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BIFURCATIONS OF PERIODIC OSCILLATIONS IN DYNAMICAL SYSTEM WITH HOMOGENEOUS NONLINEARITIES

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Abstract. The paper is devoted to the study of cycle bifurcations and bifurcations at infinity for dynamical systems with a small parameter, the nonlinearities of which contain homogeneous polynomials of even or odd degree, and the unperturbed equation has a continuum of periodic solutions. We propose new necessary and sufficient conditions for these bifurcations, obtain the formulas for the approximate construction of bifurcation solutions, and analyze their stability. We show that cycle bifurcations are typical only for systems with homogeneities of odd degree, while the bifurcations at infinity are typical only for systems with homogeneities of even degree. We demonstrate the relationship between these bifurcations and the classical Andronov — Hopf bifurcation.

Keywords: bifurcation, Andronov — Hopf bifurcation, cycles, bifurcations at infinity, homogeneity.

Mathematics Subject Classification: 34C23, 37G10, 37G15

1. Introduction and formulation of problem

We consider the dynamical system depending of a small parameter α

$$\frac{dx}{dt} = U(x) + \alpha f(x), \quad x \in \mathbb{R}^N \quad (N \geqslant 2), \tag{1.1}$$

where $U(x)$ and $f(x)$ are continuously differentiable vector function defined for all x. It is supposed that $U(0) = 0$, that is, the unperturbed system

$$\frac{dx}{dt} = U(x), \quad x \in \mathbb{R}^N, \tag{1.2}$$

has a zero equilibrium $x = 0$.

In the system (1.1) various bifurcations are possible, which are related with the emergence of periodic solutions for small non–zero values of the parameter α. In the present work we consider three bifurcation scenarios.

The first is the classical Andronov — Hopf bifurcation, which is related with with the emergence of small amplitude periodic orbits in the system (1.1) branching off from the equilibrium point $x = 0$ of the unperturbed system (1.2). The value $\alpha = 0$ is called the *Andronov — Hopf bifurcation point* for the system (1.1) if there exists a number $\varepsilon_0 > 0$ and continuously differentiable functions defined for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ such that

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U1) $\alpha(0) = 0$, $T(0) = T_0$, $\alpha(\varepsilon) \neq 0$ for $\varepsilon \neq 0$;

U2) for non–zero $\alpha = \alpha(\varepsilon)$ the system (1.1) possesses a non–stationary $T(\varepsilon)$–periodic solution $x = x(t, \varepsilon)$;

U3) the relation hold

$$\max_{0 \leqslant t \leqslant T_0} \|x(t, \varepsilon)\| \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Here T_0 is some positive number determined by the spectrum of the Jacobi matrix $U'(0)$. By $\|x\|$ we denote the Euclidean norm of a vector $x \in \mathbb{R}^N$.

The second bifurcation scenario is related with the emergence of periodic orbits in the system (1.1) branching off from a certain cycle Υ_0 of the unperturbed system (1.2). This bifurcation scenario is due to the assumption that the unperturbed system (1.2) has a family of periodic solutions $x = \varphi(t, C)$.

Let $x = \varphi_0(t)$ be a non–stationary periodic solution in this family, T_0 be the period of this solution, and Υ_0 be the corresponding trajectory in the phase space $\mathbb{R}^N$ of the system (1.2).

The value $\alpha = 0$ is called the *bifurcation point of cycles* of the system (1.1), branching off from the trajectory Υ_0 of the system (1.2) if there exists a number $\varepsilon_0 > 0$ and continuously differentiable functions defined for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ such that the conditions U1 and U2 are satisfied, and instead of U3, the following condition is satisfied:

UC) the relation holds

$$\max_{0 \leqslant t \leqslant T_0} \|x(t, \varepsilon) - \varphi_0(t)\| \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Finally, the third bifurcation scenario is related with the emergence of large–amplitude periodic orbits in the system (1.1). The value $\alpha = 0$ is called the *Andronov— Hopf bifurcation point at infinity* if there exists a number $\varepsilon_0 > 0$ and continuously differentiable functions defined for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ such that conditions U1 and U2 are satisfied, and instead of U3, the following condition is satisfied:

UB) the relations hold

$$\rho(\varepsilon) = \max_t \|x(t, \varepsilon)\| \to \infty, \quad \frac{\alpha(\varepsilon)}{\rho(\varepsilon)} \max_t \|f(x(t, \varepsilon))\| \to 0, \quad \text{as} \quad \varepsilon \to 0. \tag{1.3}$$

Here T_0 is some positive number determined by the spectrum of the Jacobi matrix $U'(0)$.

We mention that, when studying the Andronov — Hopf bifurcation problem at infinity, many authors restrict themselves only to the first of the relations in (1.3). The second relation in our formulation is related with the aim to adapt the concept of Andronov — Hopf bifurcation at infinity with the classical concept of bifurcation of solutions of large norms (see, for example, [6]), according to which the solutions of large norms should be generated by solutions of the unperturbed equation in the appropriate formulation. Being applied to the system (1.1), this means that periodic orbits of large amplitudes should branch off from the large–amplitude cycles of the unperturbed system (1.2).

A vast literature was devoted to the study of these bifurcations. A special place is occupied by the problem on the classical Andronov — Hopf bifurcation, the deep studies of this problem and the developed effective methods allow us to speak about the appearance of the Andronov — Hopf bifurcation theory, see, for example, [4], [16], [17], and the references therein. The cycle bifurcation problem was studied by many authors. A fundamental result here is the Pontryagin theorem [1], [13], which proposes a method for studying the cycle bifurcation problem in systems close to Hamiltonian ones. An effective approach to studying this problem is offered by the methods of averaging theory based on the classical works of N.N. Bogolyubov and N.M. Krylov, see, for example, [14]. The research continue in various directions, see, for example, [10]–[12], [24].

The Andronov — Hopf bifurcation problem at infinity was also studied by many authors. Various questions were addressed, both theoretical and related to applications in the theory of Hamiltonian systems, control theory, mechanics, and elsewhere, see [7]–[9], [20]–[22].

It is of interest to study comprehensively all three of these bifurcation scenarios in systems of the type (1.1). Here, relevant questions include the relationships between these bifurcations, the determination of necessary and sufficient conditions for bifurcations, the approximate construction of solutions, and the analysis of their stability. The studies of systems (1.1), in which the nonlinearities contain homogeneous polynomials of even or odd degree, are especially important. In the modern nonlinear dynamics, the study of such systems attracted an increasing attention, particularly due to the fact that these systems exhibit rich bifurcation and chaotic behavior, see, for example, [15], [25]. At the same time, many issues in the study of cycle bifurcation problems and the Andronov — Hopf bifurcation at infinity in systems with homogeneous nonlinearities remain poorly understood.

In this paper we focuse on the system (1.1), in which the function $f(x)$ can be represented as $f(x) = B_1x + b_q(x)$; here B_1 is a square (of size N) real matrix, and the nonlinearity $b_q(x)$ is a homogeneous polynomial of degree q, $q \geqslant 2$. We propose new necessary and sufficient conditions for cycle bifurcations and bifurcations at infinity in such systems, which make it possible, in particular, to establish that cycle bifurcations are typical only for the systems with homogeneities of odd degree, while bifurcations at infinity are typical only for the systems with homogeneities of even degree. The proposed bifurcation features are based on new approaches that combine methods of averaging theory and operator methods for studying problems of multiparameter bifurcations, see [3], [5].

In the present work we also propose new asymptotic formulas for the approximate construction of bifurcation solutions and for the study of their stability in problems of cycle bifurcations and bifurcations at infinity in the system (1.1) of arbitrary orders q of homogeneous nonlinearity $b_q(x)$. These formulas extend the results of [18], [19], and [23], in which similar problems were studied for systems with quadratic and cubic nonlinearities.

2. Main object of study

The main object of study in this paper is the system (1.1), in which $U(x)$ is a linear function. Namely, we consider the system

$$\frac{dx}{dt} = B_0x + \alpha f(x), \quad x \in \mathbb{R}^N, \tag{2.1}$$

in which B_0 is a square (of size N) real matrix, $f(x)$ is a continuously differentiable vector function. We make the following assumptions

V1) the matrix B_0 has a pair of pure imaginary eigenvalues $\lambda = \pm\omega_0 i$, $(\omega_0 > 0)$;

V2) the other eigenvalues of matrix B_0 have non–zero real parts.

By Assumption V1 there exist non–zero vectors $e, g, e^*, g^* \in \mathbb{R}^N$ such that identities hold

$$B_0(e + ig) = i\omega_0(e + ig), \quad B_0^*(e^* + ig^*) = -i\omega_0(e^* + ig^*), \tag{2.2}$$

where B_0^* is the transposed matrix.

In what follows, for simplicity, if this cause no confusion, we shall use the same notation for a square (of order N) matrix and the linear operator generated by this matrix in the standard basis of the space $\mathbb{R}^N$.

We denote by E_0 the eigenspace of the operator B_0 associated with simple eigenvalues $\pm i\omega_0$. The space E_0 is two–dimensional; the vectors e and g can be used as its basis. The space $\mathbb{R}^N$ can be represented as a direct sum $\mathbb{R}^N = E_0 \oplus E^0$, where E^0 is an additional invariant subspace of dimension $N-2$ for B_0.

By the mentioned assumptions the phase portrait of the linear two–dimensional system

$$\frac{dx}{dt} = B_0 x, \quad x \in E_0, \tag{2.3}$$

has a type "center", all its solutions as T_0–periodic, where $T_0 = \dfrac{2\pi}{\omega_0}$. These solutions can be represented as $x = x_0(t, C) = C\varphi_0(t)$, where C is an arbitrary constant, and the function $\varphi_0(t)$ is defined by the identity

$$\varphi_0(t) = e\cos\omega_0 t - g\sin\omega_0 t. \tag{2.4}$$

For simplicity of presentation, most of the constructions and main results will be discussed for the case when the system (2.1) is two–dimensional, that is, for the system

$$\frac{dx}{dt} = B_0 x + \alpha f(x), \quad x \in \mathbb{R}^2. \tag{2.5}$$

Hence, the unperturbed system reads

$$\frac{dx}{dt} = B_0 x, \quad x \in \mathbb{R}^2. \tag{2.6}$$

The general multi–dimensional case $N \geqslant 3$ is briefly discussed in the concluding part of paper.

3. Study of problem on bifurcation of cycles

We first discuss the problem on bifurcations of cycles in the system (2.5). We make the non–degenerate T_0–periodic change

$$y = e^{-B_0 t} x \tag{3.1}$$

in this system. As a result, the system (2.5) becomes

$$\frac{dy}{dt} = \alpha e^{-B_0 t} f(e^{B_0 t} y), \quad y \in \mathbb{R}^2, \tag{3.2}$$

where the right hand side is T_0–periodic.

Together with (3.2) we consider the averaged system

$$\frac{du}{dt} = \alpha F(u), \quad u \in \mathbb{R}^2, \tag{3.3}$$

where

$$F(u) = \frac{1}{T_0}\int_0^{T_0} e^{-B_0 t} f(e^{B_0 t} u)\, dt. \tag{3.4}$$

3.1. Necessary condition for bifurcation of cycles. The next statement provides a necessary condition for bifurcation of cycles of the system (2.5).

Theorem 3.1. *Let the value $\alpha = 0$ be the point of bifurcation of cycles of the system (2.5), which branch off from some trajectory Υ_0 of the linear system (2.6). Then each vector $u_0 \in \Upsilon_0$ is the equilibrium of the averaged system (3.3), that is, $F(u_0) = 0$.*

Proof. Let the value $\alpha = 0$ be the point of bifurcation of cycles of the system (2.5), that is, there exist continuous functions $\alpha(\varepsilon)$ and $T(\varepsilon)$ such that the conditions U1 and U2 hold as well as the condition UC, which can be represented as

$$\max_{0\leqslant t\leqslant T_0} \|x(t,\varepsilon) - C_0\varphi_0(t)\| \to 0 \quad \text{as} \quad \varepsilon \to 0; \tag{3.5}$$

here $\varphi_0(t)$ is the function (2.4), C_0 is some positive number. Then Equation (3.2) has the solution

$$y = y(t,\varepsilon) = e^{-B_0 t}x(t,\varepsilon), \tag{3.6}$$

that is,

$$\frac{dy(t,\varepsilon)}{dt} \equiv \alpha(\varepsilon)e^{-B_0 t}f(e^{B_0 t}y(t,\varepsilon)), \quad y \in \mathbb{R}^2. \tag{3.7}$$

The function (3.6) is almost periodic in t. This is why there exists a sequence $T_k \to \infty$ such that $\|y(0,\varepsilon) - y(T_k,\varepsilon)\| \to 0$ as $k \to \infty$. Integrating the identity (3.7) over the segment $[0, T_k]$, in view of the condition U1 we find that as $k \to \infty$, the identity holds

$$\int_0^{T_k} e^{-B_0 t} f(e^{B_0 t}y(t,\varepsilon))\,dt \to 0,$$

or, in view of (3.6), the identity

$$\int_0^{T_k} e^{-B_0 t} f(x(t,\varepsilon))\,dt \to 0. \tag{3.8}$$

Since in this identity the integrand is almost periodic in t, the limit

$$\lim_{T\to\infty} \frac{1}{T}\int_0^{T} e^{-B_0 t} f(x(t,\varepsilon))\,dt$$

is well–defined. By (3.8) this implies

$$\lim_{T\to\infty} \frac{1}{T}\int_0^{T} e^{-B_0 t} f(x(t,\varepsilon))\,dt = 0. \tag{3.9}$$

By (3.5) the function $x(t,\varepsilon)$ can be represented as

$$x(t,\varepsilon) = C_0\varphi_0(t) + \delta(t,\varepsilon),$$

where the function $\delta(t,\varepsilon)$ is almost periodic in t, smooth in ε and satisfies the relation

$$\max_t \|\delta(t,\varepsilon)\| \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Then the identity (3.9) casts into the form

$$\lim_{T\to\infty} \frac{1}{T}\int_0^{T} e^{-B_0 t} f(C_0\varphi_0(t) + \delta(t,\varepsilon))\,dt = 0.$$

This identity holds for all small ε. This is why

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_0^T e^{-B_0 t} f(C_0 \varphi_0(t))\, dt = 0.$$

In the obtained identity the integrand is T_0–periodic. Hence,

$$\int\limits_0^{T_0} e^{-B_0 t} f(C_0 \varphi_0(t))\, dt = 0.$$

We observe that the function (2.4) we have $\varphi_0(t) = e^{B_0 t} e$. Therefore, letting $u_0 = C_0 e$, we obtain

$$\int\limits_0^{T_0} e^{-B_0 t} f(e^{B_0 t} u_0)\, dt = 0.$$

Due to the arbitrary choice of the vector e, this completes the proof. □

3.2. Study of systems with homogeneous nonlinearities. There arises a natural question for which systems of form (2.5) the necessary condition of bifurcation of cycles hold, that is, in which case the averaged system (3.3) has non–zero equilibria. In order to study this issue, we point out the following facts.

Since the matrix B_0 is non–degenerate, there exist $\delta_1, \delta_2 > 0$ such that for $|\alpha| < \delta_2$ the system (2.5) has a unique equilibrium $x = x^*(\alpha)$ in the ball $\|x\| < \delta_1$ such that $x^*(0) = 0$ and the function $x^*(\alpha)$ is smooth. We can suppose that $x^*(\alpha) \equiv 0$, that is, the function $f(x)$ satisfies the identity $f(0) = 0$. Then the function $f(x)$ can be represented as

$$f(x) = B_1 x + b(x),$$

where B_1 is a square (of size 2) real matrix, and then nonlinearity $b(x)$ satisfies the relation

$$\|b(x)\| = O(\|x\|^2) \quad \text{as} \quad x \to 0.$$

In what follows we study the systems with homogeneous nonlinearities, namely, we shall suppose that the function $f(x)$ can be represented as

$$f(x) = B_1 x + b_q(x),$$

where the nonlinearity $b_q(x)$ is a homogeneous polynomial of degree q, $q \geqslant 2$, and therefore it satisfies the condition

$$b_q(\lambda x) \equiv \lambda^q b_q(x).$$

Thus, the system (2.5) casts into the form

$$\frac{dx}{dt} = B_0 x + \alpha[B_1 x + b_q(x)], \quad x \in \mathbb{R}^2. \tag{3.10}$$

We also suppose that the vectors e, g, e^*, g^* defined by the identities (2.2) are normalized as

$$(e, e^*) = (g, g^*) = 1, \qquad (e, g^*) = (g, e^*) = 0. \tag{3.11}$$

By the matrix B_1 and vectors e, g, e^*, g^* in (2.2) we define the numbers

$$\gamma_1 = (B_1 e, e^*) + (B_1 g, g^*), \qquad \gamma_2 = (B_1 e, g^*) - (B_1 g, e^*). \tag{3.12}$$

Theorem 3.2. *Let $\gamma_1 \neq 0$ and the value $\alpha = 0$ is the point of bifurcation of cycles of the system (3.10). Then q is odd, that is, in the system (3.10) the nonlinearity $b_q(x)$ is a homogeneous polynomial of odd degree.*

Proof. It is sufficient to show that if q, that is, if in the system (3.10) the nonlinearity $b_q(x)$ is a homogeneous polynomial of an even degree, then the necessary condition for bifurcation of cycles provided in Theorem 3.1 fails.

Let q be even. By Theorem 3.1, for the bifurcation of cycles the equation $F(u) = 0$ necessarily has a non–zero solution; here $F(u)$ is the function (3.4), which in our case reads

$$F(u) = \frac{1}{T_0} \int\limits_0^{T_0} e^{-B_0 t}[B_1 e^{B_0 t} u + b_q(e^{B_0 t} u)]\, dt. \tag{3.13}$$

In what follows we shall employ two auxiliary statements.

Lemma 3.1. *Let q be even. Then for each vector $u \in \mathbb{R}^2$ the identity holds*

$$\int\limits_0^{T_0} e^{-B_0 t} b_q(e^{B_0 t} u)\, dt = 0. \tag{3.14}$$

Indeed, since the matrix B_0 has a pair of purely imaginary eigenvalues $\lambda = \pm\omega_0 i$, $\omega_0 > 0$, we can suppose that the matrix $e^{B_0 t}$ is of the form

$$e^{B_0 t} = \begin{bmatrix} \cos\omega_0 t & -\sin\omega_0 t \\ \sin\omega_0 t & \cos\omega_0 t \end{bmatrix}.$$

Then the function $b_q(e^{B_0 t} u)$ contains only even powers of the functions $\cos\omega_0 t$ and $\sin\omega_0 t$, while the product $e^{-B_0 t} \cdot b_q(e^{B_0 t} u)$ forms odd powers of these trigonometric functions. This fact ensures the identity (3.14).

We define the matrix

$$D = \frac{1}{T_0} \int\limits_0^{T_0} e^{-B_0 t} B_1 e^{B_0 t}\, dt. \tag{3.15}$$

It is easy to establish the next lemma.

Lemma 3.2. *Let $\gamma_1 \neq 0$, where γ_1 is the number from (3.12). Then* $\det D \neq 0$, *that is, the matrix (3.15) is invertible.*

It follows from Lemma 3.1 that the function (3.13) is linear and is of the form

$$F(u) = \frac{1}{T_0} \int\limits_0^{T_0} e^{-B_0 t} B_1 e^{B_0 t} u\, dt.$$

Then by Lemma 3.2 the equation $F(u) = 0$ has only the zero solution. The proof is complete. □

3.3. Sufficient condition for bifurcation of cycles. In what follows we discuss the problem on bifurcation of cycles only in the case, when the nonlinearity $b_q(x)$ in the system (3.10) is a homogeneous polynomial of an odd degree.

The sufficient condition for the bifurcation of cycles given above is based on operator methods for studying problems on multi–parametric bifurcations, see [3], [18]. Following these works, we define the vectors

$$e(t) = e\cos 2\pi t - g\sin 2\pi t, \qquad \xi_3 = T_0 \int\limits_0^1 e^{-tT_0 B_0} b_q(e(t))\, dt, \tag{3.16}$$

and the numbers

$$\alpha_2 = -\frac{\omega_0}{\pi\gamma_1}(\xi_3, e^*), \qquad T_2 = \frac{1}{\omega_0}\Big[(\xi_3, g^*) - \frac{\gamma_2}{\gamma_1}(\xi_3, e^*)\Big]. \tag{3.17}$$

Theorem 3.3. *Let*

$$\gamma_1 \neq 0, \qquad \alpha_2 \neq 0. \tag{3.18}$$

Then $\alpha = 0$ is the point of bifurcation of cycles of the system (3.10), which branch off from the trajectory Υ_0 of the linear system (2.6) containing the vector u^*. Here $u^* = (\alpha_2)^{\frac{1}{(1-q)}} e$ (if $\alpha_2 > 0$) or $u^* = (-\alpha_2)^{\frac{1}{(1-q)}} e$ (if $\alpha_2 < 0$).

Theorem 3.3 is a continuation of a similar result obtained in [18].

Proof. For the sake of definiteness we suppose that $\alpha_2 > 0$; the case $\alpha_2 < 0$ can be considered similarly. The next lemma is obvious.

Lemma 3.3. *Let q be odd and $\alpha > 0$. Then the change $y = \alpha^{\frac{1}{(1-q)}} x$ reduces the system (3.10) to the form*

$$y' = (B_0 + \alpha B_1)y + b_q(y), \qquad y \in \mathbb{R}^2. \tag{3.19}$$

The inverse change reduces the system (3.19) to the system (3.10).

The condition $\gamma_1 \neq 0$ in (3.18) means [18] that the value $\alpha = 0$ is point of Andronov — Hopf of the system (3.19). Namely, for $\alpha = \alpha(\varepsilon)$ the system (3.19) has a non–stationary $T(\varepsilon)$–periodic solution $y(t,\varepsilon)$ with a small amplitude, and the functions $\alpha(\varepsilon)$, $T(\varepsilon)$ and $y(t,\varepsilon)$ can be represented as

$$\alpha(\varepsilon) = \alpha_2\varepsilon^{q-1} + O(\varepsilon^{q+1}), \quad T(\varepsilon) = T_0 + T_2\varepsilon^{q-1} + O(\varepsilon^{q+1}), \quad y(0,\varepsilon) = \varepsilon e + O(\varepsilon^3). \tag{3.20}$$

By these identities the bifurcation solutions $y(t,\varepsilon)$ of the system (3.19) emerge for $\alpha > 0$. By Lemma 3.3 this implies that the system (3.10) for $\alpha = \alpha(\varepsilon)$ has a non–stationary $T(\varepsilon)$–periodic solution

$$x(t,\varepsilon) = (\alpha(\varepsilon))^{\frac{1}{(1-q)}} y(t,\varepsilon).$$

This is why by the identities (3.20) we get the relation

$$x(0,\varepsilon) = (\alpha_2)^{\frac{1}{(1-q)}} e + O(\varepsilon^2).$$

This means that the value $\alpha = 0$ is the point of bifurcations of cycles of the system (3.10), which branch off from the trajectory Υ_0 of the linear system (2.6) containing the vector $u^* = (\alpha_2)^{\frac{1}{(1-q)}} e$. We note that the vector u^* is the equilibrium of the averaged system (3.3). The proof is complete. □

Theorem 3.3 and the results of the work [18] imply the next statement.

Theorem 3.4. *Under the assumptions of Theorem 3.3 the bifurcation solutions $x(t,\varepsilon)$ of the system (3.10) emerge for $\alpha > 0$ (if $\alpha_2 > 0$) or for $\alpha < 0$ (if $\alpha_2 < 0$). These solutions are orbitally asymptotically stable (unstable) if $(\xi_3, e^*) < 0$ (if $(\xi_3, e^*) > 0$).*

3.4. Example 1: Van der Pol equation. We consider the Van der Pol equation of form, see, for instance, [14],

$$y'' + \alpha(y^2 - 1)y' + y = 0. \tag{3.21}$$

The change $x_1 = y, x_2 = y'$ reduces this equation to a system of form (3.10) with $q = 3$,

$$B_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad b_3(x) = \begin{bmatrix} 0 \\ -x_1^2 x_2 \end{bmatrix}.$$

The matrix B_0 has the eigenvalues $\pm i$. As the vectors e, e^*, g, g^* in (2.2) here we can take the vectors

$$e = e^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad g = g^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We first analyze the averaged system (3.3). Simple calculations show that the equation $F(u) = 0$ leads us to the system

$$\begin{cases} 4u_1 - u_1 u_2^2 - u_1^3 = 0, \\ 4u_2 - u_1^2 u_2 - u_2^3 = 0, \end{cases}$$

the non–zero solution of which describe a circumference of radius $2 : u_1^2 + u_2^2 = 4$. By Theorem 3.1 we then find that the cycles of Equation (3.21) can branch off only from the mentioned circumference.

Now we are going to show that the value $\alpha = 0$ is indeed the bifurcation point of cycles of Equation (3.21). In order to do this, we employ Theorems 3.3 and 3.4. Here the calculations show that

$$\gamma_1 = 1, \qquad \alpha_2 = \frac{1}{4}, \qquad (\xi_3, e^*) = -\frac{\pi}{4}.$$

By Theorems 3.3 and 3.4 this implies that $\alpha = 0$ is the bifurcation point of cycles of Equation (3.21). These cycles appear for $\alpha > 0$ and are stable.

4. Study of problem on Andronov — Hopf bifurcation at infinity

We proceed to discussing problem on Andronov — Hopf bifurcation at infinity in the system (3.10).

4.1. On properties of bifurcation at infinity. The above definition of the bifurcation at infinity, namely, the conditions U1, U2 and UB imply the next statement.

Theorem 4.1. *Let the value $\alpha = 0$ be the point of Andronov — Hopf bifurcation at infinity of the system (3.10). Then*

- *the matrix B_0 has a pair of eigenvalues $\pm\omega_0 i$ ($\omega_0 > 0$), and the period T_0 mentioned in the definition of bifurcation at infinity is equal to $T_0 = \dfrac{2\pi}{\omega_0}$;*
- *the bifurcation solutions $x(t, \varepsilon)$ of the system (3.10) satisfy the relation*

$$x(0, \varepsilon) = \rho(\varepsilon) h_0 + o(\rho(\varepsilon)) \quad as \quad \varepsilon \to 0, \tag{4.1}$$

where $\rho(\varepsilon) = \max_t \|x(t, \varepsilon)\|$, and $h_0 \in E_0$ is a non–zero vector such that $\|h_0\| \leqslant 1$.

This theorem confirms the fact that under the Andronov — Hopf bifurcation at infinity, the large amplitude periodic orbits of the system (3.10) branch off from the large–amplitude cycles of the unperturbed system (2.6). We note that if we weaken the definition of bifurcation at infinity, leaving only the first relation in (1.3), then Theorem 4.1 no longer holds.

4.1.1. Example 2. As an illustration we consider two systems

$$\begin{cases} x' = x - y - \alpha(x^2 + y^2)x, \\ y' = x + y - \alpha(x^2 + y^2)y, \end{cases} \tag{4.2}$$

$$\begin{cases} x' = ky - (1 + k)\alpha(x^2 + y^2)x, \\ y' = -kx + (1 + k)\alpha(x^2 + y^2)y, \end{cases} \tag{4.3}$$

where $k > 0$. For positive α both systems have the limit cycle

$$x^2 + y^2 = \frac{1}{\alpha},$$

which tends to infinity as $\alpha \to 0$. This cycle $\Upsilon(\alpha)$ is associated with a periodic solution

$$x = \frac{1}{\sqrt{\alpha}} \cos t, \qquad y = \frac{1}{\sqrt{\alpha}} \sin t.$$

Here the second relation in (1.3) fails. The statements of Theorem 4.1 do not hold for these systems. We also note that the limit cycles $\Upsilon(\alpha)$ of these systems, coming from infinity, do not branch off from the large–amplitude cycles of the unperturbed system. Namely, the unperturbed system for (4.2) has no cycles at all, and although all solutions of the unperturbed system for (4.3) are cycles, their period $T(k) = \dfrac{2\pi}{k}$ (for $k \neq 1$) does not coincide with the period $T_0 = \dfrac{2}{\pi}$ of the cycles $\Upsilon(\alpha)$.

4.2. Necessary condition of bifurcation at infinity. Let us present a necessary condition for the bifurcation at infinity, which yields that this bifurcation is typical only for the case when the nonlinearity $b_q(x)$ in the system (3.10) is homogeneous of even order.

Theorem 4.2. *Let the value $\alpha = 0$ be the point of the Andronov — Hopf bifurcation at infinity of the system (3.10). Then for each vector $u \in \mathbb{R}^2$ the identity holds*

$$\int_0^{T_0} e^{-B_0 t} b_q(e^{B_0 t} u)\, dt = 0. \tag{4.4}$$

We observe the following fact. Lemma 3.1 implies that the necessary condition (4.4) of the Andronov — Hopf bifurcation at infinity holds for the system (3.10) for even q, that is, when the nonlinearity $b_q(x)$ is homogeneous of even order. At the same time, for odd q this condition for the system (3.10) usually fails.

Proof of Theorem 4.2. Let the value $\alpha = 0$ be the point of the Andronov — Hopf bifurcation at infinity of the system (3.10). Then there exist continuous functions $\alpha(\varepsilon)$ and $T(\varepsilon)$ such that for $\alpha = \alpha(\varepsilon)$ the system (3.10) has a $T(\varepsilon)$–periodic solution $x(t, \varepsilon)$, and $\alpha(0) = 0$, $T(0) = T_0$ and the relations (1.3) hold.

As in the problem on bifurcation of cycles, we make the nondegenerate T_0–periodic change (3.1) in the system (2.5). As a result, the system (3.10) is reduced to (3.2), which for $\alpha = \alpha(\varepsilon)$ has the solution

$$y = y(t, \varepsilon) = e^{-B_0 t} x(t, \varepsilon),$$

that is,

$$\frac{dy(t,\varepsilon)}{dt} \equiv \alpha(\varepsilon) e^{-B_0 t} f(e^{B_0 t} y(t, \varepsilon)), \quad y \in \mathbb{R}^2; \tag{4.5}$$

where $f(x) = B_1 x + b_q(x)$.

Reproducing for (4.5) the same arguing as above in consideration of the identity (3.7), we arrive at the relation

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{-B_0t}f(x(t,\varepsilon))\,dt=0. \tag{4.6}$$

By Theorem 4.1, the function $x(t,\varepsilon)$ can be represented in the form (4.1). Then the identity (4.6) reads

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{-B_0t}[B_1(\rho(\varepsilon)\varphi_0(t)+o(\rho(\varepsilon)))+b_q(\rho(\varepsilon)\varphi_0(t)+o(\rho(\varepsilon)))]\,dt=0,$$

or, in view of the homogeneous property of $b_q(x)$,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{-B_0t}[\rho(\varepsilon)B_1(\varphi_0(t)+o(1))+(\rho(\varepsilon))^q b_q(\varphi_0(t)+o(1))]\,dt=0.$$

We divide this identity by $(\rho(\varepsilon))^q$ and pass to the limit at $\varepsilon\to 0$ to obtain

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{-B_0t}b_q(\varphi_0(t))\,dt=0.$$

In this identity the integrand is a T_0–periodic function and this is why

$$\int_0^{T_0} e^{-B_0t}b_q(\varphi_0(t))\,dt=0.$$

Since the solution $\varphi_0(t)$ of the unperturbed system (2.6) can be represented as $\varphi_0(t)=e^{B_0t}u_0$ for some nonzero vector $u_0\in\mathbb{R}^2$, we get

$$\int_0^{T_0} e^{-B_0t}b_q(e^{B_0t}u_0)\,dt=0.$$

By the arbitrariness of solution $\varphi_0(t)$ we get the identity (4.4). The proof is complete. □

4.3. Sufficient condition for bifurcation at infinity. In what follows, we discuss the problem on bifurcation at infinity only for the case when the nonlinearity $b_q(x)$ in system (3.10) is homogeneous of even order.

Like Theorem 3.3, the sufficient condition for bifurcation at infinity given below is based on operator methods for studying problems on multiparametric bifurcations, see [3], [18]. Following these works, we define the vectors

$$e(t)=e\cos 2\pi t-g\sin 2\pi t,\qquad \xi_2=\int_0^1 e^{-tT_0B_0}\beta_2(t)\,dt,$$

where

$$\beta_2(t)=T_0F_2(t)\int_0^t e^{-\tau T_0B_0}b_q(e(\tau))\,d\tau,\qquad F_2(t)=T_0b'_{qx}(e(t))e^{T_0B_0t};$$

$b'_{qx}(x)$ is the Jacobi matrix of the function $b_q(x)$.

We also define the numbers

$$\alpha_2 = -\frac{\omega_0}{\pi\gamma_1}(\xi_2, e^*), \qquad T_2 = \frac{1}{\omega_0}\Big[(\xi_2, g^*) - \frac{\gamma_2}{\gamma_1}(\xi_2, e^*)\Big]. \tag{4.7}$$

Theorem 4.3. *Let q be even and the conditions hold:*

$$\gamma_1 \neq 0, \qquad \alpha_2 \neq 0. \tag{4.8}$$

Then $\alpha = 0$ is the point of Andronov — Hopf bifurcation at infinity of the system (3.10).

Theorem 4.3 is a continuation of a similar result obtained in [18].
The next lemma is obvious.

Lemma 4.1. *Let q be even. For $\alpha \neq 0$ the change $y = \alpha^{\frac{1}{(q-1)}}x$ reduces the system (3.10) to*

$$y' = (B_0 + \alpha B_1)y + b_q(y), \quad y \in \mathbb{R}^2. \tag{4.9}$$

The inverse change reduces the system (4.9) to the system (3.10).

Proof of Theorem 4.3. The first of conditions (4.8) means that $\alpha = 0$ is the point of Andronov — Hopf bifurcation of the system (4.9). Namely, for $\alpha = \alpha(\varepsilon)$ the system (4.9) has a non–stationary $T(\varepsilon)$–periodic solution $y(t, \varepsilon)$ with a small amplitude, and the functions $\alpha(\varepsilon)$, $T(\varepsilon)$ and $y(t, \varepsilon)$ can be represented as

$$\alpha(\varepsilon) = \alpha_2\varepsilon^q + O(\varepsilon^{q+2}), \qquad T(\varepsilon) = T_0 + T_2\varepsilon^q + O(\varepsilon^{q+2}), \qquad y(0, \varepsilon) = \varepsilon e + O(\varepsilon^3). \tag{4.10}$$

It follows from Lemma 4.1 that for $\alpha = \alpha(\varepsilon)$ the system (3.10) has a non–stationary $T(\varepsilon)$–periodic solution

$$x(t, \varepsilon) = (\alpha(\varepsilon))^{\frac{1}{(1-q)}} y(t, \varepsilon).$$

By the identities (4.10) this implies that the solution $x(t, \varepsilon)$ satisfies both relations (1.3). This is why the value $\alpha = 0$ is the point Andronov — Hopf bifurcation at infinity for the system (3.10). The proof is complete. □

Theorem 4.3 and the results of the work [18] imply the next statement.

Theorem 4.4. *Under the assumptions of Theorem 4.3 the bifurcation solutions $x(t, \varepsilon)$ of system (3.10) appear for $\alpha > 0$ (if $\alpha_2 > 0$) or for $\alpha < 0$ (if $\alpha_2 < 0$). The asymptotics in the small parameter ε for the solutions $x(t, \varepsilon)$ is given by the identity*

$$x(0, \varepsilon) = \frac{e}{(\alpha_2\varepsilon)^{\frac{1}{(q-1)}}} + o(\varepsilon^{\frac{1}{(1-q)}}),$$

where e is the vector from (2.2) and (3.11). These solutions are orbitally asymptotically stable (unstable) if $(\xi_2, e^*) < 0$ (if $(\xi_2, e^*) > 0$).

4.4. Example 3. As an illustration we consider the system

$$x' = A(\alpha)x + \alpha a(x), \qquad x \in \mathbb{R}^2, \tag{4.11}$$

where

$$A(\alpha) = \begin{bmatrix} -1-\alpha & 12(1+\alpha) \\ -\dfrac{1}{2} & 1 \end{bmatrix}, \qquad a(x) = \begin{bmatrix} -\dfrac{x_1^2}{12} - 12x_2^2 + 2x_1x_2 \\ -\dfrac{x_2^2}{2} - \dfrac{x_1x_2}{4} \end{bmatrix}.$$

The system (4.11) is obtained by transformation of Holling — Tanner model, see, for instance, [2], namely, by the translation of the origin at the equilibrium of the model and an appropriate cutting of the right hand side in the obtained system.

The system (4.11) can be represented in the form (3.10) for $q = 2$,

$$B_0 = \begin{bmatrix} -1 & 12 \\ -\frac{1}{2} & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -1 & 12 \\ 0 & 0 \end{bmatrix}, \quad b_2(x) = a(x).$$

The matrix B_0 possesses the eigenvalues $\pm\omega_0 i$, where $\omega_0 = \sqrt{5}$. As the vectors e, e^*, g, g^* in (2.2) and (3.11) we can take the vectors

$$e = \begin{bmatrix} 1 \\ \dfrac{(1+\sqrt{5})}{12} \end{bmatrix}, \quad g = \begin{bmatrix} -1 \\ \dfrac{(\sqrt{5}-1)}{12} \end{bmatrix}, \quad e^* = \begin{bmatrix} \dfrac{(5-\sqrt{5})}{10} \\ \dfrac{6\sqrt{5}}{5} \end{bmatrix}, \quad g^* = \begin{bmatrix} -\dfrac{(5+\sqrt{5})}{10} \\ \dfrac{6\sqrt{5}}{5} \end{bmatrix}.$$

The numbers γ_1, α_2 and (ξ_2, e^*) from (3.12) and (4.7) read

$$\gamma_1 = -1, \qquad \alpha_2 \approx 0{,}027, \qquad (\xi_2, e^*) = \frac{\pi\alpha_2}{\sqrt{5}}.$$

By Theorems 4.3 and 4.4 this implies that $\alpha = 0$ is the point of Andronov — Hopf bifurcation at infinity of the system (4.11). These cycles emerge for $\alpha > 0$ and are unstable.

5. Multidimensional system

We return back to the discussion of problems on bifurcations of cycles and bifurcation at infinity for the multidimensional system (2.1) for $N \geqslant 3$. These problems can be studied by the same scheme as for the two–dimensional system with natural modifications. We restrict ourselves by providing the scheme for studying the problem on bifurcation of cycles.

5.1. Problem on bifurcation of cycles: necessary conditions. We recall that above the conditions V1 and V2 were assumed to be satisfied. We also recall that the space $\mathbb{R}^N$ can be represented as the direct sum $\mathbb{R}^N = E_0 \oplus E^0$, where E_0 is a two–dimensional eigenspace of the operator B_0 associated with the simple eigenvalues $\pm i\omega_0$, and E^0 is an additional subspace of dimension $N-2$ invariant for B_0.

The identity $\mathbb{R}^N = E_0 \oplus E^0$ defines the projection operators

$$P_0 : \mathbb{R}^N \to E_0 \quad \text{and} \quad P^0 : \mathbb{R}^N \to E^0$$

such that $P^0 = I - P_0$, and the operator P_0 can be represented as

$$P_0 x = (x, e^*)e + (x, g^*)g,$$

where $e, g, e^*, g^* \in \mathbb{R}^N$ are the vectors chosen by the identities (2.2) and (3.11).

We consider the two–dimensional autonomous system

$$u' = \alpha F(u), \qquad u \in E_0, \tag{5.1}$$

where

$$F(u) = \frac{1}{T_0} \int\limits_0^{T_0} P_0 e^{-B_0 t} f(e^{B_0 t} u)\, dt.$$

The next statement provides a necessary condition for the bifurcation of cycles of (2.1) for $N \geqslant 3$.

Theorem 5.1. *Let the value $\alpha = 0$ be a point of bifurcation of cycles of the system (2.1), which branch off from some trajectory Υ_0 of the linear system (2.3). Then each vector $u_0 \in \Upsilon_0$ is an equilibrium of the system (5.1), that is, $F(u_0) = 0$.*

This statement can be proved in the same way as its analogue, Theorem 3.1. We describe the scheme of the proof of Theorem 5.1.

Since $\mathbb{R}^N = E_0 \oplus E^0$, each vector $x \in R^N$ is uniquely represented as $x = x_0 + x^0$, where $x_0 = P_0 x$ and $x^0 = P^0 x$. Therefore, the system (2.1) can be equivalently represented as

$$\begin{cases} (x_0)' = B_0 x_0 + \alpha P_0 f(x_0 + x^0), \\ (x^0)' = B_0 x^0 + \alpha P^0 f(x_0 + x^0). \end{cases} \tag{5.2}$$

In the system (5.2) we make the non–degenerate T_0–periodic change

$$y_0 = e^{-B_0 t} x_0, \quad y^0 = x^0. \tag{5.3}$$

As a result, the system (5.2) is transformed to

$$\begin{cases} (y_0)' = \alpha P_0 e^{-B_0 t} f(e^{B_0 t} y_0 + y^0), \\ (y^0)' = B_0 y^0 + \alpha P^0 f(e^{B_0 t} y_0 + y^0) \end{cases} \tag{5.4}$$

with a T_0–periodic right hand side.

Let the value $\alpha = 0$ be a point of bifurcation of cycles of the system (2.1), that is, there exist continuous functions $\alpha(\varepsilon)$ and $T(\varepsilon)$ such that the conditions U1 and U2 hold, as well as the condition UC, which can be presented as the relation (3.5):

$$\max_{0 \leqslant t \leqslant T_0} \|x(t,\varepsilon) - C_0 \varphi_0(t)\| \to 0 \quad \text{as} \quad \varepsilon \to 0, \tag{5.5}$$

where $\varphi_0(t)$ is the function (2.4), C_0 is some positive number. The relation (5.5) implies that the solution $x(t,\varepsilon)$ of system (2.1) can be represented as

$$x(t,\varepsilon) = C_0 \varphi_0(t) + \delta_0(t,\varepsilon) + \delta^0(t,\varepsilon), \tag{5.6}$$

where the functions $\delta_0(t,\varepsilon) \in E_0$ and $\delta^0(t,\varepsilon) \in E^0$ are almost periodic in t, smooth in ε and satisfy the relations

$$\max_t \|\delta_0(t,\varepsilon)\| \to 0, \quad \max_t \|\delta^0(t,\varepsilon)\| \to 0$$

as $\varepsilon \to 0$. Then the system (5.4) has the solution

$$y = y(t,\varepsilon) = e^{-B_0 t}[C_0 \varphi_0(t) + \delta_0(t,\varepsilon)] + \delta^0(t,\varepsilon).$$

Further arguing in similar to that in the proof of Theorem 3.1 while considering the function (3.6). Here the identity (5.6) is essential. We also note that the right hand side of the system (5.1) is obtained by averaging the right hand side of the first equation in the system (5.4) with $y^0 = 0$.

5.2. Problem on bifurcation of cycles: systems with homogeneous nonlinearities. We consider the case when the system (2.1) reads

$$\frac{dx}{dt} = B_0 x + \alpha[B_1 x + b_q(x)], \quad x \in \mathbb{R}^N, \tag{5.7}$$

where the nonlinearity $b_q(x)$ is a homogeneous polynomial of degree q.

As in the two–dimensional case, we define the numbers and vectors (3.12), (3.16) and (3.17). The following analogues of Theorems 3.2–3.4.

Theorem 5.2. *Let $\gamma_1 \neq 0$ and the value $\alpha = 0$ be the point of bifurcation of cycles of the system (5.7). Then q is odd, that is, in the system (5.7) the nonlinearity $b_q(x)$ is a homogeneous polynomial of odd degree.*

Theorem 5.3. *Let* $\gamma_1 \neq 0$, $\alpha_2 \neq 0$. *Then* $\alpha = 0$ *is the point of bifurcation of cycles of the system (5.7), which branch off from the trajectory* Υ_0 *of the linear system (2.3) containing the vector* u^*. *Here* $u^* = (\alpha_2)^{\frac{1}{(1-q)}} e$ *(if* $\alpha_2 > 0$*) or* $u^* = (-\alpha_2)^{\frac{1}{(1-q)}} e$ *(if* $\alpha_2 < 0$*). At the same time, the bifurcation solutions* $x(t, \varepsilon)$ *of the system (5.7) appear for* $\alpha > 0$ *(if* $\alpha_2 > 0$*) or for* $\alpha < 0$ *(if* $\alpha_2 < 0$*).*

Theorem 5.4. *Let the eigenvalues of the matrix* B_0 *not coinciding with* $\lambda = \pm\omega_0 i$ *have negative real parts. Then the bifurcation solutions* $x(t, \varepsilon)$ *of the system (5.7), which appear under the assumptions of Theorem 5.3 are orbitally asymptotically stable (unstable) if* $(\xi_3, e^*) < 0$ *(if* $(\xi_3, e^*) > 0$*).*

6. Conclusion

In the paper we study in detail the problems on bifurcation of cycles and the Andronov — Hopf bifurcation at infinity for dynamical systems with a small parameter with the nonlinearities containing homogeneous polynomials of even or odd degree, and the unperturbed equation has a continuum of periodic solutions. We propose new necessary and sufficient conditions for these bifurcations, obtain the formulas for the approximate construction of bifurcation solutions, and analyze their stability. We show that the cycle bifurcation of cycles is typical only for systems with the homogeneities of odd degree, while the bifurcation at infinity is typical only for systems with homogeneities of even degree. A relationship between these bifurcations and the classical Andronov — Hopf bifurcation is demonstrated.

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