

EXISTENCE OF PROPAGATION CONE FOR ONE-DIMENSIONAL WAVE INTEGRO-DIFFERENTIAL OPERATOR WITH FRACTIONAL-EXPONENTIAL MEMORY FUNCTION

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Abstract. We study a linear Volterra integro-differential operator, which is a one-dimensional wave linear partial differential operator perturbed by an integral operator of the Volterra convolution. The kernel of integral operator is the sum of fractional-exponential functions (Rabotnov functions) with positive coefficients. We establish that the support of fundamental solution of the considered integro-differential operator is localized in the propagation cone of the corresponding one-dimensional wave differential operator. The corresponding Volterra integro-differential equation describes the oscillations of one-dimensional viscous-elastic rod, the heat propagation in media with memory (Gurtin — Pipkin equation) and a series of other important applications.

Keywords: Volterra integro-partial differential operator, fundamental solution, Fourier — Laplace transform, fractional-exponential function.

Mathematics Subject Classification: 47G20, 45K05, 35R09

1. INTRODUCTION

In this paper we investigate a linear Volterra integro-partial differential operator being a one-dimensional wave linear partial differential operator perturbed by a Volterra convolution integral operator. Operators of this type have numerous applications in problems of hereditary mechanics, the theory of highly inhomogeneous media, heat conduction in media with memory, the kinetic theory of gases, biology, medicine, and other fields.

Nowadays, there is a rich literature devoted to the study of Volterra integro-differential equations and related problems arising in numerous applications, see, for example, [3], [4], [6]–[14] and the references therein.

The studied integro-differential operator studied in this paper is called the one-dimensional wave integro-differential operator with a fractional-exponential memory function. The kernel of the integral convolution operator is a sum of fractional-exponential functions (Rabotnov functions, see [7]) with positive coefficients.

In the paper we establish the existence and uniqueness of a fundamental solution supported in a cone for the studied integro-differential operator. The proof of the main assertions of the paper is based on the application of the Paley — Wiener — Vladimirov criterion, see [2], [5], which establishes an isomorphism between the space of tempered distributions supported in a cone and the space of functions analytic in a tubular domain.

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The paper consists of nine sections. The first section is an introduction. The second section contains the formulation of the problem. The third and fourth sections are devoted to the formulation of definitions of cones, tubular domains, the Fourier transform, spaces of generalized functions supported in a cone, spaces of functions analytic in a tubular domain, and the formulation of Paley — Wiener theorem, along with references to the relevant sources. The fifth section contains the formulations of main results of paper (three theorems and one lemma). The remaining four sections are devoted to the proofs of main results, as well as to the proofs of auxiliary lemmas.

The presented results are a continuation and development of the studies published in [3], [8], [9], [10]–[14]. Using the obtained results, one can establish a finite propagation velocity of perturbations for the corresponding one-dimensional wave integro-differential equation with a fractional-exponential memory function.

2. FORMULATION OF PROBLEM

We denote by $\mathcal{D} := \mathcal{D}(\mathbb{R}^n) := C_0^\infty(\mathbb{R}^n)$ the space of all compactly supported infinitely differentiable in $\mathbb{R}^n$ functions, $\mathcal{D}' = \mathcal{D}'(\mathbb{R}^n)$ is the space of all generalized functions defined on the space $\mathcal{D}(\mathbb{R}^n)$, $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ is the space of fast decaying functions, $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions [1, Ch. 2].

We consider the second order linear Volterra integro-differential equation

$$\mathcal{L}_1(D_t, D_x)u(t, x) := \frac{\partial^2}{\partial t^2}u(t, x) - a^2 \frac{\partial^2}{\partial x^2}u(t, x) + K(t) * \frac{\partial^2}{\partial x^2}u(t, x) = f(t, x), \quad (2.1)$$

where $f \in \mathcal{D}'(\mathbb{R}^2)$, $a > 0$, the symbol $*$ stands for the convolution of generalized function in the variable t [1, Ch. 2, Sect. 7.4], the function $K(t)$ can be represented as the sum

$$K(t) = \sum_{i=1}^N c_i K_i(t), \quad c_i > 0, \quad i = 1, \dots, N, \quad (2.2)$$

$K_i(t)$ are fractional-exponential functions (Rabotnov functions) [7, Ch. I], which read

$$K_i(t) = \begin{cases} t^{-\alpha} \sum_{n=0}^{\infty} \frac{(-\beta_i)^n t^{n(1-\alpha)}}{\Gamma[(n+1)(1-\alpha)]}, & t > 0, \\ 0, & t < 0, \end{cases} \quad (2.3)$$

where $0 < \alpha < 1$, $\Gamma(\cdot)$ is the Euler Gamma function, $\beta_i > 0$, $i = 1, \dots, N$.

Remark 2.1. For $t > 0$, the function $K_i(t)$ defined by the formula (2.3) is the generalized Mittag-Leffler function [7, Ch. I]:

$$K_i(t) = t^{-\alpha} E_{1-\alpha, 1-\alpha}(-\beta_i t^{1-\alpha}), \quad t > 0.$$

Equation (2.1) is the one-dimensional integro-differential equation with a fractional-exponential memore function (wave equation with memory).

The second order Volterra integro-differential operator

$$\mathcal{L}_1(D_t, D_x) := \frac{\partial^2}{\partial t^2} - a^2 \frac{\partial^2}{\partial x^2} + K(t) * \frac{\partial^2}{\partial x^2} \quad (2.4)$$

in the left hand side of Equation (2.1) is called the one-dimensional wave integro-differential operator with a fractional-exponential memore function (wave operator with memory).

Definition 2.1. A generalized function $\mathcal{E}_1(t, x) \in \mathcal{D}'(\mathbb{R}^2)$ is called the fundamental solution of operator $\mathcal{L}_1(D_t, D_x)$ if

$$\mathcal{L}_1(D_t, D_x)\mathcal{E}_1(t, x) = \delta(t, x). \quad (2.5)$$

3. FOURIER — LAPLACE TRANSFORM IN TUBULAR DOMAIN

In $\mathbb{R}^2$ we consider a closed cone

$$\Gamma_1 = \{(t, x) \in \mathbb{R}^2 \mid t \geq 0, x \in \mathbb{R}, at \geq |x|\}, \quad (3.1)$$

where $a > 0$. Then the cone

$$\Gamma_1^* = \{(p, q) \in \mathbb{R}^2 \mid pt + qx \geq 0, \forall (t, x) \in \Gamma_1\} \quad (3.2)$$

is called the *dual cone* of the cone Γ_1 . We denote

$$C_1 := \text{int } \Gamma_1^* = \{(p, q) \in \mathbb{R}^2 \mid pt + qx > 0, \forall (t, x) \in \Gamma_1\}. \quad (3.3)$$

Similarly, we can consider the closed cone

$$\Gamma_0 := \{t \in \mathbb{R} \mid t \geq 0\} =: \Gamma_0^* \quad (3.4)$$

in $\mathbb{R}$ and we denote

$$C_0 := \text{int } \Gamma_0^* = \{p \in \mathbb{R} \mid p > 0\}. \quad (3.5)$$

Definition 3.1 ([2, Ch. II], [5, Sect. 12]). *The set*

$$T^{C_1} := \mathbb{R}^2 + iC_1 = \{(\lambda, \xi) \in \mathbb{C}^2 \mid (\text{Re } \lambda, \text{Re } \xi) \in \mathbb{R}^2, (\text{Im } \lambda, \text{Im } \xi) \in C_1\}. \quad (3.6)$$

is called the tubular domain in $\mathbb{C}^2$ with the base C_1 .

Remark 3.1. *The definition of tubular domain can be also formulate for the cone C_0 defined by the formula (3.5). In this case*

$$T^{C_0} := \mathbb{R} + iC_0 = \{\lambda \in \mathbb{C} \mid \text{Re } \lambda \in \mathbb{R}, \text{Im } \lambda > 0\}. \quad (3.7)$$

Definition 3.2 ([5, Sect. 9, 10]). *The set of generalized functions in $\mathcal{D}'$, the supports of which are located in the cone Γ_1 , is denoted by $\mathcal{D}'(\Gamma_1)$.*

By $\mathcal{S}'(\Gamma_1)$ we denote the set of generalized functions in $\mathcal{S}'$ with the supports in Γ_1 . By S'_{Γ_1} we denote the space of tempered distributions in the cone Γ_1 .

The set of generalized functions $f(t) \in \mathcal{D}'(\mathbb{R})$ vanishing for $t < 0$ is denoted by $\mathcal{D}'_+$ [1, Ch. 7.7].

Statement 3.1 ([5, Sect. 10]). *The spaces $\mathcal{S}'(\Gamma_1)$ and S'_{Γ_1} are isomorphic.*

Definition 3.3 ([2, Ch. II]). *Let $f \in \mathcal{S}'(\Gamma_1)$. The Fourier — Laplace transform $L[f]$ of a generalized function f is the functions of the variables $(\lambda, \xi) \in T^{C_1}$ defined by the formula*

$$L[f](\lambda, \xi) = F[f(t, x)e^{-(t\text{Im } \lambda + x\text{Im } \xi)}](\text{Re } \lambda, \text{Re } \xi), \quad (t, x) \in \Gamma_1, \quad (\lambda, \xi) \in T^{C_1}, \quad (3.8)$$

where F is the Fourier transform of the generalized function.

Remark 3.2 ([5, Sect. 12]). *The Fourier — Laplace transform of a generalized function $f \in \mathcal{S}'(\Gamma_1)$ can be represented as*

$$L[f](\lambda, \xi) = (f(t, x), e^{i(t\lambda + x\xi)}), \quad (t, x) \in \Gamma_1. \quad (3.9)$$

The formula (3.9) makes sense since $e^{i(\lambda t + x\xi)} \in S_{\Gamma_1}$ for $(\lambda, \xi) \in T^{C_1}$.

Definition 3.4 ([5, Sect. 12]). *Let $f(t, x) \in \mathcal{S}'(\Gamma_0)$ in the variable t for each fixed value of x . The Fourier — Laplace transform $L_1[f]$ of the generalized function $f(t, x)$ in the variable t for a fixed value x is the function of variables (λ, x) , where $\lambda \in T^{C_0}$, $x \in \mathbb{R}$, which is defined by the formula*

$$L_1[f](\lambda, x) = F_1[f(t, x)e^{-t\text{Im } \lambda}](\text{Re } \lambda, x), \quad t \in \Gamma_0, \quad \lambda \in T^{C_0}, \quad x \in \mathbb{R}, \quad (3.10)$$

where F_1 is the Fourier transform of a generalized function in the variable t for a fixed value of x .

4. SPACE $H(T^{C_1})$. ISOMORPHISM OF SPACES $\mathcal{S}'(\Gamma_1)$ AND $H(T^{C_1})$

The definitions and statements of this section are borrowed from [5, §12].

We denote by $H^{(\alpha, \beta)}(T^{C_1})$, $\alpha \geq 0$, $\beta \geq 0$, $(\alpha, \beta \in \mathbb{Z})$ the set of functions analytic in the tubular domain T^{C_1} , which obey the estimate

$$|f(\lambda, \xi)| \leq M_f (1 + |\lambda|^2 + |\xi|^2)^{\frac{\alpha}{2}} [1 + \Delta^{-\beta}(\text{Im } \lambda, \text{Im } \xi)], \quad (\lambda, \xi) \in T^{C_1}, \quad (4.1)$$

where

$$\Delta(p, q) = \inf_{\substack{(t, x) \in \Gamma_1 \\ t^2 + x^2 = 1}} (tp + xq),$$

$(p, q) \in C_1$ is the distance from the point $(p, q) \in C_1$ to the boundary of cone C_1 , M_f is a positive constant.

Similarly, we denote by $H^{(\alpha, \beta)}(T^{C_0})$, $\alpha \geq 0$, $\beta \geq 0$, $(\alpha, \beta \in \mathbb{Z})$ the set of functions analytic in the tubular domain T^{C_0} , which satisfy the estimate

$$|f(\lambda)| \leq M_f (1 + |\lambda|^2)^{\frac{\alpha}{2}} [1 + (\text{Im } \lambda)^{-\beta}], \quad \lambda \in T^{C_0}. \quad (4.2)$$

We equip $H^{(\alpha, \beta)}(H(T^{C_1}))$ with the topology in accordance with the estimate (4.1) by means of the norm

$$\|f\|_1^{(\alpha, \beta)} = \sup_{(\lambda, \xi) \in T^{C_1}} \frac{|f(\lambda, \xi)|}{(1 + |\lambda|^2 + |\xi|^2)^{\frac{\alpha}{2}} [1 + \Delta^{-\beta}(\text{Im } \lambda, \text{Im } \xi)]} \quad (4.3)$$

In its turn, we equip $H^{(\alpha, \beta)}(H(T^{C_0}))$ with the topology in accordance with the estimate (4.2) by means of the norm

$$\|f\|_0^{(\alpha, \beta)} = \sup_{\lambda \in T^{C_0}} \frac{|f(\lambda)|}{(1 + |\lambda|^2)^{\frac{\alpha}{2}} [1 + (\text{Im } \lambda)^{-\beta}]} \quad (4.4)$$

Remark 4.1. The spaces $H^{(\alpha, \beta)}(T^{C_i})$, $i = 0, 1$, are Banach. Moreover, if

$$\alpha' \geq \alpha, \quad \beta' \geq \beta,$$

then

$$\|f\|_i^{(\alpha', \beta')} \leq \|f\|_i^{(\alpha, \beta)}$$

and, therefore,

$$H^{(\alpha, \beta)}(H(T^{C_i})) \subset H^{(\alpha', \beta')}(H(T^{C_i})), \quad i = 0, 1,$$

and the embedding is continuous.

We denote

$$H(T^{C_i}) := \bigcup_{\alpha \geq 0, \beta \geq 0} H^{(\alpha, \beta)}(T^{C_i}), \quad i = 0, 1 \quad (4.5)$$

Theorem 4.1 (Paley — Wiener — Vladimirov). A generalized function $f(t, x)$ belongs to the space $\mathcal{S}'(\Gamma_1)$ if and only if its Fourier — Laplace transform $L[f](\lambda, \xi)$ belongs to the space $H(T^{C_1})$. The spaces $\mathcal{S}'(\Gamma_1)$ and $H(T^{C_1})$ isomorphic and this isomorphism is established by the Fourier — Laplace transform. The function $L[f](\lambda, \xi)$ has a boundary value as $(\text{Im } \lambda, \text{Im } \xi) \rightarrow (0, 0)$, $(\text{Im } \lambda, \text{Im } \xi) \in C'$ in $\mathcal{S}'(\mathbb{R}^2)$, which is equal to $F[f](\text{Re } \lambda, \text{Re } \xi)$, that is, in $\mathcal{S}'(\mathbb{R}^2)$ the limit

$$(L[f](\lambda, \xi), \varphi(\text{Re } \lambda, \text{Re } \xi)) \xrightarrow[(\text{Im } \lambda, \text{Im } \xi) \in C'_1]{(\text{Im } \lambda, \text{Im } \xi) \rightarrow (0, 0)} (F[f](\text{Re } \lambda, \text{Re } \xi), \varphi(\text{Re } \lambda, \text{Re } \xi)) \quad (4.6)$$

is well-defined for each $\varphi(\text{Re } \lambda, \text{Re } \xi) \in S(\mathbb{R}^2)$, where F is the Fourier transform of a generalized function, C'_1 is an arbitrary subcone of the cone C_1 with the vertex at zero such that $\overline{C'_1} \subset C_1$.

Remark 4.2. *Paley — Wiener — Vladimirov theorem is true also for generalized functions $f(t, x) \in \mathcal{S}'(\Gamma_0)$ in the variable t for each fixed x , that is, the spaces $\mathcal{S}'(\Gamma_0)$ and $H(T^{C_0})$ are isomorphic for each fixed x , and this isomorphism is made by the Fourier — Laplace transform $L_t[\cdot](\lambda, x)$ defined the formula (3.10) for each fixed x .*

5. FORMULATION OF RESULTS

Definition 5.1. *The symbol of integro-differential operator (2.4) is the function*

$$\hat{\mathcal{L}}_1(\lambda, \xi) := - \left(\lambda^2 - a^2 \xi^2 + \hat{K}(\lambda) \xi^2 \right), \quad \lambda \in \mathbb{C}, \quad \xi \in \mathbb{C}, \quad (5.1)$$

where

$$\hat{K}(\lambda) := \sum_{i=1}^N c_i \hat{K}_i(\lambda), \quad \hat{K}_i(\lambda) := \frac{1}{(-i\lambda)^{(1-\alpha)} + \beta_i}$$

is the Fourier — Laplace transform of kernel of the integral operator $K_i(t)$, ($i = 1, \dots, N$), given by the formula (2.2).

Theorem 5.1. *Let the cone Γ_1 be defined by the formula (3.1), and the cone $C_1 = \text{int } \Gamma_1^*$ be defined by the formula (3.3). Then the integro-differential operator $\mathcal{L}_1(D_t, D_x)$ defined by the formula (2.4) has the fundamental solution $\mathcal{E}_1(t, x) \in \mathcal{D}'(\Gamma_1)$, which can be represented as*

$$\mathcal{E}_1(t, x) = e^{\lambda_0 t} \mathcal{E}_0(t, x), \quad (5.2)$$

where $\mathcal{E}_0(t, x) \in \mathcal{S}'(\Gamma_1)$, λ_0 is a sufficiently large positive number.

Theorem 5.2. *Let the cone Γ_1 be defined by the formula (3.1), the cone $C_1 = \text{int } \Gamma_1^*$ be defined by the formula (3.3), and $\mathcal{E}_1(t, x) \in \mathcal{D}'(\Gamma_1)$ be the fundamental solution of the integro-differential operator $\mathcal{L}_1(D_t, D_x)$. Then the support of the fundamental solution $\text{supp } \mathcal{E}_1(t, x)$ is not contained in any smaller convex cone $\tilde{\Gamma}_1 \subset \Gamma_1$ with the vertex at zero.*

Theorem 5.3. *Let the cone Γ_1 be given by the formula (3.1), and the cone $C_1 = \text{int } \Gamma_1^*$ be given by the formula (3.3). Then the fundamental solution (5.2) of the integro-differential operator $\mathcal{L}_1(D_t, D_x)$ defined by the formula (2.4) is unique.*

Lemma 5.1. *The generalized function (5.2) is a fundamental solution of the integro-differential operator (2.4) if and only if the Fourier — Laplace transform $L[\mathcal{E}_0(t, x)](\lambda, \xi)$ of the generalized function $\mathcal{E}_0(t, x) \in \mathcal{S}'(\Gamma_1)$ satisfies the equation*

$$\hat{\mathcal{L}}_1(\lambda + i\lambda_0, \xi) L[\mathcal{E}_0(t, x)](\lambda, \xi) = 1, \quad (5.3)$$

where $\hat{\mathcal{L}}_1(\lambda, \xi)$ is the symbol of integro-differential operator (2.4) defined by the formula (5.1) and the conditions hold

$$\hat{\mathcal{L}}_1(\lambda + i\lambda_0, \xi) \neq 0, \quad \frac{1}{\hat{\mathcal{L}}_1(\lambda + i\lambda_0, \xi)} \in H(T^{C_1}). \quad (5.4)$$

6. PROOF OF LEMMA 5.1 AND THEOREM 5.1

Proof of Lemma 5.1. Let $\mathcal{E}_1(t, x) = e^{\lambda_0 t} \mathcal{E}_0(t, x)$ be a fundamental solution of the operator (2.4). Applying the Fourier — Laplace transform to the both sides of identity (2.5), we get

$$\hat{\mathcal{L}}_1(\tilde{\lambda}, \xi) L[e^{\lambda_0 t} \mathcal{E}_0(t, x)](\tilde{\lambda}, \xi) = 1. \quad (6.1)$$

The properties of Fourier — Laplace transform [2, Ch. II, Sect. 9] yield

$$L[e^{\lambda_0 t} \mathcal{E}_0(t, x)](\tilde{\lambda}, \xi) = L[\mathcal{E}_0(t, x)](\tilde{\lambda} - i\lambda_0, \xi), \quad (\tilde{\lambda} - i\lambda_0, \xi) \in T^{C_1}.$$

We denote $\lambda := \tilde{\lambda} - i\lambda_0$. Then Equation (6.1) becomes (5.3). The generalized function $\mathcal{E}_0(t, x)$ belongs to the space $\mathcal{S}'(\Gamma_1)$, therefore, by Theorem 4.1, the function $L[\mathcal{E}_0(t, x)](\lambda, \xi)$ belongs to the space $H(T^{C_1})$, that is, the conditions (5.4) hold.

Vice versa, if the function $L[\mathcal{E}_0(t, x)](\lambda, \xi)$ satisfies Equation (5.3) and the conditions (5.4) hold, then, in accordance with Theorem 4.1, the generalized function $\mathcal{E}_0(t, x)$ belongs to the space $\mathcal{S}'(\Gamma_1)$ and, under the change of variables $\lambda := \tilde{\lambda} - i\lambda_0$, the function

$$L[\mathcal{E}_0(t, x)](\lambda, \xi) = L[e^{\lambda_0 t} \mathcal{E}_0(t, x)](\tilde{\lambda}, \xi)$$

satisfies Equation (6.1). Therefore, the generalized function

$$\mathcal{E}(t, x) = e^{\lambda_0 t} \mathcal{E}_0(t, x) \in \mathcal{D}'(\Gamma_1)$$

is a fundamental solution of the operator (2.4). The proof is complete. $\square$

We denote $\tilde{\lambda} := \lambda + i\lambda_0$, where λ_0 is some constant. Then the function $\hat{\mathcal{L}}_1$ defined by the formula (5.1), can be represented as

$$\hat{\mathcal{L}}_1(\tilde{\lambda}, \xi) = -\left(\tilde{\lambda}^2 - (a^2 - \hat{K}(\tilde{\lambda}))\xi^2\right) = (a^2 - \hat{K}(\tilde{\lambda}))(\xi - \xi_1(\tilde{\lambda}))(\xi + \xi_1(\tilde{\lambda})), \quad (6.2)$$

where $\pm \xi_1(\tilde{\lambda})$ are the roots of equation $\hat{\mathcal{L}}_1(\tilde{\lambda}, \xi) = 0$, that is,

$$\xi_1(\tilde{\lambda}) = \frac{\tilde{\lambda}}{\sqrt{a^2 - \hat{K}(\tilde{\lambda})}}. \quad (6.3)$$

We note that for sufficiently large $\lambda_0 > 0$ the root $\xi_1(\tilde{\lambda})$ satisfies the asymptotic representation

$$\xi_1(\tilde{\lambda}) = \frac{1}{a}\tilde{\lambda} + \frac{1}{2a^3}\tilde{\lambda}\hat{K}(\tilde{\lambda})(1 + o(1)), \quad |\tilde{\lambda}| \rightarrow +\infty. \quad (6.4)$$

Indeed, by the formula (6.3), for sufficiently large $\lambda_0 > 0$, we obtain

$$\xi_1(\tilde{\lambda}) = \frac{1}{a}\tilde{\lambda} \left(1 - \frac{1}{a^2}\hat{K}(\tilde{\lambda})\right)^{-\frac{1}{2}} = \frac{1}{a}\tilde{\lambda} \left(1 + \frac{1}{2a^2}\hat{K}(\tilde{\lambda})(1 + o(1))\right), \quad |\tilde{\lambda}| \rightarrow +\infty.$$

To prove Theorems 5.1 and 5.2, we shall need auxiliary Lemmas 6.1 and 6.2, which will be proved in Section 9.

Lemma 6.1. *The assertions hold:*

- 1) *Let the cone C_1 be given by the formula (3.3), $(\operatorname{Im} \lambda, \operatorname{Im} \xi) \in C_1$. Then $\operatorname{Im} \lambda > 0$.*
- 2) *Let $\operatorname{Im} \lambda > 0$. Then for sufficiently large $\lambda_0 > 0$ the inequalities hold*

$$\operatorname{Im} \left(\tilde{\lambda} \hat{K}(\tilde{\lambda}) \right) > 0, \quad \tan \left(\frac{\alpha\pi}{2} \right) \operatorname{Im} \left(\tilde{\lambda} \hat{K}(\tilde{\lambda}) \right) > \left| \operatorname{Re} \left(\tilde{\lambda} \hat{K}(\tilde{\lambda}) \right) \right|, \quad (6.5)$$

where $0 < \alpha < 1$, $\tilde{\lambda} := \lambda + i\lambda_0$, and the root $\xi_1(\tilde{\lambda})$ defined by the formula (6.3) satisfies the asymptotic representation

$$\operatorname{Im} \xi_1(\tilde{\lambda}) = \frac{1}{a} \operatorname{Im} \tilde{\lambda} + \frac{1}{2a^3} \operatorname{Im} \left(\tilde{\lambda} \hat{K}(\tilde{\lambda}) \right) (1 + o(1)), \quad \operatorname{Im} \tilde{\lambda} \rightarrow +\infty. \quad (6.6)$$

Lemma 6.2. *Let the cones Γ_1 and C_1 be defined by the formulas (3.1) and (3.3), respectively. Then for all $(\lambda, \xi) \in T^{C_1}$ the following inequalities hold*

$$\frac{1}{a} (\operatorname{Im} \lambda \pm a \operatorname{Im} \xi) > \Delta(\operatorname{Im} \lambda, \operatorname{Im} \xi), \quad (6.7)$$

where

$$\Delta(\operatorname{Im} \lambda, \operatorname{Im} \xi) = \inf_{\substack{(t, x) \in \Gamma_1 \\ t^2 + x^2 = 1}} (t \operatorname{Im} \lambda + x \operatorname{Im} \xi)$$

is the distance from the point $(\operatorname{Im} \lambda, \operatorname{Im} \xi) \in C_1$ to the boundary of cone C_1 , and the constant $a > 0$ is involved in the definition of cone Γ_1 .

Proof of Theorem 5.1. According to Lemma 5.1, the generalized function (5.2) is a fundamental solution of integro-differential operator (2.4) if and only if the Fourier — Laplace transform $L[\mathcal{E}_0(t, x)](\lambda, \xi)$ of the generalized function $\mathcal{E}_0(t, x) \in \mathcal{S}'(\Gamma_1)$ satisfies Equation (5.3) and the conditions (5.4) hold. Therefore, to prove the theorem, it is sufficient to show that for some sufficiently large positive number λ_0 the conditions (5.4) hold.

According to the condition (4.1), it is sufficient to show that there exist numbers $\alpha \geq 0$, $\beta \geq 0$, $(\alpha, \beta \in \mathbb{Z})$, such that the estimate holds

$$\left| \frac{1}{\hat{\mathcal{L}}_1(\lambda + i\lambda_0, \xi)} \right| \leq M (1 + |\lambda|^2 + |\xi|^2)^{\frac{\alpha}{2}} [1 + \Delta^{-\beta}(\operatorname{Im} \lambda, \operatorname{Im} \xi)], \quad (\lambda, \xi) \in T^{C_1}, \quad (6.8)$$

where

$$\Delta(p, q) = \inf_{\substack{(t, x) \in \Gamma_1 \\ t^2 + x^2 = 1}} (tp + xq)$$

is the distance from the point $(p, q) \in C_1$ to the boundary of cone C_1 , M is a positive constant.

According to the representation (6.2), to prove the condition (6.8), it is sufficient to get lower bounds for the factors $|a^2 - \hat{K}(\tilde{\lambda})|$, $|\xi - \xi_1(\tilde{\lambda})|$ and $|\xi + \xi_1(\tilde{\lambda})|$ of the function $|\hat{\mathcal{L}}_1(\lambda + i\lambda_0, \xi)|$. Let $\tilde{\lambda} = r(\cos \varphi + i \sin \varphi)$, where

$$r = |\tilde{\lambda}|, \quad \varphi = \operatorname{Arg} \tilde{\lambda} \in (-\pi + 2\pi k, \pi + 2\pi k], \quad k \in \mathbb{Z}.$$

In view of Lemmas 6.1 and 6.2 and the estimate (9.3), for sufficiently large $\lambda_0 > 0$ we get the following estimates:

$$\begin{aligned} |\xi \pm \xi_1(\tilde{\lambda})| &\geq \frac{1}{a} |a \operatorname{Im} \xi_1(\tilde{\lambda}) \pm a \operatorname{Im} \xi| = \frac{1}{a} |a \operatorname{Im} \xi_1(\tilde{\lambda}) - \operatorname{Im} \lambda + \operatorname{Im} \lambda \pm a \operatorname{Im} \xi| \\ &= \frac{1}{a} \left(\lambda_0 + \frac{1}{2a^2} \operatorname{Im}(\tilde{\lambda} \hat{K}(\tilde{\lambda}))(1 + o(1)) + \operatorname{Im} \lambda \pm a \operatorname{Im} \xi \right) \\ &> \frac{1}{a} (\operatorname{Im} \lambda \pm a \operatorname{Im} \xi) > \Delta(\operatorname{Im} \lambda, \operatorname{Im} \xi) \end{aligned}$$

and

$$\begin{aligned} |a^2 - \hat{K}(\tilde{\lambda})| &\geq |a^2 - |\hat{K}(\tilde{\lambda})|| \geq \left| a^2 - \sum_{i=1}^N \frac{1}{|(-i\tilde{\lambda})^{1-\alpha} + \beta_i|} \right| \\ &= \left| a^2 - \sum_{i=1}^N \left| r^{1-\alpha} \left(\cos \left((1-\alpha) \left(\varphi - \frac{\pi}{2} \right) \right) + i \sin \left((1-\alpha) \left(\varphi - \frac{\pi}{2} \right) \right) \right) + \beta_i \right|^{-1} \right| \\ &= \left| a^2 - \sum_{i=1}^N \left(\left(r^{1-\alpha} \cos \left((1-\alpha) \left(\varphi - \frac{\pi}{2} \right) \right) + \beta_i \right)^2 + r^{2(1-\alpha)} \sin^2 \left((1-\alpha) \left(\varphi - \frac{\pi}{2} \right) \right) \right)^{-\frac{1}{2}} \right| \\ &\geq \left| a^2 - \sum_{i=1}^N \left(r^{1-\alpha} \cos \left((1-\alpha) \left(\varphi - \frac{\pi}{2} \right) \right) + \beta_i \right)^{-1} \right| > \frac{a^2}{2} \end{aligned}$$

Finally, choosing sufficiently large $\lambda_0 > 0$, by the two latter estimates we get the desired estimate (6.8) for $\alpha = 0$, $\beta = 2$, $M = \frac{2}{a^2}$:

$$\left| \frac{1}{\hat{\mathcal{L}}_1(\lambda + i\lambda_0, \xi)} \right| \leq \frac{2}{a^2} \Delta^{-2}(\operatorname{Im} \lambda, \operatorname{Im} \xi) < M [1 + \Delta^{-2}(\operatorname{Im} \lambda, \operatorname{Im} \xi)], \quad (\lambda, \xi) \in T^{C_1}.$$

The proof is complete. $\square$

7. PROOF OF THEOREM 5.2

It follows from Theorem 5.1 that the integro-differential operator $\mathcal{L}_1(D_t, D_x)$ given by the formula (2.4) has the fundamental solution $\mathcal{E}_1(t, x) \in \mathcal{D}'(\Gamma_1)$, which can be represented in the form (5.2). We consider the Fourier — Laplace transform $L_1[\mathcal{E}_0(t, x)](\lambda, x)$ of the generalized function $\mathcal{E}_0(t, x) = e^{-\lambda_0 t} \mathcal{E}_1(t, x)$ in the variable t :

$$\hat{\mathcal{E}}_{01}(\lambda, x) := L_1[\mathcal{E}_0(t, x)](\lambda, x) = F_1 [\mathcal{E}_0(t, x) e^{-t \operatorname{Im} \lambda}] (\operatorname{Re} \lambda, x), \quad t \in \Gamma_0, \quad \lambda \in T^{C_0},$$

where F_t is the Fourier transform of a generalized function in the variable t , the cone Γ_0 and tubular domain T^{C_0} are defined by the formulas (3.4) and (3.7), respectively.

To prove Theorem 5.2, we shall need the following auxiliary Lemma 7.1, which will be proved in Section 9.

Lemma 7.1. *Let $\tilde{\lambda} := \lambda + i\lambda_0$,*

$$A(\tilde{\lambda}) := \frac{1}{2\xi_1(\tilde{\lambda}) \left(a^2 - \hat{K}(\tilde{\lambda}) \right)}, \quad (7.1)$$

where $\xi_1(\tilde{\lambda})$ is the root of the equation $\hat{\mathcal{L}}_1(\tilde{\lambda}, \xi) = 0$, the function $\hat{\mathcal{L}}_1(\tilde{\lambda}, \xi)$ is defined by the formula (6.2). Then

$$\left| \hat{\mathcal{E}}_{01}(\lambda, x) \right| = \left| A(\tilde{\lambda}) \right| e^{-|x| \operatorname{Im} \xi_1(\tilde{\lambda})}, \quad x \in \mathbb{R}. \quad (7.2)$$

Proof of Theorem 5.2. According to Theorem 5.1, $\mathcal{E}_0(t, x) \in \mathcal{S}'(\Gamma_1)$, and therefore, $\mathcal{E}_0(t, x) \in \mathcal{S}'(\Gamma_0)$, for each fixed $x \in \mathbb{R}$. It follows from Remark 4.2 to Theorem 4.1 that $L_1[\mathcal{E}_0(t, x)](\lambda, x)$ belongs to the space $H(T^{C_0})$ for each fixed $x \in \mathbb{R}$. Since $\mathcal{E}_0(t, x) \in \mathcal{S}'(\Gamma_1)$, the support of function $\mathcal{E}_0(t, x)$ is contained in the cone

$$\Gamma_1 = \{(t, x) \in \mathbb{R}^2 \mid t \geq 0, x \in \mathbb{R}, at \geq |x|\}.$$

Then the support of function

$$\tilde{\mathcal{E}}_0(t, x) := \mathcal{E}_0\left(t + \frac{|x|}{a}, x\right)$$

belong to the cone $\Gamma_0 = \{t \in \mathbb{R} \mid t \geq 0\}$ for each fixed $x \in \mathbb{R}$. Therefore, its Fourier — Laplace transform $L_1[\tilde{\mathcal{E}}_0(t, x)](\lambda, x)$ in the variable t for each fixed $x \in \mathbb{R}$ belongs to the space $H(T^{C_0})$, that is, there exist $\alpha \geq 0$ and $\beta \geq 0$, ($\alpha, \beta \in \mathbb{Z}$), such that for each fixed $x \in \mathbb{R}$ the estimate holds

$$\left| L_1[\tilde{\mathcal{E}}_0(t, x)](\lambda, x) \right| \leq M (1 + |\lambda|^2)^{\frac{\alpha}{2}} [1 + (\operatorname{Im} \lambda)^{-\beta}], \quad \lambda \in T^{C_0}, \quad (7.3)$$

where M is a positive constant. Moreover,

$$L_1[\tilde{\mathcal{E}}_0(t, x)](\lambda, x) = L_1\left[\mathcal{E}_0\left(t + \frac{|x|}{a}, x\right)\right](\lambda, x) = e^{-i \frac{|x|}{a} \operatorname{Im} \lambda} L_1[\mathcal{E}_0(t, x)](\lambda, x).$$

Therefore,

$$\left| L_1[\tilde{\mathcal{E}}_0(t, x)](\lambda, x) \right| = e^{\frac{|x|}{a} \operatorname{Im} \lambda} |L_1[\mathcal{E}_0(t, x)](\lambda, x)|$$

By (7.3) this gives the estimate

$$\begin{aligned} |L_1[\mathcal{E}_0(t, x)](\lambda, x)| &= \left| L_1[\tilde{\mathcal{E}}_0(t, x)](\lambda, x) \right| e^{-\frac{|x|}{a} \operatorname{Im} \lambda} \\ &\leq M (1 + |\lambda|^2)^{\frac{\alpha}{2}} [1 + (\operatorname{Im} \lambda)^{-\beta}] e^{-\frac{|x|}{a} \operatorname{Im} \lambda} \end{aligned} \quad (7.4)$$

for $\lambda \in T^{C_0}$.

Now we are going to show that the support of fundamental solution $\operatorname{supp} \mathcal{E}_1(t, x)$ is not contained in any smaller convex cone

$$\Gamma_{1\varepsilon} := \{(t, x) \in \mathbb{R}^2 \mid t \geq 0, x \in \mathbb{R}, (a - \varepsilon)t \geq |x|\} \subset \Gamma_1$$

with the vertex at zero, where $\varepsilon \in (0, a)$.

We suppose the opposite, then the Fourier — Laplace transform $L_1(\mathcal{E}_0(t, x))(\lambda, x)$ of the generalized function $\mathcal{E}_0(\tilde{t}, x)$ in the variable $\tilde{t}$ satisfies the following estimate similar to (7.4):

$$|L_1(\mathcal{E}_0(t, x))(\lambda, x)| \leq M_\varepsilon (1 + |\lambda|^2)^{\frac{\alpha}{2}} [1 + (\operatorname{Im} \lambda)^{-\beta}] e^{-\frac{|x|}{a-\varepsilon} \operatorname{Im} \lambda}, \quad (7.5)$$

where M_ε is a positive constant. Using the identity (7.2), we get the estimate

$$e^{-|x| \operatorname{Im} \xi_1(\tilde{\lambda})} \leq M_{1\varepsilon} (1 + |\lambda|^2)^{\frac{\alpha}{2}} [1 + (\operatorname{Im} \lambda)^{-\beta}] e^{-\frac{|x|}{a-\varepsilon} \operatorname{Im} \lambda}.$$

Letting $x \neq 0$ and substituting the asymptotic representation (6.6) instead of $\operatorname{Im} \xi_1(\tilde{\lambda})$, for sufficiently large $\operatorname{Im} \lambda > 0$ we get the following wrong inequality

$$\frac{\operatorname{Im} \lambda}{a} > \frac{\operatorname{Im} \lambda}{a - \varepsilon} (1 + o(1)), \quad \operatorname{Im} \lambda \rightarrow +\infty.$$

Thus, we arrive at the contradiction and the support of fundamental solution $\operatorname{supp} \mathcal{E}(t, x)$ is not contained in any smaller convex cone $\tilde{\Gamma}_1 \subset \Gamma_1$ with the vertex at zero. The proof is complete. $\square$

8. PROOF OF THEOREM 5.3

We introduce the notation

$$D_t := \frac{\partial}{\partial t}, \quad D_x := \frac{\partial}{\partial x}, \quad P_0(D_t, D_x) := \frac{\partial^2}{\partial t^2} - a^2 \frac{\partial^2}{\partial x^2}, \quad P_1(D_x) := \frac{\partial^2}{\partial x^2}.$$

Suppose that there exists another fundamental solution $\tilde{\mathcal{E}}(t, x)$ of the operator (2.4). We consider the generalized function

$$u(t, x) := \mathcal{E}_1(t, x) - \tilde{\mathcal{E}}(t, x),$$

which solves the equation

$$\mathcal{L}_1(D_t, D_x)u(t, x) = 0, \quad (8.1)$$

where the operator $\mathcal{L}_1(D_t, D_x)$ is defined by the formula (2.4). Let us show that Equation (8.1) has only trivial solution in the class of generalized functions, for which the convolutions

$$K(t) * P_1(D_t, D_x)\tilde{\mathcal{E}}(t, x) \quad \text{and} \quad (u * \mathcal{E}_1)(t, x)$$

are well-defined; hereinafter the convolution with the function $K(t)$ means the convolution in the variable t for a fixed value x . Indeed,

$$\begin{aligned} u(t, x) &= u(t, x) * \delta(t, x) = u(t, x) * \mathcal{L}_1(D_t, D_x)\mathcal{E}_1(t, x) \\ &= u(t, x) * P_0(D_t, D_x)\mathcal{E}_1(t, x) + u(t, x) * (K(t) * P_1(D_t, D_x)\mathcal{E}_1(t, x)). \end{aligned}$$

According to the rule of differentiation of convolution of generalized functions [1, Sect. 7.5],

$$u(t, x) * P_0(D_t, D_x)\mathcal{E}_1(t, x) = P_0(D_t, D_x)u(t, x) * \mathcal{E}_1(t, x). \quad (8.2)$$

Let us show that

$$u(t, x) * (K(t) * P_1(D_t, D_x)\mathcal{E}_1(t, x)) = (K(t) * P_1(D_t, D_x)u(t, x)) * \mathcal{E}_1(t, x). \quad (8.3)$$

Since $\mathcal{E}_1(t, x) \in \mathcal{D}'(\Gamma_1)$, then $\mathcal{E}_1(t, x) \in \mathcal{D}'_+$ in the variable t for a fixed value of x , and moreover, $K(t) \in \mathcal{D}'_+$, by the definition. Hence, the assumptions of the theorem from [1, Sect. 7.7] are satisfied, the convolution $K(t) * P_1(D_t, D_x)\mathcal{E}_1(t, x)$ is well-defined and represented by the formula (25) in [1, Sect. 7.7]:

$$(K(t) * P_1(D_t, D_x)\mathcal{E}_1(t, x), \varphi_x(t)) = (K(t) \times P_1(D_t, D_x)\mathcal{E}_1(\tau, x), \tilde{\eta}_1(t)\tilde{\eta}_2(\tau)\varphi_x(t + \tau))$$

for all $\varphi_x(t) \in \mathcal{D}(\mathbb{R})$, where $\tilde{\eta}_i(t) \in C^\infty(\mathbb{R})$ ($i = 1, 2$) are arbitrary functions, which are equal to 1 in a neighbourhood of the semi-axis $[0, +\infty)$ and 0 for sufficiently large negative t .

We consider the sequence of functions $\eta_k(t, \tau) \in \mathcal{D}(\mathbb{R}^2)$ converging to 1 in $\mathbb{R}^2$, which is employed to define the convolution of generalized functions [1, Sect. 7.4]. According to the definition of convolution of generalized functions and the theorem in [1, Sect. 7.7], for all $\varphi_x(t) \in \mathcal{D}(\mathbb{R})$ we have

$$\begin{aligned} & (u(t, x) * (K(t) * P_1(D_t, D_x)\mathcal{E}_1(t, x)), \varphi_x(t)) \\ &= \lim_{k \rightarrow \infty} (u(t, x) \times (K(\tau) * P_1(D_t, D_x)\mathcal{E}_1(\tau, x)), \eta_k(t, \tau)\varphi_x(t + \tau)) \\ &= \lim_{k \rightarrow \infty} ((K(\tau) * P_1(D_t, D_x)\mathcal{E}_1(\tau, x)), (u(t, x), \eta_k(t, \tau)\varphi_x(t + \tau))) \\ &= \lim_{k \rightarrow \infty} ((K(\tau) \times P_1(D_t, D_x)\mathcal{E}_1(\tau', x)), \tilde{\eta}_1(\tau)\tilde{\eta}_2(\tau') (u(t, x), \eta_k(t, \tau + \tau')\varphi_x(t + \tau + \tau'))) \\ &= \lim_{k \rightarrow \infty} ((K(\tau) \times P_1(D_t, D_x)\mathcal{E}_1(\tau', x)) \times u(t, x), \tilde{\eta}_1(\tau)\tilde{\eta}_2(\tau')\eta_k(t, \tau + \tau')\varphi_x(t + \tau + \tau')). \end{aligned}$$

Here we have used the lemma from in [1, Sect. 7.7], according to which

$$(u(t, x), \eta_k(t, \tau)\varphi_x(t + \tau)) \in \mathcal{D}'(\mathbb{R}).$$

We observe that $\tilde{\eta}_1(\tau)\tilde{\eta}_2(\tau')\eta_k(t, \tau + \tau')\varphi_x(t + \tau + \tau') \in \mathcal{D}(\mathbb{R}^3)$, use the commutativity and associativity of direct product of generalized functions [1, Sect. 7.2–7.3] and note that

$$(K(\tau) \times P_1(D_t, D_x)u(t, x)) = 0, \quad \mathcal{E}_1(\tau', x) = 0 \quad \text{for } \tau < 0, \quad \tau' < 0.$$

Then we apply the theorem from [1, Sect. 7.7] and pass to the limit in the chain of identities

$$\begin{aligned} & (u(t, x) * (K(t) * P_1(D_t, D_x)\mathcal{E}_1(t, x)), \varphi_x(t)) \\ &= \lim_{k \rightarrow \infty} ((K(\tau) \times P_1(D_t, D_x)u(t, x)) \times \mathcal{E}_1(\tau', x), \tilde{\eta}_1(\tau)\tilde{\eta}_2(\tau')\eta_k(t, \tau + \tau')\varphi_x(t + \tau + \tau')) \\ &= ((K(t) * P_1(D_t, D_x)u(t, x)) * \mathcal{E}_1(t, x), \varphi_x(t)). \end{aligned}$$

We note that the convolution $(K(t) * P_1(D_t, D_x)u(t, x)) * \mathcal{E}_1(t, x)$ in the variable x is well-defined since $\mathcal{E}_1(t, x) \in \mathcal{D}'(\Gamma_1)$, that is, the generalized function $\mathcal{E}_1(t, x)$ is compactly supported in the variable x for each fixed t [1, Sect. 7.6].

By the formulas (8.2) and (8.3) we obtain the following chain of identities

$$u(t, x) = u(t, x) * \delta(t, x) = u(t, x) * \mathcal{L}_1(D_t, D_x)\mathcal{E}_1(t, x) = \mathcal{L}_1(D_t, D_x)u(t, x) * \mathcal{E}_1(t, x) = 0.$$

The proof is complete.

9. PROOF OF LEMMAS 6.1, 6.2 AND 7.1

Proof of Lemma 6.1. 1) The definitions of cones Γ_1 and C_1 imply that the inequality

$$t \operatorname{Im} \lambda + x \operatorname{Im} \xi > 0$$

holds if and only if $(\operatorname{Im} \lambda, \operatorname{Im} \xi) \in C_1$ and $(t, x) \in \Gamma_1$. Therefore, if $(t_1, x) \notin \Gamma_1$, that is, $at_1 < |x|$, then for all $(\operatorname{Im} \lambda, \operatorname{Im} \xi) \in C_1$ the inequality holds $t_1 \operatorname{Im} \lambda + x \operatorname{Im} \xi \leq 0$, that is,

$$t \operatorname{Im} \lambda + x \operatorname{Im} \xi > 0 \Leftrightarrow at \geq |x|, \quad -t_1 \operatorname{Im} \lambda - x \operatorname{Im} \xi \geq 0 \Leftrightarrow -at_1 > -|x|.$$

Thus, $(t - t_1) \operatorname{Im} \lambda > 0$ and $t - t_1 > 0$, hence, $\operatorname{Im} \lambda > 0$.

2) Let $\tilde{\lambda} = r(\cos \varphi + i \sin \varphi)$, where

$$r = |\tilde{\lambda}|, \quad \varphi = \operatorname{Arg} \tilde{\lambda} \in (-\pi + 2\pi k, \pi + 2\pi k], \quad k \in \mathbb{Z}.$$

Then

$$\left(-i\tilde{\lambda}\right)^{1-\alpha} = r^{1-\alpha} \left(\cos \left((1-\alpha) \left(\varphi - \frac{\pi}{2} \right) \right) + i \sin \left((1-\alpha) \left(\varphi - \frac{\pi}{2} \right) \right) \right)$$

and

$$\tilde{\lambda}\hat{K}_i(\tilde{\lambda}) = \frac{r(\cos \varphi + i \sin \varphi)}{r^{1-\alpha} \left(\cos \left((1-\alpha) \left(\varphi - \frac{\pi}{2} \right) \right) + i \sin \left((1-\alpha) \left(\varphi - \frac{\pi}{2} \right) \right) \right) + \beta_i}, \quad i = 1, \dots, N.$$

This yields

$$\operatorname{Re} \left(\tilde{\lambda}\hat{K}_i(\tilde{\lambda}) \right) = \frac{r^{2-\alpha} \sin \left(\alpha \left(\varphi - \frac{\pi}{2} \right) \right) + \beta_i r \cos \varphi}{\left(r^{(1-\alpha)} \cos \left((1-\alpha) \left(\varphi - \frac{\pi}{2} \right) \right) + \beta_i \right)^2 + r^{2(1-\alpha)} \sin^2 \left((1-\alpha) \left(\varphi - \frac{\pi}{2} \right) \right)}, \quad (9.1)$$

$$\operatorname{Im} \left(\tilde{\lambda}\hat{K}_i(\tilde{\lambda}) \right) = \frac{r^{2-\alpha} \cos \left(\alpha \left(\varphi - \frac{\pi}{2} \right) \right) + \beta_i r \sin \varphi}{\left(r^{(1-\alpha)} \cos \left((1-\alpha) \left(\varphi - \frac{\pi}{2} \right) \right) + \beta_i \right)^2 + r^{2(1-\alpha)} \sin^2 \left((1-\alpha) \left(\varphi - \frac{\pi}{2} \right) \right)}. \quad (9.2)$$

Since $\operatorname{Im} \lambda > 0$, choosing $\lambda_0 > 0$, we get $\operatorname{Im} \tilde{\lambda} > 0$ and $\varphi \in (2\pi k, \pi + 2\pi k)$, $k \in \mathbb{Z}$. Therefore,

$$\alpha \left(\varphi - \frac{\pi}{2} \right) \subset \left(\alpha \left(-\frac{\pi}{2} + 2\pi k \right), \alpha \left(\frac{\pi}{2} + 2\pi k \right) \right) \subset \left(-\frac{\pi}{2} + 2\pi k, \frac{\pi}{2} + 2\pi k \right), \quad k \in \mathbb{Z},$$

for $\alpha \in (0, 1)$ and

$$\cos \left(\alpha \left(\varphi - \frac{\pi}{2} \right) \right) > 0, \quad \alpha \in (0, 1). \quad (9.3)$$

Thus, for sufficiently large $\lambda_0 > 0$ the inequality holds $\operatorname{Im} \left(\tilde{\lambda}\hat{K}_i(\tilde{\lambda}) \right) > 0$, $i = 1, \dots, N$, and hence

$$\operatorname{Im} \left(\tilde{\lambda}\hat{K}(\tilde{\lambda}) \right) > 0.$$

Let us show that for sufficiently large $\lambda_0 > 0$ and $0 < \alpha < 1$ the second inequality in (6.5) holds. Using the representations (9.1), (9.2), we obtain that the expression in the numerator of the difference

$$\left(\tan \left(\frac{\alpha\pi}{2} \right) \operatorname{Im} \left(\tilde{\lambda}\hat{K}_i(\tilde{\lambda}) \right) \right)^2 - \left(\operatorname{Re} \left(\tilde{\lambda}\hat{K}_i(\tilde{\lambda}) \right) \right)^2,$$

reads

$$\begin{aligned} \tan^2 \left(\frac{\alpha\pi}{2} \right) \left(r^{2-\alpha} \cos \left(\alpha \left(\varphi - \frac{\pi}{2} \right) \right) + \beta_i r \sin \varphi \right)^2 - \left(r^{2-\alpha} \sin \left(\alpha \left(\varphi - \frac{\pi}{2} \right) \right) + \beta_i r \cos \varphi \right)^2 \\ := A_1(r, \varphi) + A_2(r, \varphi) + A_3(r, \varphi), \end{aligned}$$

where

$$\begin{aligned} A_1(r, \varphi) &:= r^{2(2-\alpha)} \left[\tan^2 \left(\frac{\alpha\pi}{2} \right) \cos^2 \left(\alpha \left(\frac{\pi}{2} - \varphi \right) \right) - \sin^2 \left(\alpha \left(\frac{\pi}{2} - \varphi \right) \right) \right], \\ A_2(r, \varphi) &:= 2r^{3-\alpha} \beta_i \left(\tan^2 \left(\frac{\alpha\pi}{2} \right) \cos \left(\alpha \left(\frac{\pi}{2} - \varphi \right) \right) \sin \varphi - \sin \left(\alpha \left(\frac{\pi}{2} - \varphi \right) \right) \cos \varphi \right), \\ A_3(r, \varphi) &:= r^2 \beta_i^2 \left(\tan^2 \left(\frac{\alpha\pi}{2} \right) \sin^2 \varphi - \cos^2 \varphi \right). \end{aligned}$$

We note that for $0 < \alpha < 1$ the estimate holds

$$A_1(r, \varphi) > r^{2(2-\alpha)} \left[\sin^2 \left(\frac{\alpha\pi}{2} \right) - \sin^2 \left(\alpha \left(\frac{\pi}{2} - \varphi \right) \right) \right] = r^{2(2-\alpha)} \sin(\alpha(\pi - \varphi)) \sin \varphi > 0$$

Moreover,

$$r = \sqrt{(\operatorname{Re} \lambda)^2 + (\operatorname{Im} \lambda + \lambda_0)^2}, \quad \operatorname{Im} \lambda > 0, \quad 0 < \alpha < 1$$

and

$$2(2 - \alpha) = (1 - \alpha) + 3 - \alpha > 3 - \alpha = (1 - \alpha) + 2 > 2.$$

Thus, we find that $A_1(r, \varphi) + A_2(r, \varphi) + A_3(r, \varphi) > 0$ for sufficiently large $\lambda_0 > 0$ and this implies the second inequality in (6.5).

We consider the asymptotic representation (6.4). Let $\psi(\tilde{\lambda}) = o(1)$ as $|\tilde{\lambda}| \rightarrow +\infty$. We are going to show that the asymptotic representation (6.6) is true for sufficiently large $\lambda_0 > 0$. Indeed,

$$\operatorname{Im} \left(\psi(\tilde{\lambda}) \tilde{\lambda} \hat{K}(\tilde{\lambda}) \right) = \operatorname{Im} \left(\psi(\tilde{\lambda}) \right) \operatorname{Re} \left(\tilde{\lambda} \hat{K}(\tilde{\lambda}) \right) + \operatorname{Re} \left(\psi(\tilde{\lambda}) \right) \operatorname{Im} \left(\tilde{\lambda} \hat{K}(\tilde{\lambda}) \right).$$

Then, in view of the second inequality in (6.5), we get the estimate

$$\left| \operatorname{Im} \left(\psi(\tilde{\lambda}) \tilde{\lambda} \hat{K}(\tilde{\lambda}) \right) \right| < \operatorname{Im} \left(\tilde{\lambda} \hat{K}(\tilde{\lambda}) \right) \left(\left| \operatorname{Re} \left(\psi(\tilde{\lambda}) \right) \right| + \tan \left(\frac{\alpha\pi}{2} \right) \left| \operatorname{Im} \left(\psi(\tilde{\lambda}) \right) \right| \right)$$

which implies

$$\operatorname{Im} \left(\psi(\tilde{\lambda}) \tilde{\lambda} \hat{K}(\tilde{\lambda}) \right) = o \left(\operatorname{Im} \left(\tilde{\lambda} \hat{K}(\tilde{\lambda}) \right) \right) \quad \text{as} \quad \operatorname{Im} \tilde{\lambda} \rightarrow +\infty.$$

Thus, by the asymptotic representation (6.4) we obtain the asymptotic representation (6.6). The proof is complete. $\square$

Proof of Lemma 6.2. Taking into consideration assertion 1) of Lemma 6.1, for all $(\lambda, \xi) \in T^{C_1}$ we obtain

$$\begin{aligned} \Delta(\operatorname{Im} \lambda, \operatorname{Im} \xi) &= \inf_{\substack{at \geq |x| \\ t^2 + x^2 = 1}} (t \operatorname{Im} \lambda + x \operatorname{Im} \xi) \\ &= \inf_{at \geq |x|} \left(\frac{t}{\sqrt{t^2 + x^2}} \operatorname{Im} \lambda + \frac{x}{\sqrt{t^2 + x^2}} \operatorname{Im} \xi \right) \\ &= \inf_{at \geq |x|} \left(\frac{t}{\sqrt{t^2 + x^2}} \operatorname{Im} \lambda - \frac{|x|}{\sqrt{t^2 + x^2}} |\operatorname{Im} \xi| \right) \\ &= \inf_{at \geq |x|} \left(\frac{t}{\sqrt{t^2 + x^2}} \operatorname{Im} \lambda - \frac{at}{\sqrt{t^2 + x^2}} |\operatorname{Im} \xi| \right) \\ &= \frac{1}{\sqrt{1 + a^2}} (\operatorname{Im} \lambda - a |\operatorname{Im} \xi|) < \frac{1}{a} (\operatorname{Im} \lambda - a |\operatorname{Im} \xi|) \\ &\leq \frac{1}{a} (\operatorname{Im} \lambda + a |\operatorname{Im} \xi|), \end{aligned}$$

where the latter identity holds since

$$\frac{t}{\sqrt{t^2 + x^2}} \geq \frac{t}{\sqrt{t^2 + (at)^2}} = \frac{1}{\sqrt{1 + a^2}}.$$

We thus obtain the estimates (6.7) and complete the proof. $\square$

Proof of Lemma 7.1. By the identity (5.3) we obtain the following representation for the generalized function $L[\mathcal{E}_0(t, x)](\lambda, \xi)$:

$$L[\mathcal{E}_0(t, x)](\lambda, \xi) = \frac{1}{\hat{\mathcal{L}}_1(\lambda + i\lambda_0, \xi)} = \frac{1}{\hat{\mathcal{L}}_1(\tilde{\lambda}, \xi)}, \quad (\lambda, \xi) \in T^{C_1}. \quad (9.4)$$

By representation (6.2) we get the expansion

$$\frac{1}{\hat{\mathcal{L}}_1(\lambda + i\lambda_0, \xi)} = \frac{1}{\hat{\mathcal{L}}_1(\tilde{\lambda}, \xi)} = A(\tilde{\lambda}) \left[\frac{1}{\xi - \xi_1(\tilde{\lambda})} - \frac{1}{\xi + \xi_1(\tilde{\lambda})} \right], \quad (9.5)$$

where $\tilde{\lambda} := \lambda + i\lambda_0$, $\xi_1(\tilde{\lambda})$ is the root of equation $\hat{\mathcal{L}}_1(\tilde{\lambda}, \xi) = 0$, which is determined by the formula (6.3), $A(\tilde{\lambda})$ is determined by the formula (7.1).

The above Fourier — Laplace transform $\hat{\mathcal{E}}_{01}(\lambda, x) = L_1[\mathcal{E}_0(t, x)](\lambda, x)$ of the generalized function $\mathcal{E}_0(t, x)$ in the variable t can be represented as the inversion in the variable ξ of the Fourier — Laplace transform $L_2^{-1}[\hat{\mathcal{E}}_0(\lambda, \xi)](\lambda, x)$ of the function

$$\hat{\mathcal{E}}_0(\lambda, \xi) := L[\mathcal{E}_0(t, x)](\lambda, \xi),$$

that is,

$$\begin{aligned}\hat{\mathcal{E}}_{01}(\lambda, x) &= L_1[\mathcal{E}_0(t, x)](\lambda, x) = L_2^{-1}[L[\mathcal{E}_0(t, x)](\lambda, \xi)](\lambda, x) \\ &= e^{x \operatorname{Im} \xi} F_2^{-1}[L[\mathcal{E}_0(t, x)](\lambda, \xi)](\lambda, x) = e^{x \operatorname{Im} \xi} F_2^{-1}\left[\hat{\mathcal{L}}_1^{-1}(\tilde{\lambda}, \xi)\right](\lambda, x).\end{aligned}$$

Using the representation (9.4), the analyticity of function $\hat{\mathcal{L}}_1^{-1}(\tilde{\lambda}, \xi)$ in the tubular domain T^{C_1} , the expansion (9.5) and applying the Cauchy theorem on residues for $x \leq 0$, $\operatorname{Im} \xi > 0$, $(\lambda, \xi) \in T^{C_1}$, we obtain the following chain of identities

$$\begin{aligned}\hat{\mathcal{E}}_{01}(\lambda, x) &= e^{x \operatorname{Im} \xi} F_2^{-1}[L[\mathcal{E}_0(t, x)](\lambda, \xi)](\lambda, x) = e^{x \operatorname{Im} \xi} F_2^{-1}\left[\hat{\mathcal{L}}_1^{-1}(\tilde{\lambda}, \xi)\right](\lambda, x) \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{x \operatorname{Im} \xi - ix \operatorname{Re} \xi}}{\hat{\mathcal{L}}_1(\tilde{\lambda}, \xi)} d \operatorname{Re} \xi = \frac{1}{2\pi} \lim_{R \rightarrow +\infty} \int_{-R}^R A(\tilde{\lambda}) \left[\frac{1}{\xi - \xi_1(\tilde{\lambda})} - \frac{1}{\xi + \xi_1(\tilde{\lambda})} \right] e^{-ix \xi} d \operatorname{Re} \xi \\ &= \frac{1}{2\pi} \lim_{R \rightarrow +\infty} \left[\int_R^{R+iR} + \int_{R+iR}^{-R+iR} + \int_{-R+iR}^{-R} \right] \left(A(\tilde{\lambda}) \left[\frac{1}{\xi - \xi_1(\tilde{\lambda})} - \frac{1}{\xi + \xi_1(\tilde{\lambda})} \right] e^{-ix \xi} d \operatorname{Re} \xi \right) \\ &\quad + iA(\tilde{\lambda})e^{-ix\xi_1(\tilde{\lambda})} = iA(\tilde{\lambda})e^{-ix\xi_1(\tilde{\lambda})}.\end{aligned}$$

The latter identity is implied by the following estimates for the integrals as $x \leq 0$, $\operatorname{Im} \xi > 0$, $(\lambda, \xi) \in T^{C_1}$:

$$\begin{aligned}\left| \int_{\pm R}^{\pm R+iR} \frac{e^{-ix\xi}}{\hat{\mathcal{L}}_1(\tilde{\lambda}, \xi)} d\xi \right| &= \left| \int_0^R \frac{e^{-ix(\pm R+i \operatorname{Im} \xi)} d \operatorname{Im} \xi}{\tilde{\lambda}^2 + (a^2 - \hat{K}(\tilde{\lambda}))(\pm R + i \operatorname{Im} \xi)^2} \right| \\ &\leq \int_0^R \frac{e^{x \operatorname{Im} \xi} d \operatorname{Im} \xi}{\left| a^2 - \hat{K}(\tilde{\lambda}) \right| (R^2 + (\operatorname{Im} \xi)^2) - |\tilde{\lambda}|^2} \\ &\leq \frac{R}{\left| a^2 - \hat{K}(\tilde{\lambda}) \right| R^2 - |\tilde{\lambda}|^2} \rightarrow 0, \quad R \rightarrow +\infty, \\ \left| \int_{R+iR}^{-R+iR} \frac{e^{-ix\xi}}{\hat{\mathcal{L}}_1(\tilde{\lambda}, \xi)} d\xi \right| &= \left| \int_R^{-R} \frac{e^{-ix(\operatorname{Re} \xi + iR)} d \operatorname{Re} \xi}{\tilde{\lambda}^2 + (a^2 - \hat{K}(\tilde{\lambda}))(\operatorname{Re} \xi + iR)^2} \right| \leq \\ &\leq \int_{-R}^R \frac{e^{xR} d \operatorname{Re} \xi}{\left| a^2 - \hat{K}(\tilde{\lambda}) \right| (R^2 + (\operatorname{Re} \xi)^2) - |\tilde{\lambda}|^2} \\ &\leq \frac{2R}{\left| a^2 - \hat{K}(\tilde{\lambda}) \right| R^2 - |\tilde{\lambda}|^2} \rightarrow 0, \quad R \rightarrow +\infty.\end{aligned}$$

Similarly, for $x > 0$, $\operatorname{Im} \xi < 0$, $(\lambda, \xi) \in T^{C_1}$ we can establish the identity

$$\hat{\mathcal{E}}_{01}(\lambda, x) = -iA(\tilde{\lambda})e^{ix\xi_1(\tilde{\lambda})}.$$

The proof is complete. $\square$

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