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ABEL — GONCHAROV PROBLEM IN KERNEL OF CONVOLUTION OPERATOR

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Abstract. In the work we prove that the multiple interpolation problem is solvable, and as a corollary, the same for the Abel — Goncharov problem in the kernel of a convolution operator, when the zero sequence of the characteristic function of the convolution operator and the nodes, which are zeros of an entire function, are located in some angles in the complex plane and the nodes are multiple.

Keywords: multiple interpolation, Abel — Goncharov problem, convolution operator, entire functions.

Mathematics Subject Classification: 46A13, 30D20

1. Introduction

The Abel — Goncharov problem belongs to the theory of functions of complex variable and it consists in finding the set of all functions $f(z)$ from a given class, which satisfy the relations

$$f^{(n)}(\lambda_n) = A_n, \qquad n = 0, 1, 2, \ldots,$$

where $\{A_n\}$ and $\{\lambda_n\}$ are admissible sequences of complex numbers [1].

As it is shown below, in the kernel of the convolution operator the Abel — Goncharov problem is a particular case of the multipoint de la Vallée Poussin problem for multiple nodes (or multiple interpolation problem) in the same space. Originally the de la Vallée Poussin problem was posed for a homogeneous linear differential equation of order n [11]

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_{n-1}(x)y' + p_n(x)y = 0, \tag{1.1}$$

the coefficients $p_1(x)$, …, $p_{n-1}(x)$, $p_n(x)$ of which are continuous functions of x on the segment $[a, b]$ with some additional condition. The existence and uniqueness theorem says that, given a point x^0 in $[a, b]$ and the values y^0, y^0_1, …, y^0_{n-1}, there exists a unique solution $y(x)$ of Equation (1.1), which obeys the initial conditions

$$y(x^0) = y^0, \qquad y'(x^0) = y^0_1, \qquad \ldots, \qquad y^{(n-1)}(x^0) = y^0_{n-1}.$$

But in problems of mathematical physics and applied mathematics one often needs to find a solution to Equation (1.1), when not all initial conditions are prescribed at the same point x^0. For instance, for Equation (1.1) one can need to find a solution $y(x)$, the graph of which passes n given points. In other words, to construct a solution to (1.1), which satisfies the conditions

$$y(a_k) = A_k, \qquad k = 1, 2, \ldots, n. \tag{1.2}$$

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de la Vallée Poussin proved that if $p_k(x) \in C[a,b]$, $k = 1, 2, \dots, n$, and the inequality

$$\sum_{k=1}^{n} l_k \frac{(b-a)^k}{k!} < 1$$

holds, where $l_k \geqslant |p_k(x)|$, $k = 1, 2, \dots, n$, $x \in [a,b]$, then there exists a unique solution of the problem (1.1), (1.2) for finitely many nodes.

In 2001 in [4], the solvability of the multipoint de la Vallée Poussin problem was proved in the kernel of a convolution operator for infinitely many nodes, when the nodes belong to the set $\{0, \pm 1, \pm 2, \dots\}$. We note that the multiple interpolation problem was considered in various domains [2], [3], [8], [9].

2. Formulation of problem

Let $H(\mathbb{C})$ be the space of entire functions with the topology of uniform convergence on compact sets, $H^*(\mathbb{C})$ be the dual space for $H(\mathbb{C})$, $P_{\mathbb{C}}$ be the space of entire functions of exponential type. With a function $\varphi \in P_{\mathbb{C}}$ we associate a functional $F \in H^*(\mathbb{C})$ such that $\widehat{F}(z) = \varphi(z)$, where $\widehat{F}(z) = \langle F_\lambda, e^{\lambda z}\rangle$ is the Laplace transform of the functional F. We write the convolution operator in $H(\mathbb{C})$ as

$$M_\varphi[f](z) = (F_t, f(z+t)), \qquad f \in H(\mathbb{C}).$$

We denote by $\operatorname{Ker} M_\varphi = \{f \in H(\mathbb{C}) : M_\varphi[f] = 0\}$ the kernel of convolution operator M_φ.

We pose the multipoint de la Vallée Poussin problem (or, in other words, the multiple interpolation problem) in $\operatorname{Ker} M_\varphi$ with the nodes $\mu_j \in \mathbb{C}$ being the zeros of $\psi \in H(\mathbb{C})$, with the multiplicities q_j, $j = 0, 1, 2, \dots$, as follows: given an arbitrary sequence of complex numbers a_j^k, $j = 0, 1, 2, \dots$; $k = 0, 1, \dots, q_j - 1$, whether there exists a function $y \in \operatorname{Ker} M_\varphi$ such that

$$y^{(k)}(\mu_j) = a_j^k, \quad j = 0, 1, 2, \dots; \quad k = 0, 1, \dots, q_j - 1.$$

In [5] this problem was solved in the case, when the nodes are simple and lie on the real axis. In work [6] the interpolation problem was solved in the convolution kernel, when the nodes are complex. In this work we solve the multiple interpolation problem for complex nodes lying in some angle, its particular case is the problem

$$y^{(k)}(\mu_j) = a_j^k, \quad j = 0, 1, 2, \dots; \quad k = \overline{0, j}.$$

Hence, in the kernel of the convolution operator there exists a function $y(z)$, which, for a sequence of complex numbers a_0^0, a_1^1, $\dots$, a_n^n, $\dots$ satisfies

$$y^{(k)}(\mu_k) = a_k^k.$$

We thus obtain the Abel — Goncharov problem in the kernel of convolution operator.

In the case, when the characteristic function of convolution operator is a polynomial, the convolution operator becomes a linear differential operator of finite order with constant coefficients, and hence, as an important particular corollary we conclude that for a homogeneous linear differential operator of finite order with constant coefficients the multiple interpolation problem and Abel — Goncharov problem are solved. Moreover, a differential–difference operator, integro–differential operator, linear differential operator of infinite order with constant coefficients are also particular cases of the convolution operator and hence, the same problems are also solved for homogeneous equations.

3. Preliminary results

We introduce a series of auxiliary notions, which are needed to formulate and prove the main results.

The topology $\tau_{\mathbb{C}}$ of the space $P_{\mathbb{C}}$ is defined as the inductive limit of normed weight spaces

$$B_n = \{\varphi(\lambda) \in P_{\mathbb{C}} : \|\varphi\|_n = \sup_{\lambda \in \mathbb{C}} |\varphi(\lambda)| e^{-n|\lambda|} < \infty\}, \quad n \in \mathbb{N}.$$

Let $S \subset \mathbb{C}$ be the uniqueness set in $P_{\mathbb{C}}$. Then in $P_{\mathbb{C}}$ we can introduce the topology τ_S of inductive limit of spaces

$$B_{n,S} = \{\varphi(\lambda) \in P_{\mathbb{C}} : \|\varphi\|_{n,S} = \sup_{\lambda \in S} |\varphi(\lambda)| e^{-n|\lambda|} < \infty\}, \quad n \in \mathbb{N}.$$

In what follows we shall need the convergence to zero in the topology $\tau_{\mathbb{C}}$ [7]: let f_m be a countable sequence of function $P_{\mathbb{C}}$, then $f_m \to 0$ as $m \to \infty$ in the topology $\tau_{\mathbb{C}}$ if and only if there exist numbers $\sigma > 0$ and $M > 0$ such that

(a.1) $|f_m(z)| \leqslant M e^{\sigma|z|} \quad \forall m \in \mathbb{N}, \quad \forall z \in \mathbb{C}$;
(b.1) for each compact set $K_{\mathbb{C}} \subset \mathbb{C}$: $|f_m(z)| \rightrightarrows 0$ as $m \to \infty$, $z \in K_{\mathbb{C}}$.

Let us introduce the notion of sufficiency of a set $S \subset \mathbb{C}$ in $U \subset P_{\mathbb{C}}$ with topology the induced from $P_{\mathbb{C}}$.

Definition 3.1. *We say that S is a sufficient set on U if the conditions*

(a.2) *for each sequence of functions $q_k(z) \in U$ there exist numbers $\sigma > 0$ and $M > 0$ such that*

$$|q_k(z)| \leqslant M e^{\sigma|z|} \quad \forall k \in \mathbb{N}, \quad \forall z \in S;$$

(b.2) *for each compact set $K_S \subset S : |q_k(z)| \rightrightarrows 0$ as $k \to \infty$, $z \in K_S$*

imply the convergence of this sequence on U.

The conditions (a.2) and (b.2) define the convergence to zero in the topology τ_S.

The function $\psi(z) \in H(\mathbb{C})$ generates a linear continuous operator $M_\psi : P_{\mathbb{C}} \to P_{\mathbb{C}}$ in the space $P_{\mathbb{C}}$ [10], which acts by the rule

$$M_\psi[f](z) = \frac{1}{2\pi i} \int_C e^{z\xi} \psi(\xi) \gamma(\xi) d\xi,$$

where $\gamma(\xi)$ is a function associated with $f(z)$ in the Borel sense, C is a closed contour enveloping all singular points $\gamma(\xi)$.

We denote by N_φ the zero set of function $\varphi \in P_{\mathbb{C}}$. In [5] the next statement was proved.

Theorem 3.1. *Let $\varphi \in P_{\mathbb{C}}$, $\psi \in H(\mathbb{C})$ and N_φ is a sufficient set in* $\operatorname{Ker} M_\psi$, *then the multipoint de la Vallée Poussin problem is solvable in* $\operatorname{Ker} M_\varphi$.

4. Main result

Let $N_\varphi = \{\lambda_k\}_{k=1}^{+\infty}$ be the zero set of the function $\varphi \in P_{\mathbb{C}}$, in which each zero is repeated according to its order. In order to avoid cumbersome notation, in what follows by $\lambda_{\tilde{k}}$, $\tilde{k} = 1, 2 \ldots$, we mean some subsequence of the sequence $\{\lambda_k\}_{k=1}^{+\infty}$); by $N_\psi = \{\mu_k\}_{k=1}^{+\infty}$ we denote the set of zeros of a function $\psi \in H(\mathbb{C})$, where each zero is repeated according to its order, and q_j stands for the order of zero μ_j; and $\tilde{N}_\psi$ is an infinite set, which consists of all different zeros of the function $\psi \in H(\mathbb{C})$.

According to the result of work [10], the space Ker M_ψ consists of quasipolynomials with the exponents in the set N_ψ, that is, each element $r(z) \in$ Ker M_ψ is written as

$$r(z) = \sum_{j=1}^{N} \sum_{i=0}^{q_j - 1} C_{ji} z^i e^{\mu_j z},$$

and all coefficients C_{ji} are nonzero. We introduce a function $Q(n) : \mathbb{N} \to \mathbb{N}$, which is defined as

$$Q(n) = \sum_{j=1}^{n} q_j, \qquad q_j \in \mathbb{N},$$

the value $Q(N)$ determines the number of terms in $r(z)$.

Let us mention the properties of a quasipolynomials, which will be needed to prove the main result.

Lemma 4.1. *Let for some fixed $\alpha \in [0, +\infty)$ there exists a number $\beta \in [0, +\infty)$ such that $\alpha \cdot \beta < 1$ and the conditions hold*

(a) $N_\varphi \subset D_\alpha = \{z \in \mathbb{C} : |\operatorname{Im} z| \leqslant \alpha \operatorname{Re} z\}$ *and there exists a subsequence* $\lambda_{\tilde{k}}$ *such that*

$$\operatorname{Re}(\lambda_{\tilde{k}}) < \operatorname{Re}(\lambda_{\tilde{k}+1}), \quad \tilde{k} \in \mathbb{N}.$$

(b) $N_\psi \subset D_\beta = \{z \in \mathbb{C} : |\operatorname{Im} z| \leqslant \beta \operatorname{Re} z\}$, *and for the elements of the set* $\tilde{N}_\psi$ *we have*

$$\operatorname{Re}(\mu_k) < \frac{1 - \alpha\beta}{1 + \alpha\beta} \operatorname{Re}(\mu_{k+1}), \quad k \in \mathbb{N}. \tag{4.1}$$

Then, under the condition

$$|r(\lambda_k)| \leqslant M e^{\sigma|\lambda_k|}, M, \sigma > 0, \qquad \forall k \in \mathbb{N}, \tag{4.2}$$

1) *the estimate*

$$\operatorname{Re} \mu_j \leqslant \frac{(1 + \alpha)\sigma}{1 - \alpha\beta}, \quad j = 1, \dots, N,$$

holds;

2) *the coefficients of quasipolynomial satisfy the estimate*

$$|C_{ji}| \leqslant C := Q(N)! M \left|\lambda_{\tilde{k}_{Q(N)}}\right|^{(Q(N)-1)} e^{((Q(N)-1)\frac{1+\alpha\beta}{1-\alpha\beta}+1)\sigma(1+\alpha)\operatorname{Re}\lambda_{\tilde{k}_{Q(N)}}}, \quad j = \overline{1, N}, \quad i = \overline{0, q_j - 1}.$$

Proof. 1) We argue by contradiction. Let all zeros μ_j, $j = 1, \dots, N$ be taken in the ascending order of their real parts and suppose that

$$\operatorname{Re} \mu_N > \frac{(1 + \alpha)\sigma}{1 - \alpha\beta}.$$

We consider the quotient

$$\frac{\left|\lambda_k^{q_l - 1} e^{\mu_j \lambda_k}\right|}{\left|\lambda_k^{q_N - 1} e^{\mu_N \lambda_k}\right|} \leqslant \frac{|\lambda_k|^{q_l} e^{(1+\alpha\beta)\operatorname{Re}\mu_j \operatorname{Re}\lambda_k}}{|\lambda_k|^{q_N} e^{(1-\alpha\beta)\operatorname{Re}\mu_N \operatorname{Re}\lambda_k}} = |\lambda_k|^{q_l - q_N} e^{(1+\alpha\beta)\operatorname{Re}\mu_j \operatorname{Re}\lambda_k - (1-\alpha\beta)\operatorname{Re}\mu_N \operatorname{Re}\lambda_k}.$$

For $j = 1, 2, \dots, N - 1$ by the condition (4.1) the exponent in the exponential is negative and this is why the quotient of the absolute values tend to zero as $\operatorname{Re}\lambda_k \to +\infty$. Hence,

$$\lim_{k\to+\infty} \left| r(\lambda_k) \lambda_k^{-q_N+1} e^{-\mu_N \lambda_k} \right| = |C_{N, q_N - 1}|.$$

This is why the growth of the quasipolynomial is determined by μ_N. On the other hand, the growth $r(\lambda_k)$ is determined by (4.2). Estimating from above and below in (4.2), we obtain the chain of inequalities

$$\left| \sum_{j=1}^{N} \sum_{i=0}^{q_j - 1} C_{ji} \lambda_k^{i - q_N + 1} e^{(\mu_j - \mu_N)\lambda_k} \right| |\lambda_k|^{q_N - 1} e^{(1-\alpha\beta)\operatorname{Re}\mu_N \operatorname{Re}\lambda_k} \leqslant |r(\lambda_k)|$$
$$\leqslant M e^{\sigma|\lambda_k|} \leqslant M e^{\sigma(1+\alpha)\operatorname{Re}\lambda_k}.$$

We then get the inequality

$$\left| \sum_{j=1}^{N} \sum_{i=0}^{q_j - 1} C_{ji} \lambda_k^{i - q_N + 1} e^{(\mu_j - \mu_N)\lambda_k} \right| |\lambda_k|^{q_N - 1} e^{((1-\alpha\beta)\operatorname{Re}\mu_N - \sigma(1+\alpha))\operatorname{Re}\lambda_k} \leqslant M.$$

The estimate $\operatorname{Re}\mu_N > \frac{(1+\alpha)\sigma}{1-\alpha\beta}$ implies that the left hand side of this inequality tend to $+\infty$ as $\operatorname{Re}\lambda_k \to +\infty$ (as it has been proved above, the first factor tends to $|C_{N, q_N - 1}|$). This contradicts the above inequality. Thus, $\operatorname{Re}\mu_N \leqslant \frac{(1+\alpha)\sigma}{1-\alpha\beta}$. We have shown that the quasipolynomial $r(z)$ involves only $e^{\mu_j z}$ with the exponents μ_j obeying the estimate

$$\operatorname{Re}\mu_j \leqslant \frac{(1 + \alpha)\sigma}{1 - \alpha\beta},$$

and since μ_j are zeros of an entire function, the quasipolynomial $r(z)$ involves finitely many exponents. This proves assertion 1.

2) We employ a simple observation that if the set of $Q(N)$ zeros $\lambda_{\tilde{k}_p}$ is chosen so that the determinant of matrix

$$A = (\lambda_{\tilde{k}_p}^i e^{\mu_j \lambda_{\tilde{k}_p}}), \quad j = 1, \dots, N, \quad i = 0, \dots, q_j - 1, \quad p = 1, \dots, Q(N),$$

is non–zero, then the coefficients C_{ji} are solutions to the system of linear equations

$$\sum_{j=1}^{N} \sum_{i=0}^{q_j-1} C_{ji} \lambda_{\tilde{k}_p}^i e^{\mu_j \lambda_{\tilde{k}_p}} = r(\lambda_{\tilde{k}_p}), \qquad p = 1, \dots, Q(N).$$

Let us prove by induction in the parameter $t = 1, 2, \dots,$ that the set of zeros can be chosen so that the absolute values of determinants

$$A(t) = (\lambda_{\tilde{k}_p}^i e^{\mu_j \lambda_{\tilde{k}_p}}), \quad j = 1, \dots, N(t), \quad i = 0, \dots, q_j - 1, \quad p = 1, \dots, t,$$

exceed 1. We first consider $t = 1$:

$$|\det A(1)| = e^{\operatorname{Re}(\mu_1 \lambda_{\tilde{k}_1})} \geqslant e^{(1-\alpha\beta) \operatorname{Re} \mu_1 \operatorname{Re} \lambda_{\tilde{k}_1}} \geqslant 1.$$

As $\lambda_{\tilde{k}_1}$ we can take the first element in the sequence $\lambda_{\tilde{k}}$.

Suppose that the zeros $\lambda_{\tilde{k}_p}$, $p = 1, 2, \dots, t-1$ are chosen so that the absolute values of principal minors exceed 1 and $\operatorname{Re} \lambda_{\tilde{k}_p}$ increases in p. We expand the determinant of matrix A_t along the last row

$$\det A(t) = \sum_{j=1}^{N(t)} \sum_{i=0}^{q_j-1} (-1)^{l(i,j)+t} \det A_{l(i,j),t} \lambda_{\tilde{k}_t}^i e^{\mu_j \lambda_{\tilde{k}_t}},$$

where $(-1)^{l(i,j)+t} \det A_{l(i,j),t}$ is the cofactor of the entry $\lambda_{\tilde{k}_t}^i e^{\mu_j \lambda_{\tilde{k}_t}}$ and

$$l(i,j) = i + 1 + \sum_{s=0}^{j-1} q_s, \qquad q_0 = 0.$$

This implies

$$|\det A(t)| \geqslant \left| \sum_{i=0}^{q_{N(t)}-1} (-1)^{l(i,N(t))+t} \det A_{l(i,N(t)),t} \lambda_{\tilde{k}_t}^i e^{\mu_{N(t)} \lambda_{\tilde{k}_t}} \right| - \left| \sum_{j=1}^{N(t)-1} \sum_{i=0}^{q_j-1} (-1)^{l(i,j)+t} \det A_{l(i,j),t} \lambda_{\tilde{k}_t}^i e^{\mu_j \lambda_{\tilde{k}_t}} \right|.$$

We denote the first term in the right hand side of inequality by B_1, the second term is denoted B_2 and we are going to estimate them from below and above, respectively. But first let us estimate from above $\left|\det A_{l(i,j),t}\right|$. This determinant is the sum of $(Q(N(t)) - 1)!$ terms, each term is the product of $Q(N(t)) - 1$ factors of form

$$\lambda_{\tilde{k}_p}^i e^{\mu_j \lambda_{\tilde{k}_p}}, \quad j = 1, \dots, N(t), \quad i = 0, \dots, q_j - 1, \quad p = 1, \dots, t-1.$$

Taking into consideration the monotonicity of sets μ_j and $\lambda_{\tilde{k}_p}$, we obtain

$$\left|\det A_{l(i,j),t}\right| \leqslant (Q(N(t)) - 1)! |\lambda_{\tilde{k}_{t-1}}|^{(Q(N(t))-1)} e^{(Q(N(t))-1)(1+\alpha\beta) \operatorname{Re} \mu_{N(t)} \operatorname{Re} \lambda_{\tilde{k}_{t-1}}}. \tag{4.3}$$

We estimate B_1

$$B_1 \geqslant \left|\lambda_{\tilde{k}_t}\right|^{q_{N(t)}-1} e^{(1-\alpha\beta)\operatorname{Re}\mu_{N(t)}\operatorname{Re}\lambda_{\tilde{k}_t}} \left|\sum_{i=0}^{q_{N(t)}-1} (-1)^{l(i,N(t))+t} \det A_{l(i,N(t)),t} \lambda_{\tilde{k}_t}^{i-(q_{N(t)}-1)}\right|$$
$$\geqslant \left|\lambda_{\tilde{k}_t}\right|^{q_{N(t)}-1} e^{(1-\alpha\beta)\operatorname{Re}\mu_{N(t)}\operatorname{Re}\lambda_{\tilde{k}_t}} \left(1 - \sum_{i=0}^{q_{N(t)}-2} \left|\det A_{l(i,N(t)),t}\right| \left|\lambda_{\tilde{k}_t}\right|^{i-(q_{N(t)}-1)}\right).$$

In the latter inequality we have employed the fact that at the last element in the last row we have the principal minor, which coincides with $\det A_{t,t}$, by the induction assumption, its absolute value is not less than 1. Using the inequality (4.3), we obtain the final estimate for B_1

$$B_1 \geqslant \left|\lambda_{\tilde{k}_t}\right|^{q_{N(t)}-1} e^{(1-\alpha\beta)\operatorname{Re}\mu_{N(t)}\operatorname{Re}\lambda_{\tilde{k}_t}}$$
$$\cdot \left(1 - \left|\lambda_{\tilde{k}_t}\right|^{-1} (q_{N(t)}-1)(Q(N(t))-1)!|\lambda_{\tilde{k}_{t-1}}|^{(Q(N(t))-1)} e^{(Q(N(t))-1)(1+\alpha\beta)\operatorname{Re}\mu_{N(t)}\operatorname{Re}\lambda_{\tilde{k}_{t-1}}}\right),$$

since now $\lambda_{\tilde{k}_{t-1}}$ and $\mu_{N(t)}$ are fixed and, as it has been proved above, $\left|\lambda_{\tilde{k}_t}\right|^{-1}$ tends to zero as $\operatorname{Re}\lambda_{\tilde{k}_t} \to +\infty$, the second term in the brackets also tends to zero.

We denote $q_{max} = \max\limits_{s=1,\dots,N(t)-1} q_s$ estimate B_2 from above

$$B_2 \leqslant (Q(N(t)-1))|\det A_{l(i,j),t}| \left|\lambda_{\tilde{k}_t}\right|^{q_{max}-1} e^{(1+\alpha\beta)\operatorname{Re}\mu_{N(t)-1}\operatorname{Re}\lambda_{\tilde{k}_t}}$$
$$\leqslant (Q(N(t)-1))(Q(N(t))-1)!|\lambda_{\tilde{k}_{t-1}}|^{(Q(N(t))-1)} e^{(Q(N(t))-1)(1+\alpha\beta)\operatorname{Re}\mu_{N(t)}\operatorname{Re}\lambda_{\tilde{k}_{t-1}}}$$
$$\cdot \left|\lambda_{\tilde{k}_t}\right|^{q_{max}-1} e^{(1+\alpha\beta)\operatorname{Re}\mu_{N(t)-1}\operatorname{Re}\lambda_{\tilde{k}_t}}.$$

In the latter inequality only two last factors are varying quantities depending on $\operatorname{Re}\lambda_{\tilde{k}_t}$, while other factors are fixed. The estimate for the absolute value of determinant is written as

$$|\det A(t)| \geqslant \left|\lambda_{\tilde{k}_t}\right|^{q_{N(t)}-1} e^{(1-\alpha\beta)\operatorname{Re}\mu_{N(t)}\operatorname{Re}\lambda_{\tilde{k}_t}}$$
$$\cdot \left(1 - \left|\lambda_{\tilde{k}_t}\right|^{-1} (q_{N(t)}-1)(Q(N(t))-1)!|\lambda_{\tilde{k}_{t-1}}|^{(Q(N(t))-1)} e^{(Q(N(t))-1)(1+\alpha\beta)\operatorname{Re}\mu_{N(t)}\operatorname{Re}\lambda_{\tilde{k}_{t-1}}}\right)$$
$$- (Q(N(t)-1))(Q(N(t))-1)!|\lambda_{\tilde{k}_{t-1}}|^{(Q(N(t))-1)} e^{(Q(N(t))-1)(1+\alpha\beta)\operatorname{Re}\mu_{N(t)}\operatorname{Re}\lambda_{\tilde{k}_{t-1}}}$$
$$\cdot \left|\lambda_{\tilde{k}_t}\right|^{q_{max}-1} e^{(1+\alpha\beta)\operatorname{Re}\mu_{N(t)-1}\operatorname{Re}\lambda_{\tilde{k}_t}} = \left|\lambda_{\tilde{k}_t}\right|^{q_{N(t)}-1} e^{(1-\alpha\beta)\operatorname{Re}\mu_{N(t)}\operatorname{Re}\lambda_{\tilde{k}_t}} [(1-$$
$$\left|\lambda_{\tilde{k}_t}\right|^{-1} (q_{N(t)}-1)(Q(N(t))-1)!|\lambda_{\tilde{k}_{t-1}}|^{(Q(N(t))-1)} e^{(Q(N(t))-1)(1+\alpha\beta)\operatorname{Re}\mu_{N(t)}\operatorname{Re}\lambda_{\tilde{k}_{t-1}}})$$
$$- (Q(N(t)-1))(Q(N(t))-1)!|\lambda_{\tilde{k}_{t-1}}|^{(Q(N(t))-1)} e^{(Q(N(t))-1)(1+\alpha\beta)\operatorname{Re}\mu_{N(t)}\operatorname{Re}\lambda_{\tilde{k}_{t-1}}}$$
$$\cdot \frac{\left|\lambda_{\tilde{k}_t}\right|^{q_{max}}}{\left|\lambda_{\tilde{k}_t}\right|^{qN(t)}} e^{\left((1+\alpha\beta)\operatorname{Re}\mu_{N(t)-1} - (1-\alpha\beta)\operatorname{Re}\mu_{N(t)}\right)\operatorname{Re}\lambda_{\tilde{k}_t}}]$$

In the latter identity, as $\operatorname{Re}\lambda_{\tilde{k}_t} \to +\infty$, the expression in the round brackets tends to 1 (by the above facts), while the last factor in the square brackets tends to zero since by the condition (4.1) the exponent of exponential is negative. Thus, if we choose the zero $\lambda_{\tilde{k}_t}$ with a sufficiently large $\operatorname{Re}\lambda_{\tilde{k}_t}$, then $\Delta = |\det A(t)|$ exceeds 1.

By the Cramer's rule

$$C_{ji} = \frac{\Delta_{l(i,j)}}{\Delta},$$

where $\Delta_{l(i,j)}$ is the determinant of the matrix obtained from A_N by replacing $l(i,j)$th column by the column of right hand sides. To estimate $\Delta_{l(i,j)}$ from above, the expand the corresponding matrix along the $l(i,j)$th column

$$\Delta_{l(i,j)} = \sum_{p=1}^{Q(N)} (-1)^{l(i,j)+p} \det A_{l(i,j),p} r(\lambda_{\tilde{k}_p}),$$

As the relation (4.3), we obtain the estimate

$$\left|\det A_{l(i,j),p}\right| \leqslant (Q(N)-1)! |\lambda_{\tilde{k}_{Q(N)}}|^{(Q(N)-1)} e^{(Q(N)-1)(1+\alpha\beta)\operatorname{Re}\mu_N \operatorname{Re}\lambda_{\tilde{k}_{Q(N)}}},$$

while by the condition (4.4) we get

$$|r(\lambda_{\tilde{k}_p})| \leqslant M e^{(1+\alpha)\sigma \operatorname{Re}\lambda_{\tilde{k}_{Q(N)}}}.$$

Thus,

$$\begin{aligned} \left|\Delta_{l(i,j)}\right| &\leqslant Q(N)! M \left|\lambda_{\tilde{k}_{Q(N)}}\right|^{(Q(N)-1)} e^{(Q(N)-1)(1+\alpha\beta)\operatorname{Re}\mu_N \operatorname{Re}\lambda_{\tilde{k}_{Q(N)}}} e^{(1+\alpha)\sigma \operatorname{Re}\lambda_{\tilde{k}_{Q(N)}}} \\ &\leqslant Q(N)! M \left|\lambda_{\tilde{k}_{Q(N)}}\right|^{(Q(N)-1)} e^{((Q(N)-1)\frac{1+\alpha\beta}{1-\alpha\beta}+1)\sigma(1+\alpha)\operatorname{Re}\lambda_{\tilde{k}_{Q(N)}}} \end{aligned}$$

and

$$|C_{ji}| \leqslant C := Q(N)! M \left|\lambda_{\tilde{k}_{Q(N)}}\right|^{(Q(N)-1)} e^{((Q(N)-1)\frac{1+\alpha\beta}{1-\alpha\beta}+1)\sigma(1+\alpha)\operatorname{Re}\lambda_{\tilde{k}_{Q(N)}}}.$$

The proof is complete. □

Let the sequence

$$r_m(z) = \sum_{j=1}^{N} \sum_{i=0}^{q_j-1} C_{ji}(m) z^i e^{\mu_j z}$$

tend to zero in the topology τ_{N_φ}. By means of the conditions (a.2) and (b.2) and in view of the discreteness of set N_φ, the convergence to zero in the topology τ_{N_φ} can be written as follows: for some constants σ, $M >\ 0$ the relations hold

$$|r_m(\lambda_k)| \leqslant M e^{\sigma|\lambda_k|}, \quad \forall m \in \mathbb{N}, \quad \forall k \in \mathbb{N}, \tag{4.4}$$

and for each $k \in \mathbb{N}$ the sequence converges uniformly on compact subsets N_φ, that is,

$$|r_m(\lambda_k)| \to 0, \quad m \to \infty. \tag{4.5}$$

By Theorem 3.1, to prove the solvability of multiple interpolation problem in the kernel of convolution operator, we need to show that $r_m(z) \to 0$ in $\operatorname{Ker} M_\psi$, $z \in \mathbb{C}$. We are in position to formulate the main result.

Theorem 4.1. *Let for some fixed $\alpha \in [0,+\infty)$ there exist a number $\beta \in [0,+\infty)$ such that $\alpha \cdot \beta < 1$ and the conditions be satisfied*

(a) *$N_\varphi \subset D_\alpha = \{z \in \mathbb{C} : |\operatorname{Im} z| \leqslant \alpha \operatorname{Re} z\}$ and there exists a subsequence $\lambda_{\tilde{k}}$ such that*

$$\operatorname{Re}(\lambda_{\tilde{k}}) < \operatorname{Re}(\lambda_{\tilde{k}+1}), \quad \tilde{k} \in \mathbb{N}.$$

(b) *$N_\psi \subset D_\beta = \{z \in \mathbb{C} : |\operatorname{Im} z| \leqslant \beta \operatorname{Re} z\}$, and the elements of set $\tilde{N}_\psi$ satisfy*

$$\operatorname{Re}(\mu_k) < \frac{1-\alpha\beta}{1+\alpha\beta} \operatorname{Re}(\mu_{k+1}), \quad k \in \mathbb{N}.$$

Then under the conditions (4.4) and (4.5) the set N_φ is sufficient in $\operatorname{Ker} M_\psi$.

Proof. As it has been proved in Lemma 4.1, the absolute value of determinant of the homogeneous system

$$r_m(\lambda_{\tilde{k}_p}) = 0 \quad \forall m \in \mathbb{N}$$

exceeds 1, and this is why all coefficients of the system vanish, that is,

$$r_m(z) \equiv 0.$$

This means that N_φ is the uniqueness set in $\operatorname{Ker} M_\psi$.

Let us prove that

$$\lim_{m\to\infty} C_{ji}(m) = 0, \quad j = \overline{1, N}, \quad i = \overline{0, q_j - 1}.$$

Since it follows from Lemma 4.1 that, for $|C_{ji}(m)|$ by (4.5),

$$|C_{ji}(m)| \leqslant |\Delta_{l(i,j)}| \\ \leqslant Q(N)!|\lambda_{\tilde{k}_{Q(N)}}|^{Q(N)-1} e^{(Q(N)-1)(1+\alpha\beta)\frac{\sigma(1+\alpha)}{1-\alpha\beta}\operatorname{Re}\lambda_{\tilde{k}_{Q(N)}}} \cdot |r_m(\lambda_{\tilde{k}_{Q(N)}})| \to 0$$

as $m \to \infty$ and $\forall N \in \mathbb{N}$, we have

$$\lim_{m\to\infty} C_{ji}(m) = 0, \qquad j = \overline{1, N}, \quad i = \overline{0, q_j - 1}.$$

We are going to complete the proof of sufficiency of the set N_φ. We have shown in Lemma 4.1 that

$$\operatorname{Re}\mu_j \leqslant \frac{\sigma(1+\alpha)}{1-\alpha\beta} \qquad \text{and} \qquad |C_{ji}(m)| \leqslant C$$

for $m = 1, 2, \ldots$, $j = 1, \ldots, N$, $i = 0, \ldots, q_j - 1$. We estimate $r_m(z)$ from above under the condition that $z \neq 0$

$$|r_m(z)| \leqslant Q(N)C|z|^{\max\limits_{j=\overline{1,N}} q_j} e^{|\mu_N||z|} \leqslant Q(N)Ce^{\max\limits_{j=\overline{1,N}} q_j \ln|z| + \frac{\sigma(1+\alpha)}{1-\alpha\beta}(1+\beta)|z|}.$$

Therefore,

$$|r_m(z)| \leqslant Q(N)Ce^{2\max\left(\max\limits_{j=\overline{1,N}} q_j, \frac{\sigma(1+\alpha)(1+\beta)}{1-\alpha\beta}\right)|z|}. \tag{4.6}$$

We estimate for the case $z = 0$:

$$|r_m(0)| = \sum_{j=1}^{N} |C_{j0}(m)| \leqslant N \cdot C \leqslant Q(N) \cdot C.$$

Hence, the estimate (4.6) holds also for $z = 0$, therefore, (4.6) is true for $z \in \mathbb{C}$.

In the beginning we have shown that $C_{ji}(m) \to 0$, $m \to \infty$ for all j, i, hence,

$$\max_{j,i} |C_{ji}(m)| \to 0.$$

This is why for each compact set $K_{\mathbb{C}}$, for $z \in K_{\mathbb{C}}$,

$$|r_m(z)| \leqslant Q(N) \max_{j=\overline{1,N}, i=\overline{0,q_j}} |C_{ji}(m)| e^{2\max\left(\max\limits_{j=\overline{1,N}} q_j, \frac{\sigma(1+\alpha)(1+\beta)}{1-\alpha\beta}\right)\max\limits_{z\in K_{\mathbb{C}}}|z|} \to 0.$$

This implies that $r_m(z) \to 0$ in $\operatorname{Ker} M_\psi$, $z \in \mathbb{C}$. The proof is complete. □

Remark 4.1. *For $\beta = 0$ we obtain that the nodes are real and the coefficient in the condition (4.1) is equal to 1, that is, the nodes can be taken in the ascending order. Therefore, Theorem 4.1 can be regarded as the generalization of results of* [5].

Remark 4.2. *For $\alpha = 0$ the zeros of characteristic function are located on the real axis and the coefficient in the condition (4.1) is equal to 1. Hence, the nodes are located in the ascending order of their real parts and no additional conditions are needed for the mutual distances.*

5. Illustrative example

We consider the case when the zeros of functions φ and ψ are located on the upper boundary of the angle

$$\left\{z \in \mathbb{C} : |\operatorname{Im} z| \leqslant \frac{1}{\sqrt{3}} \operatorname{Re} z\right\}.$$

Then the constants in Theorem 4.1 have the following values: $\alpha = \beta = \frac{\pi}{6}$. The ordering coefficient of the zeros of $\psi(z)$ is equal to

$$\frac{1-\alpha\beta}{1+\alpha\beta} = \frac{1}{2}.$$

Let us construct the sequence μ_i, $i = 1, 2, \dots$ for this coefficient. We take $\operatorname{Re}\mu_1 > 0$, and as $\operatorname{Re}\mu_2$ we can take $2\operatorname{Re}\mu_1 + 1$ since

$$\frac{2\operatorname{Re}\mu_1 + 1}{\operatorname{Re}\mu_1} = 2 + \frac{1}{\operatorname{Re}\mu_1} > 2.$$

Similarly, as $\operatorname{Re}\mu_3$ we can take $2\operatorname{Re}\mu_2 + 1$, then

$$\operatorname{Re}\mu_3 = 2(2\operatorname{Re}\mu_1 + 1) + 1 = 4\operatorname{Re}\mu_1 + 3.$$

The real part of nth term in the sequence reads

$$\operatorname{Re}\mu_n = 2^{n-1}\operatorname{Re}\mu_1 + 2^{n-1} - 1.$$

Since the argument of all zeros is $\frac{\pi}{6}$, the general term of the zero sequence of $\psi(z)$ reads

$$(2^{n-1}\operatorname{Re}\mu_1 + 2^{n-1} - 1)\left(1 + \frac{1}{\sqrt{3}}i\right)$$

or, in the exponential form,

$$\mu_n = \frac{2}{\sqrt{3}}(2^{n-1}\operatorname{Re}\mu_1 + 2^{n-1} - 1)e^{i\frac{\pi}{6}}, \qquad \operatorname{Re}\mu_1 > 0, \quad n \in \mathbb{N}. \tag{5.1}$$

As λ_n we can take

$$\lambda_n = ne^{i\frac{\pi}{6}}, \quad n \in \mathbb{N}.$$

The constructed sequences λ_n and μ_n satisfy all assumptions of Theorem 4.1. Therefore, for the functions $\varphi \in P_{\mathbb{C}}$ and $\psi \in H(\mathbb{C})$, the zero sets of which coincide with the constructed sequence, the interpolation problem is solvable in the kernel of convolution operator. It should be noted that the sequence μ_n is constructed by means of real part of the first zero, which can be chosen by the only nonnegativity condition. Thus, if we take an arbitrary nonnegative number, by the formula (5.1) we can obtain the general term of sequence μ_n for the ordering coefficient 0.5. If it is needed, a similar procedure of constructing the zeros of function ψ can be made for other angles under the condition that the coefficient

$$\frac{1-\alpha\beta}{1+\alpha\beta}$$

is less than one.

The function ψ can have multiple zeros, then we can suppose that we have made the procedure of constructing the elements of set $\tilde{N}_\psi$ (N_ψ is easily obtained from $\tilde{N}_\psi$). The zeros of the function φ not necessarily to be on a single ray, some of them can be located below the ray $\arg z = \frac{\pi}{6}$, the main condition is that in this case the real parts should strongly increase with the coefficient taken into consideration. Moreover, the function $\varphi(z)$ can have multiple roots, but there should exist a sequence, the terms of which strictly increase.

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