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ON METRIZABILITY OF SOME SPECIAL FUNCTIONS

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Abstract. We study the second order ordinary differential equations satisfied by the following special functions: Bessel function, hypergeometric function, Weierstrass elliptic function. We prove that all these equations are metrizable and construct explicitly the corresponding metrics. We show that in all mentioned cases the geodesic equations admit a first integral, which is linear in momenta.

Keywords: geodesics, metrizability, special functions, integrability, first integral.

Mathematics Subject Classification: 34B30, 53C22

1. Introduction and main results

We consider a two–dimensional surface M with local coordinates $u = (u^1, u^2)$ and a Riemannian (pseudo–Riemannian) metric $ds^2 = g_{ij}(u)du^i du^j$. The *geodesics* of this metric are curves, which satisfy the following system of differential equations:

$$\ddot{u}^k + \Gamma^k_{ij}\dot{u}^i\dot{u}^j = 0, \qquad k = 1, 2, \tag{1.1}$$

hereinafter, as usually, we mean the summation over the repeating indices,

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(\frac{\partial g_{il}}{\partial u_j} + \frac{\partial g_{jl}}{\partial u_i} - \frac{\partial g_{ij}}{\partial u_l}\right)$$

are the Cristoffel symbols, and g^{ij} is the tensor inverse to the metric one: $g^{ik}g_{kj} = \delta^i_j$.

We consider one of the coordinates as the function of the other along the geodesics and rewrite the system (1.1) as a single second order differential equation

$$\frac{d^2y}{dx^2} = A_3(x,y)\left(\frac{dy}{dx}\right)^3 + A_2(x,y)\left(\frac{dy}{dx}\right)^2 + A_1(x,y)\left(\frac{dy}{dx}\right) + A_0(x,y), \tag{1.2}$$

here $u^1 = x$, $u^2 = y$, and the coefficients $A_j(x,y)$, $j = 0,\dots,3$, read

$$A_3 = \Gamma^1_{22}, \qquad A_2 = 2\Gamma^1_{12} - \Gamma^2_{22}, \qquad A_1 = \Gamma^1_{11} - 2\Gamma^2_{12}, \qquad A_0 = -\Gamma^2_{11}. \tag{1.3}$$

The search of Riemannian metrics on two–dimensional surfaces, the geodesic equations of which can be integrated by quadratures, is a classical problem. We recall the known integrability criterion, which is usually formulated in the language of Hamiltonian mechanics.

As it is known, the geodesics satisfy the Euler — Lagrange equations

$$\frac{\partial L}{\partial u^k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}^k}\right) = 0, \qquad k = 1, 2, \tag{1.4}$$

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for the Lagrangian $L = \frac{|\dot u|^2}{2} = \frac{g_{ij}\dot u^i \dot u^j}{2}$, that is, they are the extremals of the energy functional $S = \int L dt$. Applying the Legendre transform $(u, \dot u) \to (u, p)$, where $p = \frac{\partial L}{\partial \dot u}$ are conjugate momenta, we rewrite (1.4) as the Hamiltonian system

$$\dot u^k = \{u^k, H\} = \frac{\partial H}{\partial p_k}, \qquad \dot p_k = \{p_k, H\} = -\frac{\partial H}{\partial u^k}, \qquad H = \frac{1}{2} g^{ij} p_i p_j, \qquad k = 1, 2, \tag{1.5}$$

with the canonical Poisson bracket $\{u^i, u^j\} = \{p_i, p_j\} = 0$, $\{u^i, p_j\} = \delta^i_j$.

The *first integral* of the Hamiltonian system (1.5) is an arbitrary function $F(u, p)$, which is preserved along the trajectories of (1.5). The first integral F obeys the condition

$$\dot F = \{F, H\} = \frac{\partial F}{\partial u^k}\frac{\partial H}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial H}{\partial u^k} \equiv 0.$$

The Hamiltonian H is obviously a first integral of (1.5). If at the same time there exists an additional first integral F, which is functionally independent with H, then the geodesic flow (1.5) is called *completely integrable*, and in this case, by the Arnold — Liouville theorem the geodesic equations can be integrated by quadratures [2]. There is a huge amount of works devoted to the search of the first integrals; a main attention is paid to the polynomial integrals, see, for instance, [3]–[6], [10], [19] and the references therein. For instance, it is known that the integrable metrics with an additional linear or quadratic first integral in an appropriate local coordinate system read

$$\begin{aligned} &1)\ ds^2 = f(x)(dx^2 + dy^2), \qquad F_1 = p_2, \\ &2)\ ds^2 = (f(x) + g(y))(dx^2 + dy^2), \qquad F_2 = \frac{gp_1^2 - fp_2^2}{f + g}, \end{aligned} \tag{1.6}$$

here $f(x)$, $g(y)$ are arbitrary functions. We note that even with an additional integral the direct integration of geodesic equations is a separate, and generally speaking complicated problem. In the most cases the geodesic equations are not polynomially integrable, see, for instance, [15].

The problem on integrable geodesic flows and their first integrals can be approached in another way. The equations of form (1.2) with appropriate coefficients A_j appear in many various problems of mathematical physics, their solutions are often various special functions [8]. Let us pose the question whether their exist two–dimensional metrics, the geodesics of which satisfy the corresponding equation (1.2) for some *prescribed* special function. In the case of the positive answer, we generally speaking get an entire family of metrics (see the details below), the geodesic equations of which are integrated in terms of the chosen special function. To the best of our knowledge, the issue on metrizability of equations of form (1.2), which are of interest in mathematical physics, is being actively studied just recently, see [1], [9], [13].

It is clear that not each equation of form (1.2) determine the geodesics of some metric: the corresponding metrizability criterion was obtained in [16], see also [11]. It reads as follows. We consider a system of partial differential equations for the functions $\psi_j(x, y)$, $j = 1, 2, 3$:

$$\begin{aligned} &(\psi_1)_x = \frac{2}{3}A_1\psi_1 - 2A_0\psi_2, \\ &(\psi_3)_y = 2A_3\psi_2 - \frac{2}{3}A_2\psi_3, \\ &(\psi_1)_y + 2(\psi_2)_x = \frac{4}{3}A_2\psi_1 - \frac{2}{3}A_1\psi_2 - 2A_0\psi_3, \\ &(\psi_3)_x + 2(\psi_2)_y = 2A_3\psi_1 - \frac{4}{3}A_1\psi_3 + \frac{2}{3}A_2\psi_2. \end{aligned} \tag{1.7}$$

If this system has a solution obeying $\Delta = \psi_1\psi_3 - \psi_2^2 \not\equiv 0$, then Equation (1.2) determine geodesics of the metric

$$ds^2 = \frac{(\psi_1 dx^2 + 2\psi_2 dxdy + \psi_3 dy^2)}{\Delta^2}$$

and the relations (1.3) are satisfied. In this case we say that Equation (1.2) is *metrizable*.

The system (1.7) possesses the following wonderful properties.

1) The space of solutions of system (1.7), that is the space of corresponding metrics, is a finite–dimentional vector space. The geodesics of these metrics satisfy the same equation (1.2), that is, they coincide as non–parametrized curves. Such metrics are called *projectively equivalent*; they were studied in various works, see, for instance, [7], [12]. It is known for a long time that the dimension of space of geodesically equivalent metrics does not exceed 6 [17].
2) If the dimension of the space of solutions (1.7) exceeds 3, then the geodesic flow of each of corresponding metric necessarily admits a linear in momenta first integral [14], see also [18], [20].

In [1], the metrizability of Schrödinger equation

$$\frac{d^2y}{dx^2} - u(x)y = 0 \tag{1.8}$$

was studied. It was also proved that in the case of a finite gap potential $u(x)$, the metric and geodesics can be found explicitly in terms of the Baker — Akhiezer function.

In this work we study the equation of form

$$\frac{d^2y}{dx^2} + f(x)\frac{dy}{dx} + g(x)y = 0; \tag{1.9}$$

as it is known, it is satisfied by many special functions under appropriate choice of $f(x)$, $g(x)$. Our main results are as follows.

Theorem 1.1 (Bessel function)**.** *Equation* (1.9) *for*

$$f(x) = \frac{1}{x}, \qquad g(x) = \frac{x^2 - \alpha^2}{x^2}, \qquad \alpha \in \mathbb{R}, \tag{1.10}$$

is metrizable. Namely, there exists a 6–parametric family of two–dimensional metrics, the geodesics of which satisfy (1.9), (1.10) *and possess a linear in momenta first integral.*

Theorem 1.2 (Hypergeometric function)**.** *Equation* (1.9) *for*

$$f(x) = \frac{c - (a+b+1)x}{x(1-x)}, \qquad g(x) = -\frac{ab}{x(1-x)}, \qquad a, b, c \in \mathbb{R}, \tag{1.11}$$

is metrizable. Namely, there exists a 6–parametric family of two–dimensional metrics, the geodesics of which satisfy (1.9), (1.11) *and possess a linear in momenta first integral.*

We also consider the equation

$$\frac{d^2y}{dx^2} = ay^2 + by + c, \qquad a, b, c \in \mathbb{R}, \tag{1.12}$$

which is satisfied, in particular, by the Weierstrass elliptic function.

Theorem 1.3 (Weierstrass elliptic function)**.** *Equation* (1.12) *is metrizable. Namely, there exists a 2–parametric family of two–dimensional metrics, the geodesics of which satisfy* (1.12) *and possess a linear in momenta first integral. At the same time, there exists one more transcendental integral, that is, the geodesic flow is superintegrable.*

2. Bessel function

Proof of Theorem 1.1. The Bessel function [8] satisfies the equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\alpha^2)y=0, \tag{2.1}$$

here $\alpha \in \mathbb{R}$ is an arbitrary parameter called the order. For Equation (2.1) we have

$$A_3=A_2=0, \qquad A_1=-\frac{1}{x}, \qquad A_0=-\frac{x^2-\alpha^2}{x^2}y$$

and therefore, the system (1.7) becomes

$$\begin{aligned}&(\psi_1)_x+\frac{2}{3x}\psi_1+\frac{2y(\alpha^2-x^2)}{x^2}\psi_2=0, \qquad (\psi_3)_y=0,\\ &(\psi_1)_y+2(\psi_2)_x-\frac{2}{3x}\psi_2+\frac{2y(\alpha^2-x^2)}{x^2}\psi_3=0, \qquad (\psi_3)_x+2(\psi_2)_y-\frac{4}{3x}\psi_3=0.\end{aligned} \tag{2.2}$$

This immediately implies that $\psi_3(x,y)=l(x)$. Integrating the fourth equation of system (2.2), we find ψ_2 :

$$\psi_2(x,y)=\frac{2y}{3x}l(x)-\frac{y}{2}l'(x)+s(x),$$

where $s(x)$ is an arbitrary function. Integrating the third equation in (2.2), we obtain

$$\psi_1(x,y)=\frac{1}{18x^2}\{2y^2\left(9x^2-9\alpha^2+8\right)l(x)+3x(6x\gamma(x)+y(4s(x)-5yl'(x)-12xs'(x)+3xyl''(x)))\},$$

where $\gamma(x)$ is an arbitrary function. After that the first equation in (2.2) becomes

$$R_0(x)+R_1(x)y+R_2(x)y^2=0,$$

where

$$\begin{aligned}&R_0=36x^2\gamma(x)+54x^3\gamma'(x),\\ &R_1=-108x^3s''(x)-36x^2s'(x)-12x(9x^2-9\alpha^2+1)s(x),\\ &R_2=27x^3l'''(x)-27x^2l''(x)+9x(12x^2-12\alpha^2+7)l'(x)-4(9x^2-36\alpha^2+16)l(x).\end{aligned}$$

The general solution of the system $R_0=R_1=R_2=0$ is

$$\begin{aligned}&\gamma(x)=\frac{c_0}{x^{\frac{2}{3}}}, \qquad s(x)=x^{\frac{1}{3}}\left(c_1J_\alpha(x)+c_2Y_\alpha(x)\right),\\ &l(x)=x^{\frac{4}{3}}\left(c_3J_\alpha^2(x)+c_4J_\alpha(x)Y_\alpha(x)+c_5Y_\alpha^2(x)\right),\end{aligned}$$

here $J_\alpha(x)$ and $Y_\alpha(x)$ are the Bessel functions of first and second kind. Thus, all functions $\psi_1(x,y)$, $\psi_2(x,y)$, $\psi_3(x,y)$ are found and therefore, the sought metric coefficients are recovered by the formulas

$$g_{11}=\frac{\psi_1}{(\psi_1\psi_3-\psi_2^2)^2}, \qquad g_{12}=\frac{\psi_2}{(\psi_1\psi_3-\psi_2^2)^2}, \qquad g_{22}=\frac{\psi_3}{(\psi_1\psi_3-\psi_2^2)^2}. \tag{2.3}$$

The coefficients of metric depend on 6 arbitrary parameters $c_0,\dots,c_5$. Thus, we have constructed a 6–parametric family of projectively equivalent metrics. Therefore, the geodesic flow of each metric in this family possesses a linear first integral [14]. The proof is complete. □

We note that in the case of arbitrary constants $c_0,\dots,c_5$ the constructed metric coefficients are rather cumbersome and, in contrast to the metric (1.6), in this case it is not so easy to provide the linear first integral explicitly. For instance, in the partial case $c_2 = c_4 = c_5 = 0$ we have

$$g_{11}(x,y) = \frac{c_0 + y\left(xJ_{\alpha-1}(x) - \alpha J_\alpha(x)\right)\left(-2c_1 + c_3xyJ_{\alpha-1}(x) - c_3y\alpha J_\alpha(x)\right)}{x^2(c_1^2 - c_0c_3)^2 J_\alpha^4(x)},$$
$$g_{12}(x,y) = \frac{c_1 + c_3y\alpha J_\alpha(x) - c_3xyJ_{\alpha-1}(x)}{x(c_1^2 - c_0c_3)^2 J_\alpha^3(x)}, \qquad g_{22}(x,y) = \frac{c_3}{(c_1^2 - c_0c_3)^2 J_\alpha^2(x)}.$$

The Gaussian curvature of this metric is zero. It seems that generally for all values of the parameters $c_0, \dots, c_5$ all constructed metrics are planar.

3. Hypergeometric function

Proof of Theorem 1.2. The hypergeometric function ${}_2F_1(a,b;c;x)$ [8] solves the equation

$$x(1-x)\frac{d^2y}{dx^2} + (c - (a+b+1)x)\frac{dy}{dx} - aby = 0, \tag{3.1}$$

here a, b, c are arbitrary constants. Sequentially integrating the second, fourth and third equation in the system (1.7), we get

$$\begin{aligned}
\psi_3(x,y) =& l(x), \qquad \psi_2(x,y) = \frac{2(c-(a+b+1)x)}{3x(1-x)}yl(x) - \frac{y}{2}l'(x) + s(x),\\
\psi_1(x,y) =& \frac{y}{18x^2(1-x)^2}\{2y(2c(3+c) - 9abx - 4(a+b+4)cx\\
&+ (2a^2 + 13ab + 2b^2 + 10a + 10b + 8)x^2)l(x) + 3x(x-1)\\
&\cdot(4s(x)(-c + x(a+b+1)) - 5y(-c + x(a+b+1))l'(x)\\
&+ 3x(x-1)(yl''(x) - 4s'(x)))\} + \gamma(x).
\end{aligned}$$

The remaining equation in (1.7) becomes

$$R_0(x) + R_1(x)y + R_2(x)y^2 = 0.$$

Integrating the relations $R_0 = R_1 = R_2 = 0$, we obtain

$$\begin{aligned}
\gamma(x) =& d_0 x^{-\frac{2c}{3}}(x-1)^{-\frac{2(a+b+1-c)}{3}},\\
s(x) =& (-1)^{-c}x^{-\frac{2c}{3}}(x-1)^{\frac{(a+b+1-c)}{3}}\{(-1)^c d_1 x^c {}_2F_1(a,b;c;x)\\
&- d_2 x\, {}_2F_1(1+a-c, 1+b-c; 2-c; x)\},\\
l(x) =& (-1)^{-2c}x^{-\frac{2c}{3}}(x-1)^{4(a+b+1-c)/3}\{(-1)^{2c}d_3x^{2c}{}_2F_1(a,b;c;x)^2\\
&- (-1)^c d_4 x^{1+c}{}_2F_1(a,b;c;x){}_2F_1(1+a-c,1+b-c;2-c;x)\\
&+ d_5x^2{}_2F_1(1+a-c,1+b-c;2-c;x)^2,
\end{aligned}$$

here $d_0, \dots, d_5$ are arbitrary constants.

Thus, all the functions $\psi_1(x,y)$, $\psi_2(x,y)$, $\psi_3(x,y)$ have been found. The corresponding metric coefficients are found by the formulas (2.3) and, as in the case of the Bessel function, they have a rather cumbersome form. As the result we obtain a 6–parametric family of projectively equivalent metrics (which seem to be planar), and in view of the results obtained in [14] this implies the existence of the linear integral. The proof is complete. □

4. Weierstrass elliptic function

The Weierstrass elliptic function $\wp(z)$ solves the equation [8]

$$(\wp')^2 = 4\wp^3 + g_2\wp^2 + g_1\wp + g_0,$$

where g_0, g_1, g_2 are arbitrary constants. Differentiating this equation in z and simplifying, we obtain

$$\wp'' = 6\wp^2 + g_2\wp + \frac{g_1}{2}.$$

Proof of Theorem 1.3. Here we study the metrizability of a bit more general equation

$$\frac{d^2y}{dx^2} = ay^2 + by + c, \tag{4.1}$$

where a, b, c are arbitrary constants. Integrating the system (1.7), we finally get

$$\psi_1(x,y) = d_1 - \frac{d_0}{3}\left(2ay^2 + 3by + 6c\right)y, \qquad \psi_2(x,y) = 0, \qquad \psi_3(x,y) = d_0,$$

that is, by (2.3),

$$\begin{aligned} g_{11} &= d_0^{-2}\left(d_1 - \frac{d_0}{3}\left(2ay^2 + 3by + 6c\right)y\right)^{-1}, \qquad g_{12} = 0, \\ g_{22} &= d_0^{-1}\left(d_1 - \frac{d_0}{3}\left(2ay^2 + 3by + 6c\right)y\right)^{-2}. \end{aligned} \tag{4.2}$$

here d_0, d_1 are arbitrary constants. The Gaussian curvature K of this metric is

$$K = \frac{d_0^2}{3}\left(d_0(a^2y^4 + 2aby^3 + 6acy^2 - 3c^2) - 3d_1(2ay + b)\right).$$

We rewrite the geodesic equations of metric (4.2) in the Hamiltonian form (1.5). Since the components of the metric g_{ij} depend only on y, the coordinate x is cyclic and therefore, the function $F_1 = p_1$ together with the Hamiltonian H is the first integral of (1.5). We are going to show how to project F_1 onto the first integral of Equation (4.1). We employ the inverse Legendre transform and express p_1, p_2 as functions of x, y, $\dot{x}$, $\dot{y}$. The function $J = \frac{H}{F_1^2}$ is obviously also the first integral. Replacing in J the momenta p_1, p_2 by their expressions in terms of the velocities $\dot{x}$, $\dot{y}$ and using the relation $y_x = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$, we finally get the autonomous first integral

$$I_1(x,y,y_x) = 3y_x^2 - 2ay^3 - 3by^2 - 6cy$$

of Equation (4.1).

Equation (4.1) admits one more first integral. While searching for it, we follow the ideas of [13], where a similar equation was considered. By the definition, the first integral $I(x,y,u)$ of Equation (4.1) satisfies the relation

$$I_x + I_y u + I_u(ay^2 + by + c) = 0,$$

where $u = y_x$. Suppose that $I = x + f(y,u)$, then

$$uf_y + (ay^2 + by + c)f_u = -1.$$

The characteristic equation reads

$$\frac{dy}{u} = \frac{du}{ay^2 + by + c} = \frac{df}{-1}.$$

The first identity implies immediately $udu = (ay^2 + by + c)dy$, that is,

$$3y_x^2 - 2ay^3 - 3by^2 - 6cy = k_0,$$

where k_0 is an arbitrary constant; this is exactly the integral I_1 constructed above. In view of this, the relation $udf = -dy$ yields

$$f = -\int \frac{dy}{\sqrt{\frac{k_0}{3} + \frac{2}{3}ay^3 + by^2 + 2cy}}.$$

Therefore, the first integrals of Equation (4.1) are found in quadratures and they read

$$I_1 = 3y_x^2 - 2ay^3 - 3by^2 - 6cy, \qquad I_2 = x - \int \frac{dy}{\sqrt{\frac{I_1}{3} + \frac{2}{3}ay^3 + by^2 + 2cy}}.$$

Thus, the geodesic flow of metric (4.2) is superintegrable. The proof is complete. □

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