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CONVOLUTION EQUATION IN SPACE OF FAST DECAYING FUNCTIONS ON UNBOUNDED CONVEX SET IN $\mathbb{R}^n$

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Abstract. In the work we study the solvability of the convolution equation (in particular, of differential–difference equation) in the Schwartz space on an unbounded convex set in $\mathbb{R}^n$.

Keywords: Schwartz space, Fourier — Laplace transform of functionals.

Mathematics Subject Classification: 46A04

1. Introduction

1.1. On problem. Let C be an open convex acute cone in $\mathbb{R}^n$ with the vertex at the origin [1, Ch. 1, Sect. 4], b be a convex continuous positively homogeneous on degree 1 function on $\overline{C}$, the latter is the closure of C. The pair (b, C) defines a closed unbounded convex set

$$U(b, C) = \{\xi \in \mathbb{R}^n : -\langle \xi, y\rangle \leqslant b(y), \ \forall y \in C\},$$

containing no entire straight lines. We note that the interior of $U(b, C)$ is non–empty and coincides with the set

$$V(b, C) = \{\xi \in \mathbb{R}^n : -\langle \xi, y\rangle < b(y), \ \forall y \in \overline{C}\},$$

while the closure of $V(b, C)$ is $U(b, C)$. For the sake of brevity we shall sometime denote the set $U(b, C)$ by U, and the set $V(b, C)$ by V.

The most known example of the set $U(b, C)$ is obtained once we choose the function b as $b(y) = r\|y\|$ with $r \geqslant 0$ for $y \in \overline{C}$. In this case [1, Ch. 1, Sect. 4, Lm. 3]

$$U(b, C) = C^* + B_r,$$

where

$$C^* = \{\xi \in \mathbb{R}^n : \langle \xi, y\rangle \geqslant 0, \ \forall y \in C\}$$

is the dual cone, B_r is the closed ball of radius r centered at the zero. This example admits a generalization: if $\mathcal{K}$ is a convex compact set in $\mathbb{R}^n$,

$$b(y) = \sup_{t\in\mathcal{K}}(-\langle y, t\rangle) \quad \text{for} \quad y \in \overline{C},$$

then

$$U(b, C) = C^* + \mathcal{K}.$$

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For an arbitrary set $M \subset \mathbb{R}^n$ we define the function H_M in $\mathbb{R}^n$ by the formula

$$H_M(x) = \sup_{\xi \in M}(-\langle x, \xi \rangle), \quad x \in \mathbb{R}^n.$$

We let $\tilde{b}(y) = b(y)$ for $y \in \overline{C}$, $\tilde{b}(y) = +\infty$ for $y \notin \overline{C}$. Then

$$\operatorname{dom} \tilde{b} = \{y \in \mathbb{R}^n : \tilde{b}(y) < \infty\} = \overline{C}$$

is a closed set in $\mathbb{R}^n$, $\tilde{b}$ is continuous on C and hence, the function $\tilde{b}$ is closed [5, Ch. 2, Sect. 7]. Moreover, according to [5, Cor. 13.2.1],

$$\tilde{b} = H_U.$$

Let $D \subset \mathbb{R}^n$ be a bounded convex domain, K be the closure of the domain D in $\mathbb{R}^n$. We form the set

$$G = U + K;$$

then G is an unbounded convex closed set. For each $y \in \mathbb{R}^n$ we have

$$H_G(y) = \sup_{\xi_1 \in U, \xi_2 \in K}(-\langle \xi_1 + \xi_2, y \rangle) = \sup_{\xi_1 \in U}(-\langle \xi_1, y \rangle) + \max_{\xi_2 \in K}(-\langle \xi_2, y \rangle) = \tilde{b}(y) + H_K(y).$$

Hence,

$$H_G(y) = b(y) + H_K(y), \quad y \in C.$$

Thus, the function H_G is continuous on $\overline{C}$. We note that

$$G = \{\xi \in \mathbb{R}^n : -\langle \xi, y \rangle \leqslant H_G(y), \, \forall y \in C\}.$$

For $\Omega = U$ or $\Omega = G$ by $S(\Omega)$ we denote the Schwartz space of functions f in the class C^∞ on Ω such that for each $p \in \mathbb{Z}_+$

$$\|f\|_{p,\Omega} = \sup_{x \in \operatorname{int} \Omega, |\alpha| \leqslant p} |(D^\alpha f)(x)|(1 + \|x\|)^p < \infty.$$

Here $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$. The topology in $S(\Omega)$ is defined by the family of norms $\|f\|_{p,\Omega}$ ($p \in \mathbb{Z}_+$).

Let μ be a linear continuous functional on the space $C^\infty(K)$ of infinitely differentiable on K functions f with the topology defined by the system of norms

$$\|f\|_m = \max_{x \in K, |\alpha| \leqslant m} |(D^\alpha f)(x)|, \quad m \in \mathbb{Z}_+.$$

Then for some $p \in \mathbb{N}$ and $c_\mu > 0$

$$|\langle \mu, f \rangle| \leqslant c_\mu \|f\|_p, \quad f \in C^\infty(K). \tag{1.1}$$

We note that for an arbitrary function $f \in S(U+K)$ and each $x \in U$ the function $f_x(t) = f(t+x)$ is well–defined for $t \in K$ and belongs to the class $C^\infty(K)$. At the same time the linear operator $T_x : f \in S(U+K) \to f_x$ acts continuously from $S(U+K)$ into $C^\infty(K)$, see Lemma 3.1.

We define the convolution $\mu * f$ of a functional μ and a function $f \in S(U+K)$ by the rule

$$(\mu * f)(x) = \langle \mu_t, f(x+t) \rangle, \quad x \in U.$$

We note that since for each $x \in U$ the function f_x belongs to the class $C^\infty(K)$, the function $\mu * f$ is well–defined on U. Moreover, $\mu * f \in C^\infty(U)$, see Lemma 3.2, and for each $\beta \in \mathbb{Z}^n_+$

$$(D^\beta(\mu * f))(x) = (\mu * D^\beta f)(x), \quad x \in U. \tag{1.2}$$

We note that the operator $M_\mu : f \in S(U+K) \to \mu * f$ acts from $S(U+K)$ into $S(U)$. Indeed, using (1.1) and (1.2), for all $x \in U$, $m \in \mathbb{Z}_+$ and $\beta \in \mathbb{Z}^n_+$ with $|\beta| \leqslant m$ we have

$$(1 + \|x\|)^m |(D^\beta(T_\mu f)(x)| \leqslant c_\mu \sup_{t \in K, |\alpha| \leqslant p} (1 + \|x\|)^m |(D^{\alpha+\beta} f)(x+t)|.$$

This implies that for some $A = A(m, K) > 0$

$$(1 + \|x\|)^m |(D^\beta (M_\mu f)(x)| \leqslant Ac_\mu \sup_{\substack{\xi \in G, \\ |\gamma| \leqslant m+p}} (1 + \|\xi\|)^m |(D^\gamma f)(\xi)| \leqslant Ac_\mu \|f\|_{m+p,G}.$$

Thus, for each $m \in \mathbb{Z}_+$,

$$\|M_\mu f\|_{m,U} \leqslant Ac_\mu \|f\|_{m+p,G}, \quad f \in S(U+K).$$

This inequality means that $M_\mu(f) \in S(U)$ and the linear operator M_μ acts continuously from $S(U+K)$ into $S(U)$.

There arises a natural problem to find the conditions μ, which ensure the surjectivity of the operator $M_\mu : S(U+K) \to S(U)$. In this note we provide a sufficient condition for the surjectivity. Moreover, the problem on solvability of a linear partial differential equation with constant coefficients in $S(U)$ is also of interest.

1.2. Notation and definitions.

For

$$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n, \qquad u = (u_1, \dots, u_n) \in \mathbb{R}^n (\mathbb{C}^n)$$

we let

$$\|u\| = \sqrt{|u_1|^2 + \dots + |u_n|^2}, \qquad |\alpha| = \alpha_1 + \ldots + \alpha_n,$$

$$\alpha! = \alpha_1! \cdots \alpha_n!, \qquad u^\alpha = u_1^{\alpha_1} \cdots u_n^{\alpha_n},$$

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial u_1^{\alpha_1} \cdots \partial u_n^{\alpha_n}}.$$

For $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n) \in \mathbb{R}^n(\mathbb{C}^n)$ we let

$$\langle u, v \rangle := u_1 v_1 + \dots + u_n v_n.$$

$H(T_C)$ is the space of holomorphic functions in the tubular domain $T_C = \mathbb{R}^n + iC$.

$\Delta_C(y)$ is the distance from a point $y \in C$ to the boundary of the cone C.

For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $r > 0$ we let

$$\Delta(z, r) = \{w = (w_1, \dots, w_n) \in \mathbb{C}^n : |w_j - z_j| \leqslant r, \quad j = 1, \dots, n\}.$$

$[x, \xi]$ is the segment, which connects the points $x, \xi \in \mathbb{R}^n$.

For an arbitrary locally convex space X by X' we denote the space of linear continuous functionals on X, and X^* stands for the strongly dual space.

2. On dual space for $S(U)$

It is known [1], [11] that for each $z \in \mathbb{R}^n + iC$ the function $f_z(\xi) = e^{i\langle \xi, z \rangle}$ belongs to $S(U)$. This is why for an arbitrary functional $\Phi \in S'(U)$ the function

$$\hat{\Phi}(z) = (\Phi, e^{i\langle \xi, z \rangle})$$

is well–defined in $T_C = \mathbb{R}^n + iC$. It is holomorphic in T_C [1], [11]. The function $\hat{\Phi}$ is called the Fourier — Laplace transform of the functional Φ.

We define the space $V_b(T_C)$ as follows. For each $m \in \mathbb{N}$ we define the normed spaces

$$V_{b,m}(T_C) = \left\{ f \in H(T_C) : N_m(f) = \sup_{z \in T_C} \frac{|f(z)|}{(1 + \|z\|)^m (1 + \frac{1}{\Delta_C(y)})^m e^{b(y)}} < \infty \right\},$$

where $z = x + iy$, $x \in \mathbb{R}^n$, $y \in C$. Let

$$V_b(T_C) = \bigcup_{m=0}^{\infty} V_{b,m}(T_C).$$

The set $V_b(T_C)$ equipped with the usual summation and multiplication by complex number is a linear space. We also equip $V_b(T_C)$ with the topology of inductive limit of the spaces $V_{b,m}(T_C)$. $V_b(T_C)$ is the space (LN^*) [6] or, following a more modern terminology, the space DFS [3].

The following Paley — Wiener — Schwartz theorem holds for the space $S(U)$.

Theorem 2.1. *The Fourier — Laplace transform $\mathcal{F} : S^*(U) \to V_b(T_C)$ introduced by the rule $\mathcal{F}(T) = \hat{T}$ is an isomorphism.*

Theorem 2.1 was established in work [4] following the lines from [1]. For $b(y) = a\|y\|$ $(a \geqslant 0)$ Theorem 2.1 was established by Vladimirov [1, Ch. 2, Sect. 12].

Remark 2.1. *For each $\varepsilon > 0$ we define a function b_ε on $\overline{C}$ by the rule*

$$b_\varepsilon(y) = b(y) + \varepsilon\|y\|$$

and the space $V_{b_\varepsilon}(T_C)$ as the inductive limit of the normed spaces

$$V_{b_\varepsilon,k}(T_C) = \left\{ f \in H(T_C) : N_{k,\varepsilon}(f) = \sup_{z \in T_C} \frac{|f(z)|e^{-b_\varepsilon(y)}}{(1+\|z\|)^k(1+\frac{1}{\Delta_C(y)})^k} < \infty \right\},$$

where $k \in \mathbb{Z}_+$, $z = x + iy$, $x \in \mathbb{R}^n$, $y \in C$. Let $H_b(T_C)$ be the projective limit of the spaces $V_{b_\varepsilon}(T_C)$. It is known [10, Thm. D] that the spaces $V_b(T_C)$ and $H_b(T_C)$ coincide.

3. Auxiliary statements

Lemma 3.1. *Let $x \in U$. Then the linear operator $T_x : f \in S(G) \to f_x$ acts from $S(G)$ into $C^\infty(K)$ and is continuous.*

Proof. Let $f \in S(U + K)$. Then for each $m \in \mathbb{Z}_+$

$$|(D^\alpha f)(\xi)| \leqslant \frac{\|f\|_{m,G}}{(1+\|\xi\|)^m}, \quad |\alpha| \leqslant m, \quad \xi \in G. \tag{3.1}$$

Therefore,

$$\|T_x(f)\|_m = \max_{t\in K, |\alpha|\leqslant m} |(D^\alpha f)(x+t)| \leqslant \frac{\|f\|_{m,G}}{(1+\|t+x\|)^m} \leqslant \|f\|_{m,G}.$$

Thus, the linear operator T_x is continuous. The proof is complete. □

Lemma 3.2. *Let $f \in S(U + K)$. Then $\mu * f \in C^\infty(U)$.*

Proof. First we are going to show that the function $\mu * f$ is continuous on V. Let x_0 be an arbitrary point V. Then

$$(\mu * f)(x) - (\mu * f)(x_0) = \langle \mu_t, f(x+t) - f(x_0+t)\rangle, \quad x \in V.$$

Since for some $m \in \mathbb{N}$ and $c_\mu > 0$

$$|\langle \mu, g\rangle| \leqslant c_\mu \|g\|_m, \quad g \in C^\infty(K), \tag{3.2}$$

we have

$$\begin{aligned} |(\mu * f)(x) - (\mu * f)(x_0)| &= |\langle \mu, f_x - f_{x_0}\rangle| \leqslant c_\mu \|f_x - f_{x_0}\|_m \\ &= c_\mu \max_{\substack{t\in K,\\ |\alpha|\leqslant m}} |(D^\alpha(f_x(t) - f_{x_0}(t))| \\ &\leqslant 2c_\mu \max_{\substack{t\in K,\\ |\alpha|\leqslant m}} \max_{\substack{\xi\in[t+x_0,t+x],\\ \beta\in\mathbb{Z}_+^n:|\beta|=1}} |(D^{\alpha+\beta} f)(\xi)|\|x - x_0\|. \end{aligned}$$

Using the inequality (3.2), we get

$$|(\mu * f)(x) - (\mu * f)(x_0)| \leqslant 2c_\mu \|x - x_0\|\|f\|_{m+1,G}.$$

Thus, $\mu * f$ is continuous at the point $x_0 \in V$. And since x_0 is an arbitrary point in V, we conclude that $\mu * f$ is continuous in V.

Let us show that $\mu * f \in C^\infty(V)$ and that for each $\alpha \in \mathbb{Z}_+^n$

$$(D^\alpha(\mu * f))(x) = (\mu * D^\alpha f)(x), \quad x \in V.$$

We take an arbitrary point x_0 in V. Let $\alpha \in \mathbb{Z}_+^n$. For each $h \in \mathbb{R}^n$ such that $x_0 + h \in V$ we have

$$\begin{aligned}(D^\alpha(\mu * f))(x_0+h) - (D^\alpha(\mu * f))(x_0) - \sum_{\beta \in \mathbb{Z}_+^n : |\beta|=1} \langle \mu, D^{\alpha+\beta} f_{x_0} \rangle h^\beta \\ =& \langle \mu, D^\alpha f_{x_0+h} \rangle - \langle \mu, D^\alpha f_{x_0} \rangle - \sum_{\beta \in \mathbb{Z}_+^n : |\beta|=1} \langle \mu, D^{\alpha+\beta} f_{x_0} \rangle h^\beta \\ =& \langle \mu, D^\alpha f_{x_0+h} - D^\alpha f_{x_0} - \sum_{\beta \in \mathbb{Z}_+^n : |\beta|=1} D^{\alpha+\beta} f_{x_0} h^\beta \rangle.\end{aligned}$$

We note that for each $t \in K$ and each $\gamma \in \mathbb{Z}_+^n$ with $|\gamma| \leqslant m$

$$\begin{aligned}(D^{\alpha+\gamma} f_{x_0+h})(t) - (D^{\alpha+\gamma} f_{x_0})(t) - \sum_{\beta \in \mathbb{Z}_+^n : |\beta|=1} (D^{\alpha+\gamma+\beta} f_{x_0})(t) h^\beta \\ =& (D^{\alpha+\gamma} f)(t+x_0+h) - (D^{\alpha+\gamma} f)(t+x_0) - \sum_{\beta \in \mathbb{Z}_+^n : |\beta|=1} (D^{\alpha+\gamma+\beta} f)(t+x_0) h^\beta.\end{aligned}$$

This implies

$$\begin{aligned}|(D^{\alpha+\gamma} f)(t+x_0+h) - (D^{\alpha+\gamma} f)(t+x_0) - \sum_{\beta \in \mathbb{Z}_+^n : |\beta|=1} (D^{\alpha+\gamma+\beta} f)(t+x_0) h^\beta| \\ \leqslant& 2 \sum_{\beta \in \mathbb{Z}_+^n : |\beta|=2} \frac{(\max_{\xi \in [t+x_0, t+x_0+h]} |(D^{\alpha+\gamma+\beta} f)(\xi)|)|h^\beta|}{\beta!} \\ \leqslant& 2(\max_{\substack{\xi \in [t+x_0, t+x_0+h], \\ \beta \in \mathbb{Z}_+^n : |\beta|=2}} |(D^{\alpha+\gamma+\beta} f)(\xi)|) \sum_{\beta \in \mathbb{Z}_+^n : |\beta|=2} \frac{|h^\beta|}{\beta!} \\ \leqslant& \sum_{\beta \in \mathbb{Z}_+^n : |\beta|=2} \frac{2}{\beta!} \max_{\substack{\xi \in [x+y, x+y+h], \\ \beta \in \mathbb{Z}_+^n : |\beta|=2}} (|(D^{\alpha+\gamma+\beta} f)(\xi)|) \|h\|^2.\end{aligned}$$

Hence, taking into consideration the inequality (3.2), we obtain

$$\begin{aligned}|(D^{\alpha+\gamma} f)(t+x_0+h) - (D^{\alpha+\gamma} f)(t+x_0) - \sum_{\beta \in \mathbb{Z}_+^n : |\beta|=1} (D^{\alpha+\gamma+\beta} f)(t+x_0) h^\beta| \\ \leqslant& \sum_{\beta \in \mathbb{Z}_+^n : |\beta|=2} \frac{2\|f\|_{m+|\alpha|+2,G}}{\beta!} \|h\|^2.\end{aligned}$$

Now by means of the inequality (4.1) we get

$$|\langle \mu, D^\alpha f_{x_0+h} - D^\alpha f_{x_0} - \sum_{\beta \in \mathbb{Z}_+^n : |\beta|=1} D^{\alpha+\beta} f_{x_0} h^\beta \rangle| \leqslant c_\mu \sum_{\beta \in \mathbb{Z}_+^n : |\beta|=2} \frac{2\|f\|_{m+|\alpha|+2,G}}{\beta!} \|h\|^2.$$

This means that the function $D^\alpha(\mu * f)$ is differentiable at the point x_0 in V. A since x_0 is an arbitrary point in V, we conclude that $\mu * f$ is differentiable in V. We have also shown that for each $\beta \in \mathbb{Z}_+^n$ with $|\beta| = 1$

$$D^\beta(D^\alpha \mu * f)(x) = \langle \mu, (D^{\alpha+\beta} f)(x+t) \rangle, \quad x \in V.$$

Let us show that for each multi–index $\alpha \in \mathbb{Z}_+^n$ the partial derivative $D^\alpha(\mu * f)$ is continued to a continuous in U function. For all

$$x = (x_1, \ldots, x_n), \quad \xi = (\xi_1, \ldots, \xi_n) \in V, \quad t \in K$$

and each $\gamma \in \mathbb{Z}_+^n$ with $|\gamma| \leqslant m$ we have

$$\begin{aligned} |(D^{\alpha+\gamma} f)(t+x) - (D^{\alpha+\gamma} f)(t+\xi)| &\leqslant \sum_{\beta \in \mathbb{Z}_+^n : |\beta|=1} \max_{\eta \in [t+x, t+\xi]} |(D^{\alpha+\gamma+\beta} f)(\eta)|)|(x-\xi)^\beta| \\ &\leqslant n\|f\|_{m+|\alpha|+1,G}\|x-\xi\|. \end{aligned}$$

Then

$$(D^\alpha(\mu * f))(x) - (D^\alpha(\mu * f))(\xi) = \langle \mu, (D^\alpha f)(x+t) - (D^\alpha f)(\xi+t)\rangle.$$

Using the inequality (4.1) and the above inequality, we find

$$\begin{aligned} |(D^\alpha(\mu * f))(x) - (D^\alpha(\mu * f))(\xi)| &\leqslant c_\mu \max_{t \in K} |(D^{\alpha+\gamma} f)(t+x) - (D^{\alpha+\gamma} f)(t+\xi)| \\ &\leqslant c_\mu n\|f\|_{m+|\alpha|+1,G}\|x-\xi\|. \end{aligned}$$

By this estimate and the Cauchy criterion we conclude that for each point ζ on the boundary ∂V there exists a finite limit

$$\lim_{\xi \to \zeta, \xi \in V} (D^\alpha(\mu * f))(\xi).$$

We define the function F_α on U as follows

$$F_\alpha(x) = (D^\alpha(\mu * f))(x), \qquad x \in V; \qquad F_\alpha(x) = \lim_{\xi \to x, \xi \in V} (D^\alpha f)(\xi), \qquad x \in \partial V.$$

By the latter estimate we easily conclude that the function F_α is continuous on U. Moreover, it is uniformly continuous on U. Thus, for each $\alpha \in \mathbb{Z}_+^n$ the partial derivative $D^\alpha(\mu * f)$ admits a unique continuation from V to a continuous on U function. For the function F_α we keep the notation $D^\alpha(\mu * f)$. The proof is complete. □

4. Main results

Theorem 4.1. *Let μ be a linear continuous functional on the space $C^\infty(K)$. Assume that its Fourier — Laplace transform, the function*

$$\hat{\mu}(z) = (\mu, e^{i\langle \xi, z\rangle}), \quad z \in \mathbb{C}^n,$$

is such that there exist positive numbers $A, L, N > 1$ such that for each point $z \in \mathbb{C}^n$ and each $\varepsilon \in (0,1)$ there exists a number $a_\varepsilon \geqslant 1$ not exceeding $\frac{L}{\varepsilon^N}$ such that

$$e^{H_K(Im\, z) - A\ln(1+\|z\|)} \leqslant a_\varepsilon \max_{w \in \Delta(z,\varepsilon)} |\hat{\mu}(w)|.$$

Then the operator $T_\mu : S(U+K) \to S(U)$ is surjective.

Before proving Theorem 4.1, we establish the following two statements.

Lemma 4.1. *Let Γ be an open convex cone in $\mathbb{R}^n$ with the vertex at the origin. Let h be a convex continuous positively homogeneous of degree 1 function on $\overline{\Gamma}$. Then for each $\varepsilon > 0$ there exists a constant $A_\varepsilon > 0$ such that for all $y_1, y_2 \in \Gamma$ obeying $\|y_2 - y_1\| \leqslant 1$ we have*

$$|h(y_2) - h(y_1)| \leqslant \varepsilon\|y_1\| + \varepsilon\|y_2\| + A_\varepsilon.$$

Lemma 3.1 was proved in [10, Lm. 9].

Lemma 4.2. *Let Φ_1, Φ_2, $\Phi = \frac{\Phi_1}{\Phi_2}$ be holomorphic functions in the ball $B(0,R)$. Let*

$$|\Phi_1(w)| \leqslant B, \qquad |\Phi_2(w)| \leqslant A \quad \textit{for} \quad w \in B(0,R).$$

Then

$$|\Phi(w)| \leqslant BA^{\frac{2\|w\|}{R-\|w\|}} |\Phi_2(0)|^{-\frac{R+\|w\|}{R-\|w\|}}, \quad w \in B(0,R).$$

Lemma 3.2 was established in [9].

Proof of Theorem 4.1. Let

$$N_{\hat\mu} = \{z \in T_C : \hat\mu(z) = 0\}.$$

For $z \in \mathbb{C}^n$ by f_z we denote the function

$$f_z(\xi) = e^{i\langle \xi, z\rangle}, \quad x \in U.$$

For $z \in T_C \setminus N_{\hat\mu}$ we consider the equation $Lf = f_z$. It has the solution $\frac{f_z}{\hat\mu(z)}$. By the completeness of the system $\{f_z\}_{z\in T_C\setminus N_{\hat\mu}}$ this implies that the range of the operator L is dense in $S(U)$. Let us show that the range of the operator L is closed in $S(U)$.

By the Dieudonné — Schwartz theorem [2] the closedness of the range of operator L is equivalent to the closedness in $S^*(G)$ of the range of adjoint operator L^*. We define the operator $\hat L^*$ on $V_b(T_C)$ by the rule

$$\hat L^*(F) = \mathcal{F}(L^*(\mathcal{F}^{-1}(F))), \quad F \in V_b(T_C).$$

By Theorem A, the linear operator $\hat L^*$ acts from $V_b(T_C)$ into $V_{H_G}(T_C)$; the space $V_{H_G}(T_C)$ is defined by the same rule as the space; and the action of this operator is continuous. For an arbitrary function $F \in V_b(T_C)$ and each $z \in T_C$ we have

$$\begin{aligned}\hat L^*(F)(z) = (L^*(\mathcal{F}^{-1}(F)), f_z) = (\mathcal{F}^{-1}(F)), L(f_z)) = (\mathcal{F}^{-1}(F)), \hat\mu(z) f_z)\\ = \hat\mu(z)(\mathcal{F}^{-1}(F)), f_z) = \hat\mu(z)F(z).\end{aligned}$$

It follows from the results of [6] that the range $\operatorname{im}\hat L^*$ of the operator $\hat L^*$ is closed in $V_{H_G}(T_C)$ if and only if the set $\operatorname{im}\hat L^* \cap V_{H_G,m}(T_C)$ is closed in $V_{H_G,m}(T_C)$ for each $m \in \mathbb{Z}_+$. So, let $m \in \mathbb{Z}_+$ and the function F belong to the closure of the set $\operatorname{im}\hat L^* \cap V_{H_G,m}(T_C)$ in $V_{H_G,m}(T_C)$. Then there exists a sequence $(F_k)_{k=1}^\infty$ of the functions $F_k \in \operatorname{im}\hat L^* \cap V_{H_G,m}(T_C)$, which converges to F in $V_{H_G,m}(T_C)$. In particular, the sequence $(F_k)_{k=1}^\infty$ of the functions F_k converges to F uniformly on compact sets in T_C. Hence, since the functions F_k read

$$F_k(z) = \hat\mu(z)\psi_k(z), \quad \text{where} \quad \psi_k \in V_b(T_C), \quad z \in T_C,$$

the function

$$\psi(z) = \frac{F(z)}{\hat\mu(z)}, \quad z \in T_C,$$

is holomorphic in T_C. Let us estimate the growth of function ψ. Let

$$z = x + iy \in T_C, \quad \text{where} \quad x \in \mathbb{R}^n, \quad y = \operatorname{Im} z.$$

Let $\delta \in (0,1)$ be arbitrary. We define

$$\varepsilon = \min\Big(\frac14, \frac14 \Delta_C(y)\Big).$$

By the condition for $\hat\mu$ there exists a point

$$z' \in T_C : \|z - z'\| \leqslant \varepsilon$$

such that

$$e^{h_K(\operatorname{Im} z) - A\ln(1+\|z\|)} \leqslant \tilde a_\varepsilon |\hat\mu(z')|,$$

where $\tilde{a}_\varepsilon = a_{\frac{\varepsilon}{\sqrt{n}}}$. Since $F \in V_{H_G,m}(T_C)$, for some $A_F > 0$

$$\ln |F(w)| \leqslant A_F + m\ln(1+\|w\|) + m\ln\left(1+\frac{1}{\Delta_C(\operatorname{Im} w)}\right) + H_G(\operatorname{Im} w), \quad w \in T_C.$$

Since in view of Lemma 4.1) for each $\delta > 0$ we can find a constant $C_\delta > 0$ such that

$$\sup_{z''\in B(z',2\varepsilon)} H_G(\operatorname{Im} z'') \leqslant H_G(\operatorname{Im} z) + \delta|\operatorname{Im} z| + C_\delta,$$

we obtain

$$\sup_{z''\in B(z',2\varepsilon)} \ln|F(z'')| \leqslant A_F + m\ln(2+\|z\|) + m\ln\left(1+\frac{4}{\Delta_C(y)}\right) + H_G(y) + \delta|y| + C_\delta. \quad (4.1)$$

Since for some $c_\mu > 0$ and $p \in \mathbb{Z}_+$

$$|\hat{\mu}(z)| \leqslant c_\mu (1+\|z\|)^p e^{H_K(\operatorname{Im} z)},$$

for some $b_{\mu,K} > 0$ independent of ε we have

$$\sup_{z''\in B(z',2\varepsilon)} |\hat{\mu}(z'')| \leqslant b_{\mu,K} + p\ln(2+\|z\|) + H_K(\operatorname{Im} z). \quad (4.2)$$

We let

$$\Phi_1(w) = F(z'+w), \qquad \Phi_2(w) = \hat{\mu}(z'+w), \quad \text{where} \quad \|w\| < 2\varepsilon.$$

We use Lemma 4.2 with $R = 2\varepsilon$ and $w = z - z'$. We get

$$\begin{aligned}
\ln\left|\frac{F(z)}{\hat{\mu}(z)}\right| \leqslant & A_F + m\ln\Big(2+\|z\|\Big) + m\ln\left(1+\frac{4}{\Delta_C(y)}\right) + H_G(y) \\
& + \delta|y| + C_\delta + \frac{2\|z-z'\|}{R-\|z-z'\|}(b_{\mu,K} + p\ln(2+\|z\|) + H_K(y)) \\
& - \frac{R+\|z-z'\|}{R-\|z-z'\|}(H_K(y) - A\ln(1+\|z\|) - \ln\tilde{a}_\varepsilon) \\
\leqslant & A_F + m\ln(2+\|z\|) + m\ln(1+\frac{4}{\Delta_C(y)}) + b(y) + H_K(y) + \delta\|y\| + C_\delta \\
& + \frac{2\|z-z'\|}{R-\|z-z'\|}(b_{\mu,K} + p\ln(2+\|z\|)) + \frac{\|z-z'\|}{R-\|z-z'\|}H_K(y) \\
& - \frac{R}{R-\|z-z'\|}H_K(y) + \frac{R+\|z-z'\|}{R-\|z-z'\|}(A\ln(1+\|z\|) + \ln\tilde{a}_\varepsilon) \\
\leqslant & A_F + m\ln(2+\|z\|) + m\ln\left(1+\frac{4}{\Delta_C(y)}\right) + b(y) + \delta|y| + C_\delta \\
& + 2(b_{\mu,K} + p\ln(2+\|z\|)) + 3(A\ln(1+\|z\|) + \ln\tilde{a}_\varepsilon).
\end{aligned}$$

Thus,

$$\left|\frac{F(z)}{\hat{\mu}(z)}\right| \leqslant L^3 e^{A_F+2b_{\mu,K}} 4^{3N+m} e^{C_\delta} (2+\|z\|)^{m+2p+3A}\left(1+\frac{1}{\Delta_C(y)}\right)^{3N+m} e^{b(y)+\delta|y|}.$$

Therefore, for each $\delta > 0$,

$$\left|\frac{F(z)}{\hat{\mu}(z)}\right| \leqslant Ce^{C_\delta}(1+\|z\|)^{m+2p+3A}\left(1+\frac{1}{\Delta_C(y)}\right)^{3N+m} e^{b(y)+\delta|y|},$$

where $C = 2^{6N+3m+2p+3A}L^3 e^{A_F+2b_{\mu,K}}$. In view of Remark 2.1 we have $\psi \in V_b(T_C)$. Since

$$F(z) = \psi(z)g(z), \quad z \in T_C, \quad \text{then} \quad F \in \operatorname{im}\hat{L}^*.$$

Thus, the set $\operatorname{im}\hat{L}^* \cap V_{H_G,m}(T_C)$ is closed in $V_{H_G,m}(T_C)$ for each $m \in \mathbb{Z}_+$. Therefore, the range $\operatorname{im}\hat{L}^*$ of the operator $\hat{L}^*$ is closed in $V_{H_G}(T_C)$. But then the range of the operator L is closed in $S(U)$. And since the range of operator L is dense in $S(U)$, we find $\operatorname{im} L = S(U)$. This completes the proof. □

As an application of Theorem 4.1 we consider the existence of solutions to a differential–difference equation in $S(U)$; we define this equation as follows. We fix $m \in \mathbb{N}$. For

$$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n \quad \text{with} \quad |\alpha| = \alpha_1 + \dots + \alpha_n \leqslant m \quad \text{we let} \quad a_\alpha \in \mathbb{C}^n, \quad h_\alpha \in \mathbb{R}^n.$$

Let K be a convex hull of the points h_α, and the interior of K is non–empty. We observe that

$$H_K(y) = \max_{|\alpha| \leqslant m} (-\langle y, h_\alpha \rangle) \quad \forall y \in \mathbb{R}^n.$$

For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $r > 0$ we let

$$T(z, r) = \{w = (w_1, \dots, w_n) \in \mathbb{C}^n : |w_j - z_j| = r,\ j = 1, \dots, n\},$$

For a function g holomorphic $\Delta(z, r)$ we define

$$[g(z)]_r = \frac{1}{(2\pi r)^n} \int\limits_{z \in T(z,r)} |g(z)||d\,z|.$$

The following result is known [7, Prop. 3].

Theorem 4.2. *Let*

$$P(z) = \sum_{k=1}^{m} P_k(z) e^{\langle \alpha_k, z \rangle},$$

where P_k is an analytic polynomial in $\mathbb{C}^n$ for each k and $\alpha_k \in \mathbb{C}$. Let

$$h_P(z) = \max_k \operatorname{Re}\langle \alpha_k, z \rangle, \quad z \in \mathbb{C}^n.$$

Then for each $\varepsilon > 0$ there exists a constant $C(\varepsilon, P) > 0$ such that for an arbitrary function f analytic in the polydisk $\Delta(z, \varepsilon)$ we have

$$|f(z)| e^{h_P(z)} \leqslant C(\varepsilon, P)[f(z)P(z)]_\varepsilon. \tag{4.3}$$

Remark 4.1. *The analysis of the proof of Theorem 4.2 shows that for some $L > 0$ and $N \in \mathbb{N}$ independent of ε and z the inequality holds*

$$C(\varepsilon, P) \leqslant \frac{L}{\varepsilon^N}, \quad \varepsilon \in (0, 1).$$

Remark 4.2. *Under the assumptions of Theorem thB it follows from (4.2) that for each analytic in the polydisk $\Delta(z, \varepsilon)$ function f we have*

$$|f(z)| e^{h_P(z)} \leqslant C(\varepsilon, P) \max_{w \in \Delta(z,\varepsilon)} |f(w)P(w)|. \tag{4.4}$$

We note that the linear continuous functional F on the space $C^\infty(K)$ introduced by the rule

$$F(f) = \sum_{\alpha \in \mathbb{Z}_+^n : |\alpha| \leqslant m} a_\alpha (D^\alpha f)(h_\alpha),$$

defines a differential–difference operator L on $S(G)$:

$$(Lf)(x) = \sum_{\alpha \in \mathbb{Z}_+^n : |\alpha| \leqslant m} a_\alpha (D^\alpha f)(x + h_\alpha), \quad x \in U.$$

It is obvious that $Lf \in S(U)$ for each $f \in S(G)$ and the operator L is linear and continuous. Moreover,

$$\hat{F} = P, \quad h_P(z) = H_K(\operatorname{Im} z) \quad \text{for} \quad z \in \mathbb{C}^n.$$

By (4.3), F satisfies the assumptions of Theorem 4.1. Thus, by Theorem 4.1 we have the following corollary.

Corollary 4.1. *The operator L acts surjectively from $S(G)$ into $S(U)$.*

Theorem 4.3. *Let*

$$P(z) = \sum_{|\alpha|\leqslant N} a_\alpha z^\alpha$$

be a polynomial of degree N,

$$P(D) = \sum_{|\alpha|\leqslant N} a_\alpha D^\alpha$$

be the differential operator of finite order associated with the polynomial P. Then the operator $P(D)$ is surjective in the space $S(U)$.

Theorem 4.3 can be proved by the same scheme as Theorem 4.1. Here the following lemma known as the Malgrange — Ehrenpreis lemma plays an important role, see, for instance, [8].

Lemma 4.3. *Let p be a polynomial of degree m. Then there exists a number $c > 0$ such that for all $r > 0$, $z \in \mathbb{C}^n$ and each function $f \in H(B(z, r))$ such that*

$$\frac{f(\cdot)}{p(\cdot)} \in H(B(z, r))$$

the inequality holds

$$\left|\frac{f(z)}{p(z)}\right| \leqslant cr^{-m} \sup_{\zeta\in B(z,r)} |f(\zeta)|.$$

BIBLIOGRAPHY

1. V.S. Vladimirov. *Generalized Functions in Mathematical Physics.* Nauka, Moscow (1979); *English translation:* Mir Publishers, Moscow (1979).
2. J. Dieudonné, L. Schwartz. *La dualité dans les espaces* ($\mathcal{F}$) *et* ($\mathcal{LF}$) // Ann. Inst. Fourier **74**:2, 61–101 (1949). https://doi.org/10.21136/CPMF.1949.123048
3. V.V. Zharinov. *Compact families of locally convex topological vector spaces, Frechet — Schwartz and dual Frechet — Schwartz spaces* // Russ. Math. Surv. **34**:4, 105–143 (1979). https://doi.org/10.1070/RM1979v034n04ABEH002963
4. I.Kh. Musin, P.V. Fedotova. *A theorem of Paley — Wiener type for ultradistributions* // Math. Notes **85**:6, 848–867 (2009). https://doi.org/10.1134/S0001434609050265
5. R.T. Rockafellar. *Convex Analysis.* Princeton University Press. Princeton, N. J. (1970). https://doi.org/10.1515/9781400873173
6. L. Sebastião e Silva. *Su certe classi di spazi localmente convessi importanti per le applicazioni* // Rend. Mat. Appl. **14**, 388–410 (1955).
7. C.A. Berenstein, M.A. Dostal. *Some remarks on convolution equations* // Ann. Inst. Fourier **23**:1, 55–74 (1973). https://doi.org/10.5802/aif.444
8. S. Hansen. *On the «Fundamental Prinsiple» of L. Ehrenpries* // Banach Center Publications **10**:1, 185–201 (1983). https://eudml.org/doc/208496
9. L. Hörmander. *On the range of convolution operators* // Ann. Math. (2) **76**, 148–170 (1962). https://doi.org/10.2307/1970269
10. I.Kh. Musin, P.V. Yakovleva. *On a space of smooth functions on a convex unbounded set in $\mathbb{R}^n$ admitting holomorphic extension in $\mathbb{C}^n$* // Cent. Eur. J. Math. **10**:2, 665–692 (2012). https://doi.org/10.2478/s11533-011-0142-8
11. J.W. de Roever. *Analytic representations and Fourier transforms of analytic functionals in Z' carried by the real space* // SIAM J. Math. Anal. **9**:6, 996–1019 (1978). https://doi.org/10.1137/0509081

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