ISSN 2074-1871 Уфимский математический журнал. Том 17. № 4 (2025). С. 160-179.

ANALYSIS AND NUMERICAL SIMULATION OF DYNAMIC CONTACT PROBLEM WITH FRICTION IN THERMO–VISCOELASTICITY

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Abstract. The focus of our study is a dynamic frictional contact model that involves a viscoelastic body and a conductive foundation. We use Coulomb's law to describe the frictional behavior, while a normal compliance model is employed to simulate the contact. We formulate a variational formulation for the problem, and we establish the existence of its unique weak solution using the Banach fixed point theorem. We propose a fully discrete scheme, using the finite element method for the spatial approximation and the Euler scheme for the discretization of the time derivatives. The errors on the solutions are derived, and the linear convergence is obtained under suitable regularity hypotheses. Some numerical simulations are included to show the performance of method.

Keywords: Viscoelastic material, heat transfer, frictional contact, Banach fixed point, finite element approximation, error estimates, numerical simulations.

Mathematics Subject Classification: 74D99, 74M15, 37C25, 65M60, 74S05, 81T80, 74F05

1. Introduction

Contact problems can be seen everywhere in mechanics, physics, and engineering applications. Some examples from automotive industry are contact between brake pads and rotors or between pistons and cylinders. Thermal effects in contact processes affect the composition and stiffness of the contacting surfaces, and cause thermal stresses in the contacting bodies [16]. Vice versa, the current temperature may influence the elastic material response. In some works different thermomechanical frictional problems were studied and developed, see, for instance, [2], [16] and the references therein. Besides the rigorous construction of various mathematical models of contact with thermal effects, the unique weak solvability of these models was proved by using arguments of variational and hemivariational inequalities.

In [6] the authors studied a class of dynamic thermal contact problems with the normal compliance condition and friction, for viscoelastic materials, they also proposed a numerical scheme for the approximation of the solution fields, and the corresponding numerical computations. In [3], [12], [5], [13], [15], [18] numerical solutions for frictional contact problems, by taking into account the thermal effects, was presented.

Bouallala et al. [4] treated a dynamic contact problem between a thermo–viscoelastic body and a conductive foundation with normal compliance and Coulomb's friction. Here we look at the same problem and show that a weak solution exists and is unique by using the arguments of dynamic nonlinear quasi–variational inequalities, nonlinear parabolic variational equalities, and the fixed point method. We present the discrete problem using the finite element method

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M. Bouallala EL-H. Essoufi, Y. Ouafik, Analysis and numerical simulation of dynamic contact problem with friction in thermo–viscoelasticity.

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Submitted September 30, 2024.

and a backward Euler finite difference and we prove the convergence of its solution. In this study, we face a lot of difficulties with the nonlinearity of the boundary conditions and the dynamic nature of the problem. The second novelty of this work is the numerical simulations of the parameters of the problem. We note that the established results are almost similar with those in [6] in the case of a non-clamped body. But, unlike [6], in the present work, the Dirichlet boundary condition is assumed on the part of the surface of body. We also note that, unlike [6], here the model includes the heat generated by the displacement and the heat exchange condition, in which the heat exchange coefficient is not a constant but a function of the normal displacement on contact boundary.

The rest of the paper is structured as follows. The model of the dynamic process of the thermo–viscoelastic body is presented in Section 2, together with its variational formulation. In Section 3, we state and prove our main existence and uniqueness result, Theorem 3.1. The main result concerning the error estimate for fully discrete numerical scheme is presented in Sections 4. Finally, in Section 5, we present numerical simulations for a two–dimensional test problem to illustrate the theoretical error estimate and the evolution of the displacement and the temperature fields.

2. Problem statement and weak formulation

In this paper, we denote by $\mathbb{S}^d$, $(d = 1, 2, 3)$ the space of second order symmetric tensor on $\mathbb{R}^d$ and by '$\cdot$' and $\|\cdot\|$ the inner product and the Euclidean norm on the space $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively, that is for all $u,\ v \in \mathbb{R}^d$ and for all $\sigma,\ \tau \in \mathbb{S}^d$

$$u.v = u_i v_i, \qquad \|v\| = (v, v)^{\frac{1}{2}}, \qquad \sigma \cdot \tau = \sigma_{ij}\tau_{ij}, \qquad \|\tau\| = (\tau, \tau)^{\frac{1}{2}}.$$

We denote by $t \in [0, T]$ and $x \in \Omega$ the time interval where $T > 0$ and the spatial variable, respectively.

The body is made of a thermo–viscoelastic material and occupies the domain $\Omega \in \mathbb{R}^d$ with a smooth boundary $\Gamma = \partial\Omega$. This boundary is divided into three disjoint measurable parts Γ_D, Γ_N and Γ_C such that meas$(\Gamma_D) > 0$. Also, below $\nu = \{\nu_i\}$ stands for the unit outward normal.

The body is under the action of body forces of density f_0, and a volume heat source of a constant strength q_0 in $\Omega \times (0, T)$. The body is clamped on $\Gamma_D \times (0, T)$ and so the displacement field vanish there. Surface traction of density f_1 acts on $\Gamma_N \times (0, T)$. We assume that the temperature vanishes on $(\Gamma_N \cup \Gamma_D) \times (0, T)$. The body can arrive in friction contact with the foundation which is thermally conductive and its temperature is maintained at θ_F. The normal gap between Γ_C and the foundation is denoted by g.

For the displacement field $u : \Omega \times (0, T) \to \mathbb{R}^d$ and a stress tensor $\sigma : \Omega \times (0, T) \to \mathbb{S}^d$, the symbols u_ν, σ_ν, u_τ and σ_τ represent its normal and tangential components on the boundary are respectively defined by

$$u_\nu = u \cdot \nu, \qquad \sigma_\nu = (\sigma\nu), \qquad u_\tau = u - u_\nu\nu, \qquad \sigma\tau = \sigma\nu - \sigma_\nu\nu.$$

We denote by $q = (q_i) : \Omega \times (0, T) \to \mathbb{R}$ the heat flux vector, by $\theta : \Omega \times (0, T) \to \mathbb{R}$ the temperature and by $\varepsilon(u)$ the linearized strain tensor given by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \qquad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

where $u_{i,j} = \frac{\partial u_i}{\partial x_j}$. Here and below $\mathrm{Div}(\sigma) = \sigma_{ij,j}$ and $\mathrm{div}(q) = (q_{i,i})$ denote the divergence operator for tensor and vector valued function.

The classical model for the thermo–viscoelastic contact problem is as follows.

Problem 2.1. *Find a displacement field*

$$u(x, t) : \Omega \times (0, T) \longrightarrow \mathbb{R}^d,$$

a stress field

$$\sigma(x,t) : \Omega \times (0,T) \to \mathbb{S}^d,$$

a temperature field

$$\theta(x,t) : \Omega \times (0,T) \longrightarrow \mathbb{R}$$

and a heat flux

$$q(x,t) : \Omega \times (0,T) \to \mathbb{R}$$

such that

$$\sigma(t) = \mathcal{E}\varepsilon(u(t)) + \mathcal{V}\varepsilon(\dot{u}(t)) - \theta(t)\mathcal{M} \quad in \quad \Omega \times (0,T), \tag{2.1}$$
$$q(t) = -\mathcal{K}\nabla\theta(t) \quad in \quad \Omega \times (0,T), \tag{2.2}$$
$$\ddot{u}(t) = Div \quad \sigma(t) + f_0(t) \quad in \quad \Omega \times (0,T), \tag{2.3}$$
$$\dot{\theta}(t) + div \quad q(t) - \mathcal{R}\varepsilon(u(t)) = q_0(t) \quad in \quad \Omega \times (0,T), \tag{2.4}$$
$$u = 0 \quad on \quad \Gamma_D \times (0,T), \tag{2.5}$$
$$\sigma(t)\nu = f_1(t) \quad on \quad \Gamma_N \times (0,T), \tag{2.6}$$
$$\theta = 0 \quad on \quad (\Gamma_N \cup \Gamma_D) \times (0,T), \tag{2.7}$$
$$-\sigma_\nu(u(t) - g) = p_\nu(u_\nu(t) - g) \quad on \quad \Gamma_C \times (0,T), \tag{2.8}$$
$$\left.\begin{aligned} &\|\sigma_\tau(t)\| \leqslant p_\tau(u_\nu(t) - g), \\ &\dot{u}_\tau(t) \neq 0 \Rightarrow \sigma_\tau(t) = -p_\tau(u_\nu(t) - g)\frac{\dot{u}_\tau(t)}{\|\dot{u}_\tau(t)\|} \end{aligned}\right\} \quad on \quad \Gamma_C \times (0,T), \tag{2.9}$$
$$\frac{\partial q(t)}{\partial \nu} = k_c(u_\nu(t) - g)\phi_L(\theta(t) - \theta_F) \quad on \quad \Gamma_C \times (0,T), \tag{2.10}$$
$$u(0,x) = u_0, \qquad \dot{u}(0,x) = \dot{u}_0, \qquad \theta(0,x) = \theta_0 \quad in \quad \Omega. \tag{2.11}$$

Equations (2.1) and (2.2) represent the thermo–viscoelastic constitutive law in which

$$\mathcal{E} = (\mathcal{E}_{ijkl}), \qquad \mathcal{V} = (\mathcal{V}_{ijkl}), \qquad \mathcal{M} = (\mathcal{M}_{ij}), \qquad \mathcal{K} = (\mathcal{K}_{ij})$$

are respectively the elastic tensor, fourth–order viscosity tensor, thermal expansion tensor and thermal conductivity tensor. Equation (2.3) is the equation of motion where the mass density $\rho = 1$. Equation (2.4) is the Fourier law of heat conduction where the function $\mathcal{R} = (\mathcal{R}_{ij})$ describe the influence of the displacement field. In addition, (2.5)-(2.7) are the displacement and thermal boundary condition. The normal compliance contact condition is considered in (2.8), where p_ν is a prescribed function. When it is positive, $u_\nu - g$ represents the penetration of the surface as parities into those of the foundation. The relation (2.9) represents the Coulomb's law of friction, where p_τ is a prescribed non–negative function, the so–called friction bound. The relation (2.10) represents a regularized thermal contact condition where $\dfrac{\partial q}{\partial \nu}$ is the normal derivative of q such that

$$\phi_L(s) = \begin{cases} -L & \text{for} \quad s < -L, \\ s & \text{for} \quad -L \leqslant s \leqslant L, \\ L & \text{for} \quad s > L, \end{cases} \qquad \begin{cases} k_c(r) = 0 & \text{for} \quad r < 0, \\ k_c(r) > 0 & \text{for} \quad r \geqslant 0, \end{cases}$$

where L is a large positive constant [7]. Finally, the initials conditions are posed in Equation (2.11).

The variational formulation of Problem 2.1 requires some additional notation and preliminaries. First, we introduce the spaces

$$\begin{aligned}
&H = L^2\left(\Omega;\mathbb{R}^d\right) = \left\{u = (u_i):\ u_i \in L^2(\Omega)\right\},\\
&\mathcal{H} = \left\{\sigma = (\sigma_{ij}):\ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\},\\
&H_1 = L^2\left(\Omega;\mathbb{S}^d\right) = \left\{u = (u_i):\ \varepsilon(u) \in \mathcal{H}\right\},\\
&\mathcal{H}_1 = \left\{\sigma \in \mathcal{H}:\ Div\ \sigma \in H\right\}.
\end{aligned}$$

These are real Hilbert spaces endowed with the scalar product

$$\begin{aligned}
&(u,v)_H = \int\limits_\Omega u_i v_i\,dx, \qquad u,v \in H,\\
&(\sigma,\tau)_{\mathcal{H}} = \int\limits_\Omega \sigma_{ij}\tau_{ij}\,dx, \qquad \sigma,\tau \in \mathcal{H},\\
&(u,v)_{H_1} = (u,v)_H + (\varepsilon(u),\varepsilon(v))_{\mathcal{H}},\\
&(\sigma,\tau)_{\mathcal{H}_1} = (\sigma,\tau)_{\mathcal{H}} + (\mathrm{Div}\ \sigma, \mathrm{Div}\ \tau)_H,
\end{aligned}$$

and the associated norms, $\|\cdot\|_H$, $\|\cdot\|_{H_1}$, $\|\cdot\|_{\mathcal{H}}$, and $\|\cdot\|_{\mathcal{H}_1}$.

For the mechanical and the thermal unknowns, we introduce the spaces

$$\begin{aligned}
&V = \left\{v \in H_1:\ v = 0 \text{ on } \Gamma_D\right\},\\
&K = \left\{v \in V:\ v_\nu \leqslant g \text{ on } \Gamma_C\right\},\\
&Q = \left\{\eta \in H_1:\ \eta = 0 \text{ on } \Gamma_D \cup \Gamma_N\right\},
\end{aligned}$$

endowed with the inner products and norms given by

$$(u,v)_V = (\varepsilon(u),\varepsilon(v))_{\mathcal{H}}, \qquad \|v\|_V = (v,v)_V^{\frac{1}{2}}, \qquad (\theta,\eta)_Q = (\nabla\theta,\nabla\eta)_H, \qquad \|\eta\|_Q = (\eta,\eta)_Q^{\frac{1}{2}}.$$

The following Korn and Friedrichs-Poincaré inequalities hold

$$\|\varepsilon(v)\|_{\mathcal{H}} \geqslant c_k\|v\|_{H_1}, \qquad v \in V, \tag{2.12}$$

$$\|\nabla\eta\|_{L^2(\Omega)} \geqslant c_f\|\eta\|_Q, \qquad \eta \in Q. \tag{2.13}$$

where c_k and c_f are two positive constants depending on Ω and Γ_D.

By the Sobolev trace theorem,

$$\|v\|_{[L^2(\Gamma_C)]^d} \leqslant s_1\|v\|_V, \qquad v \in V, \tag{2.14}$$

$$\|\eta\|_{[L^2(\Gamma_C)]} \leqslant s_2\|\eta\|_Q, \qquad \eta \in Q, \tag{2.15}$$

where s_1 and s_2 are two positive constants depend on Ω, Γ_D and Γ_C.

We denote by V' the dual space of V and by identifying H with its own dual, we have $V \subset H = H' \subset V'$. We denote by $\langle\cdot,\cdot\rangle_{V'\times V}$ the duality pairing between V' and V and let $\|\cdot\|_{V'}$

be the norm in V'. Next, we consider the mappings

$$(f(t),v)_V := \int_\Omega f_0(t)\cdot v\,dx + \int_{\Gamma_N} f_1(t)\cdot v\,da, \tag{2.16}$$

$$(q_h(t),\eta)_Q := \int_\Omega q_0(t)\eta\,dx, \tag{2.17}$$

$$j(u(t),v) := \int_{\Gamma_C} p_\nu(u_\nu(t)-g)v_\nu\,da + \int_{\Gamma_C} p_\tau(u_\tau(t)-g)\|v_\tau\|\,da, \tag{2.18}$$

$$\chi(u(t),\theta(t),\eta) := \int_{\Gamma_C} k_c(u_\nu(t)-g)\phi_L(\theta(t)-\theta_F)\eta\,da. \tag{2.19}$$

We now introduce assumptions on the data in Problem 2.1.

(H1) The viscosity operator $\mathcal{V}=\mathcal{V}_{ijkl}:\Omega\times\mathbb{S}^d\to\mathbb{S}^d$ satisfies

i) there exists $m_b>0$ such that for all $\xi_1,\xi_2\in\mathbb{S}^d$ and almost all $x\in\Omega$

$$(\mathcal{V}(x,\xi_1)-\mathcal{V}(x,\xi_2)).(\xi_1-\xi_2)\geqslant m_b\|\xi_1-\xi_2\|^2;$$

ii) there exists $M_b>0$ such that for all $\xi_1,\xi_2\in\mathbb{S}^d$ and almost all $x\in\Omega$

$$\|\mathcal{V}(x,\xi_1)-\mathcal{V}(x,\xi_2)\|\leqslant M_b\|\xi_1-\xi_2\|;$$

iii) the mapping $x\mapsto\mathcal{V}(x,\xi)$ is measurable on Ω, for all $\xi\in\mathbb{S}^d$;

iv) $\mathcal{V}(x,0)=0$ for almost all $x\in\Omega$.

(H2) The elasticity operator $\mathcal{E}=\mathcal{E}_{ijkl}:\Omega\times\mathbb{S}^d\to\mathbb{S}^d$ satisfies

i) there exists $m_a>0$ such that for all $\xi_1,\xi_2\in\mathbb{S}^d$ and almost all $x\in\Omega$

$$(\mathcal{E}(x,\xi_1)-\mathcal{E}(x,\xi_2)).(\xi_1-\xi_2)\geqslant m_a\|\xi_1-\xi_2\|^2;$$

ii) there exists $M_a>0$ such that for all $\xi_1,\xi_2\in\mathbb{S}^d$ and almost all $x\in\Omega$

$$\|\mathcal{E}(x,\xi_1)-\mathcal{E}(x,\xi_2)\|\leqslant M_a\|\xi_1-\xi_2\|;$$

iii) the mapping $x\mapsto\mathcal{E}(x,\xi)$ is measurable on Ω, for all $\xi\in\mathbb{S}^d$;

iv) $\mathcal{E}(x,0)=0$ for almost all $x\in\Omega$.

(H3) The thermal conductivity tensor $\mathcal{K}=(\mathcal{K}_{ij}):\Omega\times\mathbb{R}^d\to\mathbb{R}^d$ satisfies

i) $\mathcal{K}_{ij}=\mathcal{K}_{ji}\in L^\infty(\Omega)$;

ii) $\mathcal{K}_{ij}(x)\xi_i\xi_j\geqslant m_d\|\xi\|^2$ with $m_d>0$ for all $\xi\in\mathbb{R}^d$, $x\in\Omega$;

iii) $\|(\mathcal{K}\nabla\theta,\nabla\eta)\|_H\leqslant M_d\|\theta\|_Q\|\eta\|_Q$ with $M_d>0$ or all $\theta,\eta\in Q$.

(H4) The thermal expansion tensor $\mathcal{M}=(\mathcal{M}_{ij}):\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies

i) $\mathcal{M}_{ij}=\mathcal{M}_{ji}\in L^\infty(\Omega)$;

ii) $\|(\mathcal{M}\theta,\varepsilon(v))\|_{\mathcal{H}}\leqslant M_m\|\theta\|_Q\|v\|_V$ with $M_m>0$ for all $\theta\in Q, v\in V$.

(H5) The influence of the displacement field tensor $\mathcal{R}=(\mathcal{R}_{ij}):\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies

i) $\mathcal{R}_{ij}=\mathcal{R}_{ji}\in L^\infty(\Omega)$;

ii) $\|(\mathcal{M}\theta,\varepsilon(v))\|_{\mathcal{H}}\leqslant M_e\|\theta\|_Q\|v\|_V$ with $M_e>0$ for all $\theta\in Q, v\in V$.

(H6) i) The forces, the traction and the thermal flux satisfy

$$f_0\in L^2(0,T;L^2(\Omega)^d),\qquad f_1\in L^2(0,T;L^2(\Gamma_N)^d),\qquad q_0\in L^2(0,T;L^2(\Omega));$$

ii) the gap function, the thermal potential and the initial data satisfy

$$g \geqslant 0, \qquad g \in L^\infty(\Gamma_C), \qquad \theta_F \in L^2(0,T;L^2(\Gamma_C)), \qquad u_0,\, \dot u_0 \in V;$$

iii) the functional j is proper, convex and lower semi–continuous on V.

(H7) The coefficient of heat exchange $k_c : \Gamma_C \times \mathbb{R} \to \mathbb{R}^+$ satisfies

i) there exists $M_{k_c} > 0$ such that $|k_c(x,u)| < M_{k_c}$ for all $u \in \mathbb{R}, x \in \Gamma_C$, $x \mapsto k_c(x,u)$ is measurable on Γ_C for all $x \in \mathbb{R}$, $k_c(x,u) = 0$ for all $x \in \Gamma_C$ and $u \leqslant 0$;

ii) there exists $L_{k_c} > 0$ such that $|k_c(x,u_1) - k_c(x,u_2)| \leqslant L_{k_c}|u_1 - u_2|$ for all $u_1, u_2 \in \mathbb{R}$.

(H8) The normal compliance function p_ν and the friction bound p_τ satisfy the following hypothesis for $r = \nu, \tau$

i) $p_r : \Gamma_C \times \mathbb{R} \to \mathbb{R}_+$;

ii) $x \to p_r(x,u)$ is measurable on Γ_C for all $u \in \mathbb{R}$;

iii) $x \to p_r(x,u) = 0$ for $u \leqslant 0$ and almost all $x \in \Gamma_C$;

iv) there exists $L_r > 0$ such that $|p_r(\cdot,u) - p_r(\cdot,v)| \leqslant L_r|u - v|$ for all $u, v \in \mathbb{R}_+$.

For the sake of simplification, we assume that

$$\begin{aligned}
&a : V \times V \to \mathbb{R}, && a(u,v) := (\mathcal{E}\varepsilon(u), \varepsilon(v))_{\mathcal{H}},\\
&b : V \times V \to \mathbb{R}, && b(u,v) := (\mathcal{V}\varepsilon(u), \varepsilon(v))_{\mathcal{H}},\\
&d : Q \times Q \to \mathbb{R}, && d(\theta,\eta) := (\mathcal{K}\nabla\theta, \nabla\eta)_{H},\\
&m : Q \times V \to \mathbb{R}, && m(\theta,v) := (\mathcal{M}\theta, \varepsilon(v))_{\mathcal{H}},\\
&e : V \times Q \to \mathbb{R}, && e(u,\eta) := (\mathcal{R}\varepsilon(v), \eta)_{L^2(\Omega)}.
\end{aligned}$$

According to this notation and through a standard derivation, we have the following variational formulation in terms of displacement field and temperature.

Problem 2.2. *Find a displacement field*

$$u(x,t) : \Omega \times (0,T) \longrightarrow \mathbb{R}^d,$$

and a temperature field

$$\theta(x,t) : \Omega \times (0,T) \longrightarrow \mathbb{R} \quad \textit{for almost all} \quad t \in]0,T[, \quad v \in V, \quad \eta \in Q$$

such that

$$\begin{aligned}
&(\ddot u(t), v - \dot u(t))_H + b(\dot u(t), v - \dot u(t)) + a(u(t), v - \dot u(t)) - m(\theta(t), v - \dot u(t))\\
&\qquad\qquad + j(u(t), v) - j(u(t), \dot u(t)) \geqslant (f(t), v - \dot u(t))_V,
\end{aligned} \tag{2.20}$$

$$(\dot\theta(t), \eta)_{L^2(\Omega)} + d(\theta(t), \eta) - e(u(t), \eta) + \chi(u(t), \theta(t), \eta) = (q_h(t), \eta)_Q, \tag{2.21}$$

$$u(0) = u_0, \qquad \dot u(0) = \dot u_0, \qquad \theta(0) = \theta_0. \tag{2.22}$$

To study this problem, we impose the condition

$$m_a < (L_\nu + L_\nu)\, s_1^2, \qquad m_d > M_{k_c} s_2^2. \tag{2.23}$$

3. Existence and uniqueness result

In this section, we present an existence and uniqueness result.

Theorem 3.1. *Assume that (H1)–(H8) and the condition (2.23) hold. Then there exists a unique solution (u,θ) to Problem 2.2, which satisfies the regularity conditions*

$$u \in L^2(0,T;V), \qquad \dot u \in L^2(0,T;V'), \qquad \theta \in L^2(0,T;Q). \tag{3.1}$$

This theorem is proved in several steps and is based on the Banach fixed point theorem.
First, let $y \in L^2(0,T;V)$ and $z \in L^2(0,T;L^2(\Omega))$ given by

$$(y(t),v)_V = m(\theta_z(t),v), \ \forall v \in V, \tag{3.2}$$
$$(z(t),\eta)_Q = -e(u_y(t),\eta), \ \forall \eta \in Q. \tag{3.3}$$

Applying Riesz representation theorem, we define the operators

$$(f_y(t),v)_V = (f(t),v)_V - (y(t),v)_V, \tag{3.4}$$
$$(q_z(t),\eta)_Q = (q_h(t),\eta)_Q - (z(t),\eta)_Q. \tag{3.5}$$

Next, we consider the intermediate problems.

Problem 3.1. *Find a displacement field $u_y \in K \times (0,T)$ for all $v \in V$ such that*

$$\begin{aligned} &\ddot{u}_y(t), v - \dot{u}_y(t))_H + b(\dot{u}_y(t), v - \dot{u}_y(t)) + a(u_y(t), v - \dot{u}_y(t)) \\ &\qquad + j(u_y(t),v) - j(u_y(t),\dot{u}_y(t)) \geqslant (f_y(t), v - \dot{u}_y(t))_V, \end{aligned} \tag{3.6}$$
$$u_y(0) = u_0, \ \dot{u}_y(0) = \dot{u}_0. \tag{3.7}$$

Problem 3.2. *Find a temperature field $\theta_z \in Q \times (0,T)$ for all $\eta \in Q$ such that*

$$\left(\dot{\theta}_z(t),\eta\right)_{L^2(\Omega)} + d(\theta_z(t),\eta) + \chi(u_y(t),\theta_z(t),\eta) = (q_z(t),\eta)_Q, \tag{3.8}$$
$$\theta_z(0) = \theta_0. \tag{3.9}$$

At the second step, we establish the unique solvability of the intermediate problems.

Lemma 3.1. *For all $v \in K$ and for almost all $t \in (0,T)$, Problem 3.1 has a unique solution u_y with the regularity*

$$u_y \in L^2(0,T;V), \qquad \dot{u}_y \in L^2(0,T;V').$$

The proof is based on similar arguments to those used in [11, Thm. 5.15].

Lemma 3.2. *For all $\eta \in Q$ and for a.e $t \in (0,T)$, Problem 3.2 has a unique solution θ_z with the regularity*

$$\theta_z \in L^2(0,T;Q).$$

The proof of this result is presented in [8, Lm. 3.3] by using the Galerkin method.
At the last step we define the operator

$$\Phi(y,z)(t) := (\Phi_1(y,z)(t), \Phi_2(y,z)(t)) \in V \times Q, \tag{3.10}$$

where

$$(\Phi_1(y,z)(t),v)_V = m(\theta_z(t),v), \tag{3.11}$$
$$(\Phi_2(y,z)(t),\eta)_Q = -e(u_y(t),\eta). \tag{3.12}$$

Lemma 3.3. *For $(y,z) \in L^2(0,T;V) \times L^2(0,T;L^2(\Omega))$, the operator Φ is continuous and has a unique fixed point $(y^*,z^*) \in L^2(0,T;V \times L^2(\Omega))$.*

Proof. Let $(y,z) \in L^2(0,T;V \times L^2(\Omega))$ and $t_1,t_2 \in [0,T]$. By the assumptions (H2) and (H5) we have

$$\|\Phi_1(y,z)(t_1) - \Phi_2(y,z)(t_2)\|_{V \times L^2(\Omega)} \leqslant M_m \|\theta_z(t_1) - \theta_z(t_2)\|_Q, \tag{3.13}$$
$$\|\Phi_2(y,z)(t_1) - \Phi_2(y,z)(t_2)\|_{V \times L^2(\Omega)} \leqslant M_e \|u_y(t_1) - u_y(t_2)\|_V, \tag{3.14}$$

and in view of the regularity of θ_z and u_z we see that Φ is continuous.
Let $(y_1,z_1),(y_2,z_2) \in L^2(0,T;V \times L^2(\Omega))$. For $t \in [0,T]$ and similar to (3.13)-(3.14) we obtain

$$\|\Phi(y_1,z_1)(t) - \Phi(y_2,z_2)(t)\|_{V \times L^2(\Omega)} \leqslant c\left(\|u_{y_1}(t) - u_{y_2}(t)\|_V + \|\theta_{z_1}(t) - \theta_{z_2}(t)\|_Q\right). \tag{3.15}$$

For $i = 1, 2$ we have

$$u_{y_i}(t) = \int_0^t \dot{y}_{y_i}(s)ds + u_0,$$

and

$$\|u_{y_1}(t) - u_{y_2}(t)\|_V \leqslant c \int_0^t \|\dot{u}_{y_1}(s) - \dot{u}_{y_2}(s)\|_V \, ds. \tag{3.16}$$

By the relations (3.4) and (3.6) we find

$$\begin{aligned}(\ddot{u}_{y_1}(t) - \ddot{u}_{y_2}(t), \dot{u}_{y_1}(t) - \dot{u}_{y_2}(t))_H &+ b\left(\dot{u}_{y_1}(t) - \dot{u}_{y_2}(t), \dot{u}_{y_1}(t) - \dot{u}_{y_2}(t)\right) \\ &+ a\left(u_{y_1}(t) - u_{y_2}(t), \dot{u}_{y_1}(t) - \dot{u}_{y_2}(t)\right) \\ &+ \left(y_1(t) - y_2(t), \dot{u}_{y_1}(t) - \dot{u}_{y_2}(t)\right) \\ &+ j\left(u_{y_1}(t), \dot{u}_{y_1}(t)\right) - j\left(u_{y_1}(t), \dot{u}_{y_2}(t)\right) \\ &- j\left(u_{y_2}(t), \dot{u}_{y_1}(t)\right) + j\left(u_{y_2}(t), \dot{u}_{y_2}(t)\right) \leqslant 0.\end{aligned} \tag{3.17}$$

By the assumptions on the operator j we have

$$\begin{aligned}&\Big| j\left(u_{y_1}(t), \dot{u}_{y_1}(t)\right) - j\left(u_{y_1}(t), \dot{u}_{y_2}(t)\right) - j\left(u_{y_2}(t), \dot{u}_{y_1}(t)\right) + j\left(u_{y_2}(t), \dot{u}_{y_2}(t)\right) \Big| \\ &\qquad\qquad \leqslant s_1^2(L_\nu + L_\tau) \|u_{y_1}(t) - u_{y_2}(t)\|_V \|\dot{u}_{y_1}(t) - \dot{u}_{y_2}(t)\|_V .\end{aligned} \tag{3.18}$$

We integrate the inequality (3.17) over $[0, T]$ and by (3.18), (H1) this yields

$$\begin{aligned}m_b \int_0^t &\|\dot{u}_{y_1}(s) - \dot{u}_{y_2}(s)\|_V^2 \, ds + \frac{1}{2} \|\dot{u}_{y_1}(t) - \dot{u}_{y_2}(t)\|_V^2 \\ &\leqslant - \int_0^t \left(y_1(s) - y_2(s), \dot{u}_{y_1}(t) - \dot{u}_{y_2}(t)\right) \, ds \\ &+ (M_a + s_1^2(L_\nu + L_\tau)) \int_0^t \|u_{y_1}(s) - u_{y_2}(s)\|_V \|\dot{u}_{y_1}(s) - \dot{u}_{y_2}(s)\|_V \, ds.\end{aligned} \tag{3.19}$$

This estimate, the Young inequality

$$\alpha\beta \leqslant \delta\alpha^2 + \frac{1}{4\delta}\beta^2, \qquad \delta > 0, \tag{3.20}$$

and the Gornwall inequality imply

$$\|u_{y_1}(t) - u_{y_2}(t)\|_{L^2(0,T;V)} \leqslant c \|y_1(t) - y_2(t)\|_{L^2(0,T;V)} . \tag{3.21}$$

Hence,

$$\begin{aligned}&\left(\dot{\theta}_{z_1}(t) - \dot{\theta}_{z_2}(t), \theta_{z_1}(t) - \theta_{z_2}(t)\right)_{L^2(\Omega)} + d\left(\theta_{z_1}(t) - \theta_{z_2}(t), \theta_{z_1}(t) - \theta_{z_2}(t)\right) \\ &+ \left(z_1(t) - z_2(t), \theta_{z_1}(t) - \theta_{z_2}(t)\right)_Q + \chi\left(\theta_{z_1}(t), u_{y_1}(t), \theta_{z_1}(t) - \theta_{z_2}(t)\right) \\ &- \chi\left(\theta_{z_2}(t), u_{y_2}(t), \theta_{z_1}(t) - \theta_{z_2}(t)\right) = 0.\end{aligned} \tag{3.22}$$

By (2.19) and (H7) we get

$$|\chi\left(\theta_{z_1}(t), u_{y_1}(t), \theta_{z_1}(t) - \theta_{z_2}(t)\right) - \chi\left(\theta_{z_2}(t), u_{y_2}(t), \theta_{z_1}(t) - \theta_{z_2}(t)\right)| \tag{3.23}$$

$$\leqslant M_{k_c} s_2 \|\theta_{z_1}(t) - \theta_{z_2}(t)\|_Q^2 + L_{k_c} L s_1 s_2 \|\theta_{z_1}(t) - \theta_{z_2}(t)\|_Q \|u_{y_1}(t) - u_{y_2}(t)\|_V . \tag{3.24}$$

After some calculation and by using condition (2.23), we see that there exists a positive constant c such that

$$\|\theta_{z_1}(t) - \theta_{z_2}(t)\|_{L^2(0,T;Q)} \leqslant c\,\|z_1(t) - z_2(t)\|_{L^2(0,T;L^2(\Omega))}. \tag{3.25}$$

Combining (3.15), (3.21) and (3.25), we find

$$\|\Phi(y_1,z_1)(t) - \Phi(y_2,z_2)(t)\|_{L^2(0,T;V\times Q)} \leqslant c\,\|(y_1,z_1)(t) - (y_2,z_2)(t)\|_{L^2(0,T;V\times L^2(\Omega))}. \tag{3.26}$$

Reiterating this inequality n times, we obtain

$$\|\Phi^n(y_1,z_1)(t) - \Phi^n(y_2,z_2)(t)\|_{L^2(0,T;V\times Q)} \leqslant \frac{(cT)^n}{n!}\,\|(y_1,z_1)(t) - (y_2,z_2)(t)\|_{L^2(0,T;V\times L^2(\Omega))}, \tag{3.27}$$

which implies that for sufficiently large n the operator Φ is a contraction in $L^2(0,T;V\times Q)$. Therefore, there exists a unique fixed point (y^*,z^*) of Φ. The proof is complete. □

Now we are in position to prove Theorem 3.1.

Proof of Theorem (3.1). *Existence.* Let $(y^*,z^*) \in L^2\left(0,T;V\times L^2(\Omega)\right)$ be the fixed point of the operator Φ and $u^*_{y^*}$, $\theta^*_{z^*}$ be the solutions of Problem 3.1 and Problem 3.2, respectively. For $(y,z) = (y^*,z^*)$, and by definition of Φ we find that the pair $\left(u^*_{y^*},\theta^*_{z^*}\right)$ is a solution of Problem 2.2.

Uniqueness. The uniqueness follows from the uniqueness of the fixed point of Φ. The proof is complete. □

4. Fully discrete scheme and error estimate

In this section, we present a fully discrete scheme for the variational formulated in Problem 2.2, and we establish a result on error estimate. Let $\{\mathcal{T}^h\}$ be a regular family of triangular finite element partition of $\overline{\Omega}$ are compatible with the boundary decomposition $\overline{\Gamma} = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_C$, where $h > 0$ denotes a spatial discretization parameter.

Let V^h and Q^h are a finite dimensional subspace of V and Q respectively given by

$$\begin{aligned}
V^h &= \left\{v^h \in \left[C(\overline{\Omega})\right]^d:\ v^h_{|Tr} \in [\mathbb{P}_1(Tr)]^d\ \forall Tr \in \mathcal{T}^h;\ v^h = 0 \text{ on } \overline{\Gamma}_C\right\} \subset V,\\
Q^h &= \left\{\eta^h \in C(\overline{\Omega}):\ \eta^h_{|Tr} \in \mathbb{P}_1(Tr)\ \forall Tr \in \mathcal{T}^h;\ \eta^h = 0 \text{ on } \overline{\Gamma}_D \cup \overline{\Gamma_N}\right\} \subset Q,\\
K^h &= K \cap V^h.
\end{aligned}$$

For a positive integer N, we define a uniform partition of $[0,T]$ given by

$$0 = t_0 < t_1 < \ldots < t_N = T,$$

and the time step size $k_n = t_n - t_{n-1}$, where $k = \max_n\{k_n\}$ be the maximal step size.
For a time continuous function $u = u(t)$, we write

$$u_n = u(t_n), \qquad \delta = \frac{1}{k}\left(u_n - u_{n-1}\right), \qquad n = 1,\ldots,N.$$

We also use the notation

$$w^{hk}_n = \delta u^{hk}_n,\ u^{hk}_n = \sum_{j=1}^{n} k_j w^{kn}_j + u^h_0, \qquad \theta^{hk}_n = \sum_{j=1}^{n} k_j \delta\theta^{hk}_j + \theta^h_0.$$

Using the backward Euler scheme, the fully discrete approximation of Problem 2.2 is as follows.

Problem 4.1. *Find a displacement field* $\{u_n^{hk}\}_{n=0}^N \subset K^h$ *and a temperature* $\{\theta_n^{hk}\}_{n=0}^N \subset Q^h$ *for all* $v^h \in V^h$ *and* $\eta^h \in Q^h$ *such that*

$$\begin{aligned}\left(\delta w_n^{hk}, v^h - w_n^{hk}\right)_H + b\left(w_n^{hk}, v^h - w_n^{hk}\right) + a\left(u_n^{hk}, v^h - w_n^{hk}\right)\\ - m\left(\theta_{n-1}^{hk}, v^h - w_n^{hk}\right) + j\left(u_n^{hk}, v^h\right) - j\left(u_n^{hk}, w_n^{hk}\right) \geqslant \left(f_n, v^h - w_n^{hk}\right)_V,\end{aligned} \tag{4.1}$$

$$\left(\delta\theta_n^{hk}, \eta^h\right)_{L^2(\Omega)} + d\left(\theta_n^{hk}, \eta^h\right) - e\left(u_{n-1}^{hk}, \eta^h\right) + \chi\left(u_n^{hk}, \theta_n^{hk}, \eta^h\right) = \left(q_{h_n}, \eta^h\right)_Q, \tag{4.2}$$

$$u_0^{hk} = u_0^h, \qquad w_0^{hk} = w_0^h, \qquad \theta_0^{hk} = \theta_0^h, \tag{4.3}$$

where $u_0^h \in V^h$, $w_0^h \in V^h$ *and* $\theta_0^h \in Q^h$ *are respectively approximates of* u_0, $\dot{u}_0$ *and* θ_0.

Under the assumptions of Theorem 3.1, we follow the same arguments asin previous section, and conclude on the unique solvability of Problem 4.1.

Next, we recall the following discrete Gornwall s inequality [17, Lm. 4.1].

Lemma 4.1. *Let* $T > 0$ *be given. For a positive integer* N, *define* $k = T/N$. *Assume that* $\{g_n\}_{n=1}^N$ *and* $\{e_n\}_{n=1}^N$ *are two sequences of nonnegative numbers satisfying*

$$e_n \leqslant c g_n + c\sum_{j=1}^n k e_j, \qquad n = 1, \dots, N,$$

for a positive constant c *independent of* N *or* k. *Then, there exists a positive constant* c, *independent of* N *or* k, *such that*

$$\max_{1\leqslant n\leqslant N} e_n \leqslant c \max_{1\leqslant n\leqslant N} g_n.$$

Now we state a result on error estimation.

Lemma 4.2. *Let* (u, θ) *and* $\left(u_n^{hk}, \theta_n^{hk}\right)$ *be solutions to Problems 2.2 and Problem 4.1, respectively. Assume (H1)–(H8) and (2.23). Then the following bound holds for all* $\{v_j^n\}_{j=1}^N \subset V^h$ *and* $\{\eta_j^n\}_{j=1}^N \subset Q^h$

$$\begin{aligned}&\left\|w_n - w_n^{hk}\right\|_H^2 + \left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}^2 + ck\sum_{j=1}^n\left(\left\|w_j - w_j^{hk}\right\|_V^2 + \left\|\theta_j - \theta_j^{hk}\right\|_Q^2\right)\\
&\leqslant c\left\{\left\|w_0 - w_0^{hk}\right\|_H^2 + \left\|u_0 - u_0^{hk}\right\|_V^2 + \left\|\theta_0 - \theta_0^{hk}\right\|_{L^2(\Omega)}^2 + \left\|\theta_0 - \theta_0^{hk}\right\|_Q^2\right\}\\
&\quad + c\left\{\left\|w_1 - v_1^h\right\|_H^2 + \left\|\theta_1 - \eta_1^h\right\|_{L^2(\Omega)}^2\right\} + c\left\{\left\|w_n - v_n^h\right\|_H^2 + \left\|\theta_n - \eta_n^h\right\|_{L^2(\Omega)}^2\right\}\\
&\quad + ck^2\left\|\theta\right\|_{H^1(0,T;Q)}^2 + ck^2\left\|u\right\|_{H^2(0,T;V)}^2 + ckR\left(w_j, v_j^h\right)\\
&\quad + ck\sum_{j=1}^n\left(\left\|\dot{w}_j - \delta w_j\right\|_H^2 + \left\|w_j - v_j^h\right\|_V^2 + \left\|\dot{\theta}_j - \delta\theta_j\right\|_{L^2(\Omega)}^2 + \left\|\theta_j - \eta_j^h\right\|_Q^2\right)\\
&\quad + \frac{1}{k}\sum_{j=1}^{n-1}\left(\left\|\left(w_j - v_j^n\right) - \left(w_{j+1} - v_{j+1}^h\right)\right\|_H^2 + \left\|\left(\theta_j - \eta_j^n\right) - \left(\theta_{j+1} - \eta_{j+1}^h\right)\right\|_{L^2(\Omega)}^2\right),\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}R\left(w_n, v^h\right) = \left(\delta w_n, v^h - w_n\right)_H + b\left(w_n, v^h - w_n\right) + a\left(u_n, v^h - w_n\right)\\ - m\left(\theta_n, v^h - w_n\right) + j\left(u_n, v^h\right) - j\left(u_n, w_n\right) - \left(f_n, v^h - w_n\right)_V,\end{aligned} \tag{4.5}$$

Proof. By the inequality (4.1) we have

$$\begin{aligned} &-\left(\delta w_n^{hk}, w_n - w_n^{hk}\right)_H + b\left(-w_n^{hk}, w_n - w_n^{hk}\right) \leqslant a\left(u_n^{hk}, w_n - w_n^{hk}\right) \\ &- m\left(\theta_{n-1}^{hk}, w_n - w_n^{hk}\right) - \left(\delta w_n^{hk}, w_n - v^h\right)_H - b\left(w_n^{hk}, w_n - v^h\right) - a\left(u_n^{hk}, w_n - v^h\right) \\ &- m\left(\theta_{n-1}^{hk}, v^h - w_n\right) + j\left(u_n^{hk}, v^h\right) - j\left(u_n^{hk}, w_n^{hk}\right) - \left(f_n, v^h - w_n^{hk}\right). \end{aligned} \tag{4.6}$$

We take $v = w_n^{hk}$ in (2.20) at time $t = t_n$

$$\begin{aligned} &\left(\dot{w}_n, w_n - w_n^{hk}\right)_H + b\left(w_n, w_n - w_n^{hk}\right) \leqslant a\left(u_n, w_n^{hk} - w_n\right) \\ &- m\left(\theta_n, w_n^{hk} - w_n\right) + j\left(u_n, w_n^{hk}\right) - j\left(u_n, w_n\right) - \left(f_n, w_n^{hk} - w_n\right)_V. \end{aligned} \tag{4.7}$$

Adding the previous inequality, after some calculations we obtain

$$\begin{aligned} \left(\dot{w}_n - \delta w_n^{hk}, w_n - w_n^{hk}\right)_H &+ b\left(w_n - w_n^{hk}, w_n - w_n^{hk}\right) \\ \leqslant &a\left(u_n - u_n^{hk}, w_n^{hk} - w_n\right) - m\left(\theta_n - \theta_{n-1}^{hk}, w_n^{hk} - w_n\right) \\ &- \left(\delta w_n^{hk}, w_n - v^n\right)_H - b\left(w_n^{hk}, w_n - v^n\right) - a\left(u_n^{hk}, w_n - v^n\right) \\ &+ m\left(\theta_{n-1}^{hk}, w_n - v^n\right) + j\left(u_n^{hk}, v^h\right) - j\left(u_n^{hk}, w_n^{hk}\right) \\ &- \left(f_n, v^h - w_n\right)_V + j\left(u_n, w_n^{hk}\right) - j\left(u_n, w_n\right), \end{aligned} \tag{4.8}$$

Using the relation

$$\begin{aligned} \left(\dot{w}_n - \delta w_n^{hk}, w_n - w_n^{hk}\right)_H = &\left(\delta w_n - \delta w_n^{hk}, w_n - w_n^{hk}\right)_H \\ &+ \left(\dot{w}_n - \delta w_n, w_n - v^h\right)_H + \left(\dot{w}_n - \delta w_n^{hk}, v^h - w_n^{hk}\right)_H, \end{aligned} \tag{4.9}$$

we get

$$\begin{aligned} \left(\delta w_n - \delta w_n^{hk}, w_n - w_n^{hk}\right)_H &+ b\left(w_n - w_n^{hk}, w_n - w_n^{hk}\right) \\ \leqslant &\left(\delta w_n - \delta w_n^{hk}, w_n - v^h\right)_H + \left(\delta w_n - \dot{w}_n, v^h - w_n^{hk}\right)_H \\ &+ a\left(u_n - u_n^{hk}, w_n^{hk} - w_n\right) - m\left(\theta_n - \theta_{n-1}^{hk}, w_n^{hk} - w_n\right) \\ &+ b\left(w_n - w_n^{hk}, w_n - v^h\right) + a\left(u_n - u_n^{hk}, w_n - v^h\right) \\ &- m\left(\theta_n - \theta_{n-1}^{hk}, w_n - v^h\right) + R\left(w_n, v^h\right) + R_1, \end{aligned} \tag{4.10}$$

where

$$R_1 = j\left(u_n, w_n^{hk}\right) - j\left(u_n, v^h\right) + j\left(u_n^{hk}, v^h\right) - j\left(u_n^{hk}, w_n^{hk}\right). \tag{4.11}$$

Using the formula

$$2(x-y,x)_H = \|x-y\|_H^2 + \|x\|_H^2 - \|y\|_H^2, \qquad x = w_n - w_n^{hk}, \qquad b = w_{n-1} - w_{n-1}^{hk},$$

we find

$$\left(\delta w_n - \delta w_n^{hk}, w_n - w_n^{hk}\right)_H \geqslant \frac{1}{2k}\left(\left\|w_n - w_n^{hk}\right\|_H^2 - \left\|w_{n-1} - w_{n-1}^{hk}\right\|_H^2\right). \tag{4.12}$$

By the Lipschitz continuity of b and the continuity of a, b and m we get

$$\begin{aligned} &b\left(w_n - w_n^{hk}, w_n - w_n^{hk}\right) \geqslant m_b\left\|w_n - w_n^{hk}\right\|_V^2, \\ &\left|b\left(w_n - w_n^{hk}, w_n - v^h\right)\right| \leqslant M_b\left\|w_n - w_n^{hk}\right\|_V\left\|w_n - v^h\right\|_V, \\ &\left|a\left(u_n - u_n^{hk}, w_n^{hk} - v^h\right)\right| \leqslant M_a\left\|u_n - u_n^{hk}\right\|_V\left\|w_n^{hk} - v^h\right\|_V, \\ &\left|a\left(u_n - u_n^{hk}, w_n^{hk} - w_n\right)\right| \leqslant M_a\left\|u_n - u_n^{hk}\right\|_V\left\|w_n - w_n^{hk}\right\|_V, \\ &\left|m\left(\theta_n - \theta_{n-1}^{hk}, w_n - v^h\right)\right| \leqslant M_m\left\|\theta_n - \theta_{n-1}^{hk}\right\|_Q\left\|w_n - v^h\right\|_V, \\ &\left|m\left(\theta_n - \theta_{n-1}^{hk}, w_n^{hk} - w_n\right)\right| \leqslant M_m\left\|\theta_n - \theta_{n-1}^{hk}\right\|_Q\left\|w_n^{hk} - w_n\right\|_V. \end{aligned} \tag{4.13}$$

Taking into account (H8) and (2.19), we find

$$|R_1| \leqslant \sqrt{\operatorname{meas}(\Gamma_C)} s_1^2 (L_\nu + L_\tau) \left\| u_n - u_n^{hk} \right\|_V \left\| w_n^{hk} - v^h \right\|_V , \tag{4.14}$$

Applying this inequality

$$\left\| w_n^{hk} - v^h \right\|_V \leqslant \left\| w_n^{hk} - w_n \right\|_V \left\| w_n - v^h \right\|_V , \tag{4.15}$$

and combing (3.2), (4.8)–(4.15), we obtain

$$\begin{aligned} &\left\| w_n - w_n^{hk} \right\|_H^2 - \left\| w_{n-1} - w_{n-1}^{hk} \right\|_H^2 + ck \left\| w_n - w_n^{hk} \right\|_V^2 \\ &\quad \leqslant ck \left\{ \left\| u_n - u_n^{hk} \right\|_V^2 + \left\| \theta_n - \theta_{n-1}^{hk} \right\|_Q^2 + \left\| w_n - v^h \right\|_V \right\} \\ &\qquad + ck \left\{ \left\| \dot{w}_n - \delta w_n \right\|_H^2 + \left\| w_n - v^h \right\|_H^2 + R\left(w_n, v^h\right)^2 \right\} \\ &\qquad + 2k \left(\delta w_n - \delta w_n^{hk}, w_n - v^h \right)_H . \end{aligned} \tag{4.16}$$

We replace n by j and sum over j from 1 to n

$$\begin{aligned} \left\| w_n - w_n^{hk} \right\|_H^2 + ck \sum_{j=1}^n \left\| w_j - w_j^{hk} \right\|_V^2 \leqslant{}& \left\| w_0 - w_0^{hk} \right\|_H^2 \\ & + ck \sum_{j=1}^n \Big(\left\| u_j - u_j^{hk} \right\|_V^2 + \left\| w_j - v_j^h \right\|_V^2 \\ & + \left\| \dot{w}_j - \delta w_j \right\|_H^2 + \left\| w_j - v_j^h \right\|_H^2 \\ & + \left\| \theta_j - \theta_{j-1}^{hk} \right\|_Q^2 + R\left(w_j, v_j^h\right) \Big) \\ & + ck \sum_{j=1}^n \left(\delta w_j - \delta w_j^{hk}, w_j - v_j^h \right)_H . \end{aligned} \tag{4.17}$$

Estimate the latter term in the right hand side of the above inequality

$$\begin{aligned} k \sum_{j=1}^n \left(\delta w_j - \delta w_j^{hk}, w_j - v_j^h \right)_H ={}& \sum_{j=1}^n \left(\left(w_j - w_j^{hk}\right) - \left(w_{j-1} - w_{j-1}^{hk}\right), w_j - v_j^h \right)_H \\ & + \left(w_n - w_n^{hk}, w_n - v_j^h \right)_H - \left(w_0 - w_0^{hk}, w_1 - v_1^h \right)_H \\ & + \sum_{j=1}^{n-1} \left(w_j - w_j^{hk}, w_j - v_j^h - \left(w_{j+1} - v_{j+1}^h\right) \right)_H \\ \leqslant{}& c \Big(\left\| w_n - w_n^{hk} \right\|_H^2 + \left\| w_n - v_n^h \right\|_H^2 + \left\| w_0 - w_0^h \right\|_H^2 \\ & + \left\| w_1 - v_1^h \right\|_H^2 \Big) + 4 \sum_{j=1}^{n-1} k \left\| w_j - w_j^{hk} \right\|_H^2 \\ & + \sum_{j=1}^{n-1} \frac{1}{k} \left\| w_j - v_j^h - \left(w_{j+1} - v_{j+1}^h\right) \right\|_H^2 . \end{aligned} \tag{4.18}$$

We recall the classical inequality

$$\left\| u_j - u_j^{hk} \right\|_V \leqslant \left\| u_0 - u_0^h \right\|_V + \sum_{l=1}^j k \left\| w_l - w_l^{hk} \right\|_V + I_1, \tag{4.19}$$

where

$$I_1 = \left\| \int_0^{t_j} w(s)ds - \sum_{l=1}^{j} k w_l \right\|_V \leqslant k\|u\|_{H^2(0,T;V)}. \tag{4.20}$$

Then

$$\left\|u_j - u_j^{hk}\right\|_V^2 \leqslant c\left(\left\|u_0 - u_0^h\right\|_V^2 + j\sum_{l=1}^{j} k^2\left\|w_l - w_l^{hk}\right\|_V^2 + k^2\|u\|_{H^2(0,T;V)}\right). \tag{4.21}$$

We use the inequality $j \leqslant n \leqslant N$ and $Nk = T$ to get

$$\sum_{j=1}^{n} k\left\|u_j - u_j^{hk}\right\|_V^2 \leqslant cT\left(\left\|u_0 - u_0^h\right\|_V^2 + k^2\|u\|_{H^2(0,T;V)}\right) + T\sum_{j=1}^{n} k\sum_{l=1}^{j}\left\|w_l - w_l^{hk}\right\|_V^2. \tag{4.22}$$

Using the previous inequalities, we obtain

$$\begin{aligned}
\left\|w_n - w_n^{hk}\right\|_H^2 + ck\sum_{j=1}^{n}\left\|w_j - w_j^{hk}\right\|_V^2 \leqslant & ck\left\|\theta_n - \theta_{n-1}^{hk}\right\|_Q^2 + ck^2\left\|u\right\|_{H^2(0,T;V)}^2 \\
& + c\Big(\left\|w_0 - w_0^{hk}\right\|_H^2 + \left\|u_0 - u_0^{hk}\right\|_V^2 \\
& + \left\|w_1 - v_1^h\right\|_H^2 + \left\|w_n - v_n^h\right\|_H^2\Big) \\
& + ck\sum_{j=1}^{n}\Big(\left\|\dot{w}_j - \delta w_j\right\|_V^2 + \left\|w_j - v_j^h\right\|_V^2 \\
& + \left\|\theta_j - \theta_{j-1}^{hk}\right\|_Q^2 + R\left(w_j, v_j^h\right)\Big) \\
& + \frac{1}{k}\sum_{j=1}^{n-1}\left\|w_j - v_j^h - \left(w_{j+1} - v_{j+1}^h\right)\right\|_H^2.
\end{aligned} \tag{4.23}$$

We take $\eta = \eta^h \in Q^h$ at time $t = t_n$ in (2.21)

$$\left(\dot{\theta}_n, \eta^h\right)_{L^2(\Omega)} + d\left(\theta_n, \eta^h\right) - e\left(u_n, \eta^h\right) + \chi\left(u_n, \theta_n, \eta^h\right) = \left(q_{h_n}, \eta^h\right). \tag{4.24}$$

We subtract (2.21) from (4.2)

$$\begin{aligned}
&\left(\dot{\theta}_n - \delta\theta_n^{hk}, \eta^h\right)_{L^2(\Omega)} + d\left(\theta_n - \theta_n^{hk}, \eta^h\right) - e\left(u_n - u_{n-1}^{hk}, \eta^h\right) \\
&+ \chi\left(u_n, \theta_n, \eta^h\right) - \chi\left(u_n^{hk}, \theta_n^{hk}, \eta^h\right) = 0.
\end{aligned} \tag{4.25}$$

We substitute η^h by $\eta^h - \theta_n^{hk}$ into (4.25)

$$\begin{aligned}
&\left(\dot{\theta}_n - \delta\theta_n^{hk}, \eta^h - \theta_n^{hk}\right)_{L^2(\Omega)} + d\left(\theta_n - \theta_n^{hk}, \eta^h - \theta_n^{hk}\right) - e\left(u_n - u_{n-1}^{hk}, \eta^h - \theta_n^{hk}\right) \\
&+ \chi\left(u_n, \theta_n, \eta^h - \theta_n^{hk}\right) - \chi\left(u_n^{hk}, \theta_n^{hk}, \eta^h - \theta_n^{hk}\right) = 0.
\end{aligned} \tag{4.26}$$

Using the identities

$$\begin{aligned}
\left(\dot{\theta}_n - \delta\theta_n^{hk}, \eta^h - \theta_n^{hk}\right)_{L^2(\Omega)} = & \left(\delta\theta_n - \delta\theta_n^{hk}, \eta^h - \theta_n\right)_{L^2(\Omega)} \\
& + \left(\delta\theta_n - \delta\theta_n^{hk}, \theta_n - \theta_n^{hk}\right)_{L^2(\Omega)} + \left(\dot{\theta} - \delta\theta_n, \eta^h - \theta_n^{hk}\right)_{L^2(\Omega)},
\end{aligned} \tag{4.27}$$

and

$$d\left(\theta_n - \theta_n^{hk}, \eta^h - \theta_n^{hk}\right) = d\left(\theta_n - \theta_n^{hk}, \eta^h - \theta_n\right) + d\left(\theta_n - \theta_n^{hk}, \theta^h - \theta_n^{hk}\right), \tag{4.28}$$

we find

$$\begin{aligned}\left(\delta\theta_n-\delta\theta_n^{hk},\theta_n-\theta_n^{hk}\right)_{L^2(\Omega)}&+d\left(\theta_n-\theta_n^{hk},\theta^h-\theta_n^{hk}\right)\\&=\left(\dot{\theta}-\delta\theta_n,\theta_n^{hk}-\eta^h\right)_{L^2(\Omega)}+\left(\delta\theta_n-\delta\theta_n^{hk},\theta_n-\eta^h\right)_{L^2(\Omega)}\\&\quad-d\left(\theta_n-\theta_n^{hk},\eta^h-\theta_n\right)+e\left(u_n-u_{n-1}^{hk},\eta^h-\theta_n\right)+R_\chi,\end{aligned}\tag{4.29}$$

where

$$R_\chi=\chi\left(u_n^{hk},\theta_n^{hk},\eta^h-\theta_n^{hk}\right)-\chi\left(u_n,\theta_n,\eta^h-\theta_n^{hk}\right).\tag{4.30}$$

Using (2.19) and (H7), we get

$$|R_\chi|\leqslant M_{k_c}L_{k_c}s_2^1\left\|\eta^h-\theta_n^{hk}\right\|_Q^2.\tag{4.31}$$

By the inequalities

$$\begin{aligned}&\left(\delta\theta_n-\delta\theta_n^{hk},\theta_n-\theta_n^{hk}\right)_{L^2(\Omega)}\geqslant\frac{1}{2k}\left(\left\|\theta_n-\theta_n^{hk}\right\|_{L^2(\Omega)}^2-\left\|\theta_{n-1}-\theta_{n-1}^{hk}\right\|_{L^2(\Omega)}^2\right),\\&\left\|\eta^h-\theta_n^{hk}\right\|_Q\leqslant\left\|\eta^h-\theta_n\right\|_Q+\left\|\theta_n-\theta_n^{hk}\right\|_Q,\end{aligned}\tag{4.32}$$

and (H1), (H3) and (H5), we conclude that there exist a positive constant c such that

$$\begin{aligned}\left\|\theta_n-\theta_n^{hk}\right\|_{L^2(\Omega)}^2&-\left\|\theta_{n-1}-\theta_{n-1}^{hk}\right\|_{L^2(\Omega)}^2+ck\left\|\theta_n-\theta_n^{hk}\right\|_Q^2\\&\leqslant ck\left(\left\|\dot{\theta}_n-\delta\theta_n\right\|_{L^2(\Omega)}^2+\left\|\theta_n-\eta^h\right\|_Q^2+\left\|u_n-u_{n-1}^{hk}\right\|_V^2\right)\\&\quad+ck\left(\delta\theta_n-\delta\theta_n^{hk},\theta_n-\eta^h\right)_{L^2(\Omega)}.\end{aligned}\tag{4.33}$$

We replace n by j and sum this inequality over j from 1 to n

$$\begin{aligned}\left\|\theta_n-\theta_n^{hk}\right\|_{L^2(\Omega)}^2&+ck\sum_{j=1}^n\left\|\theta_j-\theta_j^{hk}\right\|_Q^2\\&\leqslant\left\|\theta_0-\theta_0^{hk}\right\|_{L^2(\Omega)}^2+ck\sum_{j=1}^n\left(\left\|\dot{\theta}_j-\delta\theta_j\right\|_{L^2(\Omega)}^2+\left\|\theta_j-\eta_j^h\right\|_Q^2\right)\\&\quad+ck\sum_{j=1}^n\left\|u_j-u_{j-1}^{hk}\right\|_V^2+ck\sum_{j=1}^n\left(\delta\theta_j-\delta\theta_j^{hk},\theta_j-\eta_j^h\right)_{L^2(\Omega)}.\end{aligned}\tag{4.34}$$

Moreover,

$$\begin{aligned}k\sum_{j=1}^n\left(\delta\theta_j-\delta\theta_j^{hk},\theta_j-\eta_j^h\right)_{L^2(\Omega)}&=\sum_{j=1}^n\left(\left(\theta_j-\theta_{j-1}\right)-\left(\theta_j^{hk}-\theta_{j-1}^{hk}\right),\theta_j-\eta_j^h\right)_{L^2(\Omega)}\\&=\sum_{j=1}^n\left(\theta_j-\theta_j^{hk},\theta_j-\eta_j^h\right)_{L^2(\Omega)}\\&\quad+\sum_{j=1}^n\left(\theta_{j-1}-\theta_{j-1}^{hk},\theta_j-\eta_j^h\right)_{L^2(\Omega)}\\&=\sum_{j=1}^n\left(\theta_j-\theta_j^{hk},\left(\theta_j-\eta_j^h\right)-\left(\theta_{j+1}-\eta_{j+1}^h\right)\right)_{L^2(\Omega)}\\&\quad+\left(\theta_n-\theta_n^{hk},\theta_n-\eta_n^h\right)_{L^2(\Omega)}-\left(\theta_0-\theta_0^{hk},\theta_1-\eta_1^h\right)_{L^2(\Omega)}.\end{aligned}\tag{4.35}$$

Then

$$\begin{aligned}
&k\sum_{j=1}^{n}\left(\delta\theta_j-\delta\theta_j^{hk},\theta_j-\eta_j^h\right)_{L^2(\Omega)}\\
&\quad\leqslant +c\left(\left\|\theta_n-\theta_n^{hk}\right\|_{L^2(\Omega)}^2+\left\|\theta_n-\eta_n^h\right\|_{L^2(\Omega)}^2+\left\|\theta_0-\theta_0^h\right\|_{L^2(\Omega)}^2+\left\|\theta_1-\eta_1^h\right\|_{L^2(\Omega)}^2\right)\\
&\qquad+\sum_{j=1}^{n-1}\left\|\theta_j-\theta_j^{hk}\right\|_{L^2(\Omega)}\left\|\left(\theta_j-\eta_j^h\right)-\left(\theta_{j+1}-\eta_{j+1}^h\right)\right\|_{L^2(\Omega)}\\
&\quad\leqslant c\left(\left\|\theta_n-\theta_n^{hk}\right\|_{L^2(\Omega)}^2+\left\|\theta_n-\eta_n^h\right\|_{L^2(\Omega)}^2+\left\|\theta_0-\theta_0^h\right\|_{L^2(\Omega)}^2+\left\|\theta_1-\eta_1^h\right\|_{L^2(\Omega)}^2\right)\\
&\qquad+k\sum_{j=1}^{n-1}\left\|\theta_j-\theta_j^{hk}\right\|_{L^2(\Omega)}^2+\frac{1}{k}\sum_{j=1}^{n-1}\left\|\left(\theta_j-\eta_j^h\right)-\left(\theta_{j+1}-\eta_{j+1}^h\right)\right\|_{L^2(\Omega)}^2.
\end{aligned}\tag{4.36}$$

By (4.34), and previous inequality we get

$$\begin{aligned}
&\left\|\theta_n-\theta_n^{hk}\right\|_{L^2(\Omega)}^2+ck\sum_{j=1}^{n}\left\|\theta_j-\theta_j^{hk}\right\|_{Q}^2\\
&\qquad\leqslant c\left\{\left\|\theta_0-\theta_0^{hk}\right\|_{L^2(\Omega)}^2+\left\|\theta_1-\eta_1^h\right\|_{L^2(\Omega)}^2+\left\|\theta_n-\eta_n^h\right\|_{L^2(\Omega)}^2\right\}\\
&\qquad\quad+ck\sum_{j=1}^{n}\left(\left\|u_j-u_{j-1}^{hk}\right\|_V^2+\left\|\dot\theta_j-\delta\theta_j\right\|_{L^2(\Omega)}^2+\left\|\theta_j-\eta_j^h\right\|_Q^2\right)\\
&\qquad\quad+k\sum_{j=1}^{n-1}\left\|\theta_j-\theta_j^{hk}\right\|_{L^2(\Omega)}^2+\frac{1}{k}\sum_{j=1}^{n}\left\|\left(\theta_j-\eta_j^h\right)-\left(\theta_{j+1}-\eta_{j+1}^h\right)\right\|_{L^2(\Omega)}^2.
\end{aligned}\tag{4.37}$$

In the same way, by using in (4.22), we obtain

$$\begin{aligned}
&\sum_{j=1}^{n}k\left\|u_j-u_{j-1}^{hk}\right\|_V^2\leqslant cT\left(\left\|u_0-u_0^h\right\|_V^2+k^2\|u\|_{H^2(0,T;V)}\right)+T\sum_{j=1}^{n-1}k\sum_{l=1}^{j}\left\|w_l-w_l^{hk}\right\|_V^2,\\
&\sum_{j=1}^{n}k\left\|\theta_j-\theta_{j-1}^{hk}\right\|_Q^2\leqslant cT\left(\left\|\theta_0-\theta_0^h\right\|_Q^2+k^2\|\theta\|_{H^1(0,T;Q)}\right)+T\sum_{j=1}^{n-1}k\sum_{l=1}^{j}\left\|\delta\theta_l-\delta\theta_l^{hk}\right\|_Q^2.
\end{aligned}\tag{4.38}$$

Finally, we combine (4.23), (4.37) and (4.38) and arrive at (4.4). The proof is complete. □

The main result of this section is the next theorem.

Theorem 4.1. *Under the assumptions of Theorem 3.1 and the regularity conditions*

$$\begin{aligned}
&u\in C^1\left(0,T;H^2\left(\Omega;\mathbb{R}^d\right)\right)\cap H^3\left(0,T;H\right),\\
&\dot u_{|\Gamma_C}\in C\left(0,T;H^2\left(\Gamma_C,\mathbb{R}^d\right)\right),\\
&\theta\in C\left(0,T;H^2(\Omega)\right)\cap H^2\left(0,T;L^2(\Omega)\right),\ \dot\theta\in L^2\left(0,T;H^1(\Omega)\right),
\end{aligned}\tag{4.39}$$

we have the estimate

$$\max_{1\leqslant n\leqslant N}\left\{\left\|w_n-w_n^{hk}\right\|_H+\left\|u_n-u_n^{hk}\right\|_V+\left\|\theta_n-\theta_n^{hk}\right\|_{L^2(\Omega)}\right\}\leqslant c(h+k).\tag{4.40}$$

Proof. Applying the discrete Gronwall inequality 4.1, we find

$$\begin{aligned}
\max_{1\leqslant n}\{e_n\}\leqslant&c\left\{\left\|w_0-w_0^{hk}\right\|_H^2+\left\|u_0-u_0^{hk}\right\|_V^2+\left\|\theta_0-\theta_0^{hk}\right\|_{L^2(\Omega)}^2\right\}\\
&+ck^2\left\|\theta\right\|_{H^1(0,T;Q)}^2+ck^2\left\|u\right\|_{H^2(0,T;V)}^2+c\max_{1\leqslant n}\{g_n\},
\end{aligned}\tag{4.41}$$

where

$$e_n = \left\|w_n - w_n^{hk}\right\|_H^2 + \left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)}^2 + ck \sum_{j=1}^{n} \left(\left\|w_j - w_j^{hk}\right\|_V^2 + \left\|\theta_j - \theta_j^{hk}\right\|_Q^2\right) \tag{4.42}$$

and

$$\begin{aligned} g_n = \inf_{\substack{v_j^h \in V^h \\ \eta_j^h \in Q^h}} \Bigg(& k \sum_{j=1}^{n} \left(\|\dot{w}_j - \delta w_j\|_H^2 + \left\|w_j - v_j^h\right\|_V^2\right) \\ & + \sum_{j=1}^{n} \left(\left\|\dot{\theta}_j - \delta\theta_j\right\|_{L^2(\Omega)}^2 + \left\|\theta_j - \eta_j^h\right\|_Q^2 + R\left(w_j, v_j^h\right)\right) \\ & + \frac{1}{k} \sum_{j=1}^{n-1} \left(\left\|\left(w_j - v_j^n\right) - \left(w_{j+1} - v_{j+1}^h\right)\right\|_H^2 + \left\|\left(\theta_j - \eta_j^n\right) - \left(\theta_{j+1} - \eta_{j+1}^h\right)\right\|_{L^2(\Omega)}^2\right) \\ & + \left\|w_1 - v_1^h\right\|_H^2 + \left\|\theta_1 - \eta_1^h\right\|_{L^2(\Omega)}^2 + \left\|w_n - v_n^h\right\|_H^2 + \left\|\theta_n - \eta_n^h\right\|_{L^2(\Omega)}^2 \Bigg). \end{aligned} \tag{4.43}$$

Let $v_j^h \in V^h$, $\eta_j^h \in Q^h$ be the finite element interpolation of u_j and θ_j, respectively. The following approximation properties hold [9], [10]

$$\begin{aligned} &\max_{1 \leqslant n \leqslant N} \left\|w_n - v_n^h\right\|_V \leqslant ch \|w\|_{C\left(0,T;H^2(\Omega)^d\right)}, \\ &\max_{1 \leqslant n \leqslant N} \left\|\theta_n - \eta_n^h\right\|_Q \leqslant ch \|\theta\|_{C(0,T;H^2(\Omega))}, \end{aligned} \tag{4.44}$$

$$\begin{aligned} &\left\|w_0 - w_0^h\right\|_V \leqslant ch \|w_0\|_{H^2\left(\Omega,\mathbb{R}^d\right)}, \\ &\left\|u_0 - u_0^h\right\|_H \leqslant ch \|u_0\|_{H^1\left(\Omega,\mathbb{R}^d\right)}, \\ &\left\|\theta_0 - \theta_0^h\right\|_{L^2(\Omega)} \leqslant ch \|\theta_0\|_{L^2(\Omega)}, \end{aligned} \tag{4.45}$$

and

$$\begin{aligned} & k \sum_{j=1}^{n} \left(\|\dot{w}_j - \delta w_j\|_H + \left\|\dot{\theta}_j - \delta\theta_j\right\|_{L^2(\Omega)}\right) \leqslant ck^2 \|u\|_{H^2(0,T;L^2(\Omega))} + ck^2 \|\theta\|_{H^2(0,T;L^2(\Omega)}^2, \\ & \frac{1}{k} \sum_{j=1}^{n-1} \left(\left\|\left(w_j - v_j^n\right) - \left(w_{j+1} - v_{j+1}^h\right)\right\|_H^2 + \left\|\left(\theta_j - \eta_j^n\right) - \left(\theta_{j+1} - \eta_{j+1}^h\right)\right\|_{L^2(\Omega)}^2\right) \\ & \qquad \leqslant ch^2 \|u\|_{H^2(0,T;V)}^2 + ch^2 \|\theta\|_{H^2(0,T;Q)}. \end{aligned} \tag{4.46}$$

Similarly to the proof presented in [10], [20] we obtain

$$\left|R\left(w_j, v_j^h\right)\right| \leqslant c \left\|w_n - v_n^h\right\|_{L^2(\Gamma_C)^d} \leqslant ch^2 \|w_n\|_{C\left(0,T;H^2(\Omega)^d\right)}. \tag{4.47}$$

Finally, we combine the previous estimates (4.42), (4.44)–(4.47) and arrive at (4.40). The proof is complete. □

5. Numerical simulations

This section provides computer simulation results on the contact Problem 4.1, including numerical evidence of the theoretical error estimates obtained in the previous section for the discrete approximation of the variational problem. The solution of Problem 4.1 is based on numerical methods described in [1], [14]. For more considerations about computational contact mechanics, see [19].

The physical setting used for Problem 4.1 is shown in Figure 1. In this case the body $\Omega = (0,1)\times(0,1) \subset \mathbb{R}^2$ is clamped on $\Gamma_D = [0,1]\times\{0\}$. The tractions f_N^1 and f_N^2 are prescribed on the lateral parts Γ_N^1, Γ_N^2, respectively, that is, $\Gamma_N := \Gamma_N^1 \cup \Gamma_N^2$). The body is in contact with a thermally conductive foundation on its lower boundary $\Gamma_C = [0,1]\times\{0\}$.

The material response is governed by a linear viscoelastic constitutive law, in which the elasticity tensor $\mathcal{E}$ and the viscosity tensor $\mathcal{V}$ read

$$\begin{aligned}
(\mathcal{E}\tau)_{ij} &= \frac{E\nu}{1-\nu^2}(\tau_{11}+\tau_{22})\delta_{ij} + \frac{E}{1+\nu}\tau_{ij}, && 1 \leqslant i,j \leqslant 2, \quad \tau \in \mathbb{S}^2,\\
(\mathcal{V}\tau)_{ij} &= \mu_1(\tau_{11}+\tau_{22})\delta_{ij} + \mu_2\tau_{ij}, && 1 \leqslant i,j \leqslant 2, \quad \tau \in \mathbb{S}^2,
\end{aligned} \tag{5.1}$$

where E is the Young modulus, ν is the Poisson ratio of the material, δ_{ij} denotes the Kronecker delta, and μ_1 and μ_2 are viscosity constants.

The functions p_ν and p_τ in the frictional contact conditions (2.8) and (2.9) are given by $p_\nu(r) = c_\nu r_+$ and $p_\tau = \mu_\tau p_\nu$, where c_ν is a large positive constant and μ_τ is the friction coefficient. The truncation function ϕ_L and the conductivity functions k_c in the conditions (2.10) are given by $\phi_L(s) = s$ and

$$k_c(s) = \begin{cases} \bar{k}_c \dfrac{s}{\epsilon_c}, & 0 \leqslant s \leqslant \epsilon_c,\\ \bar{k}_c, & s > \epsilon_c,\\ 0, & \text{otherwise}, \end{cases}$$

where $\bar{k}_c$ and ϵ_c are positive constants.

For computation we use the following data (IS unity):

$E = 2, \quad \nu = 0.1, \quad \mu_1 = 10, \quad \mu_2 = 10, \quad \mathcal{M}_{ij} = \mathcal{K}_{ij} = \mathcal{R}_{ij} = 1, \quad 1 \leqslant i,j \leqslant 2,$
$f_0 = (0,-1), \quad q_0 = 1, \quad f_N^1 = (1.4, 0.4),\ f_N^2 = (-0.8, 0.4), \quad c_\nu = 10^4, \quad \mu_\tau = 0.2,$
$g = 0, \quad \bar{k}_c = 1,\ \epsilon_c = 10^{-5}, \quad T = 1, \quad u_0 = 0,\ \dot{u}_0 = 0, \quad \theta_0 = 0.$

Our interest in this example is to study the influence of the thermal conductivity of the

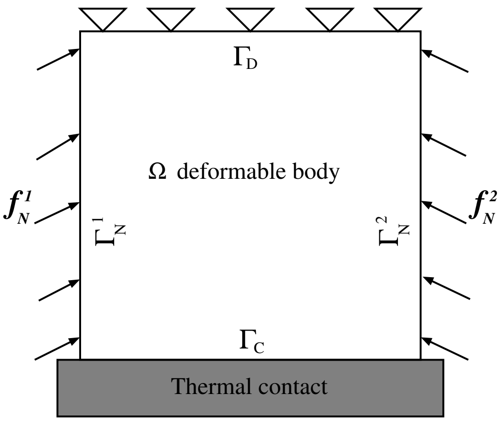

FIGURE 1. Physical setting.

foundation on the contact process. Thus, in Figure 2, we show the deformed configurations at final time, and in Figures 4 and 3, the corresponding norm of the temperature and stress fields through the body for two different values of the temperatures of the foundation. These figures show that, as the temperature of the foundation is more important then the deformations, the norm of the stress and the temperature are larger. To see the convergence behaviour of the fully discrete scheme, we compute a sequence of numerical solutions based on uniform partitions of the time interval $[0,T]$, and uniform triangulations of the body. Then, we provide the estimated

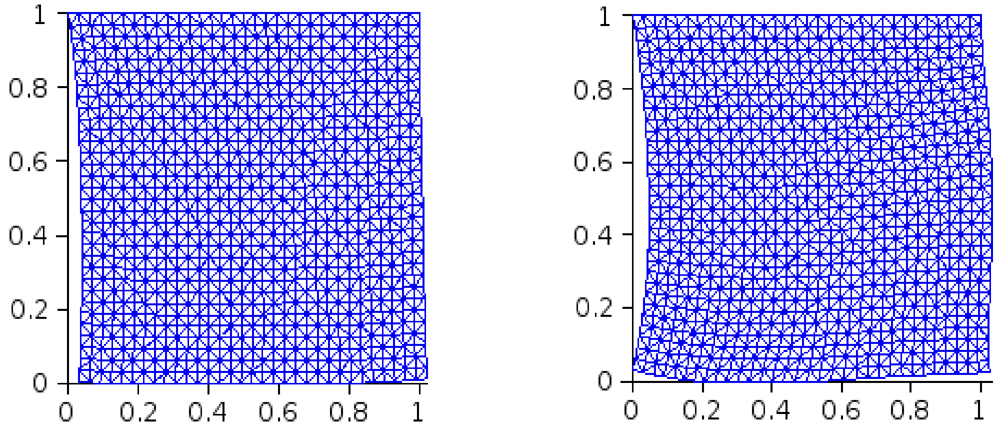

FIGURE 2. Deformed configuration for $\theta_F = 0$ (left) and $\theta_F = 10$ (right).

error values for several discretization parameters h and k. Here, the sides of the square are divided into $1/h$ equal parts and the time interval $[0, T]$ is divided into $1/k$ time steps. We start with $h = 1/16$ and $k = 1/16$ which are successively halved. The numerical solution corresponding to $h = 1/256$ and $k = 1/256$ has been considered as the "exact" solution in order to compute the numerical errors given by

$$E^{hk} = \max_{1\leqslant n\leqslant N} \left\{ \left\|w_n - w_n^{hk}\right\|_H + \left\|u_n - u_n^{hk}\right\|_V + \left\|\theta_n - \theta_n^{hk}\right\|_{L^2(\Omega)} \right\}. \tag{5.2}$$

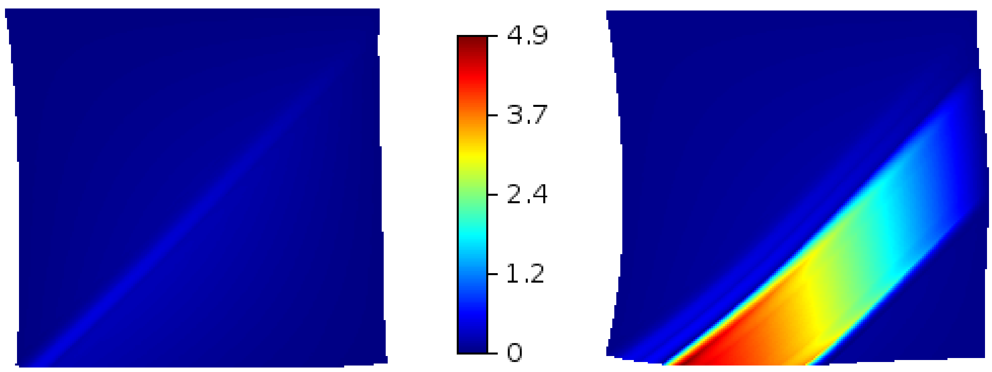

FIGURE 3. Temperature field for $\theta_F = 0$ (left) and $\theta_F = 10$ (right).

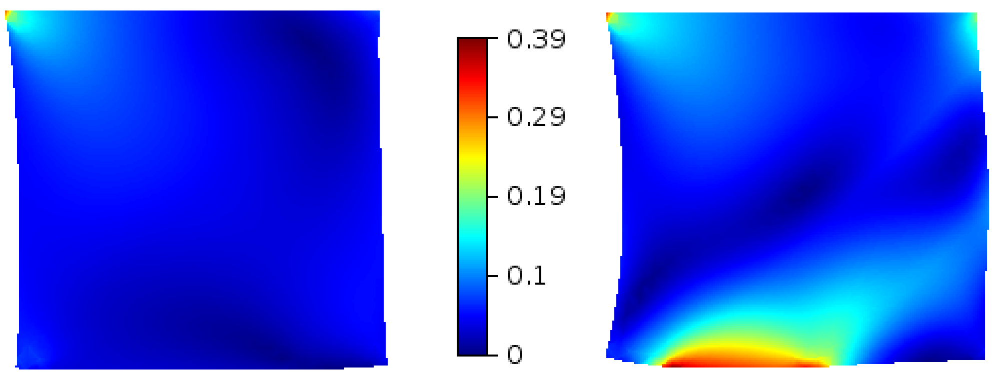

FIGURE 4. Von Mises stress norm for $\theta_F = 0$ (left) and $\theta_F = 10$ (right).

The linear asymptotic convergence behaviour obtained in (4.40) is almost observed (see Figure 5).

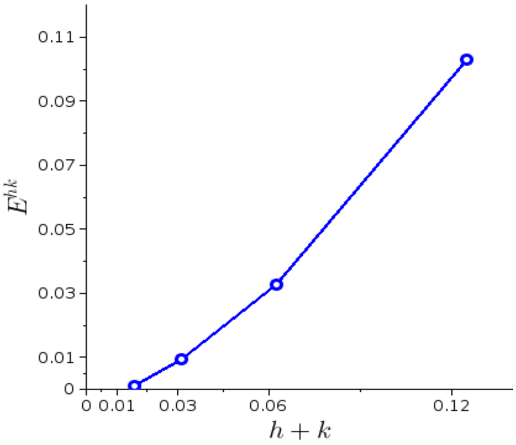

FIGURE 5. Estimated errors.

ACKNOWLEDGEMENTS

The authors thank the referees for their careful reading of the paper and for their useful remarks, which helped improve the presentation of the results.

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