ISSN 2304-0122 Ufa Mathematical Journal. Vol. 17. No 4 (2025). P. 23-36.

doi:10.13108/2025-17-4-23

EQUIVALENT CONDITIONS OF STRONG INCOMPLETENESS OF EXPONENTIAL SYSTEM

A.M. GAISIN, R.A. GAISIN

Abstract. We study interpolating sequences in the Pavlov — Korevaar — Dixon sense (Ω–interpolation sequences) and generalizations, as well as the approximative properties of exponential systems with corresponding exponents. For instance, the interpolation problem is of interest in the class of entire functions of exponential type determined by some growing majorant in the convergent class (non–quasianalytic weight). In a narrower class, when the majorant possessed the concavity property, a similar problem was completely solved Berndtsson, but in the case when the interpolation nodes are natural numbers. He obtained the solvability criterion of this interpolation problem. The corresponding criterion for an arbitrary increasing sequence of positive nodes was recently obtained by R.A. Gaisin. In 2021 he also proved the criterion of the interpolation (W–interpolating) in the case of an arbitrary non–quasianalytic weight. As in works by A.I. Pavlov, J. Korevaar and M. Dixon, we found a close relation between the interpolation property of sequences and Macintyre problem. It was also shown that if the sequence of real numbers is Ω–interpolating, then the corresponding exponential system is strongly incomplete (minimal) with respect to the rectangles; in the case of the W–interpolation property the strong incompleteness (minimality) holds with respect to the vertical strips. However the conditions of Ω–interpolation property proposed by A.M. Gaisin in 1991 were a bit unsatisfactory since there were not visual enough.

In the present paper in the terms of weight concentration index we obtain the required conditions for strong incompleteness (minimality) of exponential system with respect to the rectangles.

Keywords: interpolation sequence, strongly incomplete (minimal) exponential systems, weight concentration index, Macintyre sequence.

Mathematics Subject Classification: 30E05, 30E10

1. Introduction

The paper is devoted to generalizations of series of results by J. Korevaar and M. Dixon on interpolation sequences from the survey [18].

Let $\Lambda = \{\lambda_n\}$, $0 < \lambda_n \uparrow \infty$. If $\Lambda \subset \mathbb{N}$, then instead of Λ we shall employ the notation P, $P = \{p_n\}$, where $p_n \in \mathbb{N}$, $p_n \uparrow \infty$. In what follows $S(P) = \{z^{p_n}\}$ is the system of exponents.

We introduce the following classes of sequences $P = \{p_n\}$:

CC is the convergence class. This a class of sequences P possessing Fejer lacunas, that is,

$$\sum_{n=1}^{\infty} \frac{1}{p_n} < \infty. \tag{1.1}$$

A.M. Gaisin, R.A. Gaisin, Equivalent conditions of strong incompleteness of exponential system.

© Gaisin A.M., Gaisin R.A. 2025.

The work of A.M. Gaisin (Section 1) is made in the framework of State Task of Ministry of Science and Higher Education of Russian Federation (code FMRS-2025-0010). The work of A.R. Gaisin (Section 2, 3) is supported by the Russian Science Foundation (grant no. 25-21-00044).

Submitted May 23, 2025.

M (Macintyre sequences) is the class of sequences P, for which each transcendental function

$$f(z)=\sum_{n=1}^{\infty}a_n z^{p_n} \tag{1.2}$$

is unbounded on each curve going to infinity.

Macintyre showed that $M\subset CC$ [21]. In [19] there were studied strongly free (minimal according to L. Schwartz) and strongly incomplete systems of exponents $S(P)$.

The system of exponents $\{z^{p_n}\}$ is called *strongly free (minimal)* if for all $a>1$, $k=1,2,\dots,$

$$\inf_{\gamma_a}\inf_{c}\left\|z^{p_k}-\sum_{n\neq k}{}' c_n z^{p_n}\right\|_{\gamma_a}=\delta_k(a)>0, \tag{1.3}$$

where $\sum\limits_{n\neq k}{}'$ is a polynomial, γ_a is a curve connecting the circumferences $C(0,1)=\{z:|z|=1\}$ and $C(0,a)=\{z:|z|=a\}$, $\|g\|_{\gamma_a}=\max\limits_{z\in\gamma_a}|g(z)|$, where the internal infimum is taken over all finite sums $\sum\limits_{n\neq k}{}'$, while the external is taken over all curves γ_a [18].

We note that in this definition instead of $C(0,1)$ we can consider an arbitrary circumference $C(0,a')$, $0<a'<a$.

The system of exponents $\{z^{p_n}\}$ is called *strongly incomplete* if for each $\nu\in\mathbb{N}\setminus P$

$$\inf_{\gamma_a}\inf_{c}\left\|z^{\nu}-\sum_{n}{}' c_n z^{p_n}\right\|_{\gamma_a}=\varepsilon_\nu(a)>0. \tag{1.4}$$

Here $\sum\limits_{n}{}'$ is also a linear combination of exponents.

The class of sequences P obeying the relation (1.3) is denoted by PSF (we use the same notation as in [18]). If the condition (1.4) holds, then the corresponding class of sequences P is denoted by PSN. As it was shown in [19], $PSF\subset M$, $PSN\subset M$.

Let W be the class of positive unboundedly growing and continuous on $\mathbb{R}_+$ functions w such that

$$\int_1^{\infty}\frac{w(x)}{x^2}dx<\infty. \tag{1.5}$$

The convergence of series (1.1) is equivalent to the fact that the function $n_P(t)t^{-2}$ belongs to $L^1(\mathbb{R}_+)$, where $n_P(t)=\sum\limits_{p_n\leqslant t}1$. Hence, $P\in CC$ if and only if there exists a function $w\in W$ such that $n_P(t)\leqslant w(t)$. The set W is usually called the convergence class.

We introduce extra two classes of functions:

$$\Omega=\{\omega\in W:\omega \text{ is concave}\},\qquad \Omega_0=\left\{\omega\in W:\frac{\omega(t)}{t}\downarrow\ \text{as } t\uparrow\infty\right\}.$$

It is clear that $\Omega\subset\Omega_0$. On the other hand, for each function $\omega\in\Omega_0$ the estimates hold

$$\omega(t)\leqslant m_\omega(t)\leqslant 2\omega(t).$$

Here $m_\omega(t)$ is the minimal concave majorant of the function $\omega(t)$ see [17, Ch. VII, Sect. 2]. From this point of view the classes Ω and Ω_0 can be treated as in fact coinciding; we shall make sure later.

We provide one more important definition, which was introduced in [19].

A sequence P is called *interpolating (in the sense of Pavlov — Korevaar — Dixon or Ω_0–interpolating)* if there exists a function $\omega_P=\omega_P(r)$, $0<\omega_P(r)\uparrow\infty$ as $r\to\infty$, belonging to the class Ω_0, such that for each sequence $\{b_n\}$, $b_n \in \mathbb{C}$, $|b_n| \leqslant 1$, there exists an entire function $g(z)$ with the properties:

$$\begin{aligned} &1)\ g(p_n) = b_n, \quad n = 1, 2, \dots; \\ &2)\ M_g(r) = \max_{|z|=r} |g(z)| \leqslant e^{\omega_P(r)}. \end{aligned} \tag{1.6}$$

Following [18], we denote the class of interpolating sequences by I. It was shown in [18] that $I \subset PSF$ and $I \subset PSN$. Thus,

$$I \subset PSF \subset M \subset CC.$$

Open question (Macintyre problem). *Whether the identity $M = CC$ is valid [21]?*

We note that in the above chain of inclusions this problem has a special place, for more detail see [2, Ch. I, Sect. 1], where the regular growth of series (1.2) is studied as well.

The interpolation issues for the sequence $P_0 = \{p_n\}$ $(p_0 = 0, p_n \in \mathbb{N}, n \geqslant 1)$, and for the sequence $\{\pm p_n\}$ where somehow studied in [18], [19], [20] by Korevaar and Dixon. Howerver, these authors did not describe the class I, and the interpolation was proved only for the Pavlov and Kovari sequences [19]. The matter is that they succeeded to construct the interpolating function as a series of Lagrange type [19]

$$g(z) = \sum_{|k|=0}^{\infty} b_k \frac{Q(z)}{Q'(p_k)(z - p_k)} \left(\frac{z}{p_k}\right)^{2km_k}, \qquad p_{-k} = -p_k,$$

where

$$Q(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{p_n^2}\right),$$

and the natural numbers m_k are to be chosen in a special way. In the general case, a series of Lagrange type is not suitable. Because of this, the authors of [18], [19], [20] limited themselves to studying specific examples. Somewhat later, in [16], Berndson obtained an interpolation criterion for the sequence $P = \{p_n\}$. We present this result in the next section.

The concept of interpolation sequences is easily extended to arbitrary sequences $\Lambda = \{\lambda_n\}$, $0 < \lambda_n \uparrow \infty$ [1]. In what follows we shall show that the interpolation criterion for this sequence coincides with the Berndson's interpolation criterion for $P = \{p_n\}$. A further generalization of the interpolation property of the sequence $\Lambda = \{\lambda_n\}$ was obtained in [8], and for symmetric sequences in [9]. Unlike the Pavlov — Korevar — Dixon type interpolation problem, in [8], [9] the majorant ω_Λ for $\ln M_g(r)$ for the interpolating function $g(\lambda)$ in a problem of the type (1.6) need not be, say, concave; it belongs only to the convergence class W. In other words, $\omega_\Lambda(r)$ is simply a non–quasianalytic weight.

In [10] the concept of so–called generalized interpolation was introduced. This will be discussed in more detail in the next section.

2. Results on interpolating sequences

In [16] Berndtsson proved the following criterion.

Theorem 2.1. *The sequence $P = \{p_n\}$ is interpolating in the sense of Pavlov — Korevaar — Dixon if and only if there exists a function $\omega_P \in \Omega_0$ such that*

$$\begin{aligned} &a) \quad n_P(p_n) \leqslant \omega_P(p_n), \quad n = 1, 2, \dots, \qquad n_P(t) = \sum_{p_n \leqslant t} 1; \\ &b) \quad -\ln \prod_{\substack{k \neq n \\ \frac{p_n}{2} \leqslant p_k \leqslant 2p_n}} \left|1 - \frac{p_n}{p_k}\right| \leqslant \omega_P(p_n), \quad n = 1, 2, \dots. \end{aligned} \tag{2.1}$$

As it has been said already, this theorem is completely extended to arbitrary real sequences of numbers λ_n, $0 < \lambda_n \uparrow \infty$. This fact (see Theorem 2.3 below) will be employed in Section 3, where we shall discuss the strong incompleteness of exponential system. Now we briefly recall other generalizations of the Berndtsson theorem.

The sequence $\Lambda = \{\lambda_n\}$ is called *interpolating* (*W–interpolating*) if there exists a function $w_\Lambda \in W$ depending only on Λ such that for each sequence $\{b_n\}$, $|b_n| \leqslant 1$, there exists an entire function $f(\lambda)$ with the properties

$$\begin{aligned} &1) \quad f(\lambda_n) = b_n, \quad n = 1, 2, \dots; \\ &2) \quad M_f(r) \leqslant e^{w_\Lambda(r)}. \end{aligned} \tag{2.2}$$

Thus, if a sequence Λ is interpolating in the sense of Pavlov — Korevaar — Dixon, then it is W–interpolating.

The criterion of W–interpolation was proved [8], where a modified Berndtsson method was employed, which was based on an idea by Hörmander for solving a $\overline{\partial}$–problem in the multidimensional complex analysis. We note that almost simultaneously this method was also used in [15].

Taking into consideration the estimate [8, Lm. 3]

$$\begin{gathered} \left| -\ln \prod_{\substack{k \neq n \\ \frac{\lambda_n}{2} \leqslant \lambda_k \leqslant 2\lambda_n}} \left| 1 - \frac{\lambda_n}{\lambda_k} \right| - \int_0^{\lambda_n} \frac{\nu(\lambda_n;\, t)}{t} dt \right| \\ \leqslant n_\Lambda(2\lambda_n) + N(2\lambda_n) + \ln M_L(\lambda_n), \quad n = 1, 2, \dots, \end{gathered} \tag{2.3}$$

where $\nu(\lambda_n; t)$ is the number of points $\lambda_k \neq \lambda_n$ in the segment $\{h : |h - \lambda_n| \leqslant t\}$,

$$n_\Lambda(t) = \sum_{\lambda_n \leqslant t} 1, \qquad t > 0,$$

$$N(t) = \int_0^t \frac{n_\Lambda(x)}{x} dx, \qquad L(\lambda) = \prod_{n=1}^{\infty} \left(1 - \frac{\lambda^2}{\lambda_n^2}\right), \tag{2.4}$$

we rewrite the interpolation criterion for the sequence Λ from [8] as follows; for the symmetric sequence $\{\pm\lambda_n\}$ the criterion is the same, see [9].

Theorem 2.2. *The sequence Λ is W–interpolating if and only if there exists a function $w_\Lambda \in W$ such that*

$$A. \ \sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty; \qquad B. \ \int_0^{\lambda_n} \frac{\nu(\lambda_n;\, t)}{t} dt \leqslant w_\Lambda(\lambda_n), \quad n = 1, 2, \dots .$$

Here we have taken into consideration that under the condition A

$$\int_1^{\infty} \frac{n_\Lambda(t)}{t^2} dt < \infty,$$

and the functions $N(t)$ and $\ln M_L(t)$, $t > 0$, belong to the class W, see [13, Ch. I, Sect. 1, Subsect. 3], [5, Sect. 2, Subsect. 2.4].

We note that the estimates (2.3) and the conditions A, B imply

$$\ln \frac{1}{h_n} \leqslant w_0(\lambda_n), \quad n = 1, 2, \dots, \tag{2.5}$$

where $h_n = \min\left(\min\limits_{k\neq n}|\lambda_n - \lambda_k|, 1\right)$, w_0 is some function in W.

Thus, for a W–interpolating sequence Λ we necessarily have

$$h_n \geqslant e^{-w_0(\lambda_n)}, \qquad n = 1, 2, \dots, \qquad w_0 \in W.$$

We shortly dwell on one more extension of W–interpolation.

Let $\beta = \beta(t)$ be some fixed function in the class W. The sequence $\Lambda = \{\lambda_n\}$ is called *W–interpolating in the wide sense* if there exists a function $w_\Lambda \in W$ depending on the function $\beta(t)$ and the sequence Λ such that for each sequence of complex numbers b_n, $|b_n| \leqslant e^{\beta(\lambda_n)}$, $n = 1, 2, \dots$, there exists an entire function $f(\lambda)$, which possesses the properties (2.2).

The corresponding problem (2.2) is called the *generalized interpolation problem*. The criterion of generalized interpolation is the same as in Theorem 2.2, see [10].

Berndtsson Theorem 2.1 admits a generalization for the case of arbitrary nodes $\lambda_n > 0$[1]. We formulate this result in more convenient terms.

Theorem 2.3. *A sequence $\Lambda = \{\lambda_n\}$, $0 < \lambda_n \uparrow \infty$, is interpolating in the sense of Pavlov – Korevaar – Dixon if and only if there exists a function $\omega_\Lambda \in \Omega_0$ such that*

$$C.\quad n_\Lambda(t) \leqslant \omega_\Lambda(t);$$

$$D.\quad \int\limits_0^{\lambda_n} \frac{\nu(\lambda_n;\, t)}{t}dt \leqslant \omega_\Lambda(\lambda_n), \quad n = 1, 2, \dots.$$

3. Strong incompleteness of exponential system

The notion of strong incompleteness of system of exponents $S(P) = \{z^{p_n}\}_{n=1}^\infty$ was first introduced by Korevaar and Dixon in [19]; in [3] this notion was extended to the system of exponents $e_\Lambda = \{e^{\lambda_n z}\}$, $0 < \lambda_n \uparrow \infty$, and later in [9] to the system $\{e^{\pm\lambda_n z}\}$.

The system of exponents $\{e^{\pm\lambda_n z}\}$ is called *strongly incomplete (with respect to rectangles)* if for all a, b $(0 < a < \infty,\ 0 < b < \infty)$ and β, $\beta \neq \pm\lambda_n$, $n = 1, 2, \dots$,

$$\inf_{\gamma(-a,a)} \inf_{c_n} \left\| e^{\beta z} - \sum_{n\in\mathbb{Z}\setminus\{0\}} c_n e^{\mu_n z} \right\|_{\gamma(-a,a)} = \varepsilon_\beta(a, b) > 0.$$

Here $\|g\|_\gamma = \max\limits_{z\in\gamma}|g(z)|$, the internal infimum is taken over all quasipolynomials

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} c_n e^{\mu_n z}, \qquad \mu_n = \lambda_n, \qquad \mu_{-n} = -\lambda_n, \qquad n \in \mathbb{N};$$

the external infimum is taken over all rectified curves $\gamma = \gamma(-a, a)$ in the rectangle

$$P(a, b) = \{z = x + iy : |x| \leqslant a, |y| < b\},$$

connecting its vertical sides.

For the system $\{e^{\lambda_n z}\}$ a similar notion was considered in [3].

In [9] the following theorem was proved.

Theorem 3.1. *Let the conditions be satisfied:*

$$1)\ \sum_{n=1}^\infty \frac{1}{\lambda_n} < \infty; \qquad 2)\ \int\limits_0^{\lambda_n} \frac{\nu(\lambda_n;\, t)}{t}dt \leqslant w(\lambda_n), \quad n = 1, 2, \dots, \tag{3.1}$$

[1]The proof of this result will be published in a separate paper.

where $\nu(\lambda_n;\ t)$ is the number of points $\lambda_k \neq \lambda_n$ in the segment $\{h : |h-\lambda_n| \leqslant t\}$, and w is some function in the convergence class W.

Then the system of exponents $\{e^{\pm\lambda_n z}\}$ is strongly incomplete with respect to the vertical strips $P(a,\infty)$[1].

As it is known, the pair of conditions 1) and 2) in (3.1) is equivalent to conditions 1) and 3) or 1) and 4), where

$$3)\ \ -\ln \prod_{\substack{k\neq n\\ \frac{\lambda_n}{2}\leqslant\lambda_k\leqslant 2\lambda_n}} \left|1-\frac{\lambda_n}{\lambda_k}\right| \leqslant w(\lambda_n), \quad n=1,2,\ldots, \quad w\in W;$$

$$4)\ \ -\ln|L'(\lambda_n)| \leqslant w(\lambda_n), \quad n=1,2,\ldots, \quad w\in W.$$

This is why each of three equivalent pairs of conditions 1) and 2), 1) and 3), 1) and 4) is necessary and sufficient for the sequence $M=\{\mu_{\pm n}\}$, $\mu_n=\lambda_n$, $\mu_{-n}=-\lambda_n$, $n\in\mathbb{N}$ to be W–interpolating, see [9].

Thus, for each interpolating sequence $M=\{\pm\lambda_n\}$ the corresponding system of exponentials $\{e^{\pm\lambda_n z}\}$ is strongly incomplete on the family of rectified curves $\gamma(-a,a)$ in $P(a,\infty)$ with respect to the uniform norm. In particular, under the assumptions of Theorem 3.1, the system of exponentials $\{e^{\pm\lambda_n z}\}$ is incomplete on each rectified curve γ, that is, it is incomplete in the space of continuous functions $C(\gamma)$. This means that there exists a non–zero complex Borel measure μ on γ, the Laplace transform of which

$$\hat{\mu}(s)=\int\limits_{\gamma} e^{sz}d\mu(z)$$

vanishes at the points $\pm\lambda_n$, $n=1,2,\ldots$. It is important to note that the assumptions of Theorem 3.1 are formulated in terms of main distribution characteristics of the points of sequence $\{\pm\lambda_n\}$.

In this regard, we pay an attention to the corresponding result in [3] on strong incompleteness with respect to rectangles $P(a,b)$, $b\neq\infty$. In this result, the dependence of the conditions for strong incompleteness of the exponentials system of known distribution characteristics of the points λ_n, $n=1,2,\ldots$, is not at all obvious. Indeed, the following statement was proved in [3].

Theorem 3.2. *Let $n=o(\lambda_n)$ as $n\to\infty$, $h(\delta)=h_-(\delta)h_+(\delta)$, where*

$$h_+(\delta)=\int\limits_0^\infty |L(ir)|e^{-\delta r}dr, \qquad h_-(\delta)=\int\limits_0^\infty |L(re^{i\delta})|^{-1}e^{-\delta r}dr, \quad \delta>0.$$

If the function $h(\delta)$ satisfies the Levinson bilogarithmic condition

$$\int\limits_0^d \ln\ln h(\delta)d\delta<\infty, \quad h(d)=e, \tag{3.2}$$

then the exponential system $\{e^{\lambda_n z}\}$ is strongly incomplete with respect to the rectangles.

The functions $h_+(\delta)$ and $h_-(\delta)$ decrease on $(0,\infty)$, $h_+(\delta)\uparrow\infty$, $h_-(\delta)\uparrow\infty$ as $\delta\downarrow 0$ [3]. It is also well–known that the condition (3.2) is equivalent to the Levinson condition for each of the

[1]The strong incompleteness of the system $\{e^{\pm\lambda_n z}\}$ with respect to the strip $P(a,\infty)$ is formally understood as the strong incompleteness with respect the rectangle $P(a,b)$, for which $b=\infty$.

functions $h_+(\delta)$ and $h_-(\delta)$; in what follows, the condition

$$\int\limits_0^{d_+} \ln\ln h_+(\delta)d\delta < \infty, \qquad h_+(d_+) \geqslant e, \tag{3.3}$$

is equivalent to the condition C of Theorem 2.3, see [3], [4][1].

Our goal is to decipher the condition

$$\int\limits_0^{d_-} \ln\ln h_-(\delta)d\delta < \infty, \quad h_-(d_-) \geqslant e, \tag{3.4}$$

and transform it in a more understandable and natural form, taking into account the explicit dependence on the sequence Λ. To do this, we first find out under what conditions the sequence $\Lambda = \{\lambda_n\}$ obeys

$$\sup_{\theta\neq 0,\pi} \varlimsup_{r\to\infty} \frac{1}{\omega(r)} \ln\left|\frac{1}{L(re^{i\theta})}\right| < \infty,$$

where $\omega(r)$ is some majorant of the function $\ln M_L(r)$, which belongs to the class Ω and satisfies

$$0 < \varlimsup_{r\to\infty} \frac{\ln M_L(r)}{\omega(r)} < \infty.$$

For the case $\omega(r) \equiv r$ this question was posed by A.F. Leontiev in 1956 in relation with the problem on continuation of convergence of an arbitrary sequence of Dirichlet polynomials

$$P_n(z) = \sum_{k=1}^{q_n} a_k^{(n)} e^{\lambda_k z}, \qquad n = 1, 2, \dots,$$

which converges uniformly in some domain, where the exponential system $\{e^{\lambda_k z}\}$ is incomplete, into the half–plane $\{z : \operatorname{Re} z < c\}$ (the same for all sequences $\{P_n(z)\}$) [12].

In [11] Krasichkov gave an answer to the question by A.F. Leontiev. Krasichkov considered the case

$$\omega(r) = V(r),$$

where $V(r) = r^{\rho(r)}$, $\rho(r)$ is the proximate order, $\rho(r) \to \rho$, $\rho > 0$ (in our case $0 < \rho \leqslant 1$), such that

$$0 < \varlimsup_{r\to\infty} \frac{\ln M_L(r)}{V(r)} < \infty.$$

In [11] a necessary and sufficient condition for the sequence $\Lambda = \{\lambda_n\}$ was obtained, which ensured $\sup\limits_{\theta\neq 0,\pi} H_L(\theta) < \infty$, where

$$H_L(\theta) = \varlimsup_{r\to\infty} \frac{1}{V(r)} \ln\left|\frac{1}{L(re^{i\theta})}\right|, \qquad V(r) = r^{\rho(r)},$$

is the indicator of the function $|L(\lambda)|^{-1}$ at the proximate order $\rho(r)$. Our aim is to replace, in this result, the function of exact growth $V(r) = r^{\rho(r)}$ by an appropriate function $\omega(r)$, $\omega \in \Omega_0$. As we can guess, this would given an opportunity to formulate the convergence the integral (3.4) of the iterated logarithm of the function $h_-(\delta)$ in terms of the so–called weight concentration index (ω–concentration) of the sequence Λ. As we see, such formulation of the problem is topical, especially since the answer cannot be obtained as a simple consequence of Krasichkov's result mentioned in [11]. The point is that the function $\omega(r)$, $\omega \in \Omega_0$, unlike $V(r)$,

[1]In the condition C as the function $\omega_\Lambda(t)$ we can take the smallest concave majorant of the function $n_\Lambda(t)$ [4].

is not required to be regularly varying at infinity [14]. And in [11] this fact is essentially used, namely the properties of proximate order in evaluating integrals, see [11].

In [6] the following theorem on the finiteness of the weight indicator for the function $|L(\lambda)|^{-1}$, where $L(\lambda)$ is the entire function of exponential type in (2.4).

Theorem 3.3 ([6]). *Let the smallest concave majorant ω of the function $\ln M_L(r)$ belong to the convergence class*[1] *W,*

$$H_\omega(\theta) = \varlimsup_{r\to\infty} \frac{1}{\omega(r)} \ln \left| \frac{1}{L(re^{i\theta})} \right| \tag{3.5}$$

be the weight indicator of the function $|L(\lambda)|^{-1}$, $\lambda = re^{i\theta}$.

Then

$$\sup_{\theta\neq 0,\pi} H_\omega(\theta) < \infty, \tag{3.6}$$

if and only if the weight concentration index

$$I_\Lambda(\omega, \mathbb{R}_+) = \lim_{\varepsilon\to 0} \varlimsup_{x\to+\infty} \frac{1}{\omega(x)} \int\limits_{\varepsilon}^{1} \frac{n_\sigma(x)}{\sigma} d\sigma, \tag{3.7}$$

is finite, where $n_\sigma(x)$ is the number of points λ_n in the circle $\Delta_\sigma(x) = \{t : |t - x| \leqslant \sigma|x|\}$, $x \in \mathbb{R}$.

As it has been said, the case $\omega(r) = V(r)$, $V(r) = r^{\rho(r)}$, $\rho(r)$ is the proximate order $\rho(r) \to \rho > 0$, was studied in [11].

This result is based on the following fact, which was also essentially employed in [11] for arbitrary entire functions of proximate order $\rho(r)$, $\rho(r) \to \rho$, $0 < \rho < \infty$.

Under the assumptions of Theorem 3.3 the representation holds [6]: for all $\lambda \neq 0$

$$\ln |L(\lambda)| = -\int\limits_0^1 \frac{n_\sigma(\lambda)}{\sigma} d\sigma + R(|\lambda|), \tag{3.8}$$

where $R(|\lambda|) = O(1)\omega(|\lambda|)$, and $O(1)$ is some function bounded outside each circle $\{z : |z| \leqslant \rho\}$, $\rho > 0$; the function $n_\sigma(\lambda)$ for complex λ is defined in the same way.

As it was shown in [6],

$$|R(|\lambda|)| \leqslant A_0 + A_1\omega(|\lambda|), \qquad |\lambda| \geqslant 0.$$

This is why, Theorem 3.3 and the representation (3.8) imply the statement: if $\omega(r)$ is the smallest concave majorant of the function $\ln M_L(r)$, then the condition (3.6) is equivalent to the condition

$$\sup_{\theta\neq 0,\pi} \varlimsup_{r\to\infty} \frac{1}{\omega(r)} \int\limits_0^1 \frac{n_\sigma(re^{i\theta})}{\sigma} d\sigma < \infty. \tag{3.9}$$

As in Theorem 3.3, we suppose that $\omega \in \Omega$.

Let us estimate the function $h_-(\delta)$ from above and below. We denote

$$I(\lambda) = \int\limits_0^1 \frac{n_\sigma(\lambda)}{\sigma} d\sigma, \quad \lambda = re^{i\delta}.$$

Then, using the identity (3.8), we get

$$h_-(\delta) \leqslant h^* \left(\frac{\delta}{2}\right) e^{A_0 + m\left(\frac{\delta}{2}\right)}, \tag{3.10}$$

[1]This is obviouslyt equivalent to the condition (3.3), that is, the condition C of Theorem 2.3.

where

$$m(\xi) = \sup_{r>0}(A_1\omega(r) - \xi r), \qquad \xi > 0,$$

$$h^*(\xi) = \int_0^\infty \exp(I(\lambda) - \xi r)dr, \qquad \lambda = re^{i\delta}.$$

By (3.10) for $\delta \leqslant \delta_0$ we obtain the inequality

$$\ln h_-(\delta) \leqslant \ln h^*\left(\frac{\delta}{2}\right) + 2m\left(\frac{\delta}{2}\right).$$

Using the elementary inequality

$$\ln^+(a+b) \leqslant \ln^+ a + \ln^+ b + \ln 2, \qquad a > 0, \quad b > 0,$$

we find

$$\ln\ln h_-(\delta) \leqslant \ln\ln h^*\left(\frac{\delta}{2}\right) + 2\ln 2 + \ln m\left(\frac{\delta}{2}\right), \quad 0 < \delta \leqslant \delta_1 < \delta_0.$$

Since $\omega \in \Omega$, we have [19]

$$\int_0^{d_0} \ln m(\xi)d\xi < \infty, \quad m(d_0) \geqslant 1.$$

This is why the convergence of integral (3.4) for the function $h^*(\xi)$ implies the convergence of the same integral for $h_-(\xi)$.

On the other hand, as it is easy to confirm,

$$h_-(\delta) \geqslant h^*(2\delta)e^{-A_0 - m(\delta)},$$

that is,

$$h_-(\delta)e^{A_0 + m(\delta)} \geqslant h^*(2\delta).$$

The same arguing shows that if $h_-(\xi)$ satisfies the Levinson condition (3.4), then the function $h^*(\xi)$ does the same condition.

Thus, we obtain the following statement.

Theorem 3.4. *Let the smallest concave majorant of the function $\ln M_L(r)$ belong to the class W. Then the following statements hold:*

I. *The integrals*

$$\int_0^{d_-} \ln\ln h_-(\delta)d\delta, \qquad \int_0^{d^*} \ln\ln h^*(\delta)d\delta$$

are equiconvergent; the functions $h_-(\delta)$ and $h^*(\delta)$ were defined above.

II. *The equivalent conditions (3.6) and (3.9) hold if and only if $I_\Lambda(\omega, \mathbb{R}_+) < \infty$; here $I_\Lambda(\omega, \mathbb{R}_+)$ is the weight concentration index of sequence Λ defined by the formula.*

Remark 3.1. *We note that*

$$\int_0^1 \frac{n_\sigma(z)}{\sigma}d\sigma = \int_0^{|z|} \frac{\mu(z;t)}{t}dt,$$

where $\mu(z;t)$ *is the number of points* $\lambda \in \Lambda$ *in the circle* $\{h : |z-h| \leqslant |t|\}$. *Indeed, we make that change* $t = \sigma z$ *and get*

$$\int_0^{|z|} \frac{\mu(z;t)}{t} dt = \int_0^1 \frac{\mu(z;\sigma|z|)}{\sigma} d\sigma = \int_0^1 \frac{n_\sigma(z)}{\sigma} d\sigma.$$

Remark 3.2. *The condition (3.2) (the sufficient condition of strong incompleteness of the system* $\{e^{\lambda_n z}\}$ *with respect to rectangles) is equivalent to*

$$1^0. \quad n_\Lambda(t) \leqslant \omega_\Lambda(t), \qquad \omega_\Lambda \in \Omega_0;$$

$$2^0. \quad \int_0^{d^*} \ln\ln h^*(\delta) d\delta < \infty, \qquad h^*(d^*) \geqslant e.$$

However, the pairs of conditions 1^0 and 2^0, 1) and 2) in Theorem 3.1, the sufficient conditions of strong incompleteness of the exponential system $\{e^{\pm\lambda_n z}\}$ and hence, of the system $\{e^{\lambda_n z}\}$ with respect to vertical strips, are independent. Indeed, we consider the system of segments $\{\Delta_j\}$, where

$$\Delta_j = \left[2^{j^2} - \left[\frac{2^{j^2}}{j^2}\right], 2^{j^2}\right], \qquad j \geqslant 1,$$

where $[a]$ is the integer part of a. Let $\Lambda = \{\lambda_n\}$ be an increasing sequence of all natural numbers in $\bigcup_{j\geqslant 1} \Delta_j$. This sequence Λ obeys conditions 1) and 2) of Theorem 3.1 but [7]

$$\int_0^{d_+} \ln\ln h_+(\delta) d\delta = \infty.$$

On the other hand, let $\Lambda = \{\lambda_n\}$ be the union of two sequences $\{p_n\}$ and $\{q_n\}$, where $\{p_n\}$, $p_n \in \mathbb{N}$, is an interpolating in the sense of Pavlov — Korevaar — Dixon sequence, $q_n = p_n + \exp(-p_n \ln p_n)$. This sequence satisfies the conditions 1^0 and 2^0, and the bilogarithmic condition for the function $h(\delta) = h_+(\delta)h_-(\delta)$ obviously holds. However, the condensation index satisfies

$$\delta(\Lambda) = \overline{\lim_{n\to\infty}} \frac{1}{\lambda_n} \ln\left|\frac{1}{L'(\lambda_n)}\right| = \infty,$$

and the condition 2) of Theorem 3.1 fails [3].

We proceed to the next theorem; for natural numbers λ_n the corresponding statement was proved in [3].

Theorem 3.5. *Let* $\Lambda = \{\lambda_n\}$, $0 < \lambda_n \uparrow \infty$, *be an interpolating in the sense of Pavlov — Korevaar — Dixon sequence. Then the function* $h(\delta) = h_+(\delta)h_-(\delta)$ *satisfies the Levinson condition (3.2).*

Proof. According to Theorem 2.3 we have $n_\Lambda(t) \leqslant \omega_\Lambda(t)$, $\omega_\Lambda \in \Omega_0$. Hence, the integral (3.3) converges [4]. In the proof of the sufficiency in Theorem 2.3 (the criterion of interpolation in sense of Pavlov — Korevaar — Dixon for arbitrary real nodes $\lambda_n > 0$) we have shown that for all $z \in K_n$, $n = 1, 2, \ldots$,

$$K_n = \left\{z : \frac{h_n}{4} \leqslant |z-\lambda_n| \leqslant \frac{h_n}{2}\right\}, \quad h_n = \min\left(\min_{k\neq n} |\lambda_k - \lambda_n|, 1\right),$$

the estimate

$$\left|\frac{1}{L(z)}\right| \leqslant e^{\omega_1(\lambda_n)}, \quad n = 1, 2, \ldots, \quad \omega_1 \in \Omega_0 \tag{3.11}$$

holds.

Let $z = re^{i\delta}$, $0 < \delta < \frac{\pi}{4}$, belong to the circle $D_n = \left\{z : |z - \lambda_n| \leqslant \frac{h_n}{4}\right\}$. Then

$$\left|\frac{z-\lambda_n}{L(z)}\right| \leqslant \max_{t\in C_n}\left|\frac{t-\lambda_n}{L(t)}\right| \leqslant \frac{h_n}{4}e^{\omega_1(\lambda_n)} \leqslant \frac{1}{4}e^{\omega_1(\lambda_n)},$$

$C_n = \partial D_n$, $n = 1, 2, \dots$ Since $|z - \lambda_n| \geqslant \lambda_n \sin\delta \geqslant \frac{2\delta}{\pi}$, for $z \in D_n$, $z = re^{i\delta}$, we have

$$\left|\frac{1}{L(z)}\right| \leqslant \frac{\pi}{8\lambda_1\delta}e^{\omega_1(\lambda_n)} \leqslant \frac{1}{\lambda_1\delta}e^{\omega_1(r+1)}. \tag{3.12}$$

Since $|L(re^{i\delta})|^{-1} \uparrow$ as $\delta \downarrow$ that can be verified straightforwardly, and $\omega_1 \in \Omega_0$, in view of (3.11), (3.12) we obtain that for all $r \geqslant 1$ and $r \in \left[\lambda_n - \frac{h_n}{2}, \lambda_n + \frac{h_n}{2}\right]$,

$$\left|\frac{1}{L(re^{i\delta})}\right| \leqslant \frac{1}{\lambda_1\delta}e^{2\omega_1(r)}. \tag{3.13}$$

Let $r \in \left[\lambda_n + \frac{h_n}{2}, \lambda_{n+1} - \frac{h_{n+1}}{2}\right]$. It was shown in [2, Ch. I, Sect. 3] that for $n \geqslant n_0$ each segment $[2^{n-1}, 2^n]$ contains a point x_n such that

$$\left|\frac{1}{L(x_n)}\right| \leqslant e^{-20\ln M_L(x_n)}.$$

However, if we take into consideration the condition C of Theorem 2.3, we get

$$\left|\frac{1}{L(x_n)}\right| \leqslant e^{\omega_2(x_n)}, \quad \omega_2 \in \Omega_0. \tag{3.14}$$

The circles $B_n = \left\{z : |z - \lambda_n| \leqslant \frac{h_n}{2}\right\}$ are pairwise disjoint. If $\frac{\lambda_{n+1}}{\lambda_n} \leqslant 2$, then by the maximum principle, the estimates (3.11) and the increase of function $|L(re^{i\delta})|^{-1}$ as $\delta \downarrow 0$ we get that for $r \in \left[\lambda_n + \frac{h_n}{2}, \lambda_{n+1} - \frac{h_{n+1}}{2}\right]$

$$\left|\frac{1}{L(re^{i\delta})}\right| \leqslant e^{\omega_1(2r)} \leqslant e^{2\omega_1(r)}. \tag{3.15}$$

If $\frac{\lambda_{n+1}}{\lambda_n} > 2$, then there exists a finite set of points x'_n,

$$\lambda_n + \frac{h_n}{2} = x'_0 < x'_1 < \ldots < x'_N = \lambda_{n+1} - \frac{h_{n+1}}{2}, \qquad \frac{x'_{i+1}}{x'_i} \leqslant 4, \qquad i = 0, 1, \ldots, N,$$

in which the estimate of type (3.14) holds. Applying the previous arguing for this partition points, we again obtain the estimate of type (3.15)

$$\left|\frac{1}{L(re^{i\delta})}\right| \leqslant e^{\omega_2(4r)} \leqslant e^{4\omega_2(r)}, \quad r \in [x'_i, x'_{i+1}].$$

Thus, in view of (3.13), we finally have: for all $r \geqslant 1$

$$\left|\frac{1}{L(re^{i\delta})}\right| \leqslant \frac{\text{const}}{\delta}e^{\omega_3(r)},$$

where $\omega_3(r) = 2\omega_1(r) + 4\omega_2(r)$. Hence,

$$h_-(\delta) \leqslant \frac{\text{const}}{\delta}\int\limits_0^\infty e^{\omega_3(r)-\delta r}dr, \quad \omega_3 \in \Omega_0.$$

This implies the convergence of bilogarithmic integral for this function. The proof is complete. $\square$

Remark 3.3. *The union of finitely many interpolating in the considered sense sequences does not spoil the convergence of integrals (3.3) and (3.4). If $\Lambda = \{\lambda_n\}$ is the union of two interpolating sequences $\{\lambda'_n\}$ and $\{\lambda''_n\}$, then the exponential system $\{e^{\lambda z}\}_{\lambda\in\Lambda}$ is strongly incomplete with respect to the rectangles as the systems $\{e^{\lambda'_n z}\}$ and $\{e^{\lambda''_n z}\}$. At the same time, as we have seen, the sequence Λ is not necessarily interpolating since the numbers h_n can tend to zero arbitrarily fast for the interpolating sequences*

$$\ln\frac{1}{h_n} \leqslant e^{\omega(\lambda_n)}, \qquad n \geqslant 1,$$

where the function ω belongs at least to the convergence class W.

Remark 3.4. *The condition*

$$\sup_{\theta\neq 0,\pi} H_\omega(\theta) < \infty$$

is equivalent to the condition (3.9), and this implies that for some $K<\infty$ and each $\theta\in\left(0,\frac{\pi}{2}\right]$ for $r\geqslant r(\theta)$ we have

$$I(z) \stackrel{def}{=} \int\limits_0^1 \frac{n_\sigma(z)}{\sigma}d\sigma < K\omega(r), \qquad z = re^{i\theta}.$$

Hence, for an arbitrary $z = re^{i\theta}$

$$I(z) \leqslant \max(m_0(\theta), K\omega(r)), \qquad m_0(\theta) = \max_{0\leqslant r\leqslant r(\theta)} I(z).$$

This yields $h^*(\theta) \leqslant M(\theta)h_0(\theta)$, where the function $h^*(\theta)$ is the same as in (3.10), and

$$M(\theta) = e^{m_0(\theta)}, \qquad h_0(\theta) = \int\limits_0^\infty e^{K\omega(r)-\theta r}dr, \qquad 0<\theta\leqslant\theta_0<\frac{\pi}{4}.$$

This shows that the convergence of integral

$$\int\limits_0^{d^*} \ln\ln h^*(\theta)d\theta, \qquad h^*(d^*) \geqslant e,$$

is ensured by

$$\int\limits_0^{d_M} \ln\ln M(\theta)d\theta < \infty, \qquad M(d_M) \geqslant e.$$

As we have seen, for the interpolating sequences we have

$$M(\theta) = \frac{\text{const}}{\theta}.$$

If $\Lambda = \bigcup_{i=1}^n \Lambda^{(i)}$, where $\Lambda^{(i)}$ are interpolating sequences, then

$$M(\theta) = \text{const}\left(\frac{1}{\theta}\right)^N.$$

On the other hand we can write

$$h^*(\theta) = \left(\int\limits_0^{r(\theta)} + \int\limits_{r(\theta)}^\infty\right)[\exp(I(z)-\theta r)]\,dr \leqslant h_1(\theta)+h_0(\theta),$$

where $h_0(\theta)$ is the same function as above and

$$h_1(\theta) = \int_0^{r(\theta)} e^{I(z)-\theta r} dr, \quad z = re^{i\theta}.$$

It is easy to see that

$$h_1(\theta) \leqslant h^*(\theta) \leqslant h_1(\theta) + h_0(\theta).$$

As one can easily confirm, this means that the bilogarithmic integrals of the functions $h^*(\theta)$ and $h_1(\theta)$ are equiconvergent.

Question. How to characterise the convergence of the integral

$$\int_0^{d_1} \ln\ln h_1(\theta) d\theta, \quad h_1(d_1) \geqslant e,$$

in terms of the distribution of sequence?

BIBLIOGRAPHY

1. A.M. Gaisin. *Asymptotic behavior of sum of entire Dirichlet series on curves* // in: *Studies in Approximation Theory*, Proc. Inst. Math. Bashkir Sci. Center Ural Branch of AS USSR, Ufa, 3–15 (1990). (in Russian).
2. A.M. Gaisin. *Regular Growth of Entire Functions Represented by Dirichlet Series*. Regular and Chaotic Dynamics, Inst. Comput. Stud., Izhevsk (2024).
3. A.M. Gaisin. *Strong incompleteness of a system of exponentials, and Macintyre's problem* // Math. USSR–Sb. **73**:2, 305–318 (1992). https://doi.org/10.1070/SM1992v073n02ABEH002547
4. A.M. Gaisin. *Levinson's condition in the theory of entire functions: equivalent statements* // Math. Notes **83**:3, 317–326 (2008). https://doi.org/10.1134/S0001434608030036
5. A.M. Gaisin. *Entire Functions: Foundations of Classical Theory With Applications to Studies in Complex Analysis*. Bashkir State Univ. Publ., Ufa (2016). (in Russian).
6. A.M. Gaisin, R.A. Gaisin. *Weight concentration index* // Vladikavkaz Math. J. **27**:1, 21–35 (2025). (in Russian). https://doi.org/10.46698/y0305-5846-4678-h
7. A.M. Gaisin, Zh.G. Rakhmatullina. *Real sequences that are lacunary in the Fejér sense* // Ufim. Mat. Zh. **2**:2, 27–40 (2010). (in Russian). http://mathnet.ru/eng/ufa/v2/i2/p27
8. R.A. Gaĭsin. *Pavlov – Korevaar – Dixon interpolation problem with majorant in convergence class* // Ufa Math. J. **9**:4, 22–34 (2017). https://doi.org/10.13108/2017-9-4-22
9. R.A. Gaĭsin. *Interpolation sequences and nonspanning systems of exponentials on curves* // Sb. Math. **212**:5, 655–675 (2021). https://doi.org/10.1070/SM9370
10. R.A. Gaisin. *Generalized interpolation problem of the Korevaar – Dixon type* // J. Math. Sci., New York **257**:3, 296–304 (2021). https://doi.org/10.1007/s10958-021-05483-3
11. I.F. Krasichkov. *Lower bounds for entire functions of finite order* // Sib. Math. J. **6**:4, 840–861 (1965). (in Russian). http://mi.mathnet.ru/smj5178
12. A.F. Leont'ev. *On the convergence of a sequence of Dirichlet polynomials.* // Dokl. Akad. Nauk SSSR **108**:1, 23–26 (1956). (in Russian). https://zbmath.org/?q=an:0073.29302
13. A.F. Leont'ev. *Exponential series*. Nauka, Moscow (1976). (in Russian).
14. V.B. Sherstyukov. *Distribution of the zeros of canonical products and weighted condensation index* // Sb. Math. **206**:9, 1299–1339 (2015). https://doi.org/10.1070/SM2015v206n09ABEH004497
15. C.A. Berenstein, B.A. Taylor. *A new look at interpolation theory for the entire functions of one variable* // Adv. Math. **33**:2, 109–143 (1979). https://doi.org/10.1016/S0001-8708(79)80002-X
16. B. Berndtsson. *A note on Pavlov – Korevaar – Dixon interpolation* // Indag. Math. **40**, 409–414 (1978).
17. P. Koosis. *The logarithmic integral. I.* Cambridge Univ. Press, Cambridge (1988).
18. J. Korevaar. *Müntz approximation on arcs and Macintyre exponents* // in: "Complex Analysis", Lect. Notes Math. **747**, 205–218 (1979).

19. J. Korevaar, M. Dixon. *Interpolation, strongly nonspanning powers and Macintyre exponents* // Nederl. Akad. Wet., Proc., Ser. A **81**, 243–258 (1978).
20. J. Korevaar, M. Dixon. *Nonspanning sets of exponentials on curves* // Acta Math. Acad. Sci. Hung. **33**, 89–100 (1979). https://doi.org/10.1007/BF01903384
21. A.J. Macintyre. *Asymptotic paths of integral functions with gap power series* // Proc. Lond. Math. Soc., III. Ser. **2**, 286–296 (1952). https://doi.org/10.1112/plms/s3-2.1.286

Akhtyar Magazovich Gaisin,
Institute of Mathematics,
Ufa Federal Research Center, RAS
Chernyshevsky str. 112,
450076, Ufa, Russia
Ufa University of Science and Technology,
Zaki Validi str. 32,
450077, Ufa, Russia
E-mail: `gaisinam@mail.ru`

Rashit Akhtyarovich Gaisin,
Institute of Mathematics,
Ufa Federal Research Center, RAS
Chernyshevsky str. 112,
450076, Ufa, Russia
E-mail: `rashit.gajsin@mail.ru`