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ON BEST APPROXIMATION OF FUNCTIONS IN BERGMAN SPACE B_2

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Abstract. In the paper we study extremal problems related to the best polynomial approximation of functions analytic in the unit disk and belonging to the Hilbert Bergman space B_2. We find exact inequalities for the best approximation of an arbitrary function $f \in B_2$, analytic in the unit disk, by algebraic complex polynomials $p_n \in \mathcal{P}_n$ by means of the averaged value of the modulus of continuity $\omega(f^{(r)}, t)_{B_2}$ of the rth derivative $f^{(r)}$ in the space B_2. We introduce the class $W_2^{(r)}(\omega, \Phi)$ of functions analytic in the unit disk whose averaged value of the modulus of continuity of the derivative $f^{(r)}$ satisfies the inequality

$$\int\limits_0^u \omega^2(f^{(r)}, t)_{B_2} \sin\frac{\pi}{u} t\, dt \leqslant \Phi^2(u), \qquad 0 \leqslant u \leqslant 2\pi.$$

For certain restrictions for majorant Φ, we calculate exact values of various n–widths for the introduced class of functions. To solve the mentioned problems, we use the methods of solving extremal problems in normed spaces and we use the method for estimating n–widths developed by V.M. Tikhomirov.

Keywords: extremal problems, approximation of functions, modulus of continuity, suprema, n–widths, Bergman space.

Mathematics Subject Classification: 41A17, 41A25

1. Introduction and preliminary results

Extremal problems of best polynomial approximation of analytic in circle functions in various spaces were studied, for instance, in the works [1], [3]–[7], [9]–[13], [15], [16], [19], [20], [22]–[29], [31] and many others. In this work our aim is to find the suprema of best approximations of functions by complex algebraic polynomials in the Bergman space B_2.

Let $\mathbb{N}$, $\mathbb{Z}_+$ be respectively the set of natural and nonnegative integer numbers. Let $\mathbb{C}$ be the complex plane, $U := \{z \in \mathbb{C} : |z| < 1\}$ be the unit circle in $\mathbb{C}$, $A(U)$ be the set of functions analytic in the circle U.

Definition 1.1 ([6]). *We say that an analytic in the unit circle U function*

$$f(z) = \sum_{k=0}^{\infty} c_k(f) z^k, \qquad z = \rho e^{it}, \quad 0 \leqslant \rho < 1, \quad 0 \leqslant t \leqslant 2\pi, \tag{1.1}$$

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belongs to the Bergman space B_2 if

$$\|f\|_2 := \|f\|_{B_2} = \left(\frac{1}{\pi}\iint\limits_{(U)} |f(z)|^2\, d\sigma\right)^{\frac{1}{2}} < \infty. \tag{1.2}$$

The derivative of rth order of a function $f \in A(U)$ is defined as usually

$$f^{(r)}(z) := \frac{d^r f(z)}{dz^r} = \sum_{k=r}^{\infty} k(k-1)\cdots(k-r+1)c_k(f)z^{k-r}, \quad r \in \mathbb{N}. \tag{1.3}$$

For the sake of brevity, we introduce the notation

$$\alpha_{k,r} := k(k-1)\cdots(k-r+1) = \frac{k!}{(k-r)!}, \quad k,r \in \mathbb{N}, \quad k > r. \tag{1.4}$$

In what follows by the symbol $B_2^{(r)}$ ($r \in \mathbb{Z}_+$, $B_2^{(0)} = B_2$) we denote the set of functions $f \in A(U)$, belonging to the space B_2, whose derivative of rth order $f^{(r)}(z)$ also belongs to B_2, that is,

$$B_2^{(r)} := \left\{f \in B_2 : \|f^{(r)}\|_2 < \infty\right\}.$$

Let $\mathcal{P}_n$ be the subspace of complex algebraic polynomials of degree n of form

$$p_n(z) = \sum_{k=0}^{n} a_k z^k, \qquad a_k \in \mathbb{C}.$$

The quantity

$$E_n(f)_2 := E(f, \mathcal{P}_n)_{B_2} = \inf\left\{\|f - p_n\|_2 : p_n \in \mathcal{P}_n\right\} \tag{1.5}$$

is called the best polynomial root–mean–square of the function $f \in B_2$ by the subspace $\mathcal{P}_n$.

It is well–known [14] that an arbitrary function $f \in B_2$ satisfies the relation

$$E_{n-1}(f_2) = \|f - T_{n-1}(f)\|_2 = \left\{\sum_{k=n}^{\infty} \frac{|c_k(f)|^2}{k+1}\right\}^{\frac{1}{2}}, \tag{1.6}$$

where $T_{n-1}(f)$ is the $(n-1)$th partial sum of series (1.1).

We write the norm (1.1) in a more convenient form

$$\|f\|_2 := \left(\frac{1}{\pi}\int_0^1\int_0^{2\pi} |f(\rho e^{it})|^2 \rho\, d\rho\, dt\right)^{\frac{1}{2}},$$

and by the symbol

$$\Delta_h^1 f(\rho e^{it}) = f(\rho e^{i(t+h)}) - f(\rho e^{it})$$

we denote the first order finite difference of a function $f \in B_2$ in the variable t with the step h. By the identity

$$\begin{aligned} \omega(f,\tau)_{B_2} &:= \sup\left\{\|\Delta_h^1(f)\|_{B_2} : |h| \leqslant \tau\right\} \\ &= \sup_{|h|\leqslant\tau} \frac{1}{\pi}\int_0^1\int_0^{2\pi} |f(\rho e^{i(t+h)} - f(\rho e^{it})|^2\, d\rho\, dt \end{aligned}$$

we define the first order modulus of the function $f \in B_2$. Using the relations (1.3) and (1.4), for each $r \in \mathbb{Z}_+$ we have

$$\Delta_h^1 f^{(r)}(\rho e^{it}) = \sum_{k=r+1}^{\infty} \alpha_{k,r} c_k(f) \rho^{k-r} e^{i(k-r)t}\big(1 - e^{i(k-r)h}\big).$$

By the Parseval identity we get

$$\|\Delta_h^1 f^{(r)}\|^2 = 2\sum_{k=r+1}^{\infty} \alpha_{k,r}^2 \frac{|c_k(f)|^2}{k-r+1}\big(1-\cos(k-r)h\big) \tag{1.7}$$

and therefore,

$$\omega^2(f^{(r)},\tau)_{B_2} = 2\sup_{|h|\leqslant\tau}\sum_{k=r+1}^{\infty} \alpha_{k,r}^2 \frac{|c_k(f)|^2}{k-r+1}\big(1-\cos(k-r)h\big). \tag{1.8}$$

2. Main results

In this section we present main results obtained in this paper. The next theorem holds true.

Theorem 2.1. *For an arbitrary function $f\in B_2$ and a given $n\in\mathbb{N}$ for each $h\in\big(0,\frac{\pi}{n}\big]$ the inequality holds*

$$E_{n-1}^2(f)_{B_2} \leqslant \frac{\displaystyle\int_0^h \omega^2(f,t)_{B_2}\sin\frac{\pi}{h}t\,dt}{2\left[\dfrac{2h}{\pi} - \displaystyle\int_0^h \cos nt\sin\frac{\pi}{h}t\,dt\right]}. \tag{2.1}$$

For the function $f_0(z)=z^n\in B_2$ the inequality (2.1) becomes the identity for all $h\in\big(0,\frac{\pi}{n}\big]$.

Proof. Using the definition of the modulus of continuity we write

$$\begin{aligned}\omega^2(f,t)_{B_2} &\geqslant \|f(\cdot+t)-f(\cdot)\|_{B_2} = \frac{1}{\pi}\int_0^1\int_0^{2\pi}\rho|f(\rho e^{i(x+t)})-f(\rho e^{ix})|^2\,d\rho\,dx \\ &= 2\sum_{k=1}^{\infty}\frac{|c_k(f)|^2}{k+1}(1-\cos kt) \geqslant 2\sum_{k=n}^{\infty}\frac{|c_k(f)|^2}{k+1}(1-\cos kt).\end{aligned} \tag{2.2}$$

Supposing that $h\in\big(0,\frac{\pi}{n}\big]$, we multiply both sides of the inequality (2.2) by the function $\sin\frac{\pi}{h}t$ and integrate in t from 0 to h. As a result we get

$$\begin{aligned}\int_0^h \omega^2(f,t)_{B_2}\sin\frac{\pi}{h}t &\geqslant 2\sum_{k=n}^{\infty}\frac{|c_k(f)|^2}{k+1}\int_0^h(1-\cos kt)\sin\frac{\pi}{h}t\,dt \\ &= 2\sum_{k=n}^{\infty}\frac{|c_k(f)|^2}{k+1}\int_0^h\sin\frac{\pi}{h}t\,dt - 2\sum_{k=n}^{\infty}\frac{|c_k(f)|^2}{k+1}\int_0^h\cos kt\sin\frac{\pi}{h}t\,dt.\end{aligned} \tag{2.3}$$

We now mention that the function of natural variable

$$\varphi(k) = \int_0^h \cos kt\sin\frac{\pi}{h}t\,dt$$

decreases in $k\in\mathbb{N}$ as $h\in\big(0,\frac{\pi}{k}\big]$ since

$$\varphi'(k) = -\int_0^h t\sin kt\sin\frac{\pi}{h}t\,dt < 0.$$

This is why for $h \in \left(0, \frac{\pi}{k}\right]$, $t \in (0, h)$ and $k \geqslant n$

$$\int_0^h \cos kt \sin \frac{\pi}{h} t \, dt \leqslant \int_0^h \cos nt \sin \frac{\pi}{h} t \, dt. \tag{2.4}$$

As $h \in \left(\frac{\pi}{k}, \frac{\pi}{n}\right]$, $t \in (0, h)$ and $k \geqslant n$ the inequality (2.4) again holds since

$$\int_0^h \cos kt \sin \frac{\pi}{h} t \, dt = \frac{2\pi h}{\pi^2 - h^2 k^2} \cos^2 \frac{kh}{2} \leqslant 0,$$

$$\int_0^h \cos nt \sin \frac{\pi}{h} t \, dt = \frac{2\pi h}{\pi^2 - h^2 n^2} \cos^2 \frac{nh}{2} \geqslant 0.$$

Thus, for all $h \in \left(0, \frac{\pi}{n}\right]$, $t \in (0, h)$ and $k \geqslant n$

$$\int_0^h \cos kt \sin \frac{\pi}{h} t \, dt \leqslant \int_0^h \cos nt \sin \frac{\pi}{h} t \, dt.$$

By (2.3) this implies

$$\int_0^h \omega^2(f,t)_{B_2} \sin \frac{\pi}{h} t \geqslant \frac{4h}{\pi} E_{n-1}^2(f)_{B_2} - 2E_{n-1}^2(f)_{B_2} \int_0^h \cos nt \sin \frac{\pi}{h} t \, dt$$

$$= E_{n-1}^2(f)_{B_2} \left[\frac{4h}{\pi} - 2 \int_0^h \cos nt \sin \frac{\pi}{h} t \, dt \right],$$

which yields the inequality (2.1). The identity for $f_0(z) = z^n \in B_2$ can be verified by direct calculations. The proof is complete. □

Remark 2.1. *Since for $h = \frac{\pi}{n}$*

$$\int_0^{\frac{\pi}{n}} \cos nt \sin nt \, dt = 0,$$

by (2.1) we obtain

$$E_{n-1}(f)_{B_2} \leqslant \frac{1}{\sqrt{2}} \left(\frac{n}{2} \int_0^{\frac{\pi}{n}} \omega^2(f,t)_{B_2} \sin nt \, dt \right)^{\frac{1}{2}}. \tag{2.5}$$

The inequality (2.5) is an analogue the well–known Chernykh inequality [21] proven for the class of periodic functions $L_2 := L_2[0, 2\pi]$ to the case of analytic in the unit circle functions in the Bergman space B_2.

Theorem 2.2. *For each function $f \in B_2^{(r)}$, $r \in \mathbb{Z}_+$, and each $n \in \mathbb{N}$, $n > r$, the inequality*

$$E_{n-1}^2(f)_{B_2} \leqslant \frac{1}{2}\sqrt{\frac{n-r+1}{n+1}}\frac{1}{\alpha_{n,r}} \frac{\int\limits_0^h \omega^2(f^{(r)},t)_{B_2} \sin\frac{\pi}{h}t\,dt}{\int\limits_0^h (1-\cos(n-r)t)\sin\frac{\pi}{h}t\,dt} \tag{2.6}$$

holds.

Proof. It was proved in [30] that for an arbitrary function $f \in B_2^{(r)}$ for all $n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $n > r$ the inequality holds

$$E_{n-1}(f)_{B_2} \leqslant \sqrt{\frac{n-r+1}{n+1}}\frac{1}{\alpha_{n,r}} E_{n-r-1}(f^{(r)})_{B_2}. \tag{2.7}$$

By Theorem 2.1 we have

$$\begin{aligned} E_{n-r-1}(f^{(r)})_{B_2} &\leqslant \frac{\left\{\int\limits_0^h \omega^2(f^{(r)},t)_{B_2}\sin\frac{\pi}{h}t\,dt\right\}^{\frac{1}{2}}}{\left\{2\left[\frac{2h}{\pi} - \int\limits_0^h \cos(n-r)t\sin\frac{\pi}{h}t\,dt\right]\right\}^{\frac{1}{2}}} \\ &= \frac{\left\{\int\limits_0^h \omega^2(f^{(r)},t)_{B_2}\sin\frac{\pi}{h}t\,dt\right\}^{\frac{1}{2}}}{\left\{2\int\limits_0^h (1-\cos(n-r)t)\sin\frac{\pi}{h}t\,dt\right\}^{\frac{1}{2}}}. \end{aligned} \tag{2.8}$$

In view of the inequality (2.8), by (2.7) we obtain

$$E_{n-1}(f)_{B_2} \leqslant \sqrt{\frac{n-r+1}{n+1}}\frac{1}{\alpha_{n,r}} \frac{\left\{\int\limits_0^h \omega^2(f^{(r)},t)_{B_2}\sin\frac{\pi}{h}t\,dt\right\}^{\frac{1}{2}}}{\left\{2\int\limits_0^h (1-\cos(n-r)t)\sin\frac{\pi}{h}t\,dt\right\}^{\frac{1}{2}}} \tag{2.9}$$

and thus, the inequality (2.6) is proven. It is easy to verify that the inequality (2.6) for the function $f_0(z) = z^n \in B_2^{(r)}$, $n > r$, $n \in \mathbb{N}$, $r \in \mathbb{Z}_+$ becomes the identity. The proof is complete. $\square$

Corollary 2.1. *Under the assumptions of Theorem 2.2 for $h = \frac{\pi}{(n-r)}$, $n > r$ the inequality holds*

$$E_{n-1}(f)_{B_2} \leqslant \frac{1}{\sqrt{2}}\sqrt{\frac{n-r+1}{n+1}}\frac{1}{\alpha_{n,r}} \left\{\frac{n-r}{2}\int\limits_0^{\frac{\pi}{(n-r)}} \omega^2(f^{(r)},t)_{B_2}\sin(n-r)t\,dt\right\}^{\frac{1}{2}}. \tag{2.10}$$

Corollary 2.2. *For an arbitrary function $f_0 \in B_2^{(r)}$ for all $n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $n > r$, the Jackson type inequality holds*

$$E_{n-1}(f)_{B_2} \leqslant \frac{1}{\sqrt{2}}\sqrt{\frac{n-r+1}{n+1}}\frac{1}{\alpha_{n,r}}\omega\left(f^{(r)}, \frac{\pi}{n-r}\right)_{B_2}. \tag{2.11}$$

The inequality (2.11) is implied by the monotone increasing of the modulus of continuity $\omega(f^{(r)}, t)_{B_2}$ on the segment $\left[0, \frac{\pi}{(n-r)}\right]$. But if the modulus of continuity $\omega(f^{(r)}, t)_{B_2}$ is convex on the segment $\left[0, \frac{\pi}{(n-r)}\right]$, that is, for all $t \in \left[0, \frac{\pi}{(n-r)}\right]$ it satisfies the condition

$$\omega^2(f^{(r)}, t)_{B_2} + \omega^2\left(f^{(r)}, \frac{\pi}{n-r} - t\right)_{B_2} \leqslant 2\omega^2\left(f^{(r)}, \frac{\pi}{n-r}\right)_{B_2}, \tag{2.12}$$

then the inequality (2.11) can be specified.

Corollary 2.3. *On the set of functions $f \in B_2^{(r)}$, the function $\omega(f^{(r)}, t)_{B_2}$ of which satisfies the condition (2.12), the inequality*

$$E_{n-1}(f)_{B_2} \leqslant \frac{1}{\sqrt{2}}\sqrt{\frac{n-r+1}{n+1}}\frac{1}{\alpha_{n,r}}\omega\left(f^{(r)}, \frac{\pi}{2(n-r)}\right)_{B_2} \tag{2.13}$$

holds. There exists a function $f_0 \in B_2^{(r)}$, which turns (2.13) into the identity.

Proof. In view of the inequality (2.12), for an arbitrary function $f \in B_2^{(r)}$ we have

$$\begin{aligned}
\int_0^{\frac{\pi}{(n-r)}} \omega^2(f^{(r)}, t)_{B_2} \sin(n-r)t\,dt &= \int_0^{\frac{\pi}{2(n-r)}} \omega^2(f^{(r)}, t)_{B_2} \sin(n-r)t\,dt \\
&\quad + \int_{\frac{\pi}{2(n-r)}}^{\frac{\pi}{(n-r)}} \omega^2(f^{(r)}, t)_{B_2} \sin(n-r)t\,dt \\
&= \int_0^{\frac{\pi}{2(n-r)}} \omega^2(f^{(r)}, t)_{B_2} \sin(n-r)t\,dt \\
&\quad + \int_0^{\frac{\pi}{2(n-r)}} \omega^2\left(f^{(r)}, \frac{\pi}{n-r} - t\right)_{B_2} \sin(n-r)t\,dt \\
&= \int_0^{\frac{\pi}{2(n-r)}} \left[\omega^2(f^{(r)}, t)_{B_2} + \omega^2\left(f^{(r)}, \frac{\pi}{n-r} - t\right)_{B_2}\right] \sin(n-r)t\,dt \\
&\leqslant \int_0^{\frac{\pi}{2(n-r)}} 2\omega^2\left(f^{(r)}, \frac{\pi}{2(n-r)}\right)_{B_2} \sin(n-r)t\,dt \\
&= \frac{2}{n-r}\omega^2\left(f^{(r)}, \frac{\pi}{2(n-r)}\right)_{B_2},
\end{aligned}$$

which implies immediately that

$$E_{n-1}(f)_{B_2} \leqslant \frac{1}{\sqrt{2}}\sqrt{\frac{n-r+1}{n+1}}\frac{1}{\alpha_{n,r}}\omega\left(f^{(r)}, \frac{\pi}{2(n-r)}\right)_{B_2}$$

and this proves the inequality (2.13).

Let us show that for the function $f_0(z) = z^n \in B_2^{(r)}$ the inequality (2.13) becomes the identity. For this function we have

$$E_{n-1}(f_0)_{B_2} = \frac{1}{\sqrt{n+1}},$$

and since

$$f_0^{(r)}(z) = \alpha_{n,r}z^{n-r}, \quad n > r,$$

by the formula (1.8) we obtain

$$\omega\left(f_0^{(r)}, t\right)_{B_2} = \frac{\sqrt{2}\alpha_{n,r}}{\sqrt{n-r+1}}\left(1-\cos(n-r)t\right)^{\frac{1}{2}},$$

$$\omega\left(f_0^{(r)}, \frac{\pi}{2(n-r)}\right)_{B_2} = \frac{\sqrt{2}\alpha_{n,r}}{\sqrt{n-r+1}}.$$

Using these identities, we write

$$\sqrt{\frac{n-r+1}{n+1}}\frac{1}{\alpha_{n,r}}\frac{1}{\sqrt{2}}\omega\left(f_0^{(r)}, \frac{\pi}{2(n-r)}\right)_{B_2} = \sqrt{\frac{n-r+1}{n+1}}\frac{1}{\alpha_{n,r}}\frac{1}{\sqrt{2}}\frac{\sqrt{2}\alpha_{n,r}}{\sqrt{n-r+1}} = \frac{1}{\sqrt{n+1}} = E_{n-1}(f_0)_{B_2}$$

and this completes the proof. □

3. Exact values of n–widths of classes of functions $W_2(\omega, \Phi)$ in B_2

To formulate the results of this section, we recall needed notions and notation. Let $S := \{f : \|f\| \leqslant 1\}$ be the unit ball in B_2; $\mathfrak{M}$ be a convex centrally symmetric subset in B_2; $\mathcal{L}_n \subset B_2$ be a n–dimensional subspace; $\mathcal{L}^n \subset B_2$ be a subspace of codimension n; $\Lambda : B_2 \to \mathcal{L}_n$ be a continuous linear operator; $\Lambda^{\perp} : B_2 \to \mathcal{L}^n$ be a continuous operator of linear projection. The quantities

$$b_n(\mathfrak{M}, B_2) = \sup\left\{\sup\left\{\varepsilon > 0 : \varepsilon S \cap \mathcal{L}_{n+1} \subset \mathfrak{M}\right\} : \mathcal{L}_{n+1} \subset B_2\right\},$$
$$d_n(\mathfrak{M}, B_2) = \inf\left\{\sup\left\{\inf\left\{\|f-\varphi\|_{B_2} : \varphi \in \mathcal{L}_n\right\} : f \in \mathfrak{M}\right\} : \mathcal{L}_n \subset B_2\right\},$$
$$\delta_n(\mathfrak{M}, B_2) = \inf\left\{\inf\left\{\sup\left\{\|f-\Lambda f\|B_2 : f \in \mathfrak{M}\right\} : \Lambda B_2 \subset \mathcal{L}_n\right\} : \mathcal{L}_n \subset B_2\right\},$$
$$d^n(\mathfrak{M}, B_2) = \inf\left\{\sup\left\{\|f\|_{B_2} : f \in \mathfrak{M} \cap \mathcal{L}^n\right\} : \mathcal{L}^n \subset B_2\right\},$$
$$\Pi_n(\mathfrak{M}, B_2) = \inf\left\{\inf\left\{\sup\left\{\|f-\Lambda^{\perp}f\|_{B_2} : f \in \mathfrak{M}\right\} : \Lambda^{\perp}B_2 \subset \mathcal{L}_n\right\} : \mathcal{L}_n \subset B_2\right\},$$

are respectively called *Bernstein, Kolmogorov, linear, Gelfand, projecting* n*–widths* of the set $\mathfrak{M}$ in the space B_2.

Since B_2 is a Hilbert space, the aforementioned n–widths satisfy the relations [17], [32]:

$$b_n(\mathfrak{M}, B_2) \leqslant d^n(\mathfrak{M}, B_2) \leqslant d_n(\mathfrak{M}, B_2) = \delta_n(\mathfrak{M}, B_2) = \Pi_n(\mathfrak{M}, B_2). \tag{3.1}$$

We recall that the calculations of exact values of n–widths in the space B_2 of classes of analytic in the unit circle functions defined by means moduluses of continuity and other characteristics was studied in works by Vakarchuk [4]–[7], [9], Shabozov and his pupils [22]–[24], Pinkus [32], Farkov [19], Langarshoev [12], [13], Vakarchuk and Shabozov [8] and many others.

Using the definition of modulus of continuity, we consider the following class of functions. Let $\Phi(u)$, where $0 \leqslant u \leqslant 2\pi$, be a continuous increasing function such that $\Phi(0) = 0$.

By the symbol $W_2^{(r)}(\omega, \Phi)$ we denote the class of functions $f \in B_2^{(r)}$, $r \in \mathbb{Z}_+$, which for all $u \in (0, \pi]$ satisfy the inequality

$$\int_0^u \omega^2(f^{(r)}, t)_{B_2} \sin \frac{\pi}{u} t \, dt \leqslant \Phi^2(u).$$

We are going to calculate the exact values of aforementioned n–widths under some restrictions for the majorant $\Phi^2(u)$.

We note that similar classes of functions appeared first in works by Taĭkov [15], [16] and his pupil Ainulloev [1] while calculating exact values for widths of classes of periodic functions in $L_2 := L_2[0, 2\pi]$ and analytic in the unit circle functions belonging to the Hardy space H_q, $q \geqslant 1$.

A natural idea arises to employ these classes of functions in solving a series of extremal problems in the Bergman space.

Theorem 3.1. *If for a given $\lambda \in (0, 1)$ and all $\mu > 0$, $u \in (0, \pi]$ the majorant Φ satisfies the condition*

$$\Phi^2\left(\frac{u}{\mu}\lambda\right) \int_0^{\pi\mu} (1 - \cos t)_* \sin \frac{t}{\mu} \, dt \leqslant \Phi^2(u) \int_0^{\pi\lambda} (1 - \cos t) \sin \frac{t}{\lambda} \, dt, \tag{3.2}$$

where

$$(1 - \cos t)_* = \begin{cases} 1 - \cos t, & t \leqslant \pi, \\ 2, & t \geqslant \pi, \end{cases}$$

then for all $n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $n > r$ the inequality

$$\begin{aligned} \lambda_n(W_2^{(r)}(\omega, \Phi), B_2) &= E_{n-1}(W_2^{(r)}(\omega, \Phi))_{B_2} \\ &= \sqrt{\frac{n - r + 1}{n + 1}} \frac{n - r}{\alpha_{n,r}} \frac{\Phi\left(\dfrac{\pi\lambda}{n - r}\right)}{\sqrt{2}\left(\displaystyle\int_0^{\pi\lambda} (1 - \cos t) \sin \frac{t}{\lambda} \, dt\right)^{\frac{1}{2}}} \end{aligned} \tag{3.3}$$

holds, where $\lambda_n(\cdot)$ is any of the aforementioned n–widths, while for $\mathfrak{N} \subset B_2$ we let

$$E_{n-1}(\mathfrak{N})_{B_2} := \sup \left\{ E_{n-1}(f)_{B_2} : f \in B_2 \right\}. \tag{3.4}$$

Proof. In the right hand side of (2.9) we let $h = \frac{\pi\lambda}{(n-r)}$, $\lambda \in (0, 1)$, $n \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $n > r$ and employ the definition of class $W_2^{(r)}(\omega, \Phi)$. Then by the relation (3.1) we obtain the upper bound for all n–widths and the quantity (3.4):

$$\begin{aligned}\lambda_n(W_2^{(r)}(\omega,\Phi),B_2) &\leqslant E_{n-1}(W_2^{(r)}(\omega,\Phi)_{B_2}\\ &\leqslant \sqrt{\frac{n-r+1}{n+1}}\frac{1}{\alpha_{n,r}}\frac{\Phi\left(\dfrac{\pi\lambda}{n-r}\right)}{\sqrt{2}\left(\displaystyle\int_0^{\pi\lambda/(n-r)}(1-\cos(n-r)t)\sin\frac{n-r}{\lambda}t\,dt\right)^{\frac{1}{2}}}\\ &=\frac{1}{\sqrt{2}}\sqrt{\frac{n-r+1}{n+1}}\frac{\sqrt{n-r}}{\alpha_{n,r}}\frac{\Phi\left(\dfrac{\pi\lambda}{n-r}\right)}{\left(\displaystyle\int_0^{\pi\lambda}(1-\cos t)\sin\frac{t}{\lambda}\,dt\right)^{\frac{1}{2}}}.\end{aligned}\tag{3.5}$$

In view of (3.1), to prove the relation (3.3), it is sufficient to estimate the *Bernstein* n–width by the right hand side of (3.5). In order to do this, for an arbitrary polynomial

$$p_n(z)=\sum_{k=0}^{n}a_kz^k\in\mathcal{P}_n$$

we estimate $\omega(p_n^{(r)},t)_{B_2}$ for $t\in\left(0,\frac{\pi}{(n-r)}\right]$. By the Parseval identity we have

$$\begin{aligned}\|p_n^{(r)}(\rho e^{i(x+t)})-p_n^{(r)}(\rho e^{ix})\|_{B_2}^2&=2\sum_{k=r}^{n}\alpha_{k,r}^2\frac{|a_k|^2}{k-r+1}(1-\cos(k-r)t)\\ &=2\sum_{k=r}^{n}\alpha_{k,r}^2\frac{k+1}{k-r+1}\frac{|a_k|^2}{k+1}(1-\cos(k-r)t).\end{aligned}\tag{3.6}$$

Since

$$\max_{r\leqslant k\leqslant n}\alpha_{k,r}^2\frac{k+1}{k-r+1}=\alpha_{n,r}^2\frac{n+1}{n-r+1}$$

and for all $k,n\in\mathbb{N}$, $r\in\mathbb{Z}_+$ and all $t\geqslant 0$ and $k\leqslant n$, $\cos(k-r)t\geqslant\cos(n-r)t$, by (3.6) and the definition (1.8) of the modulus of continuity we have

$$\begin{aligned}\omega^2(p_n^{(r)},t)_{B_2}&\leqslant 2\alpha_{n,r}^2\frac{n+1}{n-r+1}(1-\cos(n-r)t)_*\sum_{k=r}^{n}\frac{|a_k|^2}{k+1}\\ &\leqslant 2\alpha_{n,r}^2\frac{n+1}{n-r+1}(1-\cos(n-r)t)_*\|p_n\|_{B_2}^2.\end{aligned}$$

We multiply both sides of the obtained inequality by the function $\sin\dfrac{\pi}{u}t$ and integrate in t from 0 to u; this gives

$$\int_0^u\omega^2(p_n^{(r)},t)_{B_2}\sin\frac{\pi}{u}t\,dt\leqslant 2\alpha_{n,r}^2\frac{n+1}{n-r+1}\|p_n\|_{B_2}^2\int_0^u(1-\cos(n-r)t)_*\sin\frac{\pi}{u}t\,dt.\tag{3.7}$$

We introduce the sphere of $(n+1)$–dimensional polynomials

$$S_{n+1} := \left\{ p_n \in \mathcal{P}_n : \|p_n\|^2_{B_2} = \frac{n-r+1}{n+1} \frac{n-r}{\alpha^2_{n,r}} \frac{\Phi^2\left(\dfrac{\pi\lambda}{n-r}\right)}{2\displaystyle\int_0^{2\lambda} (1-\cos t)\sin\frac{t}{\lambda}\,dt} \right\},$$

and we are going to show that this sphere is contained in the class $W_2^{(r)}(\omega,\Phi)$. We take an arbitrary polynomial $p_n \in S_{n+1}$ and let us show that $p_n \in W_2^{(r)}(\omega,\Phi)$. Let $p_n \in S_{n+1}$. Then by (3.7) we get

$$\int_0^u \omega^2(p_n^{(r)},t)_{B_2}\sin\frac{\pi}{u}t\,dt \leqslant \Phi^2\left(\frac{\pi\lambda}{n-r}\right)\frac{(n-r)\displaystyle\int_0^u (1-\cos(n-r)t)_*\sin\frac{\pi}{u}t\,dt}{\displaystyle\int_0^{2\lambda}(1-\cos t)\sin\frac{t}{\lambda}\,dt}. \tag{3.8}$$

Letting $u = \frac{\pi\mu}{(n-r)}$, $\mu > 0$ in the right hand side of (3.8) and making the change of variable, by the condition (3.2) we find

$$\int_0^u \omega^2(p_n^{(r)},t)_{B_2}\sin\frac{\pi}{u}t\,dt \leqslant \Phi^2\left(\frac{u}{\mu}\lambda\right)\frac{\displaystyle\int_0^{\pi\mu}(1-\cos t)_*\sin\frac{t}{\mu}\,dt}{\displaystyle\int_0^{\pi\lambda}(1-\cos t)\sin\frac{t}{\lambda}\,dt} \leqslant \Phi^2(u).$$

Therefore, $S_{n+1} \in W_2^{(r)}(\omega,\Phi)$ and by the definition of *Bernstein* n–width

$$b_n(W_2^{(r)}(\omega,\Phi),B_2) \geqslant b_n(S_{n+1},B_2)$$

$$= \sqrt{\frac{n-r+1}{n+1}}\frac{\sqrt{n-r}}{\alpha_{n,r}}\frac{\Phi\left(\dfrac{\pi\lambda}{n-r}\right)}{\sqrt{2}\left(\displaystyle\int_0^{\pi\lambda}(1-\cos t)\sin\frac{t}{\lambda}\,dt\right)^{\frac{1}{2}}}. \tag{3.9}$$

In view of the relation (3.1), the required identity (3.3) is obtained by comparing the upper and lower estimates (3.5) and (3.9). The proof is complete. □

It was shown in [2] that the function $\Phi_*^2(u) = u^\alpha$ satisfies the inequalities (3.2) for α ranging in $\dfrac{\pi}{8}+1 < \alpha < 3$.

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