}{2}} + \sum_{k=1}^n |a_k| \left(\frac{\xi^{m+1}}{m+1} e^{-a_0 t} + C_2 \left(\frac{2}{t}\right)^{\frac{m+1}{2}}\right)$$
$$\leqslant \widetilde{C} e^{-a_0 t} + \widetilde{C}_1 \frac{1}{t^{\frac{m+3}{2}}} + \widetilde{C}_0 \frac{1}{t^{\frac{m+1}{2}}}.$$

And passing to the derivative of order $m+n$, we obtain the estimate

$$\left|\frac{\partial^{m+n}\mathcal{E}}{\partial x^m \partial t^n}\right| \leqslant \widetilde{C} e^{-a_0 t} + \sum_{j=0}^n \widetilde{C}_j \frac{1}{t^{\frac{(m+2j+1)}{2}}}.$$

We see that the additional differentiation of the function $\mathcal{E}(x,t)$ (as of the function $u(x,t)$) in the variable t does not change the convergence rate.

Theorem 3.2. *Let the condition* (3.1) *hold. Then the partial derivatives of solution* (2.5) *of the Cauchy problem* (2.1), (2.2) *of order $m+n$ converge to zero as $t \to \infty$ uniformly in $x \in \mathbb{R}$. The convergence rate is estimated as follows*

$$\left|\frac{\partial^{m+n}u}{\partial x^m \partial t^n}\right| \leqslant \frac{\|u_0\|_{L_1(\mathbb{R})}}{\pi} \left[\widetilde{C} e^{-a_0 t} + \sum_{j=0}^n \widetilde{C}_j \frac{1}{t^{\frac{(m+2j+1)}{2}}}\right],$$

where $\widetilde{C}$, $\widetilde{C}_j$, $j = 0, \dots, n$, are constants.

4. Conclusion

This work is a generalization and continuation of the work [17], in which only one lower order term was considered, to the case of multiple spatial translations. We have constructed the solution to the Cauchy problem (2.1), (2.2) as the convolution of the Poisson kernel (2.3) and the function in the initial condition (2.2). Then we showed that the introduced function (2.5) indeed solves the original problem by proving that the formal differentiation is valid.

We then carefully examined the properties of the resulting solution. After obtaining the Poisson kernel, the attempt to estimate the integrand in the kernel led us to a transcendental equation, from which, however, we obtained a sufficient condition for convergence for our solution. We also obtained a sufficient condition for convergence of our integral, slightly rougher than the one described above, but clearly linking the coefficients of the lower order terms and the spatial translations in the lower order terms.

Acknowledgments

The author sincerely thanks A.B. Muravnik for the scientific supervision and support, as well as the organizers of "Ufa Autumn Mathematical School — 2024" for the opportunity to give a talk, and its participants for constructive and interesting questions.

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