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ON DEGREE OF SMOOTH MAPS BETWEEN ORBIFOLDS

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Abstract. In the present work we develop the degree theory for proper maps between orbifolds of same dimension. The definition of degree for the mentioned maps was introduced by Pasquoto and Rot (2020). We propose a new, simpler definition for the degree of proper maps between smooth oriented orbifolds of the same dimension and show that it is equivalent to the definition by Pasquotto and Rot. Using this new approach, we establish a connection between the degree of a map and the integration of exterior forms on orbifolds, which is important for physical applications. We obtain an integral formula for the degree of a map between orbifolds, which is a generalization of the corresponding formula for manifolds. We also reveal the specificity of degree of a map for compact orbifolds.

Keywords: orbifold, proper map, volume form, orbifold stratification.

Mathematics Subject Classification: 57R18, 57R35, 57R45

1. Introduction. Main results

In this paper, we develop a degree theory for proper maps between smooth orbifolds of the same dimension. The concept of the degree of a map between smooth manifolds was introduced by Brouwer [5]. Brouwer showed that the degree of a map is a homotopy invariant and he applied this fact to the proof of a fixed point theorem.

The degree of a mapping is widely used in many areas of geometry and topology [10]. In particular, the degree of a map is used to prove the well–known Gauss theorem on the existence of a root of an arbitrary complex polynomial. Gauss — Bonnet and Poincaré — Hopf theorems can be proved by using the concept of the degree of a map [10], [14].

In physics, the degree of a map is considered as a topological charge (a topological quantum number). This is why the degree theory is used to develop topological methods for analyzing the structure of solutions to nonlinear equations in mathematical physics [3], [6], [8], [9]. In [14] there was proposed a new approach to characterization of monopole configurations in the Yang— Mills — Higgs theory with the gauge group $SU(2)$ by using the degree of map between smooth manifolds.

Pasquotto and Rot introduced the definition of degree of map between the orbifolds [12].

The orbifold can be treated as a natural generalization of the concept of the manifold. The concept of the orbifold was introduced by Satake under the name V–manifold [13], and the term "orbifold" was proposed by Thurston [15], who classified two–dimensional compact orbifolds and applied it for the classification of closed smooth three-dimensional manifolds. Locally, n–dimensional orbifolds are homeomorphic to the quotient space of the n–dimensional arithmetic space $\mathbb{R}^n$ by a finite group of diffeomorphisms. This group may change when passing from one point of the orbifold to another. Rigorous definitions of the category of orbifolds and integration over them are given in Section 2.

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The orbifolds arise naturally in solving various problems in mathematics and physics [1]. For example, in mathematical physics, the orbifolds are used as string propagation spaces. In foliation theory, the orbifolds appear as spaces of leaves.

The goal of this paper is to develop the theory of degree of maps between orbifolds. In Section 3.1, we recall the definition of degree due to Pasquotto and Rot. In Section 3.2, we introduce a new, simpler definition of the degree of a proper map between connected oriented smooth orbifolds of the same dimension. The novelty of our approach is that to determine the degree of a map between orbifolds, it suffices to consider only the regular values of this map, which are regular points of the orbifold. We prove that our definition is equivalent to the corresponding definition of Pasquotto and Rot.

By using the proposed approach, in Section 3.3 we prove Theorem 1.1, which is a generalization to orbifolds of a well–known statement for manifolds.

Theorem 1.1. *Let $f\colon \mathcal{N}_1 \to \mathcal{N}_2$ be a proper map between connected oriented smooth orbifolds of same dimension $n \in \mathbb{N}$. Then for each n–form ω with a compact support on $\mathcal{N}_2$ the identity holds*

$$\int\limits_{\mathcal{N}_1} f^*\omega = \deg(f) \int\limits_{\mathcal{N}_2} \omega, \tag{1.1}$$

where $\deg(f)$ is the degree of the map f.

In Section 3.4 we obtain the following integral formula.

Theorem 1.2. *Let $f\colon \mathcal{N}_1 \to \mathcal{N}_2$ be a proper map of connected oriented orbifolds of same dimension $n \in \mathbb{N}$, Ω be the volume form on a compact orbifold $\mathcal{N}_2$ defined by the Riemannian metrics g, $\mathrm{Vol}(\mathcal{N}_2)$ be the volume of orbifold $\mathcal{N}_2$. Then the degree of map f satisfies the identity*

$$\deg(f) = \frac{1}{\mathrm{Vol}(\mathcal{N}_2)} \int\limits_{\mathcal{N}_1} f^*\Omega. \tag{1.2}$$

Moreover, the right hand side of the identity (1.2) is independent of the choice of the Riemannian metrics g on $\mathcal{N}_2$.

The formula (1.2) allows us to extend the notion of topological charge to vector fields defined on orbifolds.

In Section 3.5 we recall the definition of the covering and regular covering map for the orbifolds. The number of sheets of the covering map $f\colon \mathcal{N}_1 \to \mathcal{N}_2$ is defined as the number of pre–images of a regular point in $\mathcal{N}_2$.

Theorem 1.2 implies the following statement.

Corollary 1.1. *Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be connected compact oriented orbifolds of same dimension, $f\colon \mathcal{N}_1 \to \mathcal{N}_2$ be a surjective regular mapping. Then*

1) *$f\colon \mathcal{N}_1 \to \mathcal{N}_2$ is a k–sheeted regular covering map, where $k = \deg(f)$;*
2) *the degree of map f satisfies the identity*

$$\deg(f) = \frac{\mathrm{Vol}(\mathcal{N}_1)}{\mathrm{Vol}(\mathcal{N}_2)}.$$

Assumptions. The manifolds, orbifolds, vector fields, forms have the smoothness C^∞. The orbifolds (in particular, manifolds) are supposed to be connected if else is not said. We consider only open neighbourhoods. We also suppose that all groups of diffeomorphisms act effectively.

Notations. The quotient spaces of X by both left and right action of a group G are denoted by X/G. The symbol $|G|$ stands for the order of a finite group G. By $\mathbb{N}$ we denote the set of natural numbers.

2. Orbifolds

2.1. Categories of orbifolds. Let $\mathcal{N}$ be a connect Hausdorff topological space with a countable base. Let $\tilde{U}$ be a connected open subspace in the n–dimensional arithmetical space $\mathbb{R}^n$, Γ_U be a finite group of diffeomorphisms $\tilde{U}$, $\varphi_U\colon \tilde{U} \to \mathcal{N}$ be a Γ_U–invariant map, which induces the homeomorphism q_U from $\tilde{U}/\Gamma_U$ into the open subset $U = \varphi_U(\tilde{U})$ in $\mathcal{N}$. Then the triple $(\tilde{U}, \Gamma_U, \varphi_U)$ is called the *orbifold chart* on $\mathcal{N}$ with the *coordinate neighbourhood* U.

We consider two orbifold charts $(\tilde{U}, \Gamma_U, \varphi_U)$ and $(\tilde{V}, \Gamma_V, \varphi_V)$ with the coordinate neighbourhoods U and V, and $U \subset V$. The smooth map $\phi_{VU}\colon \tilde{U} \to \tilde{V}$ is called the *embedding* of the chart $(\tilde{U}, \Gamma_U, \varphi_U)$ into the chart $(\tilde{V}, \Gamma_V, \varphi_V)$ if $\varphi_U = \varphi_V \circ \phi_{VU}$. We observe that each map ϕ_{VU} induces the monomorphism of groups $\psi_{VU}\colon \Gamma_U \to \Gamma_V$, for which the identity holds

$$\varphi_{VU}(g(u)) = \psi_{VU}(g)(\varphi_{VU}(u)) \quad \forall u \in \tilde{U}, \quad g \in \Gamma_U.$$

The family of charts $\mathcal{A} = \{(\tilde{U}_\alpha, \Gamma_\alpha, \varphi_\alpha) | \alpha \in J\}$ is called the *orbifold atlas* if the following two conditions are satisfied:

1) the set of coordinate neighbourhoods $\{U_\alpha = \varphi_\alpha(\tilde{U}_\alpha)|\ \alpha \in J\}$ of the charts in $\mathcal{A}$ forms an open covering of the topological space $\mathcal{N}$;
2) each two charts in the atlas $\mathcal{A}$ are compatible in the following sense: if $(\tilde{U}_\alpha, \Gamma_\alpha, \varphi_\alpha) \in \mathcal{A}$ and $(\tilde{U}_\beta, \Gamma_\beta, \varphi_\beta) \in \mathcal{A}$ are two charts with the coordinate neighbourhoods U_α and U_β, $U_\alpha \cap U_\beta \neq \emptyset$, then for each point $x \in U_\alpha \cap U_\beta$ there exists a chart $(\tilde{U}_\gamma, \Gamma_\gamma, \varphi_\gamma) \in \mathcal{A}$ with the coordinate neighbourhood U_γ such that $x \in U_\gamma \subset U_\alpha \cap U_\beta$, and two embeddings $\varphi_{\alpha\gamma}$ and $\varphi_{\beta\gamma}$ of the chart $(\tilde{U}_\gamma, \Gamma_\gamma, \varphi_\gamma)$ into the charts $(\tilde{U}_\alpha, \Gamma_\alpha, \varphi_\alpha)$ and $(\tilde{U}_\beta, \Gamma_\beta, \varphi_\beta)$, respectively.

The pair $(\mathcal{N}, \mathcal{A})$, where $\mathcal{A}$ is the maximal (by embedding) orbifold atlas $\mathcal{A}$, is called the *smooth n–dimensional orbifold*. In what follows we shortly denote the orbifold $(\mathcal{N}, \mathcal{A})$ by $\mathcal{N}$.

We note that for each point $x \in \mathcal{N}$ of a smooth n–dimensional orbifold $(\mathcal{N}, \mathcal{A})$ there exists a chart $(\tilde{U}, \Gamma_U, \varphi_U) \in \mathcal{A}$ such that $\tilde{U}$ is an n–dimensional arithmetical space $\mathbb{R}^n$, $\varphi_U(0) = x$, $0 \in \mathbb{R}^n$, and Γ_U is a finite group of orthogonal transformations of $\mathbb{R}^n$. Such chart is called the *linearized chart at x*.

For the orbifold charts $(\tilde{U}, \Gamma_U, \varphi_U)$ and $(\tilde{V}, \Gamma_V, \varphi_V)$ in $\mathcal{A}$ with the coordinate neighbourhoods containing $x \in \mathcal{N}$, the isotropy subgroups $(\Gamma_U)_y$ and $(\Gamma_V)_z$ at the points $y \in \varphi_U^{-1}(x)$ and $z \in \varphi_V^{-1}(x)$ are isomorphic. Therefore, for each point $x \in \mathcal{N}$ the group $\Gamma_{(x)}$ is defined, which is unique up to the group isomorphism. The group $\Gamma_{(x)}$ is called the *orbifold group* at x. The point x is called *regular* if its orbifold group $\Gamma_{(x)}$ is trivial; otherwise the point x is called *singular*.

As it is known [2, Ex. 1], in contrast to the case $n = 2$, for $n \geqslant 3$ the smooth n–dimensional orbifold $\mathcal{N}$ is not, generally speaking, locally Euclidean space. The topological space of the orbifold $\mathcal{N}$ is called the *underlying space*.

Let $\mathcal{N}$ and $\mathcal{N}'$ be smooth orbifolds with atlases $\mathcal{A}$ and $\mathcal{A}'$, respectively. A continuous map $f\colon \mathcal{N} \to \mathcal{N}'$ is called *smooth* if for each $x \in \mathcal{N}$ there exist charts $(\tilde{V}_\beta, G_\beta, \psi_\beta) \in \mathcal{A}$ and $(\tilde{U}_\alpha, \Gamma_\alpha, \varphi_\alpha) \in \mathcal{A}'$ such that $x \in V_\beta = \psi_\beta(\tilde{V}_\beta)$, $f(V_\beta) \subset U_\alpha = \varphi_\alpha(\tilde{U}_\alpha)$, and the smooth map $f_{\alpha\beta}\colon \tilde{V}_\beta \to \tilde{U}_\alpha$ satisfying the identity $\varphi_\alpha \circ f_{\alpha\beta} = f \circ \psi_\beta$. The map $f_{\alpha\beta}$ is called the *representative* of map f in the charts $(\tilde{V}_\beta, G_\beta, \psi_\beta)$ and $(\tilde{U}_\alpha, \Gamma_\alpha, \varphi_\alpha)$. The map $f_{\alpha\beta}$ is defined up to the composition with the elements in G_β and Γ_α, respectively.

We denote by $\mathfrak{Orb}$ the category of orbifolds, the objects of which are smooth orbifolds, and the morphisms are smooth maps of orbifolds. We note that the category of manifolds is a complete subcategory of the category $\mathfrak{Orb}$.

2.2. Stratification of orbifolds. We say that two points of a smooth orbifold $\mathcal{N}$ possess the *same orbifold type* if there exist the neighbourhoods of these points isomorphic in the category $\mathfrak{Orb}$. The set of points of the same orbifold type with the induced topology has a natural structure, generally speaking, of a non–connected smooth manifold [2]. We denote by Δ_k the union of such manifolds of the dimension k. We note that the manifolds of points of different orbifold type can have the same dimension. We stress that each connected component $\Delta_k^{c_j}$ of the manifold Δ_k consists of the points of same orbifold type. It is possible that $\Delta_k = \emptyset$ for some $k \in \{0, \dots, n-1\}$, where n is the dimension of the smooth orbifold $\mathcal{N}$.

The family $\Delta(\mathcal{N}) = \{\Delta_k^{c_j} \mid j \in J_k,\ k \in \{0, \dots, n\}\}$ is called the *stratification* of the smooth n–dimensional orbifold $\mathcal{N}$, and $\Delta_k^{c_j}$ are called its *stratum* [7].

The set of regular points defines the strata Δ_n, which is a connected open everywhere dense subset in $\mathcal{N}$ and with respect to the induced smooth structure it is a smooth n–dimensional manifold.

2.3. Integration on orbifolds. We say that on a smooth n–dimensional orbifold $(\mathcal{N}, \mathcal{A})$ an exterior p–form is defined if for each chart $(\tilde{U}_\alpha, \Gamma_\alpha, \varphi_\alpha) \in \mathcal{A}$ on $\tilde{U}_\alpha$ an exterior Γ_α–invariant p–form ω_α is defined and for each embedding $\phi_{\beta\alpha} \colon \tilde{U}_\alpha \to \tilde{U}_\beta$ of the chart $(\tilde{U}_\alpha, \Gamma_\alpha, \varphi_\alpha)$ into the chart $(\tilde{U}_\beta, \Gamma_\beta, \varphi_\beta)$ the identity $\phi_{\beta\alpha}^* \omega_\beta = \omega_\alpha$ holds.

Let $\omega = \{\omega_\alpha\}_{\alpha \in J}$ be an exterior p–form on an orbifold $(\mathcal{N}, \mathcal{A})$. We note that $\omega = 0$ at $x \in \mathcal{N}$ if and only if there exists a linearized chart $(\tilde{U}_\alpha, \Gamma_\alpha, \varphi_\alpha) \in \mathcal{A}$ at $x \in U_\alpha = \varphi_\alpha(\tilde{U}_\alpha)$ such that $\omega_\alpha = 0$ at $0 \in \tilde{U}_\alpha = \mathbb{R}^n$. The closure of the set of points of the orbifold $\mathcal{N}$, at which ω is non–zero, is called the support of form ω and is denoted by $\operatorname{supp} \omega$. We denote by $\Gamma_c^p(\mathcal{N})$ the set of all exterior p–forms with a compact support on $\mathcal{N}$. The set $\Gamma_c^p(\mathcal{N})$ with pointwise summation and multiplication by the real numbers is a vector space.

Let $f \colon \mathcal{N} \to \mathcal{N}'$ be a smooth map of orbifolds, $\omega = \{\omega_\alpha\}_{\alpha \in J}$ be an exterior p–form on the orbifold $(\mathcal{N}', \mathcal{A}')$. For each $x \in \mathcal{N}$ there exist charts $(\tilde{V}_\beta, G_\beta, \psi_\beta) \in \mathcal{A}$ and $(\tilde{U}_\alpha, \Gamma_\alpha, \varphi_\alpha) \in \mathcal{A}'$, such that $x \in V_\beta = \psi_\beta(\tilde{V}_\beta)$, $f(V_\beta) \subset U_\alpha = \varphi_\alpha(\tilde{U}_\alpha)$, and a representative $f_{\alpha\beta} \colon \tilde{V}_\beta \to \tilde{U}_\alpha$ of the map f in charts $(\tilde{V}_\beta, G_\beta, \psi_\beta)$ and $(\tilde{U}_\alpha, \Gamma_\alpha, \varphi_\alpha)$, such that $\varphi_\alpha \circ f_{\alpha\beta} = f \circ \psi_\beta$. We denote by $f_{\alpha\beta}^* \omega_\alpha$ the preimage of the form ω_α under the map $f_{\alpha\beta}$. Then the set $\{f_{\alpha\beta}^* \omega_\alpha\}_{x \in \mathcal{N}}$ generates the p–form $f^* \omega$ on the orbifold $(\mathcal{N}, \mathcal{A})$, which is called the preimage of the form ω under the map f. Thus, the linear map is defined

$$f^* \colon \Gamma_c^p(\mathcal{N}') \to \Gamma_c^p(\mathcal{N}) \colon \omega \mapsto f^* \omega.$$

We recall that the orbifold $(\mathcal{N}, \mathcal{A})$ is called oriented if for each for $\alpha \in J$ the manifolds $\tilde{U}_\alpha$ are oriented so that each map $\gamma \in \Gamma_\alpha$ and each embedding $\phi_{\alpha\beta} \colon \tilde{U}_\beta \to \tilde{U}_\alpha$, $\alpha, \beta \in J$, preserves the orientation.

Let $(\mathcal{N}, \mathcal{A})$ be an oriented n–dimensional orbifold, ω be an exterior n–form with a compact support on $\mathcal{N}$. If the support $\operatorname{supp} \omega$ lies inside the coordinate neighbourhood U_α of some chart $(\tilde{U}_\alpha, \Gamma_\alpha, \varphi_\alpha) \in \mathcal{A}$, then by the definition we let

$$\int\limits_{U_\alpha} \omega = \frac{1}{|\Gamma_\alpha|} \int\limits_{\tilde{U}_\alpha} \omega_\alpha,$$

where $|\Gamma_\alpha|$ is the order of group Γ_α. In the general case the compactness of support $\operatorname{supp} \omega$ implies the existence of a finite open covering $\xi = \{U_k \mid k = 1, \dots, m\}$ of the support $\operatorname{supp} \omega$ by the coordinate neighbourhoods U_k of charts in $\mathcal{A}$ and a finite unity partition relative to ξ; that is, of the family $\{f_k \mid k = 1, \dots, m\}$ of smooth functions on $\mathcal{N}$ such that

(a) $0 \leqslant f_k(x) \leqslant 1$ for all $x \in \mathcal{N}$ and $k \in \{1, \dots, m\}$;
(b) $\operatorname{supp} f_k \subset U_k$ for all $k \in \{1, \dots, m\}$;
(c) $\sum\limits_{k=1}^{m} f_k(x) = 1$ for all $x \in \operatorname{supp} \omega$.

Then the integral of an exterior n–form with a compact support ω on the orbifold $\mathcal{N}$ is defined by the following identity

$$\int\limits_{\mathcal{N}} \omega = \sum_{k=1}^{m} \int\limits_{U_k} f_k \omega. \tag{2.1}$$

It is known that the number $\int\limits_{\mathcal{N}} \omega$ defined by the formula (2.1) is independent of the choice of the covering ξ of the support $\operatorname{supp}\omega$ and the unity partition relative to ξ. Thus, for each smooth n–dimensional orbifold $\mathcal{N}$ the linear operator is well–defined

$$\int\limits_{\mathcal{N}} : \Gamma_c^p(\mathcal{N}) \to \mathbb{R} \colon \ \omega \mapsto \int\limits_{\mathcal{N}} \omega.$$

If $\mathcal{N}$ is compact, then the support of each form ω on $\mathcal{N}$ is a compact set. Therefore, the integral is well–defined for each external n–form ω on a compact n–dimensional orbifold $\mathcal{N}$.

Theorem 2.1. *Let ω be an external n–form with a compact support on an oriented n–dimensional orbifold $\mathcal{N}$. Then the identity*

$$\int\limits_{\mathcal{N}} \omega = \int\limits_{\Delta_n} \omega$$

holds, where Δ_n is the n–dimensional strata of the orbifold $\mathcal{N}$.

Proof. Owing to the identity (2.1) it is sufficient to show that

$$\int\limits_{U_k} f_k \omega = \int\limits_{U_k \cap \Delta_n} f_k \omega \tag{2.2}$$

for all $k \in \{1, \dots, m\}$. Let $(\tilde{U}_k, \Gamma_k, \varphi_k)$ be an orbifold chart such that $\varphi_k(\tilde{U}_k) = U_k$. We observe that

$$\int\limits_{U_k} f_k \omega = \frac{1}{|\Gamma_k|} \int\limits_{\tilde{U}_k} \tilde{f}_k \omega_k, \tag{2.3}$$

where $\tilde{f}_k := f_k \circ \varphi_k$. Since Δ_n is an open and everywhere dense subset in $\mathcal{N}$, the set $\tilde{V}_k := (\varphi_k)^{-1}(U_k \cap \Delta_n)$ is open and everywhere dense in $\tilde{U}_k$. Taking this into consideration as well as the formula (2.3), we obtain the following identities

$$\int\limits_{U_k} f_k \omega = \frac{1}{|\Gamma_k|} \int\limits_{\tilde{U}_k} \tilde{f}_k \omega_k = \frac{1}{|\Gamma_k|} \int\limits_{\tilde{V}_k} \tilde{f}_k \omega_k. \tag{2.4}$$

Since $\tilde{V}_k$ is a Γ_k–invariant set and Γ_k acts on $\tilde{V}_k$ freely, the restriction

$$\varphi_k|_{\tilde{V}_k} : \tilde{V}_k \to U_k \cap \Delta_n$$

is a k–sheeted regular covering. This is why

$$\frac{1}{|\Gamma_k|} \int\limits_{\tilde{V}_k} \tilde{f}_k \omega_k = \int\limits_{U_k \cap \Delta_n} \omega. \tag{2.5}$$

The identities (2.4) and (2.5) imply (2.2). The proof is complete. $\square$

2.4. Volume form of oriented Riemannian orbifold. Let on an oriented orbifold $\mathcal{N}$ with an atlas $\mathcal{A} = \{(\tilde{U}_\alpha, \Gamma_\alpha, \varphi_\alpha) | \alpha \in J\}$ a Riemannian metrics g be defined. Then for each chart $(\tilde{U}_\alpha, \Gamma_\alpha, \varphi_\alpha)$, $\alpha \in J$, of the orbifold $\mathcal{N}$ a Γ_α–invariant Riemannian metrics g_α is defined on $\tilde{U}_\alpha$. Let (y^k_α) be local coordinates on $\tilde{U}_\alpha$. The Riemannian metrics g_α defines the volume form Ω_α. In the local coordinates (y^k_α) the form Ω_α is defined by the identity

$$\Omega_\alpha = \sqrt{\det g_\alpha} dy^1_\alpha \wedge \ldots \wedge dy^n_\alpha,$$

where $\det g_\alpha$ is the determinant of the matrix of metric tensor g_α in the local coordinates (y^k_α). Since $\mathcal{N}$ is an oriented orbifold, all isometries in the group Γ_α, as well as each embedding $\phi_{\alpha\beta} \colon \tilde{U}_\beta \to \tilde{U}_\alpha$, $\alpha, \beta \in J$, preserves the orientation. Therefore, the volume form Ω_α is Γ_α–invariant, and for each embedding $\phi_{\alpha\beta} \colon \tilde{U}_\beta \to \tilde{U}_\alpha$ the forms Ω_α and Ω_β defined on $\tilde{U}_\alpha$ and $\tilde{U}_\beta$ respectively are compatible: $\Omega_\beta = \phi^*_{\alpha\beta}\Omega_\alpha$. Thus, the family $\Omega = \{\Omega_\alpha\}$ defines an n–form on $\mathcal{N}$. This form is called the *volume form* on the orbifold $\mathcal{N}$ of the *given Riemannian metrics* g. We note that the n–form $\Omega = \{\Omega_\alpha\}$ is non–zero on $\mathcal{N}$.

The quantity

$$\mathrm{Vol}(\mathcal{N}) := \int_{\mathcal{N}} \Omega$$

is called the volume of the orbifold $\mathcal{N}$. If the orbifold $\mathcal{N}$ is compact, then $\mathrm{Vol}(\mathcal{N}) < \infty$.

3. Degree of proper maps of orbifolds: various approaches

Let $\mathcal{N}$ and $\mathcal{N}'$ be smooth orbifolds. A smooth map $f \colon \mathcal{N} \to \mathcal{N}'$ of the orbifolds $\mathcal{N}$ and $\mathcal{N}'$ is called proper if for each compact subset $K \subset \mathcal{N}'$ the preimage $f^{-1}(K)$ is compact in $\mathcal{N}$. For a compact orbifold $\mathcal{N}$ each smooth map f is proper.

3.1. Approach by Pasquotto and Rot. We recall the definition of the degree of proper maps of orbifolds given in [12].

Let $f \colon \mathcal{N} \to \mathcal{N}'$ be a smooth map of orbifolds $\mathcal{N}$ and $\mathcal{N}'$. We take an arbitrary point $x \in \mathcal{N}$. Let $f_{\alpha\beta} \colon \tilde{V}_\beta \to \tilde{U}_\alpha$ be a representative of the map f in the charts $(\tilde{V}_\beta, G_\beta, \psi_\beta)$ and $(\tilde{U}_\alpha, \Gamma_\alpha, \varphi_\alpha)$, $x \in V_\beta = \psi_\beta(\tilde{V}_\beta)$, $f(V_\beta) \subset U_\alpha = \varphi_\alpha(\tilde{U}_\alpha)$.

The point $x \in \mathcal{N}$ is called *f–regular* if the differential $df_{\alpha\beta}$ of the map $f_{\alpha\beta}$ at the point $\tilde{x} \in (\psi_\beta)^{-1}(x) \subset \tilde{V}_\beta$ is a surjective linear map. It can be shown that the definition of the f–regular point is independent of the choice of charts $(\tilde{V}_\beta, G_\beta, \psi_\beta)$ and $(\tilde{U}_\alpha, \Gamma_\alpha, \varphi_\alpha)$, the representative $f_{\alpha\beta}$ of the map f in these charts and the point $\tilde{x} \in (\psi_\beta)^{-1}(x)$. The points *$f$–singular* if they are not f–regular.

If all points of the orbifold $\mathcal{N}$ are f–regular, then the map $f \colon \mathcal{N} \to \mathcal{N}'$ is called *regular*.

A point $y \in \mathcal{N}'$ is called the *regular value* if each $x \in f^{-1}(y)$ is a f–regular point. We denote by $\mathrm{Reg}(f)$ the set of all regular values of the mapping f.

The following analogue of Sard theorem was proved in [4, Thm. 4.1].

Theorem 3.1. *Let $f \colon \mathcal{N} \to \mathcal{N}'$ be a smooth map of orbifolds. Then the set* $\mathrm{Reg}(f)$ *of regular values of the map f is everywhere dense in $\mathcal{N}'$.*

Let $f \colon \mathcal{N} \to \mathcal{N}'$ be a smooth proper map of n–dimensional oriented orbifolds $(\mathcal{N}, \mathcal{A})$ and $(\mathcal{N}', \mathcal{A}')$. Let $y \in \mathrm{Reg}(f)$ and $x \in f^{-1}(y)$. We consider linearized charts $(\tilde{V}_\beta, G_\beta, \psi_\beta)$ and $(\tilde{U}_\alpha, \Gamma_\alpha, \varphi_\alpha)$ at x and y, respectively. Let $f_{\alpha\beta}$ be the representative of the map f in these charts. Since the orbifolds $\mathcal{N}$ and $\mathcal{N}'$ are oriented, we can suppose that the Jacobians of the matrices of passage from one coordinates to the others at the points x and y are positive. If $(\widetilde{x}^m_\beta)$ and $(\widetilde{y}^l_\alpha)$ are local coordinates in these charts, then the sign of the Jacobian $\det \left(\frac{\partial \widetilde{y}^l_\alpha}{\partial \widetilde{x}^m_\beta} \right)_x$ of the map $\widetilde{y}^l_\alpha = \widetilde{y}^l_\alpha(\widetilde{x}^m_\beta)$ in $0 \in \tilde{V}_\beta = \mathbb{R}^n$ is independent of the choice of charts $(\tilde{V}_\beta, G_\beta, \psi_\beta)$ and

$(\tilde{U}_\alpha, \Gamma_\alpha, \varphi_\alpha)$, and therefore, the sign $\operatorname{sgn}\left(\det\left(\frac{\partial \widetilde{y}^l_\alpha}{\partial \widetilde{x}^m_\beta}\right)_x\right)$ is completely determined by the point x.

Let $(\Gamma_\alpha)_y$ and $(G_\beta)_x$ be stationary subgroups of the groups Γ_α and G_β at the points y and x, respectively. The number

$$\deg(f;y) := \sum_{x\in f^{-1}(y)} \operatorname{sgn}\left(\det\left(\frac{\partial \widetilde{y}^l_\alpha}{\partial \widetilde{x}^m_\beta}\right)_x\right)\frac{|(\Gamma_\alpha)_y|}{|(G_\beta)_x|} \tag{3.1}$$

is called the degree of map f at the map y.

In [12] the product $\mathcal{N}\times[0,1]$ is treated as an orbifold with the $(n+1)$–dimensional strata $\Delta_n\times[0,1]$, where Δ_n is the set of regular points in $\mathcal{N}$. Then the orbifold group $\Gamma_{(x,t)}$ of a point $(x,t)\in\mathcal{N}\times[0,1]$ is isomorphic to Γ_x. Let $f,\ g\colon \mathcal{N}\to\mathcal{N}'$ be two proper maps of orbifolds. If there exist a proper map of orbifolds $F\colon \mathcal{N}\times[0,1]\to\mathcal{N}'$ such that

$$F|_{\mathcal{N}\times\{0\}}(x,0) = f(x), \qquad F|_{\mathcal{N}\times\{1\}}(x,1) = g(x) \quad \forall x\in\mathcal{N},$$

then f and g are called smoothly homotopic.

According to [12, Thm. 3.10], for a proper map of orbifolds $f\colon\mathcal{N}\to\mathcal{N}'$ the number $\deg(f;y)$ is independent of the choice of $y\in\operatorname{Reg}(f)$, and $\deg(f;y)$ is a homotopic invariant. Thus, the degree of proper map $f\colon\mathcal{N}\to\mathcal{N}'$ of oriented orbifolds of same dimension is the number defined by the formula

$$\deg(f) = \deg(f;y),$$

where y is a regular value of the map f.

3.2. New approach. Let $f\colon\mathcal{N}_1\to\mathcal{N}_2$ be a proper map of oriented n–dimensional orbifolds $\mathcal{N}_1$ and $\mathcal{N}_2$.

We recall that the subset $S\subset X$ is everywhere dense in the topological space X if and only if $U\cap S\neq\emptyset$ for each open subset $U\subset X$.

For the sake of completeness, we provide a proof of the next statement, which is used in what follows.

Lemma 3.1. *If $f\colon\mathcal{N}_1\to\mathcal{N}_2$ is a proper map of oriented n–dimensional orbifolds, then the intersection $\operatorname{Reg}(f)\cap\Delta_n^{\mathcal{N}_2}$ is an open everywhere dense subset in $\mathcal{N}_2$.*

Proof. According to [11, Cor.], each continuous proper map $f\colon X\to Y$ of a topological space X into a metrizable space Y is closed. Since each orbifold is metrizable, the proper smooth map between orbifolds $f\colon\mathcal{N}_1\to\mathcal{N}_2$ is closed. Since the set of f–regular points of smooth map of orbifolds is open in $\mathcal{N}_1$, the set Sing of all f–singular points of the map f is closed in $\mathcal{N}_1$. Therefore, the image $f(\mathrm{Sing})$ is a closed subset in $\mathcal{N}_2$. We observe that

$$\mathcal{N}_2 = f(\mathrm{Sing})\bigsqcup\operatorname{Reg}(f),$$

and this is why $\operatorname{Reg}(f)$ is an open subset in $\mathcal{N}_2$. According to the Sard theorem for orbifolds (Theorem 3.1), the set $\operatorname{Reg}(f)$ is everywhere dense in $\mathcal{N}_2$. Since $\Delta_n^{\mathcal{N}_2}$ is open and everywhere dense in $\mathcal{N}_2$, we obtain that $\operatorname{Reg}(f)\cap\Delta_n^{\mathcal{N}_2}$ is also open and everywhere dense in $\mathcal{N}_2$. □

We take $y\in\operatorname{Reg}(f)\cap\Delta_n^{\mathcal{N}_2}$. Since $y\in\operatorname{Reg}(f)$, each point $x\in f^{-1}(y)$ is f–regular. Moreover, since $y\in\Delta_n^{\mathcal{N}_2}$, according to [4, Thm. 4.2] (see also [12, Cor. 2.7]), each point $x\in f^{-1}(y)$ is a regular point of the orbifold $\mathcal{N}_1$, that is $x\in\Delta_n^{\mathcal{N}_1}$ (in [12] such points x are called smooth). Let $f^{-1}(y)=\{x_1,x_2,\dots,x_k\}$. Then $f(x_i)=y$, $1\leqslant i\leqslant k$, where $x_i\in\Delta_n^{\mathcal{N}_1}$ and $y\in\Delta_n^{\mathcal{N}_2}$. Therefore, at each point x_i there exists a neighbourhood $U_i\subset\Delta_n^{\mathcal{N}_1}$ such that the restriction $f|_{U_i}\colon U_i\to f(U_i)\subset\operatorname{Reg}(f)\cap\Delta_n^{\mathcal{N}_2}$ is a diffeomorphism. Let (x_i^m) and (y^l) be local coordinates in neighbourhoods U_i and $f(U_i)$ of the points x_i and y, respectively. We denote by $\det\left(\frac{\partial y^l}{\partial x_i^m}\right)_{x_i}$

the Jacobian of the map $f|_{U_i}$ at the point x_i. We note that $\det\left(\frac{\partial y^l}{\partial x_i^m}\right)_{x_i} \neq 0$. We define the degree of the proper map f at an arbitrary point $y \in \mathrm{Reg}(f) \cap \Delta_n^{\mathcal{N}_2}$ by the formula

$$\mathrm{Deg}(f; y) := \sum_{i=1}^{k} \operatorname{sgn} \det \left(\frac{\partial y^l}{\partial x_i^m}\right)_{x_i}. \tag{3.2}$$

By the definition we let

$$\mathrm{Deg}(f) := \mathrm{Deg}(f; y). \tag{3.3}$$

The identities (3.1) and (3.2) imply that for each point $y \in \mathrm{Reg}(f) \cap \Delta_n^{\mathcal{N}_2}$ the identity $\mathrm{Deg}(f; y) = \deg(f; y)$ holds. According to [12, Thm. 3.10], the quantity $\deg(f) = \deg(f; y)$ is independent neither of the choice of the point $y \in \mathrm{Reg}(f)$, no of the homotopic class of the map f. By the identity $\mathrm{Deg}(f; y) = \deg(f; y)$ we have

$$\mathrm{Deg}(f) = \deg(f). \tag{3.4}$$

Therefore, the proposed definition of the degree is well–defined, that is, $\mathrm{Deg}(f)$ is independent of the choice of the point $y \in \mathrm{Reg}(f) \cap \Delta_n^{\mathcal{N}_2}$, moreover, the value $\mathrm{Deg}(f)$ is invariant with respect to the homotopic equivalence. Owing to the identity $\mathrm{Deg}(f) = \deg(f)$ in what follows we denote $\mathrm{Deg}(f)$ by $\deg(f)$.

The proposed definition of the degree of a mapping, defined by the formulas (3.2) and (3.3), is simpler than the definition by Pasquotto and Rot since it does not involve the orbifold groups. At the same time, our definition is equivalent to the definition by Pasquotto and Rot.

Thus, we have the following statement.

Proposition 3.1. *Let $f\colon \mathcal{N}_1 \to \mathcal{N}_2$ be a proper map of orbifolds of dimensions n, $M_2 := \mathrm{Reg}(f) \cap \Delta_n^{\mathcal{N}_2}$, $M_1 := f^{-1}(M_2)$ and $F := f|_{M_1}$. Then $F\colon M_1 \to M_2$ is a proper map of open submanifolds into $\mathcal{N}_1$ and $\mathcal{N}_2$, respectively, which can be non–connected, and the degrees* $\deg(f)$ *and* $\deg(F)$ *are equal*

$$\deg(f) = \deg(F). \tag{3.5}$$

3.3. Proof of Theorem 1.1. Let $M_2 := \mathrm{Reg}(f) \cap \Delta_n^{\mathcal{N}_2}$, $M_1 := f^{-1}(M_2)$ and $F := f|_{M_1}$. According to Proposition 3.1, $\deg(F) = \deg(f)$. By Lemma 3.1, M_2 is an open everywhere dense submanifold in $\mathcal{N}_2$. We observe that M_2 is open everywhere dense manifold in $\Delta_n^{\mathcal{N}_2}$. Therefore, as it is known from the integration theory on manifolds, the following identity holds:

$$\int_{\Delta_n^{\mathcal{N}_2}} \omega = \int_{M_2} \omega.$$

On the other hand, Theorem 2.1 implies

$$\int_{\mathcal{N}_2} \omega = \int_{\Delta_n^{\mathcal{N}_2}} \omega.$$

Taking this into consideration as well as the identity $\deg(F) = \deg(f)$, we obtain

$$\deg(f) \int_{\mathcal{N}_2} \omega = \deg(F) \int_{M_2} \omega. \tag{3.6}$$

We are going to prove that

$$\int_{\mathcal{N}_1} f^*\omega = \int_{M_1} f^*\omega. \tag{3.7}$$

According to Theorem 2.1, we have

$$\int\limits_{\mathcal{N}_1} f^*\omega = \int\limits_{\Delta_n^{\mathcal{N}_1}} f^*\omega.$$

Let us confirm the identity

$$\int_{\Delta_n^{\mathcal{N}_1}} f^*\omega = \int_{M_1} f^*\omega. \tag{3.8}$$

We note that the points in $\Delta_n^{\mathcal{N}_1}$ can be both f–singular and f–regular.

Let a point $x \in \Delta_n^{\mathcal{N}_1}$ be a f–singular point of the map f. Let $(\tilde{V}_\beta, G_\beta, \psi_\beta)$ and $(\tilde{U}_\alpha, \Gamma_\alpha, \varphi_\alpha)$ be linearized charts in x and $f(x)$ with the coordinate neighbourhoods V_β and U_α, respectively. Without loss of generality we suppose that $f(V_\beta) \subset U_\alpha$. We denote by $f_{\alpha\beta}\colon \tilde{V}_\beta \to \tilde{U}_\alpha$ the representative of f in these charts. We observe that $G_\beta = \{\mathrm{id}_{\tilde{V}_\beta}\}$ and the n–form ω defines the n–form ω_α on $\tilde{U}_\alpha$. Therefore, the n–form $(f_{\alpha\beta})^*\omega_\alpha$ is the representative of the n–form $f^*\omega$ on $\tilde{V}_\beta$. Since the point $x = \psi_\beta(0)$ is f–singular, the differential $df_{\alpha\beta}\colon T_0\tilde{V}_\beta \to T_0\tilde{U}_\alpha$ of the map $f_{\alpha\beta}$ at 0 is not an isomorphism of vector spaces. This means that the images $df_{\alpha\beta}(X_1)$, $df_{\alpha\beta}(X_2), \dots, df_{\alpha\beta}(X_n)$ of arbitrary n vectors $X_1, X_2, \dots, X_n \in T_0\tilde{V}_\beta$ are linearly dependent and this is why the n–form $(f_{\alpha\beta})^*\omega_\alpha$ vanishes at the point $0 \in \tilde{V}_\beta$. Therefore, the n–form $f^*\omega$ vanishes at the point x. Thus, at each f–singular point $x \in \Delta_n^{\mathcal{N}_1}$ the n–form $f^*\omega$ vanishes and we can neglect the f–singular point while integrating over $\Delta_n^{\mathcal{N}_1}$.

Let $x \in \Delta_n^{\mathcal{N}_1}$ be a f–regular point of the map f. Since the set of f–regular point is open in $\mathcal{N}_1$, there exists a chart $(\tilde{V}, \Gamma_V, \varphi_V)$ at $x \in V$ with a coordinate neighbourhood $V \subset \Delta_n^{\mathcal{N}_1}$ such that each point in V is f–regular. We denote by $\tilde{f}\colon \tilde{V} \to \tilde{U}$ the representative of the map f in the charts $(\tilde{V}, \Gamma_V, \varphi_V)$ and $(\tilde{U}, \Gamma_U, \varphi_U)$, where $(\tilde{U}, \Gamma_U, \varphi_U)$ is the chart at the point $f(x)$ with the coordinate neighbourhood U. Without loss of generality we can suppose that $f(V) = U$. Then $\tilde{f}$ is a diffeomorphism of $\tilde{V}$ onto the image $\tilde{f}(\tilde{V}) = \tilde{U}$. According to Lemma 3.1, the intersection $M_2 = \mathrm{Reg}(f) \cap \Delta_n^{\mathcal{N}_2}$ is open and everywhere dense in $\mathcal{N}_2$. Therefore, the intersection $U \cap M_2$ is open and everywhere dense in U, and it is sufficient to calculate the integral

$$\int\limits_V f^*\omega$$

only over the set of points $x \in V$, for which $f(x) \in U \cap M_2$, that is, $x \in M_1$. Thus,

$$\int\limits_V f^*\omega = \int\limits_{V \cap M_1} f^*\omega. \tag{3.9}$$

Since $x \in \Delta_n^{\mathcal{N}_1}$ is an arbitrary f–regular point, the identity (3.9) implies the identity (3.8).

Owing to the identities $\deg(f) = \deg(F)$, (3.6) and (3.7), the proof of Theorem 1.1 is reduced to confirmation of the identity

$$\int\limits_{M_1} f^*\omega = \deg(f) \int\limits_{M_2} \omega, \tag{3.10}$$

where the support $\operatorname{supp}\omega$ of the form ω lies in M_2.

Since the support $\operatorname{supp}\omega$ of the form ω is a compact subset in M_2, there exists a finite covering $\xi = \{U_k\}_{k=1,\dots,m}$ of the support $\operatorname{supp}\omega$ by the coordinate neighbourhoods U_k. Let $\{f_k\}_{k=1,\dots,m}$ be the family of functions on M_2 defining a finite unity partition relative to the covering ξ. By the defintion,

$$\int\limits_{M_2} \omega = \sum_{k=1}^{m} \int\limits_{U_k} f_k\omega,$$

where $\operatorname{supp} f_k\omega \subset U_k$.

By Proposition 3.1, the map $F = f|_{M_1}$ is proper, therefore, the support $\operatorname{supp} f^*\omega$ is also compact. Moreover, $\eta = \{V_k = f^{-1}(U_k)\}_{k=1,\ldots,m}$ is a finite covering of $\operatorname{supp} f^*\omega$.

It is easy to see that the family $\{g_k := f_k \circ f\}_{k=1,\ldots,m}$ is a finite unity partition relative to the covering η, and $\operatorname{supp}(g_k f^*\omega) \subset V_k$ $\forall k = 1,\ldots,m$. This is why

$$\int\limits_{M_1} f^*\omega = \sum_{k=1}^{m} \int\limits_{V_k} g_k f^*\omega.$$

Thus, to prove the identity (3.10), it is sufficient to confirm the identity

$$\int\limits_{V_k} g_k f^*\omega = \deg(f) \int\limits_{U_k} f_k\omega.$$

We consider an arbitrary chart $(\tilde{U}, \Gamma_U, \varphi_U)$ with the coordinate neighbourhood $U = \varphi_U(\tilde{U})$ such that $\operatorname{supp}\omega \subset U$. In this case $\operatorname{supp} f^*\omega \subset W = f^{-1}(U)$. Let us confirm the identity

$$\int\limits_{W} f^*\omega = \deg(f) \int\limits_{U} \omega. \tag{3.11}$$

Generally speaking, the manifold W is non–connected. We denote by W_i, $i = 1,\ldots,s$, the connected components of the manifold W. For each $i = 1,\ldots,s$ the restriction $f|_{W_i} \colon W_i \to U$ is a diffeomorphism of simply–connected manifolds. Without loss of generality we can suppose that W_i is the coordinate neighbourhood of the manifold M_1. Let (x_i^m) and (y^l) be local coordinates in W_i and U, respectively. We write the diffeomorphism $f|_{W_i} \colon W_i \to U$ in the local coordinates

$$y^l = y^l(x_i^m) = y^l(x_i^1, x_i^2, \ldots, x_i^n), \qquad l, m = 1,\ldots,n. \tag{3.12}$$

In the coordinates (x_i^m) and (y^l) the forms ω and $f^*\omega$ read

$$\omega = \varphi(y^l)dy^1 \wedge dy^2 \wedge \ldots \wedge dy^n,$$

$$f^*\omega = \varphi(y^l(x_i^m)) \det\left(\frac{\partial y^l}{\partial x_i^m}\right) dx_i^1 \wedge dx_i^2 \wedge \ldots \wedge dx_i^n,$$

where $\det\left(\frac{\partial y^l}{\partial x_i^m}\right)$ is the Jacobian of change of variables (3.12). We write the formula of change of variables in a multiple integral

$$\int\limits_{U} \varphi(y^l)dy^1 \wedge dy^2 \wedge \ldots \wedge dy^n = \int\limits_{W_i} \varphi(y^l(x_i^m)) \left|\det\left(\frac{\partial y^l}{\partial x_i^m}\right)\right| dx_i^1 \wedge dx_i^2 \wedge \ldots \wedge dx_i^n. \tag{3.13}$$

We note that

$$\det\left(\frac{\partial y^l}{\partial x_i^m}\right) = \operatorname{sgn}\det\left(\frac{\partial y^l}{\partial x_i^m}\right)\left|\det\left(\frac{\partial y^l}{\partial x_i^m}\right)\right|. \tag{3.14}$$

It follows from (3.13) and (3.14) that

$$\begin{aligned}
\int\limits_{W_i} f^*\omega &= \int\limits_{W_i} \varphi(y^l(x_i^m)) \det\left(\frac{\partial y^l}{\partial x_i^m}\right) dx_i^1 \wedge dx_i^2 \wedge \ldots \wedge dx_i^n \\
&= \operatorname{sgn}\det\left(\frac{\partial y^l}{\partial x_i^m}\right) \int\limits_{W_i} \varphi(y^l(x_i^m)) \left|\det\left(\frac{\partial y^l}{\partial x_i^m}\right)\right| dx_i^1 \wedge dx_i^2 \wedge \ldots \wedge dx_i^n \\
&= \operatorname{sgn}\det\left(\frac{\partial y^l}{\partial x_i^m}\right) \int\limits_{U} \varphi(y^l)dy^1 \wedge dy^2 \wedge \ldots \wedge dy^n = \operatorname{sgn}\det\left(\frac{\partial y^l}{\partial x_i^m}\right) \int\limits_{U} \omega.
\end{aligned}$$

Therefore,

$$\int\limits_{W_i} f^*\omega = \operatorname{sgn}\det\left(\frac{\partial y^l}{\partial x_i^m}\right)\int\limits_{U}\omega. \tag{3.15}$$

Summing up the identities (3.15) over $i \in \{1,\ldots,s\}$ and using the definition of degree of proper map between the smooth manifolds, we obtain the following chain of identities

$$\begin{aligned}\int\limits_{W} f^*\omega &= \sum_{i=1}^{s}\int_{W_i} f^*\omega = \sum_{i=1}^{s}\operatorname{sgn}\det\left(\frac{\partial y^l}{\partial x_i^m}\right)\int\limits_{U}\omega \\ &= \left(\sum_{i=1}^{s}\operatorname{sgn}\det\left(\frac{\partial y^l}{\partial x_i^m}\right)\right)\int\limits_{U}\omega = \deg(f)\int\limits_{U}\omega.\end{aligned}$$

Thus, the formula (3.11) is valid and this completes the proof of Theorem 1.1.

3.4. Proof of Theorem 1.2. Let $f\colon \mathcal{N}_1 \to \mathcal{N}_2$ be a proper mapping of connected oriented smooth orbifolds of the same dimension n, Ω be the volume form on a compact orbifold $\mathcal{N}_2$ defined by the Riemannian metrics g, $\mathrm{Vol}(\mathcal{N}_2)$ be the volume of the orbifold $\mathcal{N}_2$, see Section 2.4. Since the orbifold $\mathcal{N}_2$ is compact, we have $\mathrm{Vol}(\mathcal{N}_2) < \infty$.

According to the proof of Theorem 1.1, the n–form $f^*\Omega$ vanishes at each f–singular point. Therefore, generally speaking, $f^*\Omega$ is not the volume form on $\mathcal{N}_1$, see Example 3.1.

Replacing the integral in the right hand side of the identity (1.1) by $\mathrm{Vol}(\mathcal{N}_2)$, we obtain the identity from Theorem 1.2.

$$\deg(f) = \frac{1}{\mathrm{Vol}(\mathcal{N}_2)}\int\limits_{\mathcal{N}_1} f^*\Omega.$$

Owing to Theorem 1.1, the right hand side of this identity is independent on the choice of the Riemannian metrics g on $\mathcal{N}_2$. This completes the proof of Theorem 1.2.

3.5. Proof of Corollary 1.1. We recall the notion of covering map for orbifolds [15, Ch. 13]. A smooth map of orbifolds $f\colon \mathcal{N}_1 \to \mathcal{N}_2$ is called the covering if for each point $x \in \mathcal{N}_2$ there exists a chart $(\tilde{U}, \Gamma, \varphi)$ with the coordinate neighbourhood $U \ni x$ such that for each connected component U' of the preimage $f^{-1}(U)$ there exists a homeomorphism $q'\colon \tilde{U}/\Gamma' \to U'$ such that $f|_{U'} \circ \varphi' = \varphi$, where Γ' is some subgroup of the group Γ, and $\varphi'\colon \tilde{U} \to U'$ is the composition of the map q' with the quotient map $\tilde{U} \to \tilde{U}/\Gamma'$. The chart $(\tilde{U}, \Gamma, \varphi)$ in this definition is called regularly covered.

Let $f\colon \mathcal{N}_1 \to \mathcal{N}_2$ be a covering map. The covering transformation is an automorphism $h\colon \mathcal{N}_1 \to \mathcal{N}_1$ of the covering orbifold $\mathcal{N}_1$ such that $f \circ h = f$. The set $G(f)$ of all covering transformations of the covering $f\colon \mathcal{N}_1 \to \mathcal{N}_2$ defines a group. The covering $f\colon \mathcal{N}_1 \to \mathcal{N}_2$ is called regular if $\mathcal{N}_2 = \mathcal{N}_1/G(f)$.

The number of sheets of the covering map $f\colon \mathcal{N}_1 \to \mathcal{N}_2$ is defined as the number of preimages of a point in $\Delta_n^{\mathcal{N}_2}$.

We proceed to the proof of Corollary 1.1. Let $f\colon \mathcal{N}_1 \to \mathcal{N}_2$ be a surjective map between compact orbifolds $\mathcal{N}_1$ and $\mathcal{N}_2$. By the compactness of the orbifold $\mathcal{N}_1$ the map f is proper. Let Ω be the volume form on the compact orbifold $\mathcal{N}_2$ defined by the Riemannian metrics g. Since $f\colon \mathcal{N}_1 \to \mathcal{N}_2$ is a regular map, $f^*\Omega$ is the volume form on $\mathcal{N}_1$. We denote by $\mathrm{Vol}(\mathcal{N}_1)$ the volume of the compact orbifold $\mathcal{N}_1$. Then the formula (1.2) can be written in the following form

$$\deg(f) = \frac{\mathrm{Vol}(\mathcal{N}_1)}{\mathrm{Vol}(\mathcal{N}_2)}$$

that completes the proof of Statement 2) in Corollary 1.1.

Since $f\colon \mathcal{N}_1 \to \mathcal{N}_2$ is a surjective regular map, all points of the orbifold $\mathcal{N}_2$ are regular values of the map f, that is, $\mathrm{Reg}(f) = \mathcal{N}_2$, that is, $M_2 = \Delta_n^{\mathcal{N}_2}$. Thus, $f|_{M_1}\colon M_1 \to M_2$ is a finite–sheeted regular covering. We denote by k the number of sheets of this covering. By (3.10) we obtain $k = \deg(f)$. This proves Statement 1) of Corollary 1.1.

3.6. Examples.

Example 3.1. Let $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ be a map of the circumference $\mathbb{S}^1$ onto itself as it is shown in Figure 1. The map f is a surjective proper map. We note that the preimages of the points z_1 and z_2 are compact subsets in $\mathbb{S}^1$ and are segments: $f^{-1}(z_i) = [x_i, y_i]$, $i = 1, 2$. The points of these segments are f–singular, while other points of the circumference $\mathbb{S}^1$ are f–regular.

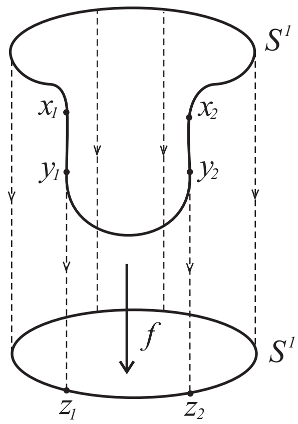

FIGURE 1. The form $f^*\Omega$ is not a volume form.

Let on $\mathbb{S}^1$ a volume form Ω be defined. According to the proof of Theorem 1.1, the form $f^*\Omega$ vanishes at each point of the segments $[x_1, y_1]$ and $[x_2, y_2]$. Thus, $f^*\Omega$ is not the volume form on $\mathbb{S}^1$. It is obvious that the degree of map f is equal to 1.

Example 3.2. Let $\mathcal{N}_1$ be a two–dimensional sphere $\mathbb{S}^2$, γ be a rotation of the sphere $\mathbb{S}^2$ by the angle $\frac{2\pi}{k}$, $k \in \mathbb{N}$, $k \geqslant 2$, around the straight line l passing the center O of the sphere $\mathbb{S}^2$. The group Γ generated by the map γ is isomorphic to the group $\mathbb{Z}_k$ of residues classes modulo k. The quotient space $\mathcal{N}_2 = \mathbb{S}^2/\Gamma$ is homemorphic to the sphere $\mathbb{S}^2$ and is a two–dimensional orbifold, see Figure 2.

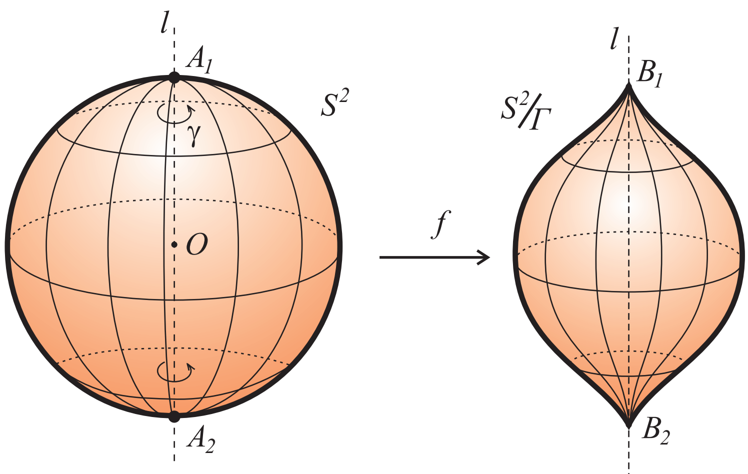

FIGURE 2. k–sheeted covering $f\colon \mathbb{S}^2 \to \mathbb{S}^2/\Gamma$.

The quotient map $f\colon \mathcal{N}_1 \to \mathcal{N}_2$ is a k–sheeted covering. Since the group Γ preserves only two points $A_1,\ A_2 \in l \cap \mathbb{S}^2$, the orbifold $\mathcal{N}_2$ has only two singular points $B_1 = f(A_1),\ B_2 = f(A_2)$, while other points of the orbifold $\mathcal{N}_2$ are regular. Thus, $\mathcal{N}_2$ has the stratification $\{\Delta_0^{\mathcal{N}_2}, \Delta_2^{\mathcal{N}_2}\}$, where $\Delta_0^{\mathcal{N}_2} = \{B_1, B_2\}$, $\Delta_2^{\mathcal{N}_2} = \mathcal{N}_2 \setminus \Delta_0^{\mathcal{N}_2}$.

We note that all points of the orbifold $\mathcal{N}_2$ are regular values: $\mathrm{Reg}(f) = \mathcal{N}_2$. Therefore, $M_2 = \Delta_2^{\mathcal{N}_2}$, $M_1 = f^{-1}(M_2) = \mathbb{S}^2 \setminus \{A_1, A_2\}$. According to Corollary 1.1, the degree of map f is equal to k.

Example 3.3. Let $\mathcal{N}_1 = \mathbb{T}^n = \{(z_1,\dots,z_n) \in \mathbb{C}^n \mid |z_k| = 1,\ k = 1,\dots,n\}$ be a n–dimensional torus. We defined the map $\gamma\colon \mathbb{T}^n \to \mathbb{T}^n$ by the formula $\gamma(z_1,\dots,z_n) = (\overline{z_1},\dots,\overline{z_n})$, where $(z_1,\dots,z_n) \in \mathbb{T}^n$, and $\overline{z_k}$ is a complex number conjugate with z_k, $k = 1,\dots,n$. The group Γ generated by the map γ is isomorphic to the group $\mathbb{Z}_2$ of residues classes modulo 2. The quotient space $\mathcal{N}_2 = \mathbb{T}^n/\Gamma$ is a n–dimensional compact oriented orbifold. The group Γ preserves 2^n points $(z_1,\dots,z_n) \in \mathbb{T}^n$, where $z_k \in \{1,-1\}$, $k = 1,\dots,n$. This is why the orbifold $\mathcal{N}_2$ has 2^n singular points, the orbifold group of which is isomorphic to $\mathbb{Z}_2$. In total, these singular points form the zero–dimensional strata $\Delta_0^{\mathcal{N}_2}$, other points of the orbifold $\mathcal{N}_2$ are regular. Thus, $\mathcal{N}_2$ has a the stratification $\{\Delta_0^{\mathcal{N}_2}, \Delta_n^{\mathcal{N}_2}\}$.

The quotient space $f\colon \mathcal{N}_1 \to \mathcal{N}_2$ is a 2–sheeted regular covering. All points of the orbifold $\mathcal{N}_2$ are regular values of the map f, that is, $\mathrm{Reg}(f) = \mathcal{N}_2$. It follows from Corollary 1.1 that the degree of the map f is equal to 2 and $\mathrm{Vol}(\mathcal{N}_1) = 2\,\mathrm{Vol}(\mathcal{N}_2)$.

For $n = 4$, the orbifold $\mathcal{N}_2$ has 16 singular points and is called the Kummer surface [1, Ex. 1.9].

For $n = 2$ the orbifold $\mathcal{N}_2$ has 4 singular points, is homemorphic to the 2–dimensional sphere $\mathbb{S}^2$ and is called "Pillow", see Figure 3.

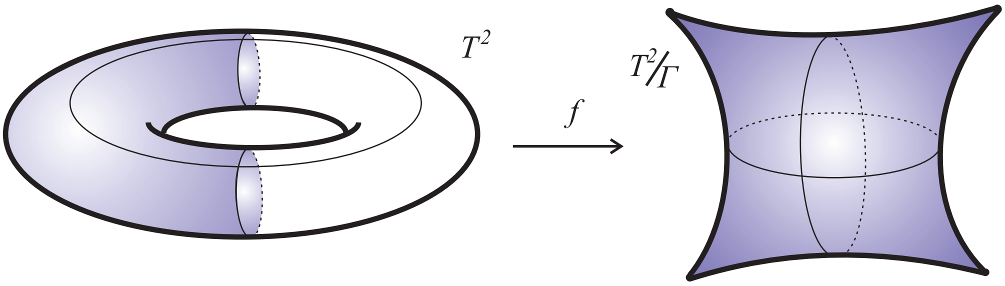

FIGURE 3. 2–sheeted covering of the orbifold $\mathcal{N}_2$ by the torus $\mathbb{T}^2$.

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