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ON NEW REPRESENTATIONS FOR VALUES OF RIEMANN ZETA FUNCTION AT ODD POINTS AND RELATED NUMBERS

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Abstract. Let $\zeta(s)$ and $\beta(s)$ be the Riemann zeta function and Dirichlet beta function. In this work, for some linear combinations of the numbers $\zeta(2n+1)$ and $\beta(2n)$, we obtain new representations by the series, the general term of which involves the logarithms. This is done by the methods of spectral theory of ordinary differential operators generated in the Hilbert space $\mathcal{L}^2[0,\pi]$ by the expression $l[y]=-y''-a^2y$ and the Dirichlet boundary condition, where a is a parameter. These representations in particular imply the known and new representations for these linear combinations as the sums of some sufficiently fast converging series, the general term of which involves $\zeta(2n)$. The obtained results are applied to various representations of Catalan constant $\beta(2)$ and Apéry constant $\zeta(3)$.

Keywords: Riemann zeta function, Dirichlet beta function, Catalan constant, Apéry constant.

Mathematics Subject Classification: 34L10, 33E20

1. Introduction

Let, as usually, $\zeta(s)$ be the Riemann zeta function defined for $\operatorname{Re} s>1$ by the identity

$$\zeta(s)=\sum_{k=1}^{+\infty}\frac{1}{k^s}.$$

Following [7, Ch. 23], by the symbols $\beta(s)$, $\lambda(s)$ and $\eta(s)$ we denote related with $\zeta(s)$ Dirichlet functions defined for $\operatorname{Re} s>0$, $\operatorname{Re} s>1$ and $\operatorname{Re} s>0$ respectively by the identities

$$\beta(s)=\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{(2k-1)^s},\qquad \lambda(s)=\sum_{k=1}^{+\infty}\frac{1}{(2k-1)^s},\qquad \eta(s)=\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{k^s}. \tag{1.1}$$

It is well–known that

$$\lambda(s)=(1-2^{-s})\zeta(s),\qquad \eta(s)=(1-2^{1-s})\zeta(s) \tag{1.2}$$

and

$$\zeta(2n)=\frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!}B_{2n},\qquad \beta(2n-1)=\frac{(-1)^{n-1}\left(\frac{\pi}{2}\right)^{2n-1}}{2(2n-2)!}E_{2(n-1)},\qquad n=1,2,\dots,$$

where B_n and E_n are the Bernulli and Euler numbers, respectively, see, for instance, [7, Ch. 23, Eqs. (23.2.20), (23.2.19), (23.2.16), (23.2.22)]. These identities imply that the numbers $\zeta(2n)$ and $\beta(2n+1)$ are transcendental. However, various known representations for the numbers

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$\zeta(2n+1)$, $\beta(2n)$, $\lambda(2n+1)$ and $\eta(2n-1)$ for $n = 1, 2, \ldots$ or their certain combinations, for instance, the integral representations

$$\zeta(2n+1) = \frac{(-1)^{n+1}(2\pi)^{2n+1}}{2(2n+1)!} \int_0^1 B_{2n+1}(x) \cot(\pi x)\, dx \tag{1.3}$$

and

$$\beta(2n) = \frac{(-1)^n \pi^{2n}}{4(2n-1)!} \int_0^1 \frac{E_{2n-1}(x)}{\cos(\pi x)}\, dx, \tag{1.4}$$

which have already become classical, see, for instance, [7, Ch. 23, Eqs. (23.2.17), (23.2.23)], and quite recent ones

$$\lambda(2n+1) = \frac{(-1)^n \pi^{2n+1}}{4(2n)!} \int_0^1 \frac{E_{2n}(x)}{\sin(\pi x)}\, dx \tag{1.5}$$

and

$$\eta(2n-1) = \frac{(-1)^n (2\pi)^{2n-1}}{2(2n-1)!} \int_0^1 B_{2n-1}(x) \tan(\pi x)\, dx, \tag{1.6}$$

see, for instance [10, Thm. 1], does not allow to study the arithmetic nature of these numbers and a little is known about this. In particular, this is the reason why the representations for these numbers or their combinations as the integrals, series and others is of a special interest, and the literature devoted to this subject is very rich, see, for instance, [10], [11], [15], [9] and the references therein.

In Section 2 of the present work we provide preliminary facts used in the proofs of the main results of this work in Sections 3–6, in which we formulate statements on new representations of certain linear combinations of the numbers $\zeta(2n+1)$, $\beta(2n)$, $\lambda(2n+1)$ and $\eta(2n-1)$ as the series, the general term of which involves the logarithm (Theorem 3.1 and Corollaries 4.1–5.1), and as the sums of sufficiently fast converging series, the general term of which involves $\zeta(2n)$ (Corollaries 6.1 and 6.2).

Throughout the paper a special attention is paid to the application of obtained results to various representations for the Catalana constant $\beta(2)$ $(=: G)$ and Apéry constant $\zeta(3)$.

2. Preliminaries

As in works [2] and [3], we consider the differential operator generated by the expression

$$l[y] = -y'' - a^2 y, \qquad -1 < a < 1,$$

and the Dirichlet boundary conditions $y(0)=y(\pi)=0$ in the Hilbert space $\mathcal{L}^2[0,\pi]$ of all classes of complex–valued measurable functions y coinciding almost everywhere such that $|y|^2$ is Lebesgue integrable over $[0,\pi]$. By the methods of spectral theory for this operator, in the above cited works, integral representations were obtained for the sequences of numbers

$$\mathcal{A}_m = \pi^{2m} \left(\sum_{n=1}^{m} \frac{(-1)^{m-n}}{2^{2(m-n)}(2m-2n)!} \frac{\beta(2n)}{\pi^{2n}} \right),$$

$$\mathcal{B}_m = \pi^{2m+1} \left(\sum_{n=1}^{m} \frac{(-1)^{m-n}}{2^{2(m-n)+1}(2m-2n+1)!} \frac{\beta(2n)}{\pi^{2n}} - \frac{\lambda(2m+1)}{\pi^{2m+1}} \right),$$

$$\mathcal{C}_m = \pi^{2m}\left(\sum_{n=1}^{m}\frac{(-1)^{m-n}}{(2m-2n+1)!}\frac{\eta(2n-1)}{\pi^{2n-1}}\right),$$
$$\mathcal{D}_m = \pi^{2m+1}\left(\sum_{n=1}^{m}\frac{(-1)^{m-n}}{(2m-2n+2)!}\frac{\eta(2n-1)}{\pi^{2n-1}} - \frac{2^{2m+1}-1}{2^{2m}}\frac{\zeta(2m+1)}{\pi^{2m+1}}\right)$$

for $m = 1, 2, \ldots$, namely, in [2, Cor. 1], the next theorem was proved.

Theorem 2.1. *For $m = 1, 2, \ldots$ the following identities hold*

$$\mathcal{A}_m = \frac{(-1)^{m-1}}{2(2m-1)!}\int_0^{\frac{\pi}{2}}\frac{x^{2m-1}}{\sin x}\,dx, \tag{2.1}$$

$$\mathcal{B}_m = \frac{(-1)^{m-1}}{2(2m)!}\int_0^{\frac{\pi}{2}}\frac{x^{2m}}{\sin x}\,dx, \tag{2.2}$$

$$\mathcal{C}_m = \frac{(-1)^{m-1}2^{2m-1}}{(2m)!}\int_0^{\frac{\pi}{2}}\frac{x^{2m}}{\sin^2 x}\,dx \tag{2.3}$$

$$\mathcal{D}_m = \frac{(-1)^{m-1}2^{2m}}{(2m+1)!}\int_0^{\frac{\pi}{2}}\frac{x^{2m+1}}{\sin^2 x}\,dx. \tag{2.4}$$

This theorem in particular implies the following known identities for the numbers G, $\lambda(3)$, $\eta(1)(= \ln 2)$, $\eta(3)$ and $\zeta(3)$

$$G = \frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{x}{\sin x}\,dx, \qquad \lambda(3) = \frac{\pi}{2}G - \frac{1}{4}\int_0^{\frac{\pi}{2}}\frac{x^2}{\sin x}\,dx, \qquad \eta(1) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\frac{x^2}{\sin^2 x}\,dx,$$

$$\eta(3) = \frac{\pi^2}{6}\ln 2 - \frac{1}{3\pi}\int_0^{\frac{\pi}{2}}\frac{x^4}{\sin^2 x}\,dx, \qquad \zeta(3) = \frac{2\pi^2}{7}\ln 2 - \frac{8}{21}\int_0^{\frac{\pi}{2}}\frac{x^3}{\sin^2 x}\,dx;$$

the formulas for G, $\eta(1)$ and $\eta(3)$ were provided, for instance, in [4, Ch. 2, Sect. 2.5.4, Eqs. (5), (7)], while the formula $\lambda(3)$ is implied by one Ramanujan identity, see [8, Entry 14].

We note that all these identities except for that for $\eta(3)$ are particular cases of the identities (1.3)–(1.6) for $n = 1$, while the identity for $\eta(3)$ is implied by (1.6) for $n = 2$.

3. Main theorem

Using the expansions

$$\frac{1}{\sin x} = \frac{1}{x} + 2x\sum_{k=1}^{+\infty}\frac{(-1)^k}{x^2-(\pi k)^2}, \qquad \cot x = \frac{1}{x} + 2x\sum_{k=1}^{+\infty}\frac{1}{x^2-(\pi k)^2}, \tag{3.1}$$

see, for instance, [16, Eqs. (4.22.5), (4.22.3)], we are going to prove the next statement.

Lemma 3.1. *For $j = 1, 2, \ldots$ the identities*

$$\int_0^{\frac{\pi}{2}}\frac{x^j}{\sin x}\,dx = \left(\frac{\pi}{2}\right)^j\left(\frac{1}{j} - 2\sum_{k=1}^{+\infty}(-1)^k\int_0^1\frac{u^{j+1}}{(2k)^2-u^2}du\right) \tag{3.2}$$

and

$$\frac{1}{j+1}\int\limits_0^{\frac{\pi}{2}} \frac{x^{j+1}}{\sin^2 x}\,dx = \left(\frac{\pi}{2}\right)^j \left(\frac{1}{j} - 2\sum_{k=1}^{+\infty}\int\limits_0^1 \frac{u^{j+1}}{(2k)^2-u^2}du\right). \tag{3.3}$$

hold.

Proof. We observe that

$$\int\limits_0^{\frac{\pi}{2}} \frac{x^j}{\sin x}\,dx = \left(\frac{\pi}{2}\right)^{j+1}\int\limits_0^1 \frac{x^j}{\sin\frac{\pi x}{2}}\,dx,$$

we replace x by $\dfrac{\pi x}{2}$ in the first identity in (3.1), multiply then both sides by x^j and integrate the obtained identity from 0 and 1. This gives

$$\int\limits_0^{\frac{\pi}{2}} \frac{x^j}{\sin x}\,dx = \left(\frac{\pi}{2}\right)^j \left(\frac{1}{j} + 2\int\limits_0^1 \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}x^{j+1}}{(2k)^2-x^2}\,dx\right).$$

The absolute value of general term of the series under the integral obviously satisfies the inequality

$$\frac{x^{j+1}}{(2k)^2-x^2} \leqslant \frac{1}{(2k)^2-1}$$

for $0\leqslant x\leqslant 1$ and $k=1,2,\dots$ This implies that this functional series converges absolutely and uniformly on $[0,1]$, and this is why we can integrate it term by term, that is, the identity (3.2) is true.

The validity of identity (3.3) can be proved in the same with the only difference that now we should start with the identity

$$\int\limits_0^{\frac{\pi}{2}} \frac{x^j}{\sin^2 x}\,dx = j\left(\frac{\pi}{2}\right)^j \int\limits_0^1 x^{j-1} ctg\frac{\pi x}{2}\,dx$$

and take into consideration the second identity in (3.1). The proof is complete. □

We then mention that the integrals in the right hand sides of identities (3.2) and (3.3) can be calculated explicitly, namely, for $m=1,2,\dots$ they satisfy the formulas

$$\int\limits_0^1 \frac{u^{2m}}{(2k)^2-u^2}du = -\frac{1}{2}\left((2k)^{2m-1}\ln\frac{2k-1}{2k+1} + 2\sum_{l=0}^{m-1}\frac{(2k)^{2l}}{2m-2l-1}\right) \tag{3.4}$$

and

$$\int\limits_0^1 \frac{u^{2m+1}}{(2k)^2-u^2}du = -\frac{1}{2}\left((2k)^{2m}\ln\left(1-\frac{1}{(2k)^2}\right) + \sum_{l=0}^{m-1}\frac{(2k)^{2l}}{m-l}\right), \tag{3.5}$$

see, for instance, [4, Ch. 1, Sect. 1.2.10, Eqs. (9), (10)].

Applying now the formulas (3.2)–(3.5) in the identities (2.1)–(2.4) of Theorem 2.1, we arrive at the next theorem.

Theorem 3.1. *For $m=1,2,\dots$ the identities hold*

$$\begin{aligned}\mathcal{A}_m =&\frac{(-1)^{m-1}\pi^{2m-1}}{4^m(2m-1)!}\\ &\cdot\left(\frac{1}{2m-1} + \sum_{k=1}^{+\infty}(-1)^k\left((2k)^{2m-1}\ln\frac{2k-1}{2k+1} + 2\sum_{l=0}^{m-1}\frac{(2k)^{2l}}{2m-2l-1}\right)\right),\end{aligned} \tag{3.6}$$

$$\mathcal{B}_m = \frac{(-1)^{m-1}\pi^{2m}}{4^m 2(2m)!} \cdot \left(\frac{1}{2m} + \sum_{k=1}^{+\infty} (-1)^k \left((2k)^{2m} \ln\left(1 - \frac{1}{(2k)^2}\right) + \sum_{l=0}^{m-1} \frac{(2k)^{2l}}{m-l} \right) \right), \tag{3.7}$$

$$\mathcal{C}_m = \frac{(-1)^{m-1}\pi^{2m-1}}{(2m-1)!} \cdot \left(\frac{1}{2m-1} + \sum_{k=1}^{+\infty} \left((2k)^{2m-1} \ln \frac{2k-1}{2k+1} + 2\sum_{l=0}^{m-1} \frac{(2k)^{2l}}{2m-2l-1} \right) \right), \tag{3.8}$$

$$\mathcal{D}_m = \frac{(-1)^{m-1}\pi^{2m}}{(2m)!} \left(\frac{1}{2m} + \sum_{k=1}^{+\infty} \left((2k)^{2m} \ln\left(1 - \frac{1}{(2k)^2}\right) + \sum_{l=0}^{m-1} \frac{(2k)^{2l}}{m-l} \right) \right). \tag{3.9}$$

We note that we obtained the identity (3.6) in [5], while to the best of our knowledge, the identities (3.7)–(3.9) are new.

4. On Catalana constant, Apéry constant and $\ln 2$

Theorem 3.1 allows us to represent the constants $\zeta(3)$, G, $\lambda(3)$, $\eta(3)$ and $\ln 2$ as the series, the general term of which involves the logarithms. Namely, letting $m = 1$ in the identities (3.6)–(3.9) and $m = 2$ in (3.8), we arrive at the following corollaries of this theorem.

Corollary 4.1. *The identities hold*

$$G = \frac{\pi}{4} \left(1 + 2\sum_{k=1}^{+\infty} (-1)^k \left(1 + k \ln \frac{2k-1}{2k+1} \right) \right), \tag{4.1}$$

$$\lambda(3) = \frac{\pi}{2} G - \frac{\pi^2}{32} \left(1 + 2\sum_{k=1}^{+\infty} (-1)^k \left(1 + 4k^2 \ln\left(1 - \frac{1}{4k^2}\right) \right) \right), \tag{4.2}$$

$$\ln 2 = 1 + 2\sum_{k=1}^{+\infty} \left(1 + k \ln \frac{2k-1}{2k+1} \right), \tag{4.3}$$

$$\eta(3) = \frac{\pi^2}{6} \ln 2 - \frac{\pi^2}{18} \left(1 + 2\sum_{k=1}^{+\infty} \left(1 + 12k^2 + 12k^3 \ln \frac{2k-1}{2k+1} \right) \right), \tag{4.4}$$

$$\zeta(3) = \frac{2\pi^2}{7} \ln 2 - \frac{\pi^2}{7} \left(1 + 2\sum_{k=1}^{+\infty} \left(1 + 4k^2 \ln\left(1 - \frac{1}{4k^2}\right) \right) \right). \tag{4.5}$$

Corollary 4.2. *The identities*

$$2\pi G - \frac{7}{2}\zeta(3) = \frac{\pi^2}{8} \left(1 + 2\sum_{k=1}^{+\infty} (-1)^k \left(1 + 4k^2 \ln\left(1 - \frac{1}{4k^2}\right) \right) \right)$$

and

$$2\pi G - \frac{35}{8}\zeta(3) = -\frac{\pi^2}{4} \ln 2 + \frac{\pi^2}{4} \left(1 + 2\sum_{k=1}^{+\infty} \left(1 + 16k^2 \ln\left(1 - \frac{1}{16k^2}\right) \right) \right)$$

hold.

Proof. The first identity is implied immediately by (4.2) once we take into consideration the first relation in (1.2). If we deduct the identity (4.5) multiplied by $\frac{7}{8}$ from the obtained identity, we arrive at the second required formula. The proof is complete. □

We note that the partial sums S_{2m} and $\overline{S}_{2m}$ of the series in the right hand sides of identities (4.1) and (4.2) are written respectively as

$$S_{2m} = \sum_{k=1}^{2m} (-1)^k k \ln \frac{2k-1}{2k+1} \qquad \text{and} \qquad \overline{S}_{2m} = 4\sum_{k=1}^{2m} (-1)^k k^2 \ln\left(1 - \frac{1}{4k^2}\right).$$

Calculating the limits of the partial sums S_{2m} and $\overline{S}_{2m}$ as $m \to +\infty$, we arrive at the next statement.

Corollary 4.3. *The identities*

$$G = \frac{\pi}{4}\left(1 + 2\lim_{m\to+\infty}\sum_{k=1}^{2m}(-1)^k k \ln\frac{2k-1}{2k+1}\right)$$

and

$$\lambda(3) = \frac{\pi}{2}G - \frac{\pi^2}{32}\left(1 + 8\lim_{m\to+\infty}\sum_{k=1}^{2m}(-1)^k k^2 \ln\left(1 - \frac{1}{4k^2}\right)\right)$$

hold.

In the integral in the right hand side of the identity (2.1) with $m = 1$ we make the change of variable $x = \frac{\pi}{2} - t$, in the obtained integral we employ the expansion

$$\frac{1}{\cos x} = 4\pi \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}(2k-1)}{(\pi(2k-1))^2 - (2x)^2}$$

(see, for instance, [1, Sect. 1.422, Eq. (1)]), integrate the obtained series term by term and calculate the appearing integral of a fractional rational function. This leads us to the following chain of identities

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{\frac{\pi}{2} - x}{\cos x}\,dx &= \pi\sum_{k=1}^{+\infty}(-1)^{k+1}(2k-1)\int_0^{\frac{\pi}{2}} \frac{x - \frac{\pi}{2}}{x^2 - \left(\frac{\pi}{2}(2k-1)\right)^2}\,dx \\
&= \pi\left(\ln 2 + \sum_{k=1}^{+\infty}(-1)^k(2k+1)\int_0^{\frac{\pi}{2}} \frac{x - \frac{\pi}{2}}{x^2 - \left(\frac{\pi}{2}(2k+1)\right)^2}\,dx\right) \\
&= \pi\left(\ln 2 + \sum_{k=1}^{+\infty}(-1)^k\left(k\ln\frac{2k}{2k+1} + (k+1)\ln\frac{2k+2}{2k+1}\right)\right) \\
&= \pi\left(\ln 2 + \frac{1}{2}\sum_{k=1}^{+\infty}(-1)^k(2k+1)\ln\left(1 - \frac{1}{(2k+1)^2}\right) + \frac{1}{2}\sum_{k=1}^{+\infty}(-1)^k\ln\frac{k+1}{k}\right)
\end{aligned}$$

and therefore, according to (2.1) with $m = 1$, we get the following identity for the Catalan constant

$$G = \frac{\pi}{2}\left(\ln 2 + \frac{1}{2}\sum_{k=1}^{+\infty}(-1)^k(2k+1)\ln\left(1 - \frac{1}{(2k+1)^2}\right) + \frac{1}{2}\sum_{k=1}^{+\infty}(-1)^k\ln\frac{k+1}{k}\right).$$

Applying the Wallis formula, we get

$$\sum_{k=1}^{+\infty}(-1)^k \ln\frac{k+1}{k} = -\ln\prod_{n=1}^{+\infty}\frac{4n^2}{(2n-1)(2n+1)} = -\ln\frac{\pi}{2}.$$

Thus,

$$G = \frac{\pi}{4}\left(\ln\frac{8}{\pi} + \sum_{k=1}^{+\infty}(-1)^k(2k+1)\ln\left(1-\frac{1}{(2k+1)^2}\right)\right). \tag{4.6}$$

This identity is due to Ramanujan [13], while the identities in the above corollaries were earlier given in [5] and [6].

In conclusion of this section, we note that Theorem 3.1 obviously yields a statement analogous to Corollary 4.1 for an arbitrary m. However, the resulting formulas are cumbersome, and here we restrict ourselves to presenting only one of them. Letting $m = 2$ in the identity (3.6), we find

$$\beta(4) = \frac{\pi^2}{8}G - \frac{\pi^3}{288}\left(1 + 2\sum_{k=1}^{+\infty}(-1)^k\left(1 + 12k^2 + 12k^3\ln\frac{2k-1}{2k+1}\right)\right).$$

Using this identity and (4.4) and taking into consideration that $\eta(3) = \dfrac{3\zeta(3)}{4}$ (see the second identity in (1.2) for $s = 3$), we obtain

$$\frac{9G}{\pi} - \frac{72\beta(4)}{\pi^3} - \frac{27\zeta(3)}{8\pi^2} + \frac{3\ln 2}{4} = \frac{1}{2} + \sum_{k=1}^{+\infty}\left(12(2k)^3\ln\frac{4k-1}{4k+1} + 12(2k)^2 + 1\right)$$

and

$$\frac{9G}{\pi} - \frac{72\beta(4)}{\pi^3} + \frac{27\zeta(3)}{8\pi^2} - \frac{3\ln 2}{4} = \sum_{k=0}^{+\infty}\left(12(2k+1)^3\ln\frac{4k+3}{4k+1} - 12(2k+1)^2 - 1\right).$$

5. Catalan constant, Apéry constant, ln 2 and infinite products

The identities (4.1)–(4.5) can be obviously written in a bit different form. Namely, the next statement is true.

Corollary 5.1. *The identities hold*

$$e^{\frac{G}{\pi}} = \frac{3\sqrt[4]{2}}{e}\prod_{k=1}^{+\infty}\left(e^{-1}\left(1+\frac{2}{4k+1}\right)^{2k+1}\right), \tag{5.1}$$

$$e^{\frac{4G}{\pi} - \frac{35\zeta(3)}{4\pi^2}} = \sqrt{\frac{e}{2}}\prod_{k=1}^{+\infty}\left(e\left(1-\frac{1}{(4k)^2}\right)^{(4k)^2}\right), \tag{5.2}$$

$$\sqrt{\frac{2}{e}} = \prod_{k=1}^{+\infty}\left(e^{-1}\left(1+\frac{2}{2k-1}\right)^{k}\right), \tag{5.3}$$

$$e^{-\frac{9\eta(3)}{\pi^2}} = \frac{1}{2}\sqrt{\frac{e}{2}}\prod_{k=1}^{+\infty}\left(e^{1+12k^2}\left(1-\frac{2}{2k+1}\right)^{12k^3}\right), \tag{5.4}$$

$$e^{-\frac{7\zeta(3)}{2\pi^2}} = \sqrt{\frac{e}{2}}\prod_{k=1}^{\infty}\left(e\left(1-\frac{1}{(2k)^2}\right)^{(2k)^2}\right). \tag{5.5}$$

Proof. We begin with the identity (5.1). In order to do this, we divide both sides of the identity (4.1) by π, and both sides of (4.3) by 4. Then we deduct the second obtained identity from the first and find

$$\frac{G}{\pi} = \frac{\ln 2}{4} + \sum_{k=0}^{+\infty} \left((2k+1) \ln \frac{4k+3}{4k+1} - 1 \right).$$

Thus,

$$e^{\frac{G}{\pi}} = \sqrt[4]{2} \exp \left(\sum_{k=0}^{+\infty} \left((2k+1) \ln \frac{4k+3}{4k+1} - 1 \right) \right) = \frac{3\sqrt[4]{2}}{e} \exp \left(\sum_{k=1}^{+\infty} \left((2k+1) \ln \frac{4k+3}{4k+1} - 1 \right) \right),$$

that is, the identity (5.1) is true. The second identity in Corollary 4.2 can be written as

$$-\frac{35\zeta(3)}{4\pi^2} + \frac{4G}{\pi} + \frac{\ln 2}{2} = \frac{1}{2} + \sum_{k=1}^{+\infty} \left(1 + (4k)^2 \ln \left(1 - \frac{1}{(4k)^2} \right) \right),$$

which implies (5.2). The identities (5.3)–(5.5) obviously follow (4.3)–(4.5). The proof is complete. □

The identity (4.1) also yields

$$\frac{G}{\pi} = \frac{1}{2} - \frac{\ln 2}{4} + \sum_{k=1}^{+\infty} \left(1 + 2k \ln \frac{4k-1}{4k+1} \right),$$

that is,

$$e^{\frac{G}{\pi}} = \frac{\sqrt{e}}{\sqrt[4]{2}} \prod_{k=1}^{+\infty} \left(e \left(1 - \frac{2}{4k+1} \right)^{2k} \right). \tag{5.6}$$

Multiplication and squaring of the identities (5.1) and (5.6) lead us to the relation

$$e \cdot e^{\frac{4G}{\pi}} = 9 \prod_{k=1}^{+\infty} \left(1 + \frac{4}{4k-1} \right) \left(1 - \frac{4}{(4k+1)^2} \right)^{4k+1},$$

which recalls the Ramanujan identity

$$\pi \cdot e^{\frac{4G}{\pi}} = 8 \prod_{k=1}^{+\infty} \left(1 - \frac{1}{(2k+1)^2} \right)^{(-1)^k (2k+1)}$$

(see identity (4.6)).

In conclusion of this section, we mention that in [12] the expansion into infinite products for some mathematical constants are given, including the numbers $\sqrt{e}$, $e^{\frac{G}{\pi}}$, $e^{\frac{7\zeta(3)}{(4\pi^2)}}$, and others. However, they were obtained by other methods and differ from the above formulas. We also note that some of the formulas in this section were given by us earlier in a slightly different form in [5] and [6].

6. Representations of $\mathcal{A}_m$, $\mathcal{B}_m$, $\mathcal{C}_m$ and $\mathcal{D}_m$ as series with general term involving $\zeta(2n)$

Theorem 3.1 allows us to represent the scalar sequences $\mathcal{A}_m$, $\mathcal{B}_m$, $\mathcal{C}_m$ and $\mathcal{D}_m$ as the sums of sufficiently fast converging series, the general term of which involves $\zeta(2n)$. Namely, the next corollary of this theorem is true.

Corollary 6.1. *For $m = 1, 2, \dots$ the identities hold*

$$\mathcal{A}_m = \frac{(-1)^{m-1}}{(2m-1)!}\left(\frac{\pi}{2}\right)^{2m-1}\sum_{n=0}^{+\infty}\frac{(2^{2n}-2)\zeta(2n)}{16^n(2n+2m-1)}, \tag{6.1}$$

$$\mathcal{B}_m = \frac{(-1)^{m-1}}{(2m)!}\left(\frac{\pi}{2}\right)^{2m}\sum_{n=0}^{+\infty}\frac{(2^{2n-1}-1)\zeta(2n)}{16^n(n+m)}, \tag{6.2}$$

$$\mathcal{C}_m = \frac{(-1)^m 2\pi^{2m-1}}{(2m-1)!}\sum_{n=0}^{+\infty}\frac{\zeta(2n)}{4^n(2n+2m-1)}, \tag{6.3}$$

$$\mathcal{D}_m = \frac{(-1)^m \pi^{2m}}{(2m)!}\sum_{n=0}^{+\infty}\frac{\zeta(2n)}{4^n(n+m)}. \tag{6.4}$$

Proof. We begin with the identity (6.1). In the right hand side of the identity (3.6) we employ the known relation

$$\ln\frac{1+x}{1-x} = 2\sum_{n=0}^{+\infty}\frac{x^{2n+1}}{2n+1}, \qquad |x|<1 \tag{6.5}$$

(see, for instance, [4, Sect. 5.2.4, Eq. (8)]), and then interchange the integration order and take into consideration the definition of eta function (see the first identity in (1.1)). This the following chain of identities for the sequence $\mathcal{A}_m$

$$\begin{aligned}
\mathcal{A}_m =&\frac{(-1)^{m-1}}{2(2m-1)!}\left(\frac{\pi}{2}\right)^{2m-1}\\
&\cdot\left(\frac{1}{2m-1}+2\sum_{k=1}^{+\infty}(-1)^k\left(\sum_{l=0}^{m-1}\frac{(2k)^{2l}}{2m-2l-1}-(2k)^{2m-1}\sum_{n=0}^{+\infty}\frac{1}{(2k)^{2n+1}(2n+1)}\right)\right)\\
=&\frac{(-1)^{m-1}}{2(2m-1)!}\left(\frac{\pi}{2}\right)^{2m-1}\left(\frac{1}{2m-1}+2\sum_{k=1}^{+\infty}(-1)^{k+1}\left(\sum_{n=m}^{+\infty}\frac{1}{(2k)^{2n-2m+2}(2n+1)}\right)\right)\\
=&\frac{(-1)^{m-1}}{2(2m-1)!}\left(\frac{\pi}{2}\right)^{2m-1}\left(\frac{1}{2m-1}+2\sum_{n=m}^{+\infty}\frac{1}{2^{2n-2m+2}(2n+1)}\left(\sum_{k=1}^{+\infty}\frac{(-1)^{k+1}}{k^{2n-2m+2}}\right)\right)\\
=&\frac{(-1)^{m-1}}{2(2m-1)!}\left(\frac{\pi}{2}\right)^{2m-1}\left(\frac{1}{2m-1}+2\sum_{n=1}^{+\infty}\frac{\eta(2n)}{4^n(2n+2m-1)}\right).
\end{aligned}$$

The analytic continuation of the function $\zeta(s)$ to the entire complex plane possesses the property $\zeta(0) = -\frac{1}{2}$. Using this property and the second identity from (1.2) in the latter identity, we arrive at (6.1).

The identities (6.2)–(6.4) can be proved in the same way once we begin with the identities (3.7)–(3.9). At the same time in the right hand of the identity (6.3) we use the same expansion (6.5), while in the right hand sides of (3.7) and (3.9) the next expansion is to be emplyoed

$$\ln(1-x) = -\sum_{n=1}^{+\infty}\frac{x^n}{n}, \qquad -1\leqslant x<1$$

(see, for instance, [4, Sect. 5.2.4, Eq. (4)]). The proof is complete. □

Letting $m=1$ in the identities (6.1)–(6.4) and $m=2$ in (6.3), we arrive at the next statement for the constants G, $\zeta(3)$, $\lambda(3)$, $\eta(3)$ and $\ln 2$.

Corollary 6.2. *The identities hold*

$$G = \pi \sum_{n=0}^{+\infty} \frac{(2^{2n-1}-1)\zeta(2n)}{16^n(2n+1)}, \tag{6.6}$$

$$\lambda(3) = \frac{\pi}{2}G - \frac{\pi^2}{8}\sum_{n=0}^{+\infty} \frac{(2^{2n-1}-1)\zeta(2n)}{16^n(n+1)} = \frac{\pi^2}{8}\sum_{n=0}^{+\infty} \frac{(2^{2n-1}-1)(2n+3)\zeta(2n)}{16^n(n+1)(2n+1)}, \tag{6.7}$$

$$\ln 2 = -2\sum_{n=0}^{+\infty} \frac{\zeta(2n)}{4^n(2n+1)}, \tag{6.8}$$

$$\eta(3) = \frac{\pi^2}{6}\left(\ln 2 + 2\sum_{n=0}^{+\infty} \frac{\zeta(2n)}{4^n(2n+3)}\right), \tag{6.9}$$

$$\zeta(3) = \frac{2\pi^2}{7}\left(\ln 2 + \sum_{n=0}^{+\infty} \frac{\zeta(2n)}{4^n(n+1)}\right). \tag{6.10}$$

We note that some of the results formulated in Corollaries 6.1 and 6.2 are well known and were obtained earlier by other authors. For example, the identities (6.3) and (6.4) were established by other methods in [14], see also [15, Eqs. (58), (59)]. Moreover, all identities in Corollaries 6.1 and 6.2 were obtained by us earlier in [2], and the formulas (6.1) and (6.2) seems to appear first here. Note also that we presented the identities (6.6)–(6.10) in [6].

In conclusion we mention that Theorem 3.1 does not cover the known formula

$$\gamma = \sum_{k=1}^{+\infty}\left(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right)\right),$$

where γ is the Euler constant, see, for instance, [4, Sect. 5.5.1, Eq. (15)], but it contains the known formula (4.3) for $\ln 2$, see, for instance, [4, Sect. 5.5.1, Eq. (21)]. It seems that the new formulas (4.1)–(4.5) can be treated as a continuation of this list. Thus, the identities in Corollary 6.2 show that Theorem 3.1 is a generalization of the identities for γ and $\ln 2$ to the case of the sequences $\mathcal{A}_m$, $\mathcal{B}_m$, $\mathcal{C}_m$ and $\mathcal{D}_m$.

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