ISSN 2304-0122 Ufa Mathematical Journal. Vol. 17. No 4 (2025). P. 52-59.

doi:10.13108/2025-17-4-52

# GENERALIZATIONS OF LINDELÖF CONDITIONS FOR DISTRIBUTION OF ZEROS OF ENTIRE FUNCTIONS

**E.G. KUDASHEVA, E.B. MENSHIKOVA, B.N. KHABIBULLIN**

**Abstract.** The Lindelöf condition is the first example of a nonradial condition for the distribution of zeros of entire functions of finite integer order. Its further development is used in the classical Rubel — Taylor theorem. It also involves negative integer powers of the complex variable. We generalize the Lindelöf condition by replacing the power test functions by arbitrary harmonic functions on concentric annuli. In particular, from this generalization, we easily deduce the necessity of the Lindelöf conditions in Rubel — Taylor theorem.

**Keywords:** holomorphic function, entire function, distribution of zeroes, subharmonich function, distribution of masses, Lindelöf condition.

**Mathematics Subject Classification:** 30D20, 31A05, 30D35, 30C15

## 1. Classical Lindelöf condition for distribution of zeros

In what follows the symbols $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{R}$ and $\mathbb{C}$ stand for the sets of respectively natural, real and complex numbers in all their algebraic, geometric and topological versions with the extensions

$$\mathbb{N}_0 := \{0\} \bigcup \mathbb{N}, \quad \overline{\mathbb{N}}_0 := \mathbb{N}_0 \bigcup \{+\infty\}, \quad \overline{\mathbb{R}} := \mathbb{R} \bigcup \{\pm\infty\},$$

the positive semi–axis $\mathbb{R}^+ := \{x \in \mathbb{R} \mid x \geqslant 0\}$ and its extension $\overline{\mathbb{R}}^+ := \mathbb{R}^+ \bigcup \{+\infty\}$.

Let $D$ be a domain, that is, an open connected domain in the complex plane $\mathbb{C}$. A function $Z$ on $D$ with values $\overline{\mathbb{N}}_0 := \{0, 1, 2, \dots, +\infty\}$ is called the *distribution of points on $D$ with the multiplicities $Z(z) \in \overline{\mathbb{N}}_0$ of points $z \in D$ in $Z$* [6, Sects. 0.1.2–0.1.3], [7, Sect. 1.2.3]. If $f$ is a holomorphic function on the domain $D$, then the distribution of points with the multiplicities at each point, which is equal to the order of zero of the function $f$ at this point is denoted by

$$\mathsf{Zero}_f \colon z \underset{z \in D}{\longmapsto} \sup\left\{ p \in \overline{\mathbb{N}}_0 \;\middle|\; \limsup_{z \neq w \to z} \frac{|f(w)|}{|w - z|^p} < +\infty \right\} \in \overline{\mathbb{N}}_0, \tag{1.1}$$

and is called the *distribution of zeros* of holomorphic function $f$ on $D$.

The following classical result was established in the beginning on XX century [11], [3, Ch. I, Sect. 11, Thm. 15], [9, Ch. 2, Sect. 2.10, Lindelöf theorem], [10, Lect. 5, Sect. 5.2, Thm. 3, 4].

---

E.G. Kudasheva, E.B. Menshikova, B.N. Khabibullin, Generalizations of Lindelöf conditions for distribution of zeros of entire functions.



The work is made in the framework of State Task of Ministry of Education and Science of Russian Federation (code of scientific theme FMRS-2025-0010).

*Submitted July 10, 2025.*

**Theorem 1.1** (Lindelöf [11]). *Let $0<\rho\in\mathbb{R}$, $Z$ be the distribution of points on $\mathbb{C}$, the radial counting function of which satisfies*

$$Z^{\mathfrak{r}}(t):=\sum_{|z|\leqslant t} Z(z)<+\infty \quad \textit{for all } t\in\mathbb{R}^+. \tag{1.2}$$

*The existence of an entire function $f$ with the distribution of zeros $\mathsf{Zero}_f=Z$ and the condition*

$$\limsup_{\mathbb{C}\ni z\to\infty}\frac{\ln|f(z)|}{|z|^{\rho}}<+\infty$$

*is equivalent to the finiteness of upper density at the order $\rho$*

$$\limsup_{0<r\to+\infty}\frac{Z^{\mathfrak{r}}(r)}{r^{\rho}}<+\infty \tag{1.3}$$

*of distribution of points $Z$ completed by the Lindelöf condition*

$$\sup_{1<R\in\mathbb{R}^+}\left|\sum_{1<|z|\leqslant R}\frac{Z(z)}{z^{\rho}}\right|<+\infty \quad \textit{for} \quad \rho\in\mathbb{N}. \tag{1.4}$$

The sufficient part of the Lindelöf theorem is easily deduced from estimates for the Weierstrass — Hadamard representation for entire functions with the distribution of zeros with a finite upper density (1.3), with the proof of necessity of Lindelöf condition (1.4) for natural $\rho\in\mathbb{N}$ required nontrivial lower bounds for polynomials and entire functions in the original approach.

For a radial function $M(z)\underset{z\in\mathbb{C}}{=}M(|z|)$ on $\mathbb{C}$, which increases on radial rays, and a class of entire functions $f$ with restrictions of form

$$\ln|f(z)|\leqslant C_f M\bigl(c_f|z|\bigr) \quad \text{for all} \quad z\in\mathbb{C}, \tag{1.5}$$

where $c_f\geqslant 0$ and $C_f\geqslant 0$ are some numbers, depending on the function $f$, the final results on description of distributions of zeros for such classes of entire functions were obtained jointly by Rubel and Taylor by the method of Fourier series [13], [14], which comes back to the studies by Akhiezer [1, Sect. 7] and was developed in works by Kondratyuk [2], Malyutin [4] and many others.

**Theorem 1.2** (Rubel — Taylor [13], [14, Ch. 14, Lm. ], [1, Sect. 7, Thm. 2]). *Let $Z$ be a distribution of points on $\mathbb{C}$ with $Z(0)=0$ and (1.2), and $M\geqslant 0$ be an increasing continuous function on $\mathbb{R}^+$. An entire function $f$ with the distribution of zeros $\mathsf{Zero}_f=Z$ and the condition (1.5) exists if and only if there exists a number $C\in\mathbb{R}^+$, for which*

$$\int_0^R\frac{Z^{\mathfrak{r}}(t)}{t}\,\mathrm{d}t\leqslant CM(CR) \quad \textit{for all} \quad R>0, \tag{1.6}$$

*and for each $k\in\mathbb{N}$ the general Lindelöf condition holds*

$$\left|\sum_{r<|z|\leqslant R}\frac{Z(z)}{kz^k}\right|\leqslant C\Bigl(\frac{M(Cr)}{r^k}+\frac{M(CR)}{R^k}\Bigr) \quad \textit{for all} \quad 0<r<R<+\infty. \tag{1.7}$$

In the particular case of the power function $M(r)\underset{r\in\mathbb{R}^+}{=}r^{\rho}$ the Rubel — Lindelöf theorem 1.2 immediately gives the classical Lindelöf theorem 1.1, in which the condition (1.7) for non–integer $\rho$ is reduced to the only first condition in (1.6), which the simplest implication of Poission — Jensen formula. For a natural $\rho$ we however need the second condition only for $k:=\rho$ [13], [14], [1, Sect. 7]. In the original proof of Rubel — Taylor theorem 1.2 by the methods of Fourier series the main difficulties were concentrated in the deducing the necessity of the generalization (1.7) of Lindelöf condition (1.4).

In the present paper, on the base of one of the integral formulas in the recent work [5] by the second coauthor, we consider a version of further developing and generalizations of analogues of Lindelöf conditions (1.4) and (1.7). Here the particular power function $\frac{1}{z^k}$ in the right hand sides of (1.4) and (1.7) is replaced by rather arbitrary harmonic functions in concentric annuli centered at zero. As the same time our generalizations in the case of power function allows us to obtain a simple proof of the necessity of Lindelöf condition (1.7) in Rubel — Taylor theorem 1.2. The proofs and original formulations of the main results are provided in the subharmonic version for the functions on a disk.

## 2. Main result for entire functions

By $\mathbb{D} := \bigl\{z \in \mathbb{C} \bigm| |z| < 1\bigr\}$ and $\overline{\mathbb{D}} := \bigl\{z \in \mathbb{C} \bigm| |z| \leqslant 1\bigr\}$, and by $\partial\mathbb{D} := \partial\overline{\mathbb{D}} := \overline{\mathbb{D}} \setminus \mathbb{D}$ we denote respectively the open and closed *unit disks*, as well as the *unit circle* centered at zero. Then $r\mathbb{D}$ and $r\overline{\mathbb{D}}$, as well as $r\partial\mathbb{D} = r\partial\overline{\mathbb{D}}$ for $r \in \mathbb{R}^+$ are respectively open and closed disks and the circle of radius $r$ centered at zero. For the supremum of a function $v$ on the circle $r\partial\mathbb{D}$ we employ the notation

$$v^{\vee r} := \sup_{|z|=r} v(z), \tag{2.1}$$

while the integral $v^{\circ r}$ of the function $v$ over the circle $r\partial\mathbb{D}$ is denoted by

$$v^{\circ r} := \frac{1}{2\pi}\int_0^{2\pi} v(re^{i\theta})\,\mathrm{d}\theta \overset{(2.1)}{\leqslant} v^{\vee r}, \tag{2.2}$$

where an appropriate integrability is assumed.

In the next section we provide the results only for entire functions in traditional notation of the Nevanlinna theory. For a holomorphic in a neighbourhood of the closed disk $r\overline{\mathbb{D}}$ function $f$ by

$$M(r,f) \overset{(2.1)}{:=} |f|^{\vee r} \overset{(2.1)}{=} \sup_{|z|=r}\bigl|f(z)\bigr| \tag{2.3}$$

we often denoted by maximum of modulus $|f|$ of the function $f$ on the circle $r\partial\mathbb{D}$,

$$T(r,f) = m(r,f) \overset{(2.2)}{:=} \bigl(\ln^+|f|\bigr)^{\circ r} \overset{(2.2)}{=} \frac{1}{2\pi}\int_0^{2\pi} \ln^+\bigl|f(re^{i\theta})\bigr|\,\mathrm{d}\theta \tag{2.4}$$

is the integral mean of the positive part $\ln^+|f| := \sup\bigl\{\ln|f|, 0\bigr\}$ of the modulus $|f|$ over the same circle, which defines the *Nevanlinna characteristics* $T(\cdot, f)$ of the function $f$.

For $k \in \mathbb{N}$ and $S \subseteq \mathbb{C}$ by $C^{(k)}(S)$ we denote the class of all functions with the values in $\mathbb{R}$ or in $\mathbb{C}$ with continuous partial derivatives of order $k$ in some open set $S$ depending on the choice of function.

Given a function $L$ on a neighbourhood of a point $re^{i\theta}$ of the circle $r\partial\mathbb{D}$ with the values in $\mathbb{C}$, its derivative in the radius, once it is well–defined, is denoted by

$$L'_{\mathfrak{r}}(re^{i\theta}) := \lim_{r \neq t \to r} \frac{L(te^{i\theta}) - L(re^{i\theta})}{t - r}. \tag{2.5}$$

**Theorem 2.1.** *Let $0 < r < R \in \mathbb{R}^+$ and a function $L \in C^{(1)}\bigl(\frac{R^2}{r}\overline{\mathbb{D}} \setminus r\mathbb{D}\bigr)$ with the values in $\mathbb{C}$ be harmonic in an open annulus $\frac{R^2}{r}\mathbb{D} \setminus r\overline{\mathbb{D}}$. Then for each entire function $f$ with the value*

$f(0)=1$ *the inequality holds*

$$\Biggl|\sum_{r<|z|\leqslant R<+\infty}\mathsf{Zero}_f(z)L(z)\Biggr|\leqslant 4\Bigl(|L|^{\vee r}+|L|^{\vee\frac{R^2}{r}}+r|L'_{\mathfrak{r}}|^{\vee r}+\frac{R^2}{r}|L'_{\mathfrak{r}}|^{\vee\frac{R^2}{r}}\Bigr)T(er,f) \\ +8\Bigl(|L|^{\vee\frac{R^2}{r}}+|L|^{\vee R}+R|L'_{\mathfrak{r}}|^{\vee R}\Bigr)T(eR,f). \tag{2.6}$$

Theorem 2.1 is immediately implied by Theorem 3.2 on subharmonic functions in the end of the paper. Let us demonstrate Theorem 2.1 by a very short comparison with [13], [14] proof of the necessity of Lindelöf condition (1.7) in Rubel — Taylor theorem 1.2.

*Proof.* We apply Theorem 2.1 to harmonic on $\mathbb{C}\setminus\{0\}$ functions

$$L\colon z\underset{0\neq z\in\mathbb{C}}{\longmapsto}\frac{1}{z^k}\in\mathbb{C},\quad k\in\mathbb{N}.$$

Under such choice of $L$ elementary calculations for $0<r<R<+\infty$ give

$$|L|^{\vee r}+|L|^{\vee\frac{R^2}{r}}+r|L'_{\mathfrak{r}}|^{\vee r}+\frac{R^2}{r}|L'_{\mathfrak{r}}|^{\vee\frac{R^2}{r}}=\frac{1}{r^k}+\frac{r^k}{R^{2k}}+r\frac{k}{r^{k+1}}+\frac{R^2}{r}\frac{kr^{k+1}}{R^{2(k+1)}}\leqslant\frac{4k}{r^k},$$
$$|L|^{\vee\frac{R^2}{r}}+|L|^{\vee R}+R|L'_{\mathfrak{r}}|^{\vee R}\leqslant\frac{r^k}{R^{2k}}+\frac{1}{R^k}+R\frac{k}{R^{k+1}}\leqslant\frac{2k}{R^k}.$$

By the inequality (2.6) for the entire function $f$ with the distribution of zeros $Z=\mathsf{Zero}_f$ and the value $f(0)=1$ we obtain

$$\Biggl|\sum_{r<|z|\leqslant R<+\infty}\frac{Z(z)}{z^k}\Biggr|\leqslant 16k\Bigl(\frac{T(er,f)}{r^k}+\frac{T(eR,f)}{R^k}\Bigr)\overset{(2.4),(2.3)}{\leqslant}16k\Bigl(\frac{M(er,f)}{r^k}+\frac{M(eR,f)}{R^k}\Bigr).$$

Under the condition (1.5) on $f$ this immediately gives the Lindelöf condition (1.7) for $k\in\mathbb{N}$. It is trivial to get rid of the condition $f(0)=1$. The proof is complete. □

## 3. Versions for subharmonic functions

The *positive part of extended scalar function* $v\colon X\to\overline{\mathbb{R}}$ is the function

$$v^+\colon x\underset{x\in X}{\longmapsto}\sup\bigl\{v(x),0\bigr\}\in\overline{\mathbb{R}}^+.$$

In what follows, we consider the Laplace operator $\triangle$ on the plane also in the sense of theory of generalized functions. If a function $u\not\equiv-\infty$ is *subharmonic* on a domain $D\subseteq\mathbb{C}$, then the *Riesz mass distribution* of the function $u$ is the positive Radon measure [12], [8]

$$\varDelta_u:=\frac{1}{2\pi}\triangle u. \tag{3.1}$$

For a subharmonic in a neighbourhood of the disk $r\overline{\mathbb{D}}$ function $u$ the *radial counting function* of its Riesz mass distribution is denoted and defined as the function

$$\varDelta_u^{\mathfrak{r}}\colon t\underset{t\in[0,r]}{\longmapsto}\varDelta_u\bigl(t\overline{\mathbb{D}}\bigr).$$

In particular, for an entire function $f$ and respectively a subharmonic function $\ln|f|$ we have

$$\mathsf{Zero}_f^{\mathfrak{r}}(r)\overset{(1.2)}{=}\varDelta_{\ln|f|}(r)\quad\text{for all}\quad r\in\mathbb{R}^+, \tag{3.2}$$

since for each bounded function $v$ and a subset $S\subset\mathbb{C}$ [12, Thm. 3.7.8] we have

$$\int_S v\,\mathrm{d}\varDelta_{\ln|f|}=\sum_{z\in S}\mathsf{Zero}_f(z)v(z). \tag{3.3}$$

The function is subharmonic on $S \subseteq \mathbb{C}$ if it is a restriction on $S$ of some subharmonic on an open neighbourhood of the set $S$ function.

**Theorem 3.1.** *Let $0 < r < R \in \mathbb{R}^+$ and a function $V \in C^{(1)}\big(R\overline{\mathbb{D}} \setminus r\mathbb{D}\big)$ be harmonic on an open annulus $R\mathbb{D} \setminus r\overline{\mathbb{D}}$, and also vanish on the circle $R\partial\overline{\mathbb{D}}$. Then for each subharmonic on the closed disk $R\overline{\mathbb{D}}$ function $u$ withe value $u(0) = 0$ the inequality*

$$\Bigg|\int\limits_{R\overline{\mathbb{D}}\setminus r\overline{\mathbb{D}}} V \,\mathrm{d}\Delta_u\Bigg| \leqslant |V|^{\vee r}\Delta_u^{\mathfrak{r}}(r) + 4r|V_{\mathfrak{r}}'|^{\vee r}(u^+)^{\circ r} + 4R|V_{\mathfrak{r}}'|^{\vee R}(u^+)^{\circ R} \tag{3.4}$$

*holds, and for each holomorphic on $R\overline{\mathbb{D}}$ function $f$ with $f(0) = 1$ the inequality*

$$\Bigg|\sum_{r<|z|\leqslant R} \mathsf{Zero}_f(z)V(z)\Bigg| \leqslant |V|^{\vee r}\,\mathsf{Zero}_f^{\mathfrak{r}}(r) + 4r|V_{\mathfrak{r}}'|^{\vee r}T(r,f) + 4R|V_{\mathfrak{r}}'|^{\vee R}T(R,f) \tag{3.5}$$

*holds.*

*Proof.* By (3.2), (3.3), (2.3), (2.4), it follows immediately from (3.4) that (3.5). This is why it is sufficient to show (3.4). We consider the symmetric continuation of the function $V$ with respect to the circle $r\partial\overline{\mathbb{D}}$, which is denoted and defined by the inversion

$$V_r^{\star}\colon z \longmapsto V\Big(\frac{r^2}{\bar z}\Big) \tag{3.6}$$

as the glued continuous function

$$V_r^{\odot}(z) \underset{z\in R\overline{\mathbb{D}}\setminus\frac{r^2}{R}\mathbb{D}}{:=} \begin{cases} V(z) & \text{for } z \in R\overline{\mathbb{D}} \setminus r\mathbb{D}, \\ V_r^{\star}(z) = V\Big(\dfrac{r^2}{\bar z}\Big) & \text{for } z \in r\mathbb{D} \setminus \frac{r^2}{R}\mathbb{D} \end{cases} \tag{3.7}$$

on the annulus $R\overline{\mathbb{D}} \setminus \frac{r^2}{R}\mathbb{D}$ and we employ just a partial "harmonic" case of one integral formula for concentric annuli in work [5].

**Lemma 3.1** ([5, Thm. 2])**.** *For each subharmonic on some neighbourhood of the annulus $R\overline{\mathbb{D}} \setminus \frac{r^2}{R}\mathbb{D}$ function $u \not\equiv -\infty$ the identity*

$$\int\limits_{R\mathbb{D}\setminus\frac{r^2}{R}\overline{\mathbb{D}}} V_r^{\odot}\,\mathrm{d}\Delta_u = \frac{r}{\pi}\int\limits_0^{2\pi} u(re^{i\theta})V_{\mathfrak{r}}'(re^{i\theta})\,\mathrm{d}\theta - \frac{R}{2\pi}\int\limits_0^{2\pi}\bigg(u(Re^{i\theta}) + u\Big(\frac{r^2}{R}e^{i\theta}\Big)\bigg)V_{\mathfrak{r}}'(Re^{i\theta})\,\mathrm{d}\theta$$

*holds.*

By this lemma, in view of vanishing on the function $V$ on the circle $R\partial\overline{\mathbb{D}}$ in the notation (2.1), (2.2) and (3.7) we obtain the identity

$$\int\limits_{R\overline{\mathbb{D}}\setminus r\overline{\mathbb{D}}} V\,\mathrm{d}\Delta_u = \int\limits_{r\mathbb{D}\setminus\frac{r^2}{R}\overline{\mathbb{D}}} V_r^{\star}\,\mathrm{d}\Delta_u + 2r(uV_{\mathfrak{r}}')^{\circ r} - R(uV_{\mathfrak{r}}')^{\circ R} + R(u_r^{\star}V_{\mathfrak{r}}')^{\circ R}.$$

According to (2.2) and the obvious inequality $|(uv)^{\circ r}| \leqslant |v|^{\vee r}|u|^{\circ r}$ this gives the inequality

$$\Bigg|\int\limits_{R\overline{\mathbb{D}}\setminus r\overline{\mathbb{D}}} V\,\mathrm{d}\Delta_u\Bigg| \leqslant \sup_{r\mathbb{D}\setminus\frac{r^2}{R}\overline{\mathbb{D}}}|V_r^{\star}|\Delta_u^{\mathfrak{r}}(r) + 2r|V_{\mathfrak{r}}'|^{\vee r}|u|^{\circ r} + R|V_{\mathfrak{r}}'|^{\vee R}|u|^{\circ R} + R|V_{\mathfrak{r}}'|^{\vee R}|u_r^{\star}|^{\circ R}. \tag{3.8}$$

For the harmonic on $r\mathbb{D}\setminus\frac{r^2}{R}\overline{\mathbb{D}}$ function $V_r^{\star}$, which vanishes on the circle $\frac{r^2}{R}\overline{\mathbb{D}}$ by the construction (3.6), by the maximum principle the inequality

$$\sup_{r\mathbb{D}\setminus\frac{r^2}{R}\overline{\mathbb{D}}}|V_r^{\star}| \leqslant |V|^{\vee r}$$

holds, while by the definitions (3.6) and (2.2) we have $|u_r^\star|^{\circ R} = |u|^{\circ \frac{r^2}{R}}$. Thus, by (3.8) we obtain

$$\Bigg|\int\limits_{R\overline{\mathbb{D}}\setminus r\overline{\mathbb{D}}} V \,\mathrm{d}\varDelta_u\Bigg| \leqslant |V|^{\vee r}\varDelta_u^{\mathfrak{r}}(r) + 2r|V_{\mathfrak{r}}'|^{\vee r}|u|^{\circ r} + R|V_{\mathfrak{r}}'|^{\vee R}|u|^{\circ R} + R|V_{\mathfrak{r}}'|^{\vee R}|u|^{\circ \frac{r^2}{R}}. \tag{3.9}$$

**Lemma 3.2.** *For each subharmonic on $r\overline{\mathbb{D}}$ function $u$ in view of the obvious identity $|u| = 2u^+ - u$, and by the inequality $u(0) \leqslant u^{\circ r}$ we have*

$$|u|^{\circ r} = 2(u^+)^{\circ r} - u^{\circ r} \leqslant 2(u^+)^{\circ r} - u(0) \overset{(2.2)}{\leqslant} 2(u^+)^{\vee r} - u(0). \tag{3.10}$$

By the inequality (3.10) for $u(0) = 0$ we obtain $|u|^{\circ r} \leqslant 2(u^+)^{\vee r}$, which by (3.9) gives

$$\Bigg|\int\limits_{R\overline{\mathbb{D}}\setminus r\overline{\mathbb{D}}} V \,\mathrm{d}\varDelta_u\Bigg| \leqslant |V|^{\vee r}\varDelta_u^{\mathfrak{r}}(r) + 4r|V_{\mathfrak{r}}'|^{\vee r}(u^+)^{\circ r} + 2R|V_{\mathfrak{r}}'|^{\vee R}(u^+)^{\circ R} + 2R|V_{\mathfrak{r}}'|^{\vee R}(u^+)^{\circ \frac{r^2}{R}}.$$

This implies the required inequality (3.4) since for the second factor with the subharmonic function $u^+$ in the latter term we have $(u^+)^{\circ \frac{r^2}{R}} \leqslant (u^+)^{\circ R}$. The proof is complete. □

**Theorem 3.2.** *Let the assumptions of Theorem 2.1 be satisfied. Then for each subharmonic on $\mathbb{C}$ function $u$ with the value $u(0) = 0$ the inequality*

$$\begin{aligned}\Bigg|\int\limits_{R\overline{\mathbb{D}}\setminus r\mathbb{D}} L \,\mathrm{d}\varDelta_u\Bigg| \leqslant&\Big(|L|^{\vee r} + |L|^{\vee \frac{R^2}{r}} + 4r|L_{\mathfrak{r}}'|^{\vee r} + 4\frac{R^2}{r}|L_{\mathfrak{r}}'|^{\vee \frac{R^2}{r}}\Big)(u^+)^{\circ er}\\ &+ \Big(|L|^{\vee \frac{R^2}{r}} + |L|^{\vee R} + 8R|L_{\mathfrak{r}}'|^{\vee R}\Big)(u^+)^{\circ eR}\end{aligned} \tag{3.11}$$

*holds.*

*Proof.* We consider the vanishing on $R\partial\mathbb{D}$ function

$$V\colon z \underset{r\leqslant|z|\leqslant R}{\longmapsto} L(z) - L\Big(\frac{R^2}{\bar{z}}\Big) \overset{(3.6)}{=} L(z) - L_R^\star(z), \tag{3.12}$$

which is obtained by deducting from the function $L$ its inversion $L_R^\star$ with respect to the circle $R\partial\mathbb{D}$. Since the inversion preserves the harmonicity, by the construction (3.12) this function $V$ satisfies the assumptions of Theorem 3.1. Thus, the concluding inequality (3.4) of Theorem 3.1 holds, which by construction (3.12) can be rewritten as

$$\begin{aligned}\Bigg|\int\limits_{R\overline{\mathbb{D}}\setminus r\overline{\mathbb{D}}} L \,\mathrm{d}\varDelta_u\Bigg| \leqslant&\Bigg|\int\limits_{R\overline{\mathbb{D}}\setminus r\overline{\mathbb{D}}} L_R^\star \,\mathrm{d}\varDelta_u\Bigg| + \big(|L|^{\vee r} + |L_R^\star|^{\vee r}\big)\varDelta_u^{\mathfrak{r}}(r)\\ &+ 4r\Big(|L_{\mathfrak{r}}'|^{\vee r} + \big|(L_R^\star)_{\mathfrak{r}}'\big|^{\vee r}\Big)(u^+)^{\circ r} + 4R\Big(|L_{\mathfrak{r}}'|^{\vee R} + \big|(L_R^\star)_{\mathfrak{r}}'\big|^{\vee R}\Big)(u^+)^{\circ R}.\end{aligned} \tag{3.13}$$

For the first term in the right hand side by minimum–maximum principle for the harmonic functions and the properties of inversion we have

$$\Bigg|\int\limits_{R\overline{\mathbb{D}}\setminus r\overline{\mathbb{D}}} L_R^\star \,\mathrm{d}\varDelta_u\Bigg| \leqslant \big(|L_R^\star|^{\vee r} + |L_R^\star|^{\vee R}\big)\varDelta_u^{\mathfrak{r}}(R) = \big(|L|^{\vee \frac{R^2}{r}} + |L|^{\vee R}\big)\varDelta_u^{\mathfrak{r}}(R),$$

Hence, by elementary relations,

$$\varDelta_u^{\mathfrak{r}}(R) = \int\limits_R^{eR} \frac{\varDelta_u^{\mathfrak{r}}(R)}{t}\,\mathrm{d}t \leqslant \int\limits_0^{eR} \frac{\varDelta_u^{\mathfrak{r}}(t)}{t}\,\mathrm{d}t = \frac{1}{2\pi}\int\limits_0^{2\pi} u\big(eRe^{i\theta}\big)\,\mathrm{d}t = u^{\circ eR} \leqslant (u^+)^{\circ eR}, \tag{3.14}$$

which gives the inequality

$$\Bigg|\int\limits_{R\overline{\mathbb D}\setminus r\overline{\mathbb D}} L_R^{\star}\,\mathrm d\varDelta_u\Bigg| \leqslant \big(|L|^{\vee\frac{R^2}{r}}+|L|^{\vee R}\big)(u^+)^{\circ eR}. \tag{3.15}$$

Similarly for the second term in the right hand side of (3.13) we have

$$\big(|L|^{\vee r}+|L_R^{\star}|^{\vee r}\big)\varDelta_u^{\mathfrak r}(r) \overset{(3.14)}{\leqslant} \big(|L|^{\vee r}+|L|^{\vee\frac{R^2}{r}}\big)(u^+)^{\circ er}. \tag{3.16}$$

The definition (3.6) of inversion for the radial derivative yields the identity

$$\big|(L_R^{\star})'_{\mathfrak r}\big|^{\vee t}=\frac{R^2}{t^2}|L'_{\mathfrak r}|^{\vee\frac{R^2}{t}}. \tag{3.17}$$

We use this with $t:=r$ for the third term in the right hand side of (3.13) and we get

$$4r\Big(|L'_{\mathfrak r}|^{\vee r}+\big|(L_R^{\star})'_{\mathfrak r}\big|^{\vee r}\Big)(u^+)^{\circ r} \overset{(3.17)}{=} 4\Big(r|L'_{\mathfrak r}|^{\vee r}+\frac{R^2}{r}|L'_{\mathfrak r}|^{\vee\frac{R^2}{r}}\Big)(u^+)^{\circ r}. \tag{3.18}$$

Similarly, by (3.17) with $t:=R$ for the latter term in the right hand side of (3.13) we have

$$4R\Big(|L'_{\mathfrak r}|^{\vee R}+\big|(L_R^{\star})'_{\mathfrak r}\big|^{\vee R}\Big)(u^+)^{\circ R} \overset{(3.17)}{=} 8R|L'_{\mathfrak r}|^{\vee R}(u^+)^{\circ R}. \tag{3.19}$$

The inequalities (3.15) and (3.16), as well as the pairs of identities (3.18) and (3.19) together with the obvious inequality $(u^+)^{\circ t}\leqslant(u^+)^{\circ et}$ for each $t>0$ by (3.13) we obtain (3.11). The proof is complete. □

## BIBLIOGRAPHY


1. A.A. Gol'dberg, B.Ya. Levin, I.V. Ostrovkij. *Entire and meromorphic functions* // Complex Analysis I. Encycl. Math. Sci. **85**, 1–193 (1995).
2. A.A. Kondratyuk. *The Fourier series method for entire and meromorphic functions of completely regular growth* // Math. USSR, Sb. **35**:1, 63–84 (1979). https://doi.org/10.1070/SM1979v035n01ABEH001452
3. B.Ya. Levin. *Distribution of Zeros of Entire Functions.* Fizmatgiz, Moscow (1956); *English translation:* Amer. Math. Soc., Providence, R.I. (1964).
4. K.G. Malyutin. *Fourier series and $\delta$–subharmonic functions of finite $\gamma$–type in a half–plane* // Sb. Math. **192**:6, 843–861 (2001). https://doi.org/10.1070/SM2001v192n06ABEH000572
5. E.B. Men'shikova. *Integral formulas of Carleman and Levin for meromorphic and subharmonic functions* // Russ. Math. **66**:6, 28–42 (2022). https://doi.org/10.3103/S1066369X22060056
6. B.N. Khabibullin. *Completeness of Exponential Systems and Uniqueness Sets*, 4th ed. Bashkir State Univ. Publ., Ufa (2012) (in Russian).
7. B.N. Khabibullin. *Distribution of zeros of entire functions with a subharmonic majorant* // Sb. Math. **216**:7, 977–1018 (2025). https://doi.org/10.4213/sm10151e
8. W.K. Hayman, P.B. Kennedy. *Subharmonic Functions. Vol. I.* Academic Press, London (1976).
9. R.P. Boas Jr. *Entire Functions.* Academic Press Inc. Publishers, New York (1954).
10. B.Ya. Levin. *Lectures on Entire Functions.* Amer. Math. Soc., Providence, R.I. (1996).
11. E. Lindelöf. *Sur les fonctions entières d'ordre entier* // Ann. de l'Éc. Norm. (3). **22**, 369–395 (1905). https://doi.org/10.24033/asens.555
12. Th. Ransford *Potential Theory in the Complex Plane.* Cambridge University Press, Cambridge (1995).
13. L.A. Rubel, B.A. Taylor. *A Fourier series method for meromorfic and entire functions* // Bull. Soc. Math. France. **96**, 53–96 (1968). https://doi.org/10.24033/bsmf.1660
14. L.A. Rubel, J.E. Colliander. *Entire and Meromorphic Functions.* Springer, New York (1996).

Elena Gennadievna Kudasheva,
Bashkir State Pedagogical University named after M. Akhmulla,
Oktyabrskoy revolutsii str. 3a,
450008, Ufa, Russia
E-mail: lena_kudasheva@mail.ru

Enzhe Bulatovna Menshikova,
Institute of Mathematics,
Ufa Federal Research Center, RAS
Chernyshevsky str. 112,
450008, Ufa, Russia
E-mail: algeom@bsu.bashedu.ru

Bulat Nurmievich Khabibullin,
Bashkir State Pedagogical University named after M. Akhmulla,
Oktyabrskoy revolutsii str. 3a,
450008, Ufa, Russia
Institute of Mathematics,
Ufa Federal Research Center, RAS
Chernyshevsky str. 112,
450008, Ufa, Russia
E-mail: khabib-bulat@mail.ru