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# ON COMPLETENESS CONDITIONS FOR SYSTEM OF ROOT FUNCTIONS OF DIFFERENTIAL OPERATOR ON SEGMENT WITH INTEGRAL CONDITIONS

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**Abstract.** In the work we study the completeness conditions for the system of root functions (SRF) of the operator $L_U$ generated by the differential expression

$$l(y) = -y'' + qy \quad (q \in L_1(0,1))$$

and the integral conditions

$$y^{(j-1)}(0) + (l(y), u_j) = 0 \quad (u_j \in L_2(0,1),\ j = 1,2)$$

in the space $H = L_2(0,1)$. We show that SRF of the operator $L_U$ is complete in its domain if there exist two rays in the upper half–plane such for all large $\lambda$ on these rays the characteristic determinant is bounded from below by the function $\lambda^m e^{-|\operatorname{Im}\lambda|}$, $m \geqslant \dfrac{1}{2}$. If the operator $L_U$ is densely defined, then to ensure the completeness of SRF in $H$, it is sufficient to have the mentioned estimate with an arbitrary $m \in \mathbb{R}$. Moreover, we obtain an integral representation for the characteristic determinant as the sine–transform of some function $A$, which is expressed via $u_1$, $u_2$ and the kernel of the transformation operator for the equation $l(y) = \lambda^2 y$. Employing this representation, we find explicit (in terms of the functions $u_1, u_2$) completeness conditions for SRF of the operator $L_U$ in $H$ or $D(L_U)$.

**Keywords:** differential operator with integral boundary conditions, spectrum, asymptotics.

**Mathematics Subject Classification:** 34L10, 47B28

## 1. Introduction

We consider the operator $L$ acting in the space $L_2(0,1)$ by the rule

$$Ly = l(y) := -y'' + qy,$$
$$D(L) = D := \{y \in L_2(0,1) :\ y, y' \in AC[0,1],\ l(y) \in L_2(0,1)\},$$

and its restriction $L_U$ defined by the conditions

$$U_j(y) := y^{(j-1)}(0) + (l(y), u_j) = 0, \qquad j = 1,2. \tag{1.1}$$

Here $AC[0,1]$ is the set of absolutely continuous functions on $[0,1]$, $(f,g)$ is the scalar product in $L_2(0,1)$, the functions $q \in L_1(0,1)$, $u_1, u_2 \in L_2(0,1)$ are complex–valued. As it is known [5], [13], the conditions (1.1) provide a complete description of all restrictions of the operator $L$, which has a non–empty resolvent set $\rho(L_U)$; for all $\mu \in \rho(L_U)$ the resolvent $(L_U - \mu)^{-1}$ is

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compact [5, Ch. III, Sect. 1, Lm. 6], $\sigma(L_U)$, the spectrum of operator $L_U$, coincides with the set $\{\lambda^2: \ \Delta_U(\lambda)=0\}$, where

$$\Delta_U = \begin{vmatrix} U_1(c) & U_1(s) \\ U_2(c) & U_2(s) \end{vmatrix}, \tag{1.2}$$

$s$, $c$ are solutions of the equation

$$-y'' + qy = \lambda^2 y, \quad x \in [0,1], \tag{1.3}$$

obeying the conditions

$$s(0,\lambda) = c'(0,\lambda) = 0, \quad s'(0,\lambda) = c(0,\lambda) = 1;$$

hereinafter $\varphi'(x,\lambda)$ is the derivative in $x$. For each fixed $x \in [0,1]$, $s(x,\cdot)$ and $c(x,\cdot)$ are entire functions of exponential type [18, Ch. I, Sect. 2], and this is why the function $\Delta_U$ is entire and $\sigma(L_U)$ is either empty or consists of finitely many or countably many eigenvalues, each having a finite algebraic multiplicity. One of the authors showed [34] that only the first and third cases are possible and in terms of some equation for the triple $(q,u_1,u_2)$, a necessary and sufficient condition for the identity $\sigma(L_U) = \varnothing$ was found. In the work [10] by the other author the following result was obtained:

*Under the conditions*

$$\lim_{\delta\to+0} \delta^{-2} \int_0^\delta \int_{1-\delta}^1 (u_1(y)u_2(x) - u_2(y)u_1(x))\, dydx \neq 0, \tag{1.4}$$

$$|\Delta_U(\lambda)| \geqslant C|\lambda|\ln(1+|\lambda|)e^{|\mathrm{Im}\lambda|}, \qquad \lambda \in \mathbb{C}\backslash A_\varepsilon, \tag{1.5}$$

*where $C = C(\varepsilon) > 0$, $A_\varepsilon$ is the union of circles of radius $\varepsilon$ centered as the zeros of $\Delta_U$, the SRF of the operator $L_U$ is complete in $L_2(0,1)$.*

In some form, the condition (1.4) is present in almost all papers devoted to the issues of completeness or basis property of the SRF of operator $L_U$. In fact, this condition serves to ensure an estimate of type (1.5), depending on the smoothness of the functions $u_j$. For instance, in [30], for a differential expression of $n$th order, Shkalikov found a class of integral conditions under which the SRF of the corresponding operator forms a Riesz basis with brackets or simply a Riesz basis. Being applied to our case, these conditions read

$$V_j(y) + \sum_{\nu=0}^{k_j} \int_0^1 y^{(\nu)}(x) du_{j\nu}(x) = 0, \quad j=1,2, \tag{1.6}$$

$$V_j(y) = \sum_{\nu=0}^{k_j} \left(\alpha_{j\nu} y^{(\nu)}(0) + \beta_{j\nu} y^{(\nu)}(1)\right),$$

where $1 \geqslant k_1 \geqslant k_2 \geqslant 0$, $u_{j\nu}$ are functions of bounded variation continuous at the points 0, 1. If $L_V$ is the operator obtained from $L_U$ by replacing (1.6) by the condition

$$V_j(y) = 0, \qquad j = 1,2, \tag{1.7}$$

which are regular in the Birkhoff sense [20, Ch. II, Sect. 4], the SRF of the $L_V$ forms a Riesz basis with brackets in $L_2(0,1)$ [29], and a usual Riesz basis in the case of the strong regularity [19], [12]. According to the main result of [30], if the conditions (1.7) are regular in the Birkhoff sense, then SRF of the operator $L_U$ forms a Riesz basis with brackets in $L_2(0,1)$ and a Riesz basis in the case of a strong regularity. An important point in the proof of the basicity is the estimate

$$|\Delta_U(\lambda)| > C(\varepsilon)|\lambda|^{k_1+k_2-1} e^{|\,\mathrm{Im}\,\lambda|}, \qquad \lambda \in \mathbb{C}\backslash A_\varepsilon, \tag{1.8}$$

where $C(\varepsilon)$, $A_\varepsilon$ are defined as in (1.5). As it was pointed out in [30, Lm. 1], this estimate is equivalent to the regularity in the Birkhoff sense of the conditions (1.8).

In the work [26] by Skubachevskii and Steblov and the followed series of works [2], [21], [23], [24], [25], [27], the case, when the conditions (1.6) read

$$(y, \varphi_j) = 0 \ (\varphi_j \in L_2(0,1)), \qquad j = 1, 2, \tag{1.9}$$

was studied in detail. If at least one of the conditions (1.1) has the form (1.9), the operator $L_U$ is not densely defined and therefore has no adjoint operator. In such a situation, the approaches of works [19], [12], [30] is not applicable. Nevertheless, assuming some regularity of the behavior of the functions $\varphi_j$ at the the points 0 and 1 so that

$$\varphi_1(1)\varphi_2(0) - \varphi_1(0)\varphi_2(1) \neq 0,$$

one succeeds to study quite in detail various spectral properties of the considered operator. For instance, in [26], [2], [27], the discreteness of the spectrum and its localization near the ray $\arg\lambda = 0$ were proved. In [21], the Abel basis property of SRF in $D(L_U)$ was established, and in [23], the Riesz basis property was established. In [24], [25], the asymptotic behavior of the spectrum was found and an estimate for the resolvent far from the spectrum was obtained by using only certain requirements on the asymptotic behavior of the Laplace transforms of the functions $\varphi_j$. We also mention the works [4], [35], in which results close to [26], [2], [27] were obtained for the operator $L_U$ with $U_j(y) = (y', \varphi_j)$ $(\varphi_j \in L_2(0,1))$ and the 4th order operator of the form $L_U$.

The conditions (1.1) can be reduced to (1.6) (in particular, to (1.9)) only if $u_j$ is smooth enough. Below, in Lemmas 2.3, 2.4, we shall obtain criteria for the equivalence of the conditions (1.1), (1.9) and (1.1), (1.6).

The question arises: is it possible to obtain some nontrivial information about the spectral properties of the operator $L_U$ under weaker conditions on the functions $u_j$?

In this paper, we find an estimate that is significantly weaker than (1.5), under which SRF of the operator $L_U$ is complete in $D(L_U)$ or in $L_2(0,1)$. Furthermore, we obtain an integral representation of the function $\Delta_U$ as the sine transform of some function $A$, which is expressed in terms of $u_1$, $u_2$ and the kernel $\mathcal{K}(\cdot,\cdot)$ of the transformation operator for equation (1.3). We note that the function $A$ was obtained in [34] in studying the spectrum of operator $L_U$. In this paper, we have succeeded to simplify the form of this function. Using the mentioned representation, we find explicit (in terms of functions $u_1, u_2$) conditions for the completeness of SRF of operator $L_U$ in $D(L_U)$.

Operators of the $L_U$ type arise in the turbulence theory [39] and in the theory of Markov processes [32], [33]. Various spectral properties of operators of the form $L_U$ (of arbitrary order), apart of the aforementioned ones, were studied in the works by Picone [37], [38], Tamarkin [28], Lubich [15], [16], Brjuns [1], Krall [36], Il'in and Moiseev [6], [7], Makin [17], Kanguzhin [9], [11], Polyakov [22] and others. A more detailed bibliography on the discussed topic can be found in the surveys [5], [27], [36].

## 2. Formulation of main results

**2.1. Preliminary statements.** It has been noted above that classical methods do not work when the operator $L_U$ is not densely defined. In this regard, it is important to know under what conditions on the function $u_j$ the operator $L_U$ is densely defined. Obviously, if at least one linear combination of conditions (1.1) is equivalent to the condition $(y, \varphi) = 0$, $(\varphi \in L_2(0,1), \varphi \neq 0)$, then the operator $L_U$ is not densely defined. The opposite is also true.

**Lemma 2.1.** *The operator $L_U$ is not densely defined if and only if there exists $\varphi \in L_2(0,1)$, $\varphi \neq 0$, such that*

$$\psi'(0)u_1 - \psi(0)u_2 = \psi,$$

*where $\psi$ is a solution to the problem*

$$-\psi'' + \overline{q}\psi = \varphi, \qquad \psi(1) = \psi'(1) = 0. \tag{2.1}$$

**Lemma 2.2.** *If there exists a sequence $\{\lambda_k\}$ escaping to infinity in the sector*

$$S_\varepsilon = \{\varepsilon \leqslant \arg\lambda \leqslant \pi - \varepsilon\}, \qquad \varepsilon > 0,$$

*and a constant $C > 0$ such that*

$$|\Delta_U(\lambda_k)| \geqslant Ce^{\operatorname{Im}\lambda_k}, \tag{2.2}$$

*then the operator $L_U$ is densely defined.*

In the next proposition we provide a complete description of the class of functions $u_1$, $u_2$, for which the conditions (1.1) can be represented in the form (1.9).

**Lemma 2.3.** *Let the functions $\varphi_1, \varphi_2 \in L_2(0,1)$ be such that $\psi_j$, solutions to the problem (2.1) for $\varphi = \varphi_j$, satisfy the condition*

$$d_1 := \psi_1(0)\psi_2'(0) - \psi_1'(0)\psi_2(0) \neq 0,$$

*and let*

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{d_1}\begin{pmatrix} \psi_2(0) & -\psi_1(0) \\ \psi_2'(0) & -\psi_1'(0) \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{2.3}$$

*Then the conditions (1.1) and (1.9) are equivalent in $D$.*

*And vice versa, let the conditions (1.1) and (1.9) be equivalent in $D$. Then $d_1 \neq 0$ and the identity (2.3) holds.*

**Remark 2.1.** *Let $c_0 = c|_{\lambda=0}$, $s_0 = s|_{\lambda=0}$, where $s$, $c$ are the functions involved in (1.2). Then the solution of problem (2.1) satisfies the formula*

$$\psi(x) = \int\limits_x^1 (\overline{s_0}(x)\overline{c_0}(t) - \overline{c_0}(x)\overline{s_0}(t))\varphi(t)dt. \tag{2.4}$$

*This is why if $\varphi$ is an arbitrary function in $L_2(0,1)$ orthogonal to $s_0$ and $c_0$, then $\psi(0) = \psi'(0) = 0$. Therefore, $d_1 = 0$ if at least one of the functions $\varphi_1$, $\varphi_2$ is orthogonal to $s_0$ and $c_0$.*

By Lemma 2.3, for the conditions (1.1) and (1.9) to be equivalent, it is necessary that the functions $u_j$, among other things, must be sufficiently smooth, namely, belong to the set

$$D^* = \{y \in L_2(0,1): \ y, y' \in AC[0,1], \ -y'' + \overline{q}y \in L_2(0,1)\}.$$

Now let us find out how smooth the functions $u_j$ must be for conditions (1.1) and (1.6) to be equivalent. To do this, we rewrite (1.6) as

$$\Phi_j(y) := \sum_{\nu=0}^{1} \int\limits_0^1 y^{(\nu)}(x)d\varphi_{j\nu}(x) = 0, \qquad j = 1, 2, \tag{2.5}$$

where[1] the functions $\varphi_{j\nu}$ differ from $u_{j\nu}$ only by the values at the points 0 and 1. Without loss of generality we can suppose that $\varphi_{j\nu}(1) = 0$. We denote by $Q$ the operator in $L_\infty(0,1)$, which

[1] If $k_j = 0$ in (1.6), then we let $\varphi_{j1} = 0$ in (2.5).

acts by the formula

$$Qf = \int_x^1 (t-x)\overline{q} f\,dt,$$

and we let

$$\psi_j = (I-Q)^{-1} f_j, \qquad j=1,2, \tag{2.6}$$

$$f_j(x) = \overline{\varphi_{j1}}(x) + \int_x^1 \overline{\varphi_{j0}}(t)dt.$$

The operator $Q$ is Volterra in the space e $L_\infty(0,1)$, and this is Equation (2.6) has the unique solution in this space, which by the identity

$$\psi_j = f_j + Q\psi_j$$

is a function of bounded variation, and it vanishes at the point 1.

**Lemma 2.4.** *Let the function $\varphi_{j\nu}$, $j,\nu = 1,2$, of bounded variation by such that*

$$d_2 := \begin{vmatrix} \overline{\varphi_{10}}(0) - \int_0^1 \overline{q}\psi_1 dt & \psi_1(0) \\ \overline{\varphi_{20}}(0) - \int_0^1 \overline{q}\psi_2 dt & \psi_2(0) \end{vmatrix} \neq 0 \tag{2.7}$$

*and let*

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{d_2} \begin{pmatrix} -\psi_2(0) & \psi_1(0) \\ \overline{\varphi_{20}}(0) - \int_0^1 \overline{q}\psi_2 dt & \overline{\varphi_{10}}(0) - \int_0^1 \overline{q}\psi_1 dt \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{2.8}$$

*Then the condition (1.1) and (2.5) are equivalent in $D$.*

*And vice versa, let (1.1) and (2.5) be equivalent in $D$. Then $d_2 \neq 0$ and the identity (2.8) holds.*

Let $L_\Gamma$ be the operator $L_U$ for

$$U_j(y) = \Gamma_j(y) := a_{j1}y(0) + a_{j2}y'(0) + a_{j3}y(1) + a_{j4}y'(1), \quad j=1,2,$$

where $a_{jk} \in \mathbb{C}$ are such that $\rho(L_\Gamma) \neq \varnothing$. In particular, if the boundary conditions $\Gamma_j(y) = 0$ are nondegenerate, then SRF $L_\Gamma$ is complete in $L_2(0,1)$ [18, Thm. 1.3.1]. We denote

$$R_U(\lambda) = (L_U - \lambda^2)^{-1} \quad \text{и} \quad R_\Gamma(\lambda) = (L_\Gamma - \lambda^2)^{-1}.$$

**Lemma 2.5.** *Let $\lambda_0^2 \in \rho(L_U)$. Then for all $\Gamma_j$ such that $\lambda_0^2 \in \rho(L_\Gamma)$ the representation holds*

$$R_U(\lambda_0)f = \frac{1}{\Delta_U(\lambda_0)} \begin{vmatrix} c & s & R_\Gamma f \\ U_1(c) & U_1(s) & U_1(R_\Gamma f) \\ U_2(c) & U_2(s) & U_2(R_\Gamma f) \end{vmatrix} (\lambda_0), \qquad f \in L_2(0,1). \tag{2.9}$$

**Corollary 2.1.** *For all $u_1, u_2 \in L_2(0,1)$ the resolvent of operator $L_U$ is a Hilbert — Schmidt operator.*

**2.2. Main results.** If the operator $L_U$ is densely defined, we can employ the general scheme [3, Ch. XI, Sect. 6, Cor. 31][1]

**Theorem 2.1** (Dunform — Schwartz). *Let $H$ be a separable Hilbert space, $T$ be a densely defined in $H$ operator with the resolvent in the Hilbert — Schmidt class. Let $p_k$, $(k = \overline{1,5})$, be the rays satisfying the conditions*

a) *the angles between all neighbouring rays are less* $\frac{\pi}{2}$,

b) *there exists a constant $M \geqslant -1$ such that*

$$\left\|(T-\lambda)^{-1}\right\| = O\left(\lambda^M\right) \quad as \quad \lambda \to \infty$$

*along each ray $p_k$.*

*Then the system of root vectors of operator $T$ is complete in $H$.*

**Theorem 2.2.** *Let the operator $L_U$ be densely defined and possess the property: there exist rays*

$$P_k = \{\arg z = \beta_k,\ |z| \geqslant R_k\} \quad (k = 1,2), \quad R_k \geqslant 0, \quad 0 < \beta_1 < \beta_2 < \pi,$$

*such that*

$$|\Delta(\lambda)| \geqslant C|\lambda|^N e^{\operatorname{Im}\lambda}, \quad \lambda \in P_1 \cup P_2, \tag{2.10}$$

*where $C > 0$ and $N \in \mathbb{R}$ are constants independent of $\lambda$. Then SRF of operator $L_U$ is complete in $L_2(0,1)$.*

**Remark 2.2.** *According to Corollary 2.1, $L_U^{-1} \in \sigma_2$, and by the Dunford — Schwartz theorem, to ensure the completeness of SRF of the operator $L_U$, it is sufficient to find 5 rays obeying the conditions a) and b). The features of operator $L_U$ allows one to deal only with two rays.*

**Corollary 2.2.** *If the function $\Delta_U$ satisfies the estimates (2.2) and (2.10), then SRF of the operator $L_U$ is complete in $L_2(0,1)$.*

According to Lemma 2.1, the domain of the operator $L_U$ may not be dense. In this situation, $L_U^*$ does not exist, so the above general scheme does not work. Using only the methods of function theory, we succeed to obtain the following result.

**Theorem 2.3.** *Let $P_k$ be the same ways as in Theorem 2.2, and*

$$|\Delta(\lambda)| \geqslant C|\lambda|^{-\frac{1}{2}} e^{\operatorname{Im}\lambda}, \quad \lambda \in P_1 \cup P_2, \tag{2.11}$$

*where $C > 0$ is independent of $\lambda$. Then SRF of the operator $L_U$ is complete in $D(L_U)$.*

To formulate the next result, we introduce a notation. Let $\mathcal{K}(\cdot,\cdot)$ be the kernel of transformation operator for the solution $e(x,\lambda)$ of Equation (1.3) with the initial conditions $e(0,\lambda) = 1$, $e'(0,\lambda) = i\lambda$ [18, Ch. 1, Sect. 2]:

$$e(x,\lambda) = e^{i\lambda x} + \int_{-x}^{x} \mathcal{K}(x,t) e^{i\lambda t} dt,$$

$K_\pm$ be the operator acting in $L_2(0,1)$ by the formulas

$$\left[K_\pm f\right](x) = \int_x^1 \mathcal{K}(t, \pm x) f(t) dt. \tag{2.12}$$

[1] There is a generalization of this statement to the operator with the resolvent in Neumann — Schatten ideal $\sigma_p$ with an arbitrary $p > 0$ [31, Thm. 4.7, Rem. 4.8].

We let

$$A_1[g,f](x) = B[(I+K_+)g,(I+K_-)f](x) - B[(I+K_+)f,(I+K_-)g](x),$$

$$B[f,g](x) = \int\limits_x^1 f(t)g(t-x)dt,$$

$$A_2[f](x) = \int\limits_x^1 \left[(I+K_+ + K_-)f\right](t)dt,$$

$$A_3[f](x) = \int\limits_x^1 (t-x)\left[(I+K_+ - K_-)f\right](t)dt,$$

$$A(x) = A_1[u_2,u_1](x) - A_2[u_1](x) - A_3[u_2](x).$$

**Theorem 2.4.** *The characteristic determinant of operator $L_U$ can be represented as*

$$\Delta_U(\lambda) = \lambda^3 \int\limits_0^1 \sin\lambda x A(x)dx + \lambda^2 A(0) + 1. \tag{2.13}$$

**Remark 2.3.** *In* [8], *for the Sturm — Liouville operator on a curve with the Dirichlet or Neumann condition, a representation similar to* (2.13) *for the characteristic determinant was found. Using this representation, we obtain a criterion for the asymptotic localization of spectrum near finitely many rays.*

**Corollary 2.3.** *Let for some $m$ the limit*

$$\lim_{x\to 1}\frac{A(x)}{(1-x)^m}$$

*is well–defined and non–zero. If $m \leqslant \dfrac{5}{2}$, then SRF of the operator $L_U$ is complete in $D(L_U)$, and for $m\leqslant 2$ it is complete in $L_2(0,1)$.*

**Corollary 2.4.** *Let $q=0$ and for some $m$ the limit*

$$\lim_{x\to 1}(1-x)^{-m}\int\limits_x^1 \left[u_2(t)u_1(t-x) - u_2(t-x)u_1(t) - u_1(t) - (t-x)u_2(t)\right]dt$$

*is well–defined and non–zero. If $m \leqslant \dfrac{5}{2}$, then SRF of the operator $L_U$ is complete in $D(L_U)$, and for $m\leqslant 2$ it is complete in $L_2(0,1)$.*

## 3. Proofs of preliminary lemmas

**3.1. Proof of Lemma 2.1.** The sufficiency can be verified by straightforward calculations. Let $D_U$ be incomplete in $L_2(0,1)$. Then there exists a non–zero element $\varphi$ in $L_2(0,1)$ such that $(y,\varphi)=0$ for all $y\in D(L_U)$. Let $\psi$ be the function in the formulation of lemma. We have

$$0 = (y, l^*(\psi)) = y(0)\overline{\psi'}(0) - y'(0)\overline{\psi}(0) + (l(y),\psi).$$

According to (1.1)

$$y(0) = -(l(y),u_1),\ \ y'(0) = -(l(y),u_2),$$

and hence

$$(l(y), \psi'(0)u_1 - \psi(0)u_2 - \psi) = 0$$

for all $y \in D(L_U)$. Since $0 \in \rho(L_U)$ and the set $\{l(y),\ y \in (D(L_U)\} = \operatorname{Ran}(L_U)$ coincides with $L_2(0,1)$, we have

$$\psi'(0)u_1 - \psi(0)u_2 - \psi = 0.$$

The proof is complete.

**3.2. Proof of Lemma 2.2.** Suppose that the operator $L_U$ is not densely defined. If $\psi$ is a solution of the problem (2.1) for some $\varphi \neq 0$, then at least one of the numbers $\psi(0)$ and $\psi'(0)$ is non–zero. Let $\psi'(0) \neq 0$. Since

$$\overline{\psi'}(0)U_1(y) - \overline{\psi}(0)U_2(y) = \overline{\psi'}(0)y(0) - \overline{\psi}(0)y'(0) + (ly, \psi) = (y, \varphi),$$

we have

$$\Delta_U(\lambda) = \frac{1}{\overline{\psi'}(0)} \begin{vmatrix} (c,\varphi) & (s,\varphi) \\ U_2(c) & U_2(s) \end{vmatrix}.$$

Let $y_1$, $y_2$ be solutions (1.3) with the asymptotics [20, Ch. II, Sect. 4]

$$y_k^{(j-1)}(x,\lambda) \sim (\omega_k\lambda)^{j-1} e^{\omega_k \lambda x}[1], \qquad x \in [0,1], \qquad \lambda \to \infty, \qquad k,j = 1,2, \tag{3.1}$$

where $\omega_1 = -i$, $\omega_2 = i$, the symbol $[1]$ stands for the expression $1 + O(\lambda^{-1})$, where the estimate $O(\lambda^{-1})$ is uniform in $x \in [0,1]$ and $0 \leqslant \arg\lambda \leqslant \pi$. We have

$$\begin{pmatrix} c \\ s \end{pmatrix} = \frac{1}{2} \operatorname{diag}\left(1, (i\lambda)^{-1}\right) \begin{pmatrix} [1] & [1] \\ -[1] & [1] \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{3.2}$$

Then

$$\Delta_U(\lambda) = \frac{[1]}{2i\overline{\psi'}(0)\lambda} \begin{vmatrix} (y_1,\varphi) & (y_2,\varphi) \\ U_2(y_1) & U_2(y_2) \end{vmatrix}.$$

Hence, since for each $f \in L_2(0,1)$

$$(y_1, f) = o\left(\lambda^{-\frac{1}{2}} e^{\operatorname{Im}\lambda}\right), \qquad (y_2, f) = o\left(\lambda^{-\frac{1}{2}}\right)$$

for large $\lambda$ in $S_\varepsilon$, we get

$$\sup_{\varepsilon \leqslant \alpha \leqslant \pi - \varepsilon} \left|\Delta_U(re^{i\alpha}) e^{-r\sin\alpha}\right| \to 0, \quad r \to +\infty.$$

This estimate contradicts the assumptions, therefore, the operator $L_U$ is densely defined. The proof is complete.

**3.3. Proof of Lemma 2.3.** Let $d_1 \neq 0$ and the identity (2.3) holds. Then

$$\psi_j'(0)u_1 - \psi_j(0)u_2 = \psi_j \quad (j = 1,2).$$

This implies, see the proof of Lemma 2.2,

$$\overline{\psi_j'}(0)U_1(y) - \overline{\psi_j}(0)U_2(y) = (y, \varphi_j).$$

And vice versa, if the conditions (1.1) and (1.9) are equivalent, then the generate the same operator. Therefore, the functions

$$\Delta_U \quad \text{and} \quad \Delta_\Phi := \begin{vmatrix} (c,\varphi_1) & (s,\varphi_1) \\ (c,\varphi_2) & (s,\varphi_2) \end{vmatrix}$$

have the same zeros. Since $\Delta_U(0) = 1$, we have $\Delta_\Phi(0) \neq 0$. According to the formula (2.4),

$$d_1 = (c_0,\varphi_1)(s_0,\varphi_2) - (s_0,\varphi_1)(c_0,\varphi_2) = \Delta_\Phi(0).$$

This is why $d_1 \neq 0$.

Arguing as in the proof of Lemma 2.1, we get

$$\psi_j'(0)u_1 - \psi_j(0)u_2 = \psi_j \quad (j = 1,2).$$

This implies (2.3). The proof is complete.

**3.4. Proof of Lemma 2.4.** Let $d_2 \neq 0$. The sufficiency (2.8) for the equivalence of conditions (1.1) and (2.5) can be proved as in the previous proof. Let the identities (1.1) and (2.5) are equivalent in $D$. Since $\psi_1$, $\psi_2$ are functions of bounded variation, we find

$$(ly, \psi_j) = \int_0^1 y'(x) d \left[ \overline{\psi_j}(x) + \int_x^1 (x-t) q \overline{\psi_j} dt \right] - y(0) \int_0^1 q \overline{\psi_j} dt + y'(0) \overline{\psi_j}(0), \quad j = 1, 2, \ y \in D.$$

On the other hand, according to (2.5),

$$\Phi_j(y) = \int_0^1 y' d \left[ \varphi_{j1}(x) + \int_x^1 \varphi_{j0}(t) dt \right] - y(0) \varphi_{j0}(0), \qquad j = 1, 2, \ y \in D.$$

Therefore,

$$\Phi_j(y) = (ly, \psi_j) - y(0) \left[ \varphi_{j0}(0) - \int_0^1 q \overline{\psi_j} dx \right] - y'(0) \overline{\psi_j}(0), \qquad j = 1, 2, \ y \in D. \tag{3.3}$$

Now we proceed as in the proof of Lemma 2.1: in view of the equivalence of identities (1.1) and (2.5) in $D$, the relation (3.3) implies

$$\left( ly, \psi_j + \left( \overline{\varphi_{j0}}(0) - \int_0^1 \overline{q} \psi_j dt \right) u_1 + \psi_j(0) u_2 \right) = 0, \qquad j = 1, 2, \ y \in D_U,$$

so that

$$\psi_j + \left( \overline{\varphi_{j0}}(0) - \int_0^1 \overline{q} \psi_j dt \right) u_1 + \psi_j(0) u_2 = 0, \qquad j = 1, 2. \tag{3.4}$$

Then according to (2.7) and (3.3), $d_2 = \overline{\Delta_\Phi}(0) \neq 0$ since $\Delta_U$ and $\Delta_\Phi$ have the same zeros and $\Delta_U(0) = 1$. Therefore, the determinant of system (3.4) is non–zero. Solving this system, we obtain (2.8). The proof is complete.

**3.5. Proof of Lemma 2.5.** We denote by $L_0$ the operator $L_U$ for $u_1 = u_2 = 0$. Since $\Delta_0(\lambda) \equiv 1$, the operator $R_0(\lambda) := (L_0 - \lambda^2)^{-1}$ is well–defined for all $\lambda \in \mathbb{C}$. This is why for all $\lambda^2 \in \rho(L_U)$ we have

$$R_U(\lambda) f = \frac{1}{\Delta_U(\lambda)} \begin{vmatrix} c & s & R_0(\lambda) f \\ U_1(c) & U_1(s) & U_1(R_0(\lambda) f) \\ U_2(c) & U_2(s) & U_2(R_0(\lambda) f) \end{vmatrix}, \qquad f \in L_2(0,1). \tag{3.5}$$

Let $\lambda_0^2 \in \rho(L_U)$. We choose the boundary conditions $\Gamma_j$ so that $\lambda_0^2 \in \rho(L_\Gamma)$. Since the functions $R_0(\lambda_0) f$ and $R_\Gamma(\lambda_0) f$ satisfy the same equation $l(y) - \lambda_0^2 y = f$, their difference is a linear combinations of the functions $s(\cdot, \lambda_0)$ and $c(\cdot, \lambda_0)$. By the identity (3.5) this implies (2.9). The proof is complete.

**3.6. Proof of Corollary 2.1.** The statement is implied by (3.5) and the relations

$$[R_0(\lambda) f](x) = -\int_0^x (s(x,\lambda) c(t,\lambda) - s(t,\lambda) c(x,\lambda)) f(t) dt,$$

$$U_j(R_0(\lambda) f) = \lambda^2 \left( f, R_0^*(\lambda) u_j \right) + (f, u_j)$$

the proof is complete.

## 4. Proofs of Theorem 2.2–2.4

**4.1. SRF of operator $L_U$.** The proof of Theorem 1 in [34] shows that the spectrum $L_U$ is empty if and only if $\Delta_U(\lambda) \equiv 1$. This is why under the assumptions of Theorems 2.2 and 2.3, the spectrum of $L_U$ consists of countably many eigenvalues $\{\mu_k\}_1^\infty$ with the multiplicities $m_k$. Let $\mu_k = \lambda_k^2$, where $\{\lambda_k\}$ are the zeros of $\Delta_U$ in $\Pi_+ = \{0 \leqslant \arg\lambda < \pi\}$ taken in ascending order of their absolute values without counting the multiplicities; without loss of generality we suppose that 0 is outside the spectrum of $L_U$. We consider the functions

$$w_j = U_j(c)s - U_j(s)c, \quad j = 1, 2. \tag{4.1}$$

Since $U_1(w_1) = 0$, $U_2(w_1) = \Delta_U$, we see that $w_1(\cdot, \lambda_k)$ is an eigenfunction of the operator $L_U$ associated with the eigenvalue $\mu_k$. The same is obviously true for $w_2$. If

$$\widetilde{w}_j(x, \mu) = w_j(x, \sqrt{\mu}), \quad 0 \leqslant \arg\mu < 2\pi,$$

then for each $j = 1, 2$ the sequence

$$W_j := \left(\frac{\partial}{\partial\mu}\right)^\nu \widetilde{w}_j(x, \mu)\bigg|_{\mu=\mu_k}, \quad k = 1, 2, \dots, \quad \nu = \overline{0, m_k - 1}, \tag{4.2}$$

forms SRF of the operator $L_U$.

**4.2. Proof of Theorem 2.2.** Suppose that the system $W_1$[1] is incomplete, that is, there exists a non–zero function $f$ in $L_2(0, 1)$ orthogonal to $W_1$. Following the idea presented in the aforementioned Remark 4.8 from [31], we consider a vector–valued (with values in $L_2(0, 1)$) function

$$w(\lambda) = \left(L_U^* - \overline{\lambda}^2\right)^{-1} f,$$

which by the made assumption is entire with respect the variable $\overline{\lambda}$. We are going to show that

$$\|w(\lambda)\| = O\left(\lambda^N\right), \quad \lambda \in P_1 \cup P_2. \tag{4.3}$$

It is sufficient to show that

$$\|\left(L_U - \lambda^2\right)^{-1}\| = O\left(\lambda^N\right), \quad \lambda \in P_1 \cup P_2. \tag{4.4}$$

As $\Gamma$ in (2.9) we choose the Dirichlet condition $y(0) = y(1) = 0$ and denote by $L_D$ and $R_D(\lambda)$ the corresponding operator and its resolvent $(L_D - \lambda^2)^{-1}$. Then

$$\|R_D(\lambda)\| = O\left(\lambda^{-2}\right), \quad \lambda \to \infty, \tag{4.5}$$

along each ray $\arg\lambda = \beta$ $(0 < \beta < \pi)$.

The functions $y_j$ defined by (3.1) are linearly independent and

$$\begin{pmatrix} c \\ s \end{pmatrix} = \Omega \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

where $\Omega$ is non–degenerate for all $\lambda$ matrix, which depends only on $\lambda$. Passing to $y_1$, $y_2$ in (2.9), we get

$$R_U(\lambda)f = R_D(\lambda)f + \frac{1}{D_U(\lambda)} \begin{vmatrix} \Psi_1(\cdot, \lambda) & \Psi_2(\cdot, \lambda) \\ U_1(R_D(\lambda)f) & U_2(R_D(\lambda)f) \end{vmatrix}, \quad f \in L_2(0, 1), \tag{4.6}$$

$$\Psi_j(x, \lambda) = \begin{vmatrix} y_1(x, \lambda) & y_2(x, \lambda) \\ U_1(R_D(\lambda)f) & U_2(R_D(\lambda)f) \end{vmatrix},$$

$$D_U = \begin{vmatrix} U_1(y_1) & U_1(y_2) \\ U_2(y_1) & U_2(y_2) \end{vmatrix}. \tag{4.7}$$

[1] The systems $W_1$ and $W_2$ are complete or incomplete simultaneously.

Without loss of generality we can suppose that the constants $R_k$ in the definition of the rays $P_k$ are such that the estimates (3.1) and (4.5) hold on both rays $P_1$ and $P_2$. Then

$$|D_U(\lambda)| = 2[1]|\lambda||\Delta(\lambda)|, \quad \lambda \in P_1 \cup P_2. \tag{4.8}$$

Using these estimates and (2.10) we see that for all $\lambda \in P_1 \cup P_2$

$$\|\Psi_j\| = O\left(\lambda e^{\operatorname{Im}\lambda}\right), \quad U_j\left(R_D(\lambda)f\right) = O(1)\|f\|,$$
$$|D_U(\lambda)| \geqslant C_1|\lambda|^{N+1}e^{\operatorname{Im}\lambda},$$

where $C_1 > 0$ is independent of $\lambda$. By the identity (4.6) this implies (4.4). The proof is complete.

According to (4.6) and the estimate (3.1), $R_U$ is a quotient of two entire functions of order at most 1, therefore [14, Ch. I, Sect. 9], it has the order at most 1. Since

$$|w(\lambda)| \leqslant \|R_U(\lambda)\|\|f\|,$$

the order of function $w$ also does not exceed 1. Applying the Phragmén — Lindelöf principle, in view of (4.3) we conclude that

$$w(\lambda) = O\left(|\lambda|^N + 1\right), \quad \lambda \in S, \tag{4.9}$$

where $S$ is a sector bounded by the rays $\arg\lambda = \beta_1$ and $\arg\lambda = \beta_2$. Since the function $w$ is even, the estimate (4.9) is also true on the vertical with $S$ sector. Applying the Phragmén — Lindelöf principle to one of adjacent with $S$ sectors, we arrive at the identity

$$w(\lambda) = f_0 + f_2\overline{\lambda}^2 + \cdots + f_M\overline{\lambda}^{2M}, \quad f_j \in D(L_U^*).$$

Therefore,

$$-f + (L_U^* - \overline{\lambda}^2)f_0 + f_2\overline{\lambda}^2 + \cdots + f_M\overline{\lambda}^{2M} = 0, \qquad \lambda \in \mathbb{C}.$$

Equating to 0 the coefficients at $\overline{\lambda}^{2k}$, we get $f_M = f_{M-1} = \cdots = f_0 = 0$, and hence $f = 0$.

**4.3. Proof of Theorem 2.3.** Suppose that the system (4.2) is incomplete in $D(L_U)$, that is, there exists a non–zero function $f \in D(L_U)$ orthogonal to (4.2). Let us show that

$$\Delta_j(\lambda) := (w_j, f)(\lambda) \equiv 0, \quad j = 1, 2. \tag{4.10}$$

We introduce the functions

$$F_j(\lambda) = \frac{\Delta_j(\lambda)}{\Delta_U(\lambda)}, \quad j = 1, 2. \tag{4.11}$$

By (4.1) and the supposed incompleteness, $F_j$ is an even entire function.

We denote by $D_j$ the function, which is obtained from $\Delta_j$ by replacing $c$, $s$ respectively by $y_1$, $y_2$, see (3.1). We have

$$F_j = \frac{D_j}{D_U}. \tag{4.12}$$

As in the previous section we suppose that the estimates (3.1) and (4.8) hold on both rays $P_1$ and $P_2$. This is why in view of (2.11) and (4.7) we have

$$|D_U(\lambda)| \geqslant C|\lambda|^{\frac{1}{2}}e^{\operatorname{Im}\lambda}, \qquad \lambda \in P_1 \cup P_2, \tag{4.13}$$

with a constant $C > 0$ independent of $\lambda$. Then

$$D_j = \begin{vmatrix} ccU_j(y_1) & U_j(y_2) \\ (y_1, f) & (y_2, f) \end{vmatrix} = \lambda^{-2}\begin{vmatrix} ccU_j(y_1) & U_j(y_2) \\ (l(y_1), f) & (l(y_2), f) \end{vmatrix}. \tag{4.14}$$

Since $u_j \in L_2(0,1)$, taking into consideration the estimates (3.1), it is easy to show that

$$U_k(y_j)(\lambda) = o\left(\lambda^{\frac{3}{2}}e^{\sigma_j \operatorname{Im}\lambda}\right) \quad (k, j = 1, 2), \quad \lambda \to \infty, \tag{4.15}$$

on each fixed ray $P(\beta, R)$ $(0 < \beta < \pi,\ R > 0)$. To estimate the elements $(l(y_j), f)$, we observe that $f \in D_U$, therefore, we can integrate once by parts in these expressions. In view of (3.1) this gives

$$(l(y_j), f) = O\left(\lambda e^{\sigma_j \operatorname{Im} \lambda}\right) \quad (\sigma_1 = 1, \sigma_2 = 0), \quad \lambda \to \infty, \tag{4.16}$$

uniformly in $0 \leqslant \arg \lambda \leqslant \pi$. Combining now the formulas (4.12), (4.14) and estimates (4.13), (4.15), (4.16), we obtain

$$F_j(\lambda) = o(1), \qquad \lambda \in P(\beta_1, R) \cup P(\beta_2, R), \quad R \to +\infty. \tag{4.17}$$

The asymptotic estimates (3.1), the relations (3.2), (4.1), (1.2) and (4.11) imply that the order of the functions $F_j$ does not exceed 1. As in the previous section, using the parity of the functions $F_j$, we successively apply the Phragmén — Lindelöf principle to the sector $S$ and the adjacent sector and verify that the functions $F_j$ are bounded on the entire plane. Taking into consideration (4.17), we find $F_j \equiv 0$. This imply the relations (4.10). They can be written in the form

$$\begin{pmatrix} U_1(c) & U_1(s) \\ U_2(c) & U_2(s) \end{pmatrix} \begin{pmatrix} (s, f) \\ -(c, f) \end{pmatrix} = 0.$$

The determinant of the matrix of this system coincides with the function $\Delta_U$. Since $\Delta_U$ is entire and $\Delta_U(0) = 1$, we have $(s, f) = (c, f) \equiv 0$ near 0, and hence everywhere on $\mathbb{C}$. The system of functions $\{c(\cdot, \lambda),\ \lambda \in \mathbb{C}\}$ is obviously complete in $L_2(0, 1)$, therefore $f \equiv 0$ on $[0, 1]$. The proof is complete.

**4.4. Proof of Theorem 2.4.** According to (1.2) and (1.1),

$$\Delta_U(\lambda) = \lambda^3 \Psi_1(\lambda) + \Psi_2(\lambda) + 1, \tag{4.18}$$

$$\Psi_1(\lambda) = (c, u_1)(s_1, u_2) - (c, u_2)(s_1, u_1), \tag{4.19}$$

$$\Psi_2(\lambda) = \lambda^2 (c, u_1) + \lambda (s_1, u_2),$$

where $s_1 = \lambda s$. Since

$$c = \frac{(e + e_1)}{2}, \quad s = \frac{(e - e_1)}{2i\lambda}, \quad e_1(x, \lambda) := e(x, -\lambda),$$

we get

$$s_1(x, \lambda) = \sin \lambda x + \int_0^x \mathcal{K}_\infty(x, t) \sin \lambda t dt, \tag{4.20}$$

$$c(x, \lambda) = \cos \lambda x + \int_0^x \mathcal{K}_0(x, t) \cos \lambda t dt, \tag{4.21}$$

$$\mathcal{K}_\infty(x, t) = \mathcal{K}(x, t) - \mathcal{K}(x, -t), \quad \mathcal{K}_0(x, t) = \mathcal{K}(x, t) + \mathcal{K}(x, -t).$$

Let $K_\infty$ and $K_0$ be the integral operators in the right hand sides of the identities (4.20) and (4.21), respectively. By (2.12)

$$(K_0 f, g) = (f, F_1 g), \quad (K_\infty f, g) = (f, F_2 g), \quad f, g \in L_2(0, 1).$$

Then

$$\begin{aligned} \Psi_1(\lambda) &= \frac{1}{2i} \left((e, u_2)(e_1, u_1) - (e, u_1)(e_1, u_2)\right) \\ &= \frac{1}{2i} \left((e_0^-, (I + K_-)u_1)(e_0^+, (I + K_+)u_2) - (e_0^+, (I + K_+)u_1)(e_0^-, (I + K_-)u_2)\right), \\ \Psi_2(\lambda) &= \lambda^2 (c_0, (I + F_1)u_1) + \lambda (s_0, (I + F_2)u_2), \\ c_0 &= \cos \lambda x, \quad s_0 = \sin \lambda x, \quad e_0^\pm = e^{\pm i\lambda x}. \end{aligned} \tag{4.22}$$

Direct calculations show that

$$(e_0^+, f)(e_0^-, g) = \int_{-1}^{1} e^{i\lambda x} C[f,g](x)dx, \qquad C[f,g](x) = \int_{\max\{0,x\}}^{\min\{1,1+x\}} f(t)g(t-x)dt.$$

This yields

$$\Psi_1(\lambda) = \frac{1}{2i}\int_{-1}^{1} e^{i\lambda x} D(x)dx,$$
$$D(x) = C[(I+K_+)u_2, (I+K_-)u_1](x) - C[(I+K_+)u_1, (I+K_-)u_2](x).$$

Up to a multiplicative constant, the function $D$ coincides with the Fourier transform of the function $\Psi_1$, which, according to (4.19), is odd. This is why $D$ is also odd and on $[0,1]$ it coincides with $A_1$, and

$$\Psi_1(\lambda) = \int_0^1 \sin\lambda x A_1(x)dx. \tag{4.23}$$

The integration by parts allows us to rewrite the expression (4.22) as

$$\Psi_2(\lambda) = -\lambda^3 \int_0^1 \sin\lambda x\,(A_2(x) + A_3(x))\,dx + \lambda^2(A_2(0) + A_3(0)). \tag{4.24}$$

Substituting (4.23) and (4.24) into (4.18), we obtain (2.13).

Corollaries 2.3 and 2.4 are implied directly by Lemma 2.2 and Theorems 2.2 — 2.4.

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