ISSN 2304-0122 Ufa Mathematical Journal. Vol. 17. No 4 (2025). P. 127-139.

doi:10.13108/2025-17-4-127

CAUCHY PROBLEM AND INVERSE PROBLEM FOR INTEGRO–DIFFERENTIAL EQUATIONS OF GERASIMOV TYPE WITH REGULAR KERNEL

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Abstract. We study the unique solvability of the Cauchy problem for a linear regular integro–differential equation of Gerasimov type in a Banach space. This allows us to obtain a well–posedness criterion for the corresponding linear inverse problem with a constant unknown coefficient in the right–hand side.

The abstract results are used to consider direct and inverse initial boundary value problems for a class of equations with a Gerasimov type integro–differential operator in time and polynomials of the Laplace operator in spatial variables, as well as to study the unique solvability of the Cauchy problem and the linear inverse problem for a system of ordinary integro–differential equations. The regular kernel of the integral operator in the system under consideration is essentially operator–valued and defines linear combinations of various integro–differential operators in the equations of the system.

Keywords: integro–differential equation of Gerasimov type, regular integral kernel, Cauchy problem, inverse coefficient problem, initial boundary value problem.

Mathematics Subject Classification: 34K30, 35R09, 35R30, 47G20, 47J05

1. Introduction

In recent decades, the problems for equations with various fractional derivatives, which are used in modeling phenomena and processes in physics, chemistry, biology, and in the technical and social sciences and humanities (see the monographs [4]–[7], [18], [22] and the references therein), have attracted sustained interest from researchers. Most fractional derivatives are integro–differential operators, in which first an integral convolution operator acts on a function, and then the ordinary differentiation operator does, as in the Riemann — Liouville fractional derivative, or vice versa, as in the Gerasimov–Caputo fractional derivative. In this case, the kernel in the integral operator is singular. The works of various authors are devoted to the study of both direct and inverse coefficient problems for fractional differential equations, see, for instance, [1], [2], [10], [15], [20].

Initial value problems for linear equations in Banach spaces resolved with respect to an integro–differential operator (of Riemann — Liouville type or Gerasimov type, depending on the order of action of the convolution operator and the differentiation operator in them) with an abstract singular integral kernel, in the case of a bounded operator at the unknown function, were studied in [14]. In [8], [13], the unique solvability was studied of initial value problems for integro–differential equations of Gerasimov type and Riemann — Liouville type, respectively, with an unbounded linear operator generating an analytic resolving family of operators of the

V.E. Fedorov, A.V. Nagumanova, A.O. Sagimbaeva, Cauchy problem and inverse problem for integro–differential equations of Gerasimov type with regular kernel.

The research is financially supported by the Russian Science Foundation, grant no. 24-21-20015, https://rscf.ru/project/24-21-20015/ and also supported by the Government of Chelyabinsk Region.

Submitted March 5, 2025.

corresponding linear homogeneous equation in the sector. Linear inverse problems for such Riemann — Liouville type equations were studied in [9].

Recently, integro–differential operators with a regular integral kernel have become objects of study, in particular, the so–called Caputo — Fabrizio derivatives [11], see also the discussion on the validity of classifying such integro–differential operators as fractional derivatives [12], [17], [21], [23]. We note that the present paper remains outside this discussion and does not use the term "fractional derivative" in relation to such integro–differential operators.

In [3] there were studied the issues on unique solvability of direct and inverse coefficient problems for evolutionary equations in Banach spaces with the Caputo — Fabrizio derivative, and in [16] there were addressed for equations with an abstract regular integro–differential operator of Riemann — Liouville type and a bounded operator at the sought function. The present work continues the studies of [3], [16] and is devoted to studying direct and inverse problems for equations in Banach spaces with an abstract regular integro–differential operator of Gerasimov type.

Let $\mathcal{Z}$, $\mathcal{U}$ be Banach space, by $\mathcal{L}(\mathcal{U};\mathcal{Z})$ we denote the Banach space of linear continuous operators from $\mathcal{U}$ into $\mathcal{Z}$. For $\mathcal{U}=\mathcal{Z}$ this notation is shortened to $\mathcal{L}(\mathcal{Z})$. We consider the Cauchy problem

$$z(0)=z_0 \tag{1.1}$$

for the evolutionary equation

$$\int_0^t K(t-s)D^1z(s)ds = Az(t)+B(t)u+g(t), \quad t\in[0,T], \tag{1.2}$$

where D^1 is the differentation operator, A is a linear closed operator, $K\in C^1(\overline{\mathbb{R}}_+;\mathcal{L}(\mathcal{Z}))$, $B:[0,T]\to\mathcal{L}(\mathcal{U};\mathcal{Z})$, $u\in\mathcal{U}$, $g:[0,T]\to\mathcal{Z}$. We obtain the conditions ensuring the unique solvability of the Cauchy problem (1.1) for Equation (1.2) with a known right hand side. This allows us to study the inverse coefficient problem for such equation. The overdetermination condition in this problem reads

$$\int_0^T z(t)d\mu(t)=z_T \tag{1.3}$$

in the case of an independent of time uknown coefficient u; here μ is a given function with a bounded variation, and $z_T\in\mathcal{Z}$.

In the second section of the present work we obtain the conditions guaranteeing the unique solvability of the Cauchy problem for a linear homogeneous integro–differential equation. The third section contains a similar result for a linear inhomogeneous integro–differential equation. In Section 4 we obtain the well–definiteness criterion of the linear inverse problem (1.1)–(1.3). The obtained general results are applied for studying direct and inverse initial boundary value problems for a class of partial differential equations with an integro–differential operator of Gerasimov type and polynomials of the Laplace operator. For the system of ordinary integro–differential equations we also study the Cauchy problem and the inverse problem with an unknown constant element. The kernel of the integral operator in the considered system is essentially operator–valued and defines linear combinations of various integro–differential operators in the equations of the system.

2. Cauchy problem for homogeneous equation

Let $\mathcal{Z}$ be a Banach space, $\mathcal{C}l(\mathcal{Z})$ be the set of all linear closed operator densely defined in the space $\mathcal{Z}$, the domain D_A of an operator $A\in\mathcal{C}l(\mathcal{Z})$ be equipped with the graph norm

$\|\cdot\|_{D_A} := \|\cdot\|_{\mathcal{Z}} + \|A\cdot\|_{\mathcal{Z}}$,

$$\rho(A) := \{\mu \in \mathbb{C} : (\mu I - A)^{-1} \in \mathcal{L}(\mathcal{Z})\}$$

be the resolvent set of the operator A, and $\sigma(A) = \mathbb{C} \setminus \rho(A)$ be its spectrum, $\mathbb{R}_+ = \{a \in \mathbb{R} : a > 0\}$, $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{0\}$, $K(t) \in \mathcal{L}(\mathcal{Z})$ for $t > 0$. We define the convolution operator

$$(J^K z)(t) := \int_0^t K(t-s)z(s)ds := (K * z)(t), \quad t > 0,$$

and the integro–differential operator of the Gerasimov type

$$(D^{K,1}z)(t) := (J^K D^1 z)(t) := \int_0^t K(t-s)D^1 z(s)ds, \quad t > 0,$$

where D^1 is the operator of the usual first derivative.

For $A \in \mathcal{C}l(\mathcal{Z})$, $K \in C([0,T];\mathcal{L}(\mathcal{Z}))$, $z_0 \in D_A$ we consider the Cauchy problem

$$z(0) = z_0 \tag{2.1}$$

for the equation

$$(D^{K,1}z)(t) = Az(t), \qquad t \in [0,T]. \tag{2.2}$$

A solution to the Cauchy problem (2.1), (2.2) is a function $z \in C([0,T];D_A) \cap W_1^1(0,T;\mathcal{Z})$ satisfying the condition (2.1) and identity (2.2).

Lemma 2.1. *Let $A \in \mathcal{C}l(\mathcal{Z})$, $K \in C([0,T];\mathcal{L}(\mathcal{Z}))$. Then if the solution to the problem (2.1), (2.2) exists, then $z_0 \in \ker A$.*

Proof. For $z \in W_1^1(0,T;\mathcal{Z})$

$$\left\| \int_0^t K(t-s)D^1 z(s)ds \right\|_{\mathcal{Z}} \leqslant \max_{s\in[0,T]} \|K(s)\|_{\mathcal{L}(\mathcal{Z})} \int_0^t \|D^1 z(s)\|_{\mathcal{Z}} ds \to 0, \quad t \to 0+.$$

By the definition of the solution z we have

$$Az_0 = \lim_{t\to 0+} Az(t) = \lim_{t\to 0+} (D^{K,1}z)(t) = \lim_{t\to 0+} \int_0^t K(t-s)D^1 z(s)ds = 0.$$

The proof is complete. □

Theorem 2.1. *Let*

$$A \in \mathcal{C}l(\mathcal{Z}), \qquad K \in C^1([0,T];\mathcal{L}(\mathcal{Z})), \qquad (K(0)-A)^{-1} \in \mathcal{L}(\mathcal{Z}), \qquad z_0 \in \ker A.$$

Then the function $z(t) \equiv z_0$ is the unique solution to the Cauchy problem (2.1), (2.2).

Proof. It is easy to confirm that $z(t) \equiv z_0$ solves the problem (2.1), (2.2). Let us prove the uniqueness. Suppose that z_1 and z_2 are solutions to the problem (2.1), (2.2), then the function $y = z_1 - z_2$ solves the Cauchy problem $y(0) = 0$ for the equation

$$(D^{K,1}y)(t) = K(0)y(t) - K(t)y(0) - (J^{K'}y)(t) = Ay(t).$$

This yields $y(t) = (K(0)-A)^{-1}(J^{K'}y)(t)$.

We consider the operator

$$By(t) = (K(0)-A)^{-1}(J^{K'}y)(t)$$

in the space $L_1(0,T_1;\mathcal{Z})$ for some $T_1 \in (0,T]$. If $K' \equiv 0$, then

$$y(t) = (K(0)-A)^{-1}0 \equiv 0.$$

If $\|K'\|_{C([0,T_1];\mathcal{L}(\mathcal{Z}))} \neq 0$, for $q \in (0,1)$ we take
$$T_1 = q\|(K(0)-A)^{-1}\|^{-1}_{\mathcal{L}(\mathcal{Z})}\|K'\|^{-1}_{C([0,T_1];\mathcal{L}(\mathcal{Z}))}$$
and we get
$$\|By\|_{L_1(0,T_1;\mathcal{Z})} \leqslant \|(K(0)-A)^{-1}\|_{\mathcal{L}(\mathcal{Z})}\int_0^{T_1}\left\|\int_0^t K'(t-s)y(s)ds\right\|_{\mathcal{Z}} dt$$
$$\leqslant T_1\|(K(0)-A)^{-1}\|_{\mathcal{L}(\mathcal{Z})}\|K'\|_{C([0,T_1];\mathcal{L}(\mathcal{Z}))}\|y\|_{L_1(0,T_1;\mathcal{Z})} = q\|y\|_{L_1(0,T_1;\mathcal{Z})}.$$
Therefore, the operator B is a contraction in the space $L_1(0,T_1;\mathcal{Z})$, and this is why the unique equation of the equation $y = By$ in this space is the function $y = 0$ almost everywhere on $(0,T_1)$.

If $T_1 < T$, we consider the space
$$L_1^{T_1}(0,2T_1;\mathcal{Z}) := \{y \in L_1(0,2T_1;\mathcal{Z}) : y(t) = 0 \quad \text{almost everywhere on} \quad (0,T_1)\}.$$
Then
$$\|B\|_{\mathcal{L}\left(L_1^{T_1}(0,2T_1;\mathcal{Z})\right)} \leqslant (2T_1 - T_1)\|(K(0)-A)^{-1}\|_{\mathcal{L}(\mathcal{Z})}\|K'\|_{C([0,2T_1];\mathcal{L}(\mathcal{Z}))} = q < 1,$$
which implies the uniqueness of the trivial solution to the equation $y = By$ on $(0,2T_1)$. If $2T_1 < T$, we take the space
$$L_1^{2T_1}(0,3T_1;\mathcal{Z}) := \{y \in L_1(0,3T_1;\mathcal{Z}) : y(t) = 0 \text{ almost everywhere on } (0,2T_1)\}$$
and prove the uniqueness of the trivial solution to the equation $y = By$ on the segment $[0,3T_1]$. Repeating the arguing, in finitely many steps we completely cover the segment $[0,T]$. Hence, the solution to the problem (2.1), (2.2) is unique on $[0,T]$. The proof is complete. □

3. Cauchy problem for inhomogeneous equation

Let $A \in \mathcal{C}l(\mathcal{Z})$, $K \in C^1([0,T];\mathcal{L}(\mathcal{Z}))$, $f \in C([0,T];\mathcal{Z})$, $z_0 \in D_A$. We consider the Cauchy problem
$$z(0) = z_0 \tag{3.1}$$
for the linear inhomogeneous equation
$$(D^{K,1}z)(t) = Az(t) + f(t), \quad t \in [0,T]. \tag{3.2}$$
A solution of the Cauchy problem (3.1), (3.2) is a function $z \in C([0,T];D_A) \cap W_1^1(0,T;\mathcal{Z})$ satisfying the condition (3.1) and identity (3.2).

Similarly to Lemma 2.1 we obtain the following statement.

Lemma 3.1. *If a solution to the problem (3.1), (3.2) exists, then* $Az_0 + f(0) = 0$.

For a function $h : \mathbb{R}_+ \to \mathcal{Z}$, by $\widehat{h}$ we denote its Laplace transform.

In what follows we suppose the following condition.

($\widehat{K}$) For a function $K \in C^1(\overline{\mathbb{R}}_+;\mathcal{L}(\mathcal{Z}))$ the Laplace transform $\widehat{K}(\lambda)$ is well–defined and it can be continued to the univalent analytic function on the set
$$\Omega_{a_K} := \{\lambda \in \mathbb{C} : |\lambda| > a_K\}.$$

Lemma 3.2. *Let* $A \in \mathcal{C}l(\mathcal{Z})$, $K \in C^1(\overline{\mathbb{R}}_+;\mathcal{L}(\mathcal{Z}))$, $(K(0)-A)^{-1} \in \mathcal{L}(\mathcal{Z})$, *and the condition* ($\widehat{K}$) *be satisfied. Then for some* $r > a_K$ *the operator–valued function*
$$Z(t) = \frac{1}{2\pi i}\int_{|\lambda|=r}(\lambda\widehat{K}(\lambda) - A)^{-1}e^{\lambda t}d\lambda, \quad t \in \mathbb{C}, \tag{3.3}$$

is well–defined and analytic.

Proof. By the initial value theorem [19], $\lim_{\lambda\to+\infty}\lambda\widehat{K}(\lambda)=K(0)$ in $\mathcal{L}(\mathcal{Z})$. Since $(K(0)-A)^{-1}\in\mathcal{L}(\mathcal{Z})$, in view of the condition $(\widehat{K})$ this means that the operators

$$(\lambda\widehat{K}(\lambda)-A)^{-1}=(K(0)-A+\lambda\widehat{K}(\lambda)-K(0))^{-1}$$
$$=(I+(K(0)-A)^{-1}(\lambda\widehat{K}(\lambda)-K(0)))^{-1}(K(0)-A)^{-1}$$

are well–defined for sufficiently large $|\lambda|>r_0\geqslant a_K$, for which

$$\|(\lambda\widehat{K}(\lambda)-K(0))\|_{\mathcal{L}(\mathcal{Z})}<\|(K(0)-A)^{-1}\|^{-1}_{\mathcal{L}(\mathcal{Z})}.$$

At the same time there exists $C>0$ such that for all $|\lambda|>r_0\geqslant a_K$ we have

$$\|(\lambda\widehat{K}(\lambda)-A)^{-1}\|_{\mathcal{L}(\mathcal{Z})}\leqslant C.$$

We also note that

$$(\lambda\widehat{K}(\lambda)-A)(\mu\widehat{K}(\mu)-A)^{-1}=I+(\lambda\widehat{K}(\lambda)-\mu\widehat{K}(\mu))(\mu\widehat{K}(\mu)-A)^{-1},$$
$$(\mu\widehat{K}(\mu)-A)^{-1}-(\lambda\widehat{K}(\lambda)-A)^{-1}=(\lambda\widehat{K}(\lambda)-A)^{-1}(\lambda\widehat{K}(\lambda)-\mu\widehat{K}(\mu))(\mu\widehat{K}(\mu)-A)^{-1},$$
$$\|(\mu\widehat{K}(\mu)-A)^{-1}-(\lambda\widehat{K}(\lambda)-A)^{-1}\|_{\mathcal{L}(\mathcal{Z})}$$
$$\leqslant C\|\lambda\widehat{K}(\lambda)-\mu\widehat{K}(\mu)\|_{\mathcal{L}(\mathcal{Z})}\|(\mu\widehat{K}(\mu)-A)^{-1}\|_{\mathcal{L}(\mathcal{Z})}\to 0,\quad \lambda\to\mu.$$

Moreover,

$$\frac{d}{d\mu}(\mu\widehat{K}(\mu)-A)^{-1}=-(\mu\widehat{K}(\mu)-A)^{-1}\left[\frac{d}{d\mu}[\mu\widehat{K}(\mu)]\right](\mu\widehat{K}(\mu)-A)^{-1}.$$

Hence, the integrand in (3.3) is analytic in Ω_{r_0}, and since the contour $\{|\lambda|=r>r_0\}$ is bounded, the function $Z(t)$ is analytic in $t\in\mathbb{C}$. The proof is complete. □

Theorem 3.1. *Let*

$$A\in\mathcal{C}l(\mathcal{Z}),\qquad K\in C^1(\overline{\mathbb{R}}_+;\mathcal{L}(\mathcal{Z})),\qquad K(0)-A)^{-1}\in\mathcal{L}(\mathcal{Z}),$$

the condition $(\widehat{K})$ be satisfied,

$$f\in W_1^1(0,T;\mathcal{Z}),\qquad Az_0+f(0)=0.$$

Then the function

$$z(t)=z_0+(K(0)-A)^{-1}(f(t)-f(0))-\int_0^t Z(s)f(0)ds+\int_0^t Z(t-s)f(s)ds,\tag{3.4}$$

is the unique solution to the problem (3.1), (3.2).

Proof. It is clear that the initial condition $z(0)=z_0$ is satisfied and

$$D^1z(t)=(K(0)-A)^{-1}D^1f(t)-Z(t)f(0)+Z(0)f(t)+\int_0^t D^1Z(t-s)f(s)ds\in L_1(0,T;\mathcal{Z}),$$
$$D^{K,1}z(t)=(K(0)-A)^{-1}D^{K,1}f(t)-J^K[Z(t)f(0)]+J^K[Z(0)f(t)]$$
$$+J^K\int_0^t D^1Z(t-s)f(s)ds\in C([0,T];\mathcal{Z}),$$
$$A(K(0)-A)^{-1}=K(0)(K(0)-A)^{-1}-I\in\mathcal{L}(\mathcal{Z}),$$

$$AZ(t) = \frac{1}{2\pi i} \int\limits_{|\lambda|=r} A(\lambda \widehat{K}(\lambda) - A)^{-1} e^{\lambda t} d\lambda$$

$$= \frac{1}{2\pi i} \int\limits_{|\lambda|=r} \lambda \widehat{K}(\lambda)(\lambda \widehat{K}(\lambda) - A)^{-1} e^{\lambda t} d\lambda \in C([0,T]; \mathcal{L}(\mathcal{Z})),$$

$$Az(t) = Az_0 + A(K(0) - A)^{-1}(f(t) - f(0)) - \int\limits_0^t AZ(s) f(0) ds$$

$$+ \int\limits_0^t AZ(t-s) f(s) ds \in C([0,T]; \mathcal{Z}).$$

For $\operatorname{Re}\mu > r$ by Cauchy integral formula we have

$$\widehat{Z}(\mu) = \frac{1}{2\pi i} \int\limits_{|\lambda|=r} \frac{1}{\mu - \lambda} (\lambda \widehat{K}(\lambda) - A)^{-1} d\lambda = \frac{1}{2\pi i} \int\limits_{|\eta|=\frac{1}{r}} \frac{1}{\mu - \frac{1}{\eta}} \left(\frac{1}{\eta} \widehat{K}\left(\frac{1}{\eta}\right) - A \right)^{-1} \frac{d\eta}{\eta^2}$$

$$= \frac{1}{2\pi i} \int\limits_{|\eta|=\frac{1}{r}} \frac{1}{\mu \eta \left(\eta - \frac{1}{\mu}\right)} \left(\frac{1}{\eta} \widehat{K}\left(\frac{1}{\eta}\right) - A \right)^{-1} d\eta = (\mu \widehat{K}(\mu) - A)^{-1} - (K(0) - A)^{-1}.$$

We define the function f in a continuous bounded way for $t > T$ and denote $z_f := Z * f$, then

$$\widehat{z}_f(\mu) = \widehat{Z}(\mu) \widehat{f}(\mu) = [(\mu \widehat{K}(\mu) - A)^{-1} - (K(0) - A)^{-1}] \widehat{f}(\mu).$$

Therefore,

$$\widehat{z}(\mu) = \frac{z_0}{\mu} + (K(0) - A)^{-1} \left(\widehat{f}(\mu) - \frac{1}{\mu} f(0) \right) - \frac{1}{\mu} [(\mu \widehat{K}(\mu) - A)^{-1} - (K(0) - A)^{-1}] f(0) +$$
$$+ [(\mu \widehat{K}(\mu) - A)^{-1} - (K(0) - A)^{-1}] \widehat{f}(\mu),$$

$$\widehat{D^{K,1} z}(\mu) - \widehat{Az}(\mu) = (\mu \widehat{K}(\mu) - A) \widehat{z}(\mu) - \widehat{K}(\mu) z_0 = -A \frac{z_0}{\mu} - \frac{1}{\mu} f(0) + \widehat{f}(\mu) = \widehat{f}(\mu).$$

Applying the inverse Laplace transform, we obtain (3.2).

The uniqueness of the solution can be established as in the proof of Theorem 2.1. The proof is complete. □

4. Inverse problem with constant coefficient

Let $\mathcal{Z}, \mathcal{U}$ be Banach spaces. We consider an inverse problem for the evolutionary equation

$$(D^{K,1} z)(t) = Az(t) + B(t)u + g(t), \quad t \in [0,T], \tag{4.1}$$

where $D^{K,1}$ is an integro–differential operator of Gerasimov type,

$$K \in C^1(\overline{\mathbb{R}}_+; \mathcal{L}(\mathcal{Z})), \qquad A \in \mathcal{C}l(\mathcal{Z}), \qquad B \in W_1^1(0,T; \mathcal{L}(\mathcal{U}; \mathcal{Z})), \qquad g \in W_1^1(0,T; \mathcal{Z}),$$

with the initial condition

$$z(0) = z_0 \tag{4.2}$$

and the overdetermination condition

$$\int\limits_0^T z(t) d\nu(t) = z_T \in D_A, \tag{4.3}$$

where the function $\nu : (0, T] \to \mathbb{R}$ has a bounded variation; shortly $\nu \in BV((0, T]; \mathbb{R})$. Here we take into consideration the fact that $Az \in C([0, T]; \mathcal{Z})$, and this is why

$$A \int_0^T z(t)d\nu(t) = \int_0^T Az(t)d\nu(t) \quad \text{and} \quad \int_0^T z(t)d\nu(t) \in D_A.$$

At the same time, the additional unknown element u in Equation (4.1) is to be found by means of the additional condition (4.3).

An element $u \in \mathcal{U}$ a solution to the problem (4.1)–(4.3) if for this u there exists a solution z to the Cauchy problem (4.1), (4.2), which satisfies the condition (4.3). The problem (4.1)–(4.3) is called well–posed if for all $z_0, z_T \in D_A$, $g \in W_1^1(0, T; \mathcal{Z})$ there exists a unique solution $u \in \mathcal{U}$ to the problem, and this solution satisfies the estimate

$$\|u\|_{\mathcal{U}} \leqslant C \left(\|z_0\|_{\mathcal{Z}} + \|z_T\|_{\mathcal{Z}} + \|g\|_{C([0,T];\mathcal{Z})} \right), \tag{4.4}$$

where $C > 0$ is independent of z_0, z_T, g.

By the representation of solution (3.4) in the case of existence of a solution to the Cauchy problem (4.1), (4.2) the element u is a solution to the problem (4.1)–(4.3) if and only if it satisfies the equation

$$\Psi u = \psi, \tag{4.5}$$

where Ψ and ψ are defined by the formulas

$$\Psi := \int_0^T (K(0) - A)^{-1}(B(t) - B(0))d\nu(t) - \int_0^T \int_0^t Z(s)B(0)dsd\nu(t)$$
$$+ \int_0^T \int_0^t Z(t-s)B(s)dsd\nu(t) \in \mathcal{L}(\mathcal{U}; \mathcal{Z})$$
$$\psi := z_T - \int_0^T d\nu(t)z_0 - \int_0^T (K(0) - A)^{-1}(g(t) - g(0))d\nu(t) + \int_0^T \int_0^t Z(s)g(0)dsd\nu(t)$$
$$- \int_0^T \int_0^t Z(t-s)g(s)dsd\nu(t) \in \mathcal{Z}.$$

Theorem 4.1. *Let*

$$A \in \mathcal{C}l(\mathcal{Z}), \qquad K \in C^1(\overline{\mathbb{R}}_+; \mathcal{L}(\mathcal{Z})), \qquad (K(0) - A)^{-1} \in \mathcal{L}(\mathcal{Z}),$$

the condition $(\widehat{K})$ *be satisfied,*

$$B \in W_1^1(0, T; \mathcal{L}(\mathcal{U}; \mathcal{Z})), \qquad B(0) = 0, \qquad g \in W_1^1(0, T; \mathcal{Z}),$$
$$z_0, z_T \in D_A, \qquad Az_0 + g(0) = 0, \qquad \nu \in BV((0, T]; \mathbb{C}).$$

Then the inverse problem (4.1)–(4.3) is well–posed if and only if there exists the inverse operator $\Psi^{-1} \in \mathcal{L}(\mathcal{Z}; \mathcal{U})$. *At the same time the solution to the problem reads* $u = \Psi^{-1}\psi$.

Proof. By Theorem 3.1, for a given element $u \in \mathcal{U}$, a solution to the Cauchy problem (4.1), (4.2) exists and it reads

$$z(t) = z_0 + (K(0) - A)^{-1}(B(t)u + g(t) - B(0)u - g(0)) - \int_0^t Z(s)(B(0)u + g(0))ds$$

$$+\int_0^t Z(t-s)(B(s)u+g(s))ds.$$

Substituting this solution into the overdetermination condition (4.3), we obtain the identity (4.5), which implies the required fact. At the same time by the identity $B(0)=0$ we have

$$\Psi := \int_0^T (K(0)-A)^{-1}B(t)d\nu(t) + \int_0^T\int_0^t Z(t-s)B(s)dsd\nu(t),$$

$$\begin{aligned}\|u\|_{\mathcal{U}} &\leqslant \|\Psi^{-1}\|_{\mathcal{L}(\mathcal{Z};\mathcal{U})}\|\psi\|_{\mathcal{Z}} \leqslant \|\Psi^{-1}\|_{\mathcal{L}(\mathcal{Z};\mathcal{U})}(\|z_T\|_{\mathcal{Z}} + V_0^T(\nu)\|z_0\|_{\mathcal{Z}} \\ &+ 2TV_0^T(\nu)\|(K(0)-A)^{-1}\|_{\mathcal{L}(\mathcal{Z})}\|g\|_{C([0,T];\mathcal{Z})} + 2T^2V_0^T(\nu)\|Z\|_{C([0,T];\mathcal{Z})}\|g\|_{C([0,T];\mathcal{Z})}) \\ &\leqslant C\left(\|z_0\|_{\mathcal{Z}} + \|z_T\|_{\mathcal{Z}} + \|g\|_{C([0,T];\mathcal{Z})}\right)\end{aligned}$$

for

$$C = \max\{1, V_0^T(\nu), (2TV_0^T(\nu)\|(K(0)-A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} + 2T^2V_0^T(\nu)\|Z\|_{C([0,T];\mathcal{Z})})\} > 0.$$

Here $V_0^T(\nu)$ is the variation of the function ν on the semi–interval $(0,T]$. The proof is complete. □

5. One class of initial boundary value problem

Let

$$P_n(\lambda) = \sum_{j=0}^n c_j\lambda^j, \qquad Q_n(\lambda) = \sum_{j=0}^n d_j\lambda^j,$$

$c_j, d_j \in \mathbb{C}$, $j = 0,1,\dots,n \in \mathbb{N}_0 : \mathbb{N}\cup\{0\}$, $d_n \neq 0$. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a smooth boundary $\partial\Omega$, in this domain the Laplace operator

$$\Delta w(\xi) = \sum_{j=1}^d \frac{\partial^2 w}{\partial \xi^2}(\xi)$$

is defined on the domain $D_\Delta = H_0^2(\Omega) := \{w \in H^2(\Omega) : w(\xi) = 0,\ \xi \in \partial\Omega\}$. As it is known, the spectrum $\sigma(\Delta)$ of the operator Δ is negative, discrete, of finite multiplicity and accumulates at $-\infty$ only. Let $\{\varphi_k : k \in \mathbb{N}\}$ be an orthonormalized in $L_2(\Omega)$ system of eigenfunctions of the operator Δ associated with the corresponding eigenvalues $\{\lambda_k : k \in \mathbb{N}\}$ taken in the ascending order counting the multitplicities.

We consider the inverse problem with a time–independent element u

$$v(\xi,s) = v_0(\xi), \qquad \xi \in \Omega, \tag{5.1}$$

$$\Delta^k v(\xi,t) = 0, \qquad k = 1,2,\dots,n-1, \qquad (\xi,t) \in \partial\Omega\times[0,T], \tag{5.2}$$

$$\begin{aligned}&P_n(\Delta)\int_0^t aE^{\beta}_{\alpha,1}(b(t-s)^\alpha)\frac{\partial v}{\partial s}(\xi,s)ds \\ &\qquad = Q_n(\Delta)v(\xi,t) + c(t)u(\xi) + h(\xi,t), \qquad (\xi,t)\in\Omega\times[0,T],\end{aligned} \tag{5.3}$$

$$v(\xi,T) = v_T(\xi), \qquad \xi\in\Omega, \tag{5.4}$$

where the functions $v_0, v_T : \Omega \to \mathbb{R}$, $c : [0,T] \to \mathbb{R}$, $h : \Omega\times[0,T] \to \mathbb{R}$ are given.

In Equation (5.3), the kernel of integro–differential operator is the generalized Mittag–Leffler [18]

$$K_{\alpha,\beta}(t) := aE^{\beta}_{\alpha,1}(bt^{\alpha})I = a\sum_{k=0}^{\infty}\frac{(\beta)_k b^k t^{\alpha k}}{\Gamma(\alpha k+1)}I, \quad a,b\in\mathbb{R}\setminus\{0\}, \quad \alpha,\beta\in\mathbb{N},$$

where $(\beta)_k$ is the Pochhammer function [18]. Its Laplace image

$$\widehat{K}_{\alpha,\beta}(\lambda) = \frac{a\lambda^{\alpha\beta-1}}{(\lambda^{\alpha}-b)^{\beta}}I$$

is a univalent analytic function on the set

$$\Omega_{|b|^{\frac{1}{\alpha}}} := \{\mu\in\mathbb{C} : |\mu| > |b|^{\frac{1}{\alpha}}\},$$

and this is why the condition $(\widehat{K})$ holds for this function with $a_K = |b|^{\frac{1}{\alpha}}$. We note that for $\alpha = \beta = 1$, $K_{1,1}(t) = ae^{bt}$ the integro–differential operator $D^{K,1}$ is the Caputo — Fabrizio derivative [11], while for $\alpha\in\mathbb{N}\setminus\{0\}$, $\beta = 1$, $K_{\alpha,1}(t) = aE_{\alpha}(bt^{\alpha})$ is the one–parametric Mittag–Leffler function [18], while $D^{K,1}$ is the so–called Atangana — Baleanu derivative.

We let

$$n_0 := \max\{j\in\{0,1,\dots,n\} : c_j\neq 0\},$$
$$\mathcal{Z} = \{w\in H^{2rn_0}(\Omega) : \Delta^k w(\xi) = 0,\ k = 0,1,\dots,n_0-1,\ \xi\in\partial\Omega\}.$$

The operator $P_n(\Delta)\in\mathcal{L}(\mathcal{Z};L_2(\Omega))$ is continuously invertible if and only if $P_n(\lambda_k)\neq 0$ for all $k\in\mathbb{N}$. In this case, on the Banach space $\mathcal{Z}$ we define the linear operator $A = P_n(\Delta)^{-1}Q_n(\Delta)$, which is bounded in $\mathcal{Z}$ if $n_0 = n$, that is, $c_n\neq 0$. If $c_n = 0$ and $n_0 < n$, we have $A\in\mathcal{C}l(\mathcal{Z})$ with the domain

$$D_A = \{w\in H^{2rn}(\Omega) : \Delta^k w(\xi) = 0,\ k = 0,1,\dots,n-1,\ \xi\in\partial\Omega\}.$$

The problem (5.1)–(5.3) is reduced to the problem (4.1), (4.2).

We denote

$$\zeta_k(t) := \frac{1}{2\pi i}\int\limits_{|\lambda|=r}\frac{e^{\lambda t}d\lambda}{\frac{a\lambda^{\alpha\beta}}{(\lambda^{\alpha}-b)^{\beta}} - \frac{Q_n(\lambda_k)}{P_n(\lambda_k)}}, \quad t\in\mathbb{C}, \quad k\in\mathbb{N}.$$

Theorem 5.1. *Let*

$$a,b\in\mathbb{R}\setminus\{0\}, \quad \alpha,\beta\in\mathbb{N}, \quad P_n(\lambda_k)\neq 0, \quad Q_n(\lambda_k)/P_n(\lambda_k)\neq a \quad \textit{for all} \quad k\in\mathbb{N},$$
$$v_0\in D_A, \quad u\in L_2(\Omega), \quad c\in W_1^1(0,T;\mathbb{R}), \quad h\in W_1^1(0,T;L_2(\Omega)),$$
$$Q_n(\Delta)v_0(\xi) + c(0)u(\xi) + h(\xi,0)\equiv 0 \quad \textit{in} \quad \Omega.$$

Then the problem (5.1)–(5.3) *has the unique solution*

$$v(\xi,t) = v_0(\xi) + \sum_{k=1}^{\infty}\frac{\langle(c(t)-c(0))u(\cdot)+h(\cdot,t)-h(\cdot,0),\varphi_k\rangle\varphi_k(\xi)}{aP_n(\lambda_k)-Q_n(\lambda_k)}$$
$$-\int\limits_0^t\sum_{k=1}^{\infty}\frac{\zeta_k(s)\langle c(0)u(\cdot)+h(\cdot,0),\varphi_k\rangle\varphi_k(\xi)}{P_n(\lambda_k)}ds$$
$$+\int\limits_0^t\sum_{k=1}^{\infty}\frac{\zeta_k(t-s)\langle c(s)u(\cdot)+h(\cdot,s),\varphi_k\rangle\varphi_k(\xi)}{P_n(\lambda_k)}ds.$$

Proof. The spectrum of operator $A = P_n(\Delta)^{-1}Q_n(\Delta)$ is the set

$$\sigma(A) = \left\{\frac{Q_n(\lambda_k)}{P_n(\lambda_k)}, \quad k\in\mathbb{N}\right\}.$$

Therefore, the inequality

$$\frac{Q_n(\lambda_k)}{P_n(\lambda_k)} \neq a \quad \text{for all} \quad k \in \mathbb{N}$$

means the existence of the inverse operator $(K(0) - A)^{-1} = (aI - A)^{-1} \in \mathcal{L}(\mathcal{Z})$.

For a sufficiently large $r > 0$ we have

$$Z(t) = \sum_{k=1}^{\infty} \frac{1}{2\pi i} \int\limits_{|\lambda|=r} \frac{\langle \cdot, \varphi_k \rangle \varphi_k(\xi) e^{\lambda t} d\lambda}{\frac{a\lambda^{\alpha\beta}}{(\lambda^\alpha - b)^\beta} - \frac{Q_n(\lambda_k)}{P_n(\lambda_k)}} = \sum_{k=1}^{\infty} \zeta_k(t) \langle \cdot, \varphi_k \rangle \varphi_k(\xi), \quad t \in \mathbb{R}.$$

In Theorem 3.1 we take $f(t) = P_n(\Delta)^{-1}(c(t)u(\cdot) + h(\cdot, t))$ and get the required statement. The proof is complete. □

For the inverse problem (5.1)–(5.4) with the unknown element u in the case $c(0) = 0$ we respectively obtain the operator and vector

$$\Psi := \sum_{k=1}^{\infty} \frac{c(T)\langle \cdot, \varphi_k \rangle \varphi_k(\xi)}{aP_n(\lambda_k) - Q_n(\lambda_k)} + \int\limits_0^T \sum_{k=1}^{\infty} \frac{\zeta_k(t-s)c(s)\langle \cdot, \varphi_k \rangle \varphi_k(\xi)}{P_n(\lambda_k)} ds,$$

$$\psi := v_T - v_0 - \sum_{k=1}^{\infty} \frac{(h(\cdot, T) - h(\cdot, 0))\langle \cdot, \varphi_k \rangle \varphi_k(\xi)}{aP_n(\lambda_k) - Q_n(\lambda_k)} + \int\limits_0^T \sum_{k=1}^{\infty} \frac{\zeta_k(s) h(\cdot, 0)\langle \cdot, \varphi_k \rangle \varphi_k(\xi)}{P_n(\lambda_k)} ds$$

$$- \int\limits_0^T \sum_{k=1}^{\infty} \frac{\zeta_k(t-s) h(\cdot, s)\langle \cdot, \varphi_k \rangle \varphi_k(\xi)}{P_n(\lambda_k)} ds.$$

Theorem 5.2. *Let*

$a, b \in \mathbb{R} \setminus \{0\}$, $\alpha, \beta \in \mathbb{N}$, $P_n(\lambda_k) \neq 0$, $Q_n(\lambda_k)/P_n(\lambda_k) \neq a$ *for all* $k \in \mathbb{N}$,
$c \in W_1^1(0, T; \mathbb{R})$, $c(0) = 0$, $v_0, v_T \in D_A$, $h \in W_1^1(0, T; L_2(\Omega))$,
$Q_n(\Delta)v_0(\xi) + h(\xi, 0) \equiv 0$ *in* Ω.

Then the inverse problem (5.1)–(5.4) *is well–posed if and only if there exists* $d > 0$ *such that, for all* $k \in \mathbb{N}$,

$$\left| \frac{c(T)}{aP_n(\lambda_k) - Q_n(\lambda_k)} + \int\limits_0^T \frac{\zeta_k(t-s)c(s)}{P_n(\lambda_k)} ds \right| \geqslant d.$$

At the same time, $u = \Psi^{-1}\psi$.

Proof. Here ν is the function of the unit jump at the point $t = T$. We take the space $\mathcal{U} = L_2(\Omega)$ and the operator function

$$B(t) = c(t)P_n(\Delta)^{-1} \in W_1^1(0, T; \mathcal{L}(L_2(\Omega); \mathcal{Z})).$$

Thus, the problem (5.1)–(5.4) is reduced to the inverse problem (4.1)–(4.3). The assumptions of the theorem imply

$$\|\Psi^{-1}\|_{\mathcal{L}(\mathcal{Z}; L_2(\Omega))} \leqslant d^{-1}.$$

The proof is complete. □

6. System of ordinary integro–differential equations

We consider the Cauchy problem

$$z_j(0) = z_{0j}, \quad j = 1, 2, \dots, n, \tag{6.1}$$

for the system of integro–differential equations

$$\sum_{j=1}^{n} D^{k_{ij},1} z_j(t) = \sum_{j=1}^{n} a_{ij} z_j(t) + \sum_{k=1}^{m} b_{ik}(t) u_k + h_i(t), \qquad i = 1, 2, \dots, n, \qquad t \in [0, T], \tag{6.2}$$

where

$$a_{ij} \in \mathbb{C}, \qquad b_{ik} : [0,T] \to \mathbb{C}, \qquad u_k \in \mathbb{C}, \qquad h_i : [0,T] \to \mathbb{C}, \qquad k_{ij}(t) := a_{ij} E^{\beta_{ij}}_{\alpha_{ij},1}(b_{ij} t^{\alpha}),$$
$$a_{ij}, b_{ij} \in \mathbb{R} \setminus \{0\}, \qquad \alpha_{ij}, \beta_{ij} \in \mathbb{N},, \qquad i, j = 1, 2, \dots, n, \qquad k = 1, 2, \dots, m.$$

We take

$$\mathcal{Z} = \mathbb{R}^n, \qquad \mathcal{U} = \mathbb{R}^m, \qquad z(t) = (z_1(t), z_2(t), \dots, z_n(t))^T : [0,T] \to \mathcal{Z},$$
$$g(t) = (g_1(t), g_2(t), \dots, g_n(t))^T : [0,T] \to \mathcal{Z}, \qquad u = (u_1, u_2, \dots, u_m)^T \in \mathcal{U}.$$

The actions of the operators $A \in \mathcal{L}(\mathcal{Z})$, $K(t) \in \mathcal{L}(\mathcal{Z})$ for $t \geqslant 0$ and $B(t) \in \mathcal{L}(\mathcal{U}; \mathcal{Z})$ for $t \in [0,T]$ are given by same–named matrices of sizes $n \times n$, $n \times n$ and $n \times m$, respectively:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \qquad K(t) = \begin{pmatrix} k_{11}(t) & k_{12}(t) & \dots & k_{1n}(t) \\ k_{21}(t) & k_{22}(t) & \dots & k_{2n}(t) \\ \dots & \dots & \dots & \dots \\ k_{n1}(t) & k_{n2}(t) & \dots & k_{nn}(t) \end{pmatrix},$$
$$B(t) = \begin{pmatrix} b_{11}(t) & b_{12}(t) & \dots & b_{1m}(t) \\ b_{21}(t) & b_{22}(t) & \dots & b_{2m}(t) \\ \dots & \dots & \dots & \dots \\ b_{n1}(t) & b_{n2}(t) & \dots & b_{nm}(t) \end{pmatrix}.$$

At the same time,

$$\widehat{K}(\lambda) = \begin{pmatrix} \frac{a_{11}\lambda^{\alpha_{11}\beta_{11}-1}}{(\lambda^{\alpha_{11}}-b_{11})^{\beta_{11}}} & \frac{a_{12}\lambda^{\alpha_{12}\beta_{12}-1}}{(\lambda^{\alpha_{12}}-b_{12})^{\beta_{12}}} & \dots & \frac{a_{1n}\lambda^{\alpha_{1n}\beta_{1n}-1}}{(\lambda^{\alpha_{1n}}-b_{1n})^{\beta_{1n}}} \\ \frac{a_{21}\lambda^{\alpha_{21}\beta_{21}-1}}{(\lambda^{\alpha_{21}}-b_{21})^{\beta_{21}}} & \frac{a_{22}\lambda^{\alpha_{22}\beta_{22}-1}}{(\lambda^{\alpha_{22}}-b_{22})^{\beta_{22}}} & \dots & \frac{a_{2n}\lambda^{\alpha_{2n}\beta_{2n}-1}}{(\lambda^{\alpha_{2n}}-b_{2n})^{\beta_{2n}}} \\ \dots & \dots & \dots & \dots \\ \frac{a_{n1}\lambda^{\alpha_{n1}\beta_{n1}-1}}{(\lambda^{\alpha_{n1}}-b_{n1})^{\beta_{n1}}} & \frac{a_{n2}\lambda^{\alpha_{n2}\beta_{n2}-1}}{(\lambda^{\alpha_{n2}}-b_{n2})^{\beta_{n2}}} & \dots & \frac{a_{nn}\lambda^{\alpha_{nn}\beta_{nn}-1}}{(\lambda^{\alpha_{nn}}-b_{nn})^{\beta_{nn}}} \end{pmatrix}.$$

Thus, we have obtained the direct problem (4.1), (4.2).

Theorem 6.1. *Let*

$$\det(K(0) - A) \neq 0, \qquad \det \widehat{K} \not\equiv 0, \qquad b_{ik} \in W^1_1(0,T;\mathbb{C}), \qquad u_k \in \mathbb{C},$$
$$h_i \in W^1_1(0,T;\mathbb{C}), \qquad z_{0i} \in \mathbb{C}, \qquad \sum_{j=1}^{n} a_{ij} z_{0j} + \sum_{k=1}^{m} b_{ik}(0) u_k + h_i(0) = 0,$$
$$i = 1, 2, \dots, n, \qquad k = 1, 2, \dots, m.$$

Then there exists a unique solution of the problem (6.1)*,* (6.2)*.*

Proof. We note that the function $\det \widehat{K}$ is the quotient of two polynomials and this is why it possesses finitely many poles, and hence, the condition $(\widehat{K})$ is satisfied. By Theorem 3.1 we get the required statement. □

We note that the solution to the problem (6.1), (6.2) is of the form (3.4) for $f(t) = B(t)u + h(t)$ (in the vector notation).

Under the assumption that u_k, $k = 1, 2, \dots, m$, are unknown, we consider the overdetermination conditions

$$z_j(T) = z_{Tj}, \quad j = 1, 2, \dots, n. \tag{6.3}$$

Theorem 6.2. *Let*

$$\det(K(0) - A) \neq 0, \qquad \det \widehat{K} \not\equiv 0, \qquad b_{ik} \in W_1^1(0, T; \mathbb{C}), \qquad b_{ik}(0) = 0,$$

$$h_i \in W_1^1(0, T; \mathbb{C}), \qquad z_{0i}, z_{Ti} \in \mathbb{C}, \qquad \sum_{j=1}^{n} a_{ij} z_{0j} + h_i(0) = 0,$$

$$i = 1, 2, \dots, n, \qquad k = 1, 2, \dots, m.$$

Then the inverse problem (6.1)–(6.3) *is well–posed if and only if*

$$\det \left((K(0) - A)^{-1} B(T) + \int_0^T Z(T - s) B(s) ds \right) \neq 0.$$

At the same time the solution of the problem in the vector notation reads

$$u = \left((K(0) - A)^{-1} B(T) + \int_0^T Z(T - s) B(s) ds \right)^{-1}$$

$$\cdot \left(z_T - z_0 - (K(0) - A)^{-1}(g(T) - g(0)) + \int_0^T Z(s) g(0) ds - \int_0^T Z(T - s) g(s) ds \right),$$

where the matrices $Z(t)$ are of the form (3.3).

Proof. By Theorem 4.1 we obtain the required statement. □

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