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ON MULTIPLE INTERPOLATION OF PERIODIC COMPLEX–VALUED FUNCTIONS

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Absract. We obtain fully constructive results on construction of trigonometric interpolation polynomials with multiple nodes. We construct polynomials interpolating periodic complex–valued functions of a real variable. The polynomials are represented in general form and in the form of expansions over fundamental polynomials. We provide examples and discuss unresolved problems.

Keywords: multiple interpolation, complex–valued functions.

Mathematics Subject Classification: 65D05

1. Introduction

Algebraic interpolation polynomials with multiple nodes, called Hermite polynomials, are well studied and successfully applied in practice to solve a wide range of problems. Their trigonometric counterpart is much worse investigated. First studies of such polynomials began apparently since the late 1930s. Lozinsky [1] considered the approximation of complex variable functions regular within the unit circle and continuous on its boundary by Hermite — Fejér interpolation polynomials with multiple nodes located on the unit circle.

Zeel [2], [3], summarizing the results of predecessors [4], [5], [6], [7], proved the existence of trigonometric interpolation polynomials of arbitrary multiplicity $m \geqslant 0$ on the system of equidistant nodes for real–valued 2π–periodic functions and indicated the way of threading the corresponding fundamental polynomials. In addition, he obtained the conditions for uniform convergence of such polynomials to the interpolated function depending on its smoothness and the parity of m. Despite the fact that Zeel announced in his works the explicit construction of trigonometric polynomials for multiple interpolation, the coefficients of such polynomials were not calculated explicitly, but only equations for the coefficients were given.

In the deep and informative article [8] for an arbitrary permissible system of nodes (not necessarily equidistant) Trigub provided an algorithm for finding, in finitely many steps, trigonometrical polynomial with given values of function and the values of all its derivatives at these nodes. These values can be complex or real numbers. Moreover, the number of given values in different nodes can vary. For the particular case when the number of nodes is m and the number of values in all these nodes are the same and equal to $r + 1$, in [8, Lm. 2] there was proved the existence of a unique interpolation trigonometric polynomial with the spectrum on $[p, n]$, where $n - p = m(r + 1) - 1$.

In this work we prove the existence of interpolation polynomials with multiple nodes. These polynomials are represented as a power decomposition of the variable and as a fundamental polynomial decomposition. All the results are completely constructive, that is, all the coefficients of the polynomials are calculated explicitly. We note that trigonometric interpolation

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polynomials with multiple nodes were applied for the approximate solution of singular integro-differential equations only in the works of the author [9], [10].

2. Formulation of problem

As usually, by $\mathbb{N}$ and $\mathbb{R}$ we denote the sets of natural and real numbers, and the symbol $\mathbb{N}_0$ stands for the non–negative integer numbers.

We fix numbers $m, n \in \mathbb{N}_0$ and denote by C^m the set of 2π–periodic complex–valued functions having continuous derivatives of orders up to m. For $m = 0$ by $C = C^0$ we denote the set of 2π–periodic continuous complex–valued functions.

The problem of constructing a trigonometric interpolation polynomial with multiple nodes consists in finding a polynomial $x_{m,n}$ for a given function $x \in C^m$ such that at grid nodes

$$\Delta_n : t_k = \frac{2\pi k}{2n+1}, \quad k = 0, 1, \dots, 2n,$$

the conditions

$$x^{(\mu)}_{m,n}(t_k) = x^{(\mu)}(t_k), \quad \mu = 0, 1, \dots, m, \quad k = 0, 1, \dots, 2n, \tag{2.1}$$

are fulfilled.

3. Existence

We seek the required polynomial in the form

$$x_{m,n}(t) = \sum_{l=0}^{(2n+1)m+2n} d_l e^{ilt}, \quad t \in \mathbb{R}, \tag{3.1}$$

and the coefficients d_l, $l = 0, 1, \dots, (2n+1)m + 2n$, are determined by the system of equations (2.1).

Theorem 3.1. *For all $m, n \in \mathbb{N}_0$ and each function $x \in C^m$ there is a unique polynomial (3.1) obeying (2.1).*

Proof. We fix $m, n \in \mathbb{N}_0$ and a function $x \in C^m$. We rewrite the system of equations (2.1) as

$$\sum_{l=0}^{(2n+1)m+2n} d_l (il)^{\mu} e^{ilt_k} = x^{(\mu)}(t_k), \quad \mu = 0, 1, \dots, m, \quad k = 0, 1, \dots, 2n. \tag{3.2}$$

It is clear that for $m = 0$ the system of equations (3.2) becomes the system of equations

$$\sum_{l=0}^{2n} d_l e^{ilt_k} = x(t_k), \qquad k = 0, 1, \dots, 2n. \tag{3.3}$$

The matrix of the above system reads

$$S^0 = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{i\frac{2\pi}{2n+1}} & \cdots & e^{i\frac{4\pi n}{2n+1}} \\ \vdots & \vdots & & \vdots \\ 1 & e^{i\frac{4\pi n}{2n+1}} & \cdots & e^{i\frac{8\pi n^2}{2n+1}} \end{pmatrix}.$$

The determinant of this matrix is the Vandermonde determinant

$$|S^0| = V(\alpha_0, \alpha_1, \dots, \alpha_{2n}) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \alpha_0 & \alpha_1 & \dots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_0^{2n} & \alpha_1^{2n} & \dots & \alpha_{2n}^{2n} \end{vmatrix} \neq 0, \quad \alpha_k = e^{i\dfrac{2\pi k}{2n+1}}, \quad k = 0, 1, \dots, 2n,$$

therefore, for $m = 0$, the system of equations (3.3) is uniquely solvable for each function $x \in C$.

For $m > 0$ the matrix of equation (3.2) has the block structure

$$S^m = \begin{pmatrix} S^m_{0,0} & S^m_{0,1} & \dots & S^m_{0,m} \\ S^m_{1,0} & S^m_{1,1} & \dots & S^m_{1,m} \\ \vdots & \vdots & \vdots & \vdots \\ S^m_{m,0} & S^m_{m,1} & \dots & S^m_{m,m} \end{pmatrix}$$

with the matrices

$$S^m_{\mu,\nu} = \begin{pmatrix} \beta^\mu_{\nu,0} & \beta^\mu_{\nu,1} & \dots & \beta^\mu_{\nu,2n} \\ \beta^\mu_{\nu,0}\alpha_0 & \beta^\mu_{\nu,1}\alpha_1 & \dots & \beta^\mu_{\nu,2n}\alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \beta^\mu_{\nu,0}\alpha_0^{2n} & \beta^\mu_{\nu,1}\alpha_1^{2n} & \dots & \beta^\mu_{\nu,2n}\alpha_{2n}^{2n} \end{pmatrix},$$

where

$$\beta_{\nu,j} = i((2n+1)\nu + j), \qquad j = 0, 1, \dots, 2n, \qquad \mu, \nu = 0, 1, \dots, m,$$

and we adopt $0^0 = 1$. Their determinants are equal to

$$|S^m_{\mu,\nu}| = V(\alpha_0, \alpha_1, \dots, \alpha_{2n}) \prod_{j=0}^{2n} \beta^\mu_{\nu,j}, \qquad \mu, \nu = 0, 1, \dots, m,$$

therefore, the determinant of S^m is equal to

$$|S^m| = V^{m+1}(\alpha_0, \alpha_1, \dots, \alpha_{2n}) V(\gamma_0, \gamma_1, \dots, \gamma_m) \neq 0, \qquad \gamma_\nu = \prod_{j=0}^{2n} \beta_{\nu,j}, \qquad \nu = 0, 1, \dots, m.$$

The arbitrary choice of the numbers m, n and the function x implies the statement of the theorem and completes the proof. □

4. Explicit form

Theorem 3.1 is a generalization of Theorem 1 from [3] to the case of complex–valued functions. Moreover, although in [3] the explicit form of the desired polynomial was declared, the coefficients were not calculated explicitly, but only their existence was proved. Here we find explicitly the coefficients of the polynomial (3.1). In order to do this, we denote

$$B_{\mu,\nu}(\eta_0, \eta_1, \dots, \eta_m), \qquad \mu, \nu = 0, 1, \dots, m.$$

The Viète's formulas express the coefficients of reduced polynomials

$$\prod_{\substack{\mu=0 \\ \mu\neq\nu}}^{m} (\eta - \eta_\mu) = \sum_{\mu=0}^{m} (-1)^{m-\mu} B_{\mu,\nu}(\eta_0, \eta_1, \dots, \eta_m) \eta^\mu, \qquad \nu = 0, 1, \dots, m,$$

in terms of its roots η_μ, $\mu = 0, 1, \dots, \nu-1, \nu+1, \dots, m$. Here the parameter ν specifies, which of the root from the list $\eta_0, \eta_1, \dots, \eta_m$ is skipped. In particular, for $\mu = 0$,

$$B_{0,\nu}(\eta_0, \eta_1, \dots, \eta_m) = \prod_{\substack{\xi=0 \\ \xi\neq\nu}}^{m} \eta_\xi, \qquad \nu = 0, 1, \dots, m, \tag{4.1}$$

for $\mu = m-1$ for we have

$$B_{m-1,\nu}(\eta_0, \eta_1, \dots, \eta_m) = \sum_{\substack{\xi=0 \\ \xi\neq\nu}}^{m} \eta_\xi, \qquad \nu = 0, 1, \dots, m,$$

and for $\mu = m$

$$B_{m,\nu}(\eta_0, \eta_1, \dots, \eta_m) = 1, \qquad \nu = 0, 1, \dots, m.$$

Using the Viète's formulas, we find particular determinants of the system of equations (3.3). In order to do this, we need the next lemma.

Lemma 4.1. *For each number $m \in \mathbb{N}_0$ and each Vandermonde determinant*

$$V(\eta_0, \eta_1, \dots, \eta_m) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \eta_0 & \eta_1 & \dots & \eta_m \\ \vdots & \vdots & \vdots & \vdots \\ \eta_0^m & \eta_1^m & \dots & \eta_m^m \end{vmatrix} \tag{4.2}$$

the identities hold

$$\begin{vmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ \eta_0 & \dots & \eta_{\nu-1} & \eta_{\nu+1} & \dots & \eta_m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \eta_0^{\mu-1} & \dots & \eta_{\nu-1}^{\mu-1} & \eta_{\nu+1}^{\mu-1} & \dots & \eta_m^{\mu-1} \\ \eta_0^{\mu+1} & \dots & \eta_{\nu-1}^{\mu+1} & \eta_{\nu+1}^{\mu+1} & \dots & \eta_m^{\mu+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \eta_0^m & \dots & \eta_{\nu-1}^m & \eta_{\nu+1}^m & \dots & \eta_m^m \end{vmatrix} = \frac{(-1)^{m-\nu} V(\eta_0, \eta_1 \dots, \eta_m)}{\omega_m'(\eta_\nu)} B_{\mu,\nu}(\eta_0, \eta_1, \dots, \eta_m), \tag{4.3}$$

where

$$\omega_m(\eta) = \prod_{\nu=0}^{m} (\eta - \eta_\nu), \qquad \mu, \nu = 0, 1, \dots, m.$$

Proof. We expand the determinant (4.2) along the νth column

$$\begin{aligned} \begin{vmatrix} 1 & 1 & \dots & 1 \\ \eta_0 & \eta_1 & \dots & \eta_m \\ \vdots & \vdots & \vdots & \vdots \\ \eta_0^m & \eta_1^m & \dots & \eta_m^m \end{vmatrix} =& (-1)^{\nu} \begin{vmatrix} \eta_0 & \dots & \eta_{\nu-1} & \eta_{\nu+1} & \dots & \eta_m \\ \eta_0^2 & \dots & \eta_{\nu-1}^2 & \eta_{\nu+1}^2 & \dots & \eta_m \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \eta_0^m & \dots & \eta_{\nu-1}^m & \eta_{\nu+1}^m & \dots & \eta_m^m \end{vmatrix} \\ &+ (-1)^{1+\nu} \begin{vmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ \eta_0^2 & \dots & \eta_{\nu-1}^2 & \eta_{\nu+1}^2 & \dots & \eta_m^2 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \eta_0^m & \dots & \eta_{\nu-1}^m & \eta_{\nu+1}^m & \dots & \eta_m^m \end{vmatrix} \eta_\nu + \dots \\ &+ (-1)^{m+\nu} \begin{vmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ \eta_0 & \dots & \eta_{\nu-1} & \eta_{\nu+1} & \dots & \eta_m \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \eta_0^{m-1} & \dots & \eta_{\nu-1}^{m-1} & \eta_{\nu+1}^{m-1} & \dots & \eta_m^{m-1} \end{vmatrix} \eta_\nu^m, \end{aligned} \tag{4.4}$$

and we get a polynomial of order m in the variable η_ν. On the other hand, the determinant (4.2) is equal to

$$\begin{aligned}\begin{pmatrix} 1 & 1 & \dots & 1 \\ \eta_0 & \eta_1 & \dots & \eta_m \\ \vdots & \vdots & \vdots & \vdots \\ \eta_0^m & \eta_1^m & \dots & \eta_m^m \end{pmatrix} &= \prod_{0\leqslant q<p\leqslant m}(\eta_p-\eta_q) = \prod_{q=0}^{\nu-1}(\eta_\nu-\eta_q)\prod_{p=\nu+1}^{m}(\eta_p-\eta_\nu)\prod_{\substack{0\leqslant q<\nu \\ \nu<q\leqslant m}}(\eta_p-\eta_q) \\ &=(-1)^{m-\nu}\prod_{q=0}^{\nu-1}(\eta_\nu-\eta_q)\prod_{p=\nu+1}^{m}(\eta_\nu-\eta_p)\prod_{\substack{0\leqslant q<\nu \\ \nu<q\leqslant m}}(\eta_p-\eta_q) \\ &=(-1)^{m-\nu}\prod_{\substack{p=0 \\ p\neq\nu}}(\eta_\nu-\eta_p)\prod_{\substack{0\leqslant q<\nu \\ \nu<q\leqslant m}}(\eta_p-\eta_q) \\ &=(-1)^{m-\nu}V(\eta_0,\dots,\eta_{\nu-1},\eta_{\nu+1},\dots,\eta_m) \\ &\quad\cdot\sum_{\mu=0}^{m}(-1)^{m-\mu}B_{\mu,\nu}(\eta_0,\eta_1,\dots,\eta_m)\eta_\nu^\mu \\ &=V(\eta_0,\dots,\eta_{\nu-1},\eta_{\nu+1},\dots,\eta_m)\sum_{\mu=0}^{m}(-1)^{\mu+\nu}B_{\mu,\nu}(\eta_0,\eta_1,\dots,\eta_m)\eta_\nu^\mu. \end{aligned} \tag{4.5}$$

Equating the coefficients in the polynomials (4.4) and (4.5) at the like degrees of η_ν, we get the identities

$$\begin{vmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ \eta_0 & \dots & \eta_{\nu-1} & \eta_{\nu+1} & \dots & \eta_m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \eta_0^{\mu-1} & \dots & \eta_{\nu-1}^{\mu-1} & \eta_{\nu+1}^{\mu-1} & \dots & \eta_m^{\mu-1} \\ \eta_0^{\mu+1} & \dots & \eta_{\nu-1}^{\mu+1} & \eta_{\nu+1}^{\mu+1} & \dots & \eta_m^{\mu+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \eta_0^{m} & \dots & \eta_{\nu-1}^{m} & \eta_{\nu+1}^{m} & \dots & \eta_m^{m} \end{vmatrix} = V(\eta_0,\dots,\eta_{\nu-1},\eta_{\nu+1},\dots,\eta_m)B_{\mu,\nu}(\eta_0,\eta_1,\dots,\eta_m), \tag{4.6}$$

where $\mu,\nu=0,1,\dots,m$. Since

$$\begin{aligned} V(\eta_0,\eta_1,\dots,\eta_m) &= V(\eta_0,\dots,\eta_{\nu-1},\eta_{\nu+1},\dots,\eta_m)\prod_{p=0}^{\nu-1}(\eta_\nu-\eta_p)\prod_{p=\nu+1}^{m}(\eta_p-\eta_\nu) \\ &=(-1)^{m-\nu}\omega_m'(\eta_\nu)V(\eta_0,\dots,\eta_{\nu-1},\eta_{\nu+1},\dots,\eta_m), \qquad \nu=0,1,\dots,m, \end{aligned}$$

we have

$$V(\eta_0,\dots,\eta_{\nu-1},\eta_{\nu+1},\dots,\eta_m) = \frac{(-1)^{m-\nu}V(\eta_0,\eta_1,\dots,\eta_m)}{\omega_m'(\eta_\nu)}, \quad \nu=0,1,\dots,m. \tag{4.7}$$

Sudstituting (4.7) to (4.6), we arrive at the identities (4.3). □

To calculate explicitly the coefficients d_l, $l=0,1,\dots,(2n+1)m+2n$, we need to solve the system of equations

$$\begin{aligned} &S^m\mathbf{d}^T=\mathbf{a}^T, \quad \mathbf{d}=(d_0,d_1,\dots,d_{(2n+1)m+2n}), \\ &\mathbf{a}=(x(t_0),x(t_1),\dots,x(t_{2n}),x'(t_0),\dots,x^m(t_{2n})), \end{aligned} \tag{4.8}$$

with respect to the vector $\mathbf{d}$.

We denote by $|S_l^m|$ the partial determinant of the system of equations (4.8), which is obtained from the determinant $|S^m|$ by replacing the column number l of the matrix S^m by the vector $\mathbf{a}^T$. Let us calculate first the necessary parameters by l

$$\nu = \left[\frac{l}{2n+1}\right], \quad 0 \leqslant \nu \leqslant m, \quad r = l - (2n+1)\nu, \quad 0 \leqslant r \leqslant 2n,$$

where $[\cdot]$ denotes the integer part of a number. The ν parameter indicates that the right hand side vector $\mathbf{a}^T$ goes through the matrices $S_{0,\nu}^m, S_{1,\nu}^m, \ldots, S_{m,\nu}^m$ of the matrix S^m, and the parameter r indicates that in these matrices the rth column is replaced by the column

$$\mathbf{a}_j^T = (x(t_0), x(t_1), \ldots, x(t_{2n}))^T$$

in the matrix $S_{j,\nu}^m$, $j = 0, 1, \ldots, m$. The resulting modified matrices are denoted by $\tilde{S}_{0,\nu}^m$, $\tilde{S}_{1,\nu}^m$, $\ldots$, $\tilde{S}_{m,\nu}^m$. The explicit form of coefficients d_l, $l = 0, 1, \ldots, (2n+1)m + 2n$, is provided by the next theorem.

Theorem 4.1. *For all numbers $m, n \in \mathbb{N}_0$ and each function $x \in C^m$ the coefficients d_l, $l = 0, 1, \ldots, (2n+1)m + 2n$, of the polynomial (3.1) are equal to*

$$\begin{aligned} d_l &= \frac{1}{\omega_{2n}'(\alpha_l)} \sum_{k=0}^{2n} (-1)^k x(t_k) B_{k,l}(\alpha_0, \alpha_1, \ldots, \alpha_{2n}), \\ \omega_{2n}(\alpha) &= \prod_{l=0}^{2n} (\alpha - \alpha_l), \quad l = 0, 1, \ldots, 2n, \end{aligned} \tag{4.9}$$

for $m = 0$, $n \geqslant 0$;

$$\begin{aligned} d_l &= \frac{1}{\omega_m'(\beta_{l,0})} \sum_{\mu=0}^{m} (-1)^{m+\mu} x^{(\mu)}(t_0) B_{\mu,l}(\beta_{0,0}, \beta_{1,0}, \ldots, \beta_{m,0}), \\ \omega_m(\beta) &= \prod_{\mu=0}^{m} (\beta - \beta_{\mu,0}), \qquad l = 0, 1, \ldots, m, \end{aligned} \tag{4.10}$$

for $m \geqslant 0$, $n = 0$;

$$\begin{aligned} d_l = \frac{1}{\omega_m'(\gamma_\nu)\omega_{2n}'(\alpha_r)} &\sum_{\mu=0}^{m} \sum_{k=0}^{2n} (-1)^{m+\mu+k} x^{(\mu)}(t_k) \\ &\cdot \prod_{\substack{j=0 \\ j \neq r}}^{2n} \beta_{\nu,k}^{\mu} B_{\mu,\nu}(\gamma_0, \gamma_1, \ldots, \gamma_m) B_{k,r}(\alpha_0, \alpha_1, \ldots, \alpha_{2n}), \end{aligned} \tag{4.11}$$

$$\omega_m(\gamma) = \prod_{\mu=0}^{m} (\gamma - \gamma_\nu), \quad l = 0, 1, \ldots, (2n+1)m + 2n,$$

for $m > 0$, $n > 0$.

Proof. For $m = 0$, $n \geqslant 0$ the determinant $|S^0|$ is equal to

$$|S^0| = V(\alpha_0, \alpha_1, \ldots, \alpha_{2n}),$$

and the determinants $|S_l^0|$ are equal to

$$\begin{aligned}|S_l^0| &= \begin{vmatrix} 1 & \dots & 1 & x(t_0) & 1 & \dots & 1 \\ \alpha_0 & \dots & \alpha_{l-1} & x(t_1) & \alpha_{l+1} & \dots & \alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \alpha_0^{2n} & \dots & \alpha_{l-1}^{2n} & x(t_{2n}) & \alpha_{l+1}^{2n} & \dots & \alpha_{2n}^{2n} \end{vmatrix} \\ &= \sum_{k=0}^{2n} (-1)^{l+k} x(t_k) V(\alpha_0, \dots, \alpha_{l-1}, \alpha_{l=1}, \dots, \alpha_{2n}) B_{k,l}(\alpha_0, \alpha_1, \dots, \alpha_{2n}) \\ &= \frac{V(\alpha_0, \alpha_1, \dots, \alpha_{2n})}{\omega_{2n}'(\alpha_l)} \sum_{k=0}^{2n} (-1)^{l+k} x(t_k) B_{k,l}(\alpha_0, \alpha_1, \dots, \alpha_{2n}).\end{aligned}$$

Hence,

$$d_l = \frac{|S_l^0|}{|S^0|} = \frac{1}{\omega_{2n}(\alpha_l)} \sum_{k=0}^{2n} (-1)^k x(t_k) B_{k,l}(\alpha_0, \alpha_1, \dots, \alpha_{2n}),$$

$$\omega(\alpha) = \prod_{l=0}^{2n} (\alpha - \alpha_l), \qquad l = 0, 1, \dots, 2n.$$

For $n = 0$, $m \geqslant 0$, the determinant $|S^m|$ is equal to

$$|S^m| = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \beta_{0,0} & \beta_{1,0} & \dots & \beta_{m,0} \\ \vdots & \vdots & \vdots & \vdots \\ \beta_{0,0} & \beta_{1,0}^m & \dots & \beta_{m,0}^m \end{pmatrix} = V(\beta_{0,0}, \beta_{1,0}, \dots, \beta_{m,0}),$$

and the determinants $|S_l^m|$ are equal to

$$\begin{aligned}|S_l^m| &= \begin{pmatrix} 1 & \dots & 1 & x(t_0) & 1 & \dots & 1 \\ \beta_{0,0} & \dots & \beta_{l-1,0} & x'(t_0) & \beta_{l+1,0} & \dots & \beta_{m,0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \beta_{0,0}^m & \dots & \beta_{l-1,0}^m & x^{(m)}(t_0) & \beta_{l+1,0}^m & \dots & \beta_{m,0}^m \end{pmatrix} \\ &= \sum_{\mu=0}^{m} (-1)^{l+\mu} x^{(\mu)}(t_0) V(\beta_{0,0}, \dots, \beta_{l-1,0}, \beta_{l+1,0}, \dots, \beta_{m,0}) B_{\mu,l}(\beta_{0,0}, \beta_{1,0}, \dots, \beta_{m,0}) \\ &= \frac{V(\beta_0, 0, \beta_{1,0}, \dots, \beta_{m,0})}{\omega_m'(\beta_{l,0})} \sum_{\mu=0}^{m} (-1)^{m+\mu} x^{(\mu)}(t_0) B_{\mu,l}(\beta_{0,0}, \beta_{1,0,\dots,\beta_{m,0}}).\end{aligned}$$

Therefore,

$$d_l = \frac{1}{\omega_m'(\beta_{l,o})} \sum_{\mu=0}^{m} (-1)^{m+\mu} x^{(\mu)}(t_0) B_{\mu,l}(\beta_{0,0}, \beta_{1,0}, \dots, \beta_{m,0}),$$

$$\omega_m(\beta) = \prod_{l=0}^{m} (\beta - \beta_{l,0}), \qquad l = 0, 1, \dots, m.$$

For $m > 0$, $n > 0$ the determinant $|S^m|$ is equal to

$$|S^m| = V^{m+1}(\alpha_0, \alpha_1, \dots, \alpha_{2n}) V(\gamma_0, \gamma_1, \dots, \gamma_m),$$

and the determinants $|S_l^m|$ are equal to

$$\begin{aligned}
|S_l^m| &= \begin{vmatrix} |S_{0,0}^m| & \dots & |\tilde{S}_{0,\nu}^m| & \dots & |S_{0,m}^m| \\ |S_{1,0}^m| & \dots & |\tilde{S}_{1,\nu}^m| & \dots & |S_{1,m}^m| \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ |S_{m,0}^m| & \dots & |\tilde{S}_{m,\nu}^m| & \dots & |S_{m,m}^m| \end{vmatrix} \\
&= \begin{vmatrix} V(\alpha_0,\alpha_1,\dots,\alpha_{2n}) & \dots & |\tilde{S}_{0,\nu}^m| & \dots & V(\alpha_0,\alpha_1,\dots,\alpha_{2n}) \\ V(\alpha_0,\alpha_1,\dots,\alpha_{2n})\gamma_0 & \dots & |\tilde{S}_{1,\nu}^m| & \dots & V(\alpha_0,\alpha_1,\dots,\alpha_{2n})\gamma_m \\ \vdots & \vdots & \vdots & & \vdots \\ V(\alpha_0,\alpha_1,\dots,\alpha_{2n})\gamma_0^m & \dots & |\tilde{S}_{1,\nu}^m| & \dots & V(\alpha_0,\alpha_1,\dots,\alpha_{2n})\gamma_m^m \end{vmatrix} \\
&= V^m(\alpha_0,\alpha_1,\dots,\alpha_{2n}) \begin{vmatrix} 1 & \dots & |\tilde{S}_{0,\nu}^m| & \dots & 1 \\ \gamma_0 & \dots & |\tilde{S}_{1,\nu}^m| & \dots & \gamma_m \\ \vdots & \vdots & \vdots & & \vdots \\ \gamma_0^m & \dots & |\tilde{S}_{1,\nu}^m| & \dots & \gamma_m^m \end{vmatrix} \\
&= V^m(\alpha_0,\alpha_1,\dots,\alpha_{2n}) \\
&\quad \cdot \sum_{\mu=0}^{m} (-1)^{\nu+\mu} |\tilde{S}_{\mu,\nu}^m| V(\gamma_0,\dots,\gamma_{\nu-1},\gamma_{\nu+1},\dots,\gamma_m) B_{\mu,\nu}(\gamma_0,\gamma_1,\dots,\gamma_m) \\
&= \frac{V^m(\alpha_0,\alpha_1,\dots,\alpha_{2n}) V(\gamma_0,\gamma_1,\dots,\gamma_m)}{\omega_m'(\gamma_\nu)} \sum_{\mu=0}^{m} (-1)^{m+\mu} |\tilde{S}_{\mu,\nu}^m| B_{\mu,\nu}(\gamma_0,\gamma_1,\dots,\gamma_m).
\end{aligned} \tag{4.12}$$

We proceed to calculating the determinants $|\tilde{S}_{\mu,\nu}^m|$, $\mu,\nu = 0,1,\dots,m$,

$$\begin{aligned}
|\tilde{S}_{\mu,\nu}^m| &= \begin{vmatrix} \beta_{\nu,o}^{\mu} & \dots & x^{(\mu)}(t_0) & \dots & \beta_{\nu,2n}^{\mu} \\ \beta_{\nu,0}^{\mu}\alpha_0 & \dots & x^{(\mu)}(t_1) & \dots & \beta_{\nu,2n}^{\mu}\alpha_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{\nu,0}^{\mu}\alpha_0^{2n} & \dots & x^{(\mu)}(t_{2n}) & \dots & \beta_{\nu,2n}^{\mu}\alpha_{2n}^{2n} \end{vmatrix} \\
&= \sum_{k=0}^{2n} (-1)^k x^{(\mu)}(t_k) \prod_{\substack{j=0 \\ j\neq r}}^{2n} \beta_{\nu,j}^{\mu} \frac{V(\alpha_0,\alpha_1,\dots,\alpha_{2n})}{\omega_{2n}'(\alpha_r)} B_{k,r}(\alpha_0,\alpha_1,\dots,\alpha_{2n}) \\
&= \frac{V(\alpha_0,\alpha_1,\dots,\alpha_{2n})}{\omega_{2n}'(\alpha_r)} \prod_{\substack{j=0 \\ j\neq r}}^{2n} \beta_{\nu,j}^{\mu} \sum_{k=0}^{2n} (-1)^k x^{(\mu)}(t_k) B_{k,r}(\alpha_0,\alpha_1,\dots,\alpha_{2n}).
\end{aligned} \tag{4.13}$$

By these identities and (4.12) we get

$$\begin{aligned}
|S_l^m| =& \frac{V^{m+1}(\alpha_0,\alpha_1,\dots,\alpha_{2n}) V(\gamma_0,\gamma_1,\dots,\gamma_m)}{\omega_m'(\gamma_\nu)\omega_{2n}'(\alpha_r)} \\
& \cdot \sum_{\mu=0}^{m} \sum_{k=0}^{2n} (-1)^{m+\mu+k} x^{(\mu)}(t_k) \prod_{\substack{j=0 \\ j\neq r}}^{2n} \beta_{\nu,j}^{\mu} B_{\mu,\nu}(\gamma_0,\gamma_1,\dots,\gamma_m) B_{k,r}(\alpha_0,\alpha_1,\dots,\alpha_{2n}),
\end{aligned}$$

and this means that coefficients d_l are equal to

$$d_l = \frac{1}{\omega_m'(\gamma_\nu)\omega_{2n}'(\alpha_r)} \sum_{\mu=0}^{m} \sum_{k=0}^{2n} (-1)^{m+\mu+k} x^{(mu)}(t_k) \prod_{\substack{j=0 \\ j\neq r}}^{2n} \beta_{\nu,j}^{\mu} B_{\mu,\nu}(\gamma_0.\gamma_1,\dots,\gamma_m) B_{k,r}(\alpha_0,\alpha_1,\dots,\alpha_{2n}),$$

where $l = 0,1,\dots,(2n+1)m+2n$. The proof is complete. □

5. Fundamental polynomials

The polynomial $x_{m,n}$, written as (3.1), is inconvenient for applications. It is better to rewrite it via fundamental polynomials, that is, the polynomials $u_{\mu,k}$, $\mu = 0,1,\dots,m$, $k = 0,1,\dots,2n$, of form (3.1) satisfying the conditions

$$u^{(\nu)}_{\mu,k}(t_r) = \begin{cases} 1, & (\mu-\nu)^2 + (k-r)^2 = 0, \\ 0, & (\mu-\nu)^2 + (k-r)^2 > 0, \end{cases} \qquad \mu,\nu = 0,1,\dots,m, \ k,r = 0,1,\dots,2n. \tag{5.1}$$

In terms of these polynomials, the polynomial $x_{m,n}$ can be written as

$$x_{m,n}(t) = \sum_{\mu=0}^{m}\sum_{k=0}^{2n} x^{(\mu)}(t_k)u_{\mu,k}(t), \quad t \in \mathbb{R}. \tag{5.2}$$

Theorem 5.1. *For all numbers $m,n \in \mathbb{N}_0$ and each function $x \in C^m$ the coefficients*

$$d_{l,\mu,k}, \quad l = 0,1,\dots,(2n+1)m+2n, \quad \mu = 0,1,\dots,m, \quad k = 0,1,\dots,2n,$$

of the fundamental polynomials

$$u_{\mu,k}(t) = \sum_{l=0}^{(2n+1)m+2n} d_{l,\mu,k}e^{ilt}, \qquad t \in \mathbb{R},$$

in the interpolation polynomials (5.2) are equal to

$$d_{l,0,k} = \frac{(-1)^k}{\omega'_{2n}(\alpha_l)}B_{k,l}(\alpha_0,\alpha_1,\dots,\alpha_{2n}), \qquad l = 0,1,\dots,2n; \tag{5.3}$$

for $m = 0$, $n \geqslant 0$;

$$d_{l,\mu,0} = \frac{(-1)^{m+\mu}}{\omega'_m(\beta_{l,0})}B_{\mu,l}(\beta_{0,0},\beta_{1,0},\dots,\beta_{m,0}), \quad l = 0,1,\dots,m, \tag{5.4}$$

for $n = 0$, $m \geqslant 0$;

$$\begin{gathered} d_{l,\mu,k} = \frac{(-1)^{m+\mu+k}}{\omega'_m(\gamma_\nu)\omega'_{2n}(\alpha_r)}\prod_{\substack{j=0\\ j\neq r}}^{2n}\beta^{\mu}_{\nu,j}B_{\mu,\nu}(\gamma_0,\gamma_1,\dots,\gamma_m)B_{k,r}(\alpha_0,\alpha_1,\dots,\alpha_{2n}), \\ \nu = \left[\frac{l}{2n+1}\right], \qquad r = l-(2n+1)\nu, \qquad l = 0,1,\dots,(2n+1)m+2n, \end{gathered} \tag{5.5}$$

for $m > 0$, $n > 0$.

Proof. The polynomials $u_{\mu,k}$ are special cases of the polynomials (3.1) under the conditions (5.1). These conditions mean that for each pair, (μ,k), $\mu = 0,1,\dots,m$, $k = 0,1,\dots,2n$, the vector in the right hand sides of the system of equations (4.8) consists of $(2n+1)m+2n$ zeros and one unity located at the $(2n+1)\mu+k$-th place.

According to Theorem 4.1 with $m = 0$, $n \geqslant 0$, we have $\mu = 0$, and the parameter k is fixed by location of the unity in a vector of right hand sides of the system of equations (4.8). At the same time, l ranges as $l = 0,1,\dots,2n$. Fixing the value of parameter k in the identity (4.9), we get the formula (5.3).

For $n = 0$, $m \geqslant 0$, we have $k = 0$, and the parameter μ is fixed by location of the unity in the vector in the right hand side of the system of equations (4.8). In this case, the parameter l ranges as $l = 0,1,\dots,m$. Fixing the value of parameter μ in the identity (4.10), we get the formula (5.4).

For $m \geqslant 0$, $n \geqslant 0$, both the parameters μ and k are fixed by the position of the unity in the vector in the right hand side of the system of equations (4.8), and parameter l ranges as

$l = 0, 1, \ldots, (2n+1)m+2n$. Fixing the parameters μ and k in the identity (4.11) and calculating the parameters ν and r by l, we get the formula (5.5). The proof is complete. □

6. Examples and discussion of results

In Theorems 4.1 and 5.1, the formulas (4.9), (5.3) for $m = 0$, $n \geqslant 0$ and the formulas (4.10), (5.4) for $n = 0$, $m \geqslant 0$ are provided separately. These cases are interesting due to the facts that the formula (4.9) generalizes the Lagrange trigonometric interpolation polynomial to the case of complex–valued functions of real variable, while the formula (4.10) allows to construct periodic analogues of the partial sums of the Maclaurin series for the same functions.

Let $m = 0$. We fix $n \in \mathbb{N}_0$ and we are going to construct the fundamental polynomials for different values of the parameter $k = 0, 1, \ldots, 2n$. By the formula (5.3), the coefficients of the polynomials $u_{0,k}$ are equal to

$$d_{l,0,k} = \frac{(-1)^k}{\omega'_{2n}(\alpha_l)} B_{k,l}(\alpha_0, \alpha_1, \ldots, \alpha_{2n}), \qquad l = 0, 1, \ldots, 2n,$$

hence,

$$u_{0,k}(t) = \sum_{l=0}^{2n} d_{l,0,k} e^{ilt}, \qquad k = 0, 1, \ldots, 2n.$$

This allows us to write the interpolation polynomial $x_{0,n}$ for the values of the function $x \in C$ at the nodes Δ_n

$$x_{0,n}(t) = \sum_{k=0}^{2n} x(t_k) u_{0,k}(t), \qquad t \in \mathbb{R}. \tag{6.1}$$

Let us calculate explicitly the polynomials $u_{0,k}$, $k = 0, 1, \ldots, 2n$. Using the formula (5.3), we find

$$u_{0,k}(t) = \frac{1}{2n+1} \sum_{j=0}^{2n} e^{ij(t-t_k)} = \frac{1 - e^{i(2n+1)(t-t_k)}}{(2n+1)\left(1 - e^{i(t-t_k)}\right)}, \qquad k = 0, 1, \ldots, 2n, \qquad t \in \mathbb{R}. \tag{6.2}$$

The expression (6.2) is a complex analogue of the Dirichlet kernel in the periodic Lagrange interpolation polynomial. Substituting (6.2) into (6.1), we get

$$x_{0,n}(t) = \frac{1}{2n+1} \sum_{k=0}^{2n} x(t_k) \frac{1 - e^{i(2n+1)(t-t_k)}}{1 - e^{i(t-t_k)}}, \qquad t \in \mathbb{R}.$$

A more interesting case is for $n = 0$, $m \geqslant 0$. In this case, the polynomials

$$x_{m,0}(t) = \sum_{\mu=0}^{m} x^{(\mu)}(t_0) u_{\mu,0}(t), \qquad m \in \mathbb{N}_0, \qquad t \in \mathbb{R}, \tag{6.3}$$

are partial sums of the Maclaurin–type series for periodic functions $x \in C^\infty$. The term "Maclaurin type" means here that the polynomials $u_{\mu,0}$, $\mu = 0, 1, \ldots, m$, are not algebraic, but periodic, like the function $x \in C^m$.

Let $n = 0$. We fix $m \in \mathbb{N}_0$ and we are going to construct the fundamental polynomials for different values of the parameter $\mu = 0, 1, \ldots, m$. By the formula (5.4), the coefficients of polynomial $u_{\mu,0}$ are equal to

$$d_{l,\mu,0} = \frac{(-1)^{m+\mu}}{\omega'_m(\beta_{l,0})} B_{\mu,l}(\beta_{0,0}, \beta_{1,0}, \ldots, \beta_{m,0}), \qquad l = 0, 1, \ldots, m,$$

and hence,

$$u_{\mu,0}(t)=\sum_{l=0}^{m} d_{l,\mu,0}e^{ilt}, \qquad \mu=0,1,\dots,m.$$

Let us calculate the explicit form of polynomials (6.1) for several values of m.

Let $m=0$, then $l=\mu=0$, $\omega_0'(\beta_{0,0})=B_{0,0}(\beta_{0,0})=1$, hence, $u_{0,0}(t)=1$, and

$$x_{0,0}(t)=x(t_0), \qquad t\in\mathbb{R}.$$

For $m=1$ we find

$$\beta_{o,o}=0, \qquad \beta_{1,0}=i, \qquad \omega_1'(\beta_{0,0})=-i, \qquad \omega_1'(\beta_{1,0})=i,$$
$$B_{0,0}(\beta_{0,0},\beta_{1,0})=i, \quad B_{1,0}(\beta_{0,0},\beta_{1,0})=1, \quad B_{0,1}(\beta_{0,0},\beta_{1,0})=0, \quad B_{1,1}(\beta_{0,0},\beta_{1,0})=1,$$

and therefore,

$$d_{0,0,0}=1, \qquad d_{1,0,0}=0, \qquad d_{0,1,0}=i, \qquad d_1,1,0=-i.$$

The polynomials $u_{\mu,0}$, $\mu=0,1$, read as

$$u_{0,0}(t)=1, \qquad u_{1,0}(t)=i-ie^{it},$$

and the polynomial $x_{1,0}$ is

$$x_{1,0}(t)=x(t_0)+x'(t_0)(i-ie^{it}), \qquad t\in\mathbb{R}.$$

For $m=2$ we get

$$\beta_{0,0}=0, \qquad \beta_{1,0}=i, \qquad \beta_{2,0}=2i,$$
$$\omega_2'(\beta_{0,0})=-2, \qquad \omega_2'(\beta_{1,0})=1, \qquad \omega_2'(\beta_{2,0})=-2,$$
$$B_{0,0}(\beta_{0,0},\beta_{1,0},\beta_{2,0})=-2, \quad B_{0,1}(\beta_{0,0},\beta_{1,0},\beta_{2,0})=0, \quad B_{0,2}(\beta_{0,0},\beta_{1,0},\beta_{2,0})=0,$$
$$B_{1,0}(\beta_{0,0},\beta_{1,0},\beta_{2,0})=3i, \quad B_{1,1}(\beta_{0,0},\beta_{1,0},\beta_{2,0})=2i, \quad B_{1,2}(\beta_{0,0},\beta_{1,0},\beta_{2,0})=i,$$
$$B_{2,0}(\beta_{0,0},\beta_{1,0},\beta_{2,0})=1, \quad B_{2,1}(\beta_{0,0},\beta_{1,0},\beta_{2,0})=1, \quad B_{2,2}(\beta_{0,0},\beta_{1,0},\beta_{2,0})=1,$$

and therefore,

$$d_{0,0,0}=1, \qquad d_{1,0,0}=0, \qquad d_{2,0,0}=0, \qquad d_{0,1,0}=\frac{3}{2}i, \qquad d_{1,1,0}=-2i,$$
$$d_{2,1,0}=\frac{1}{2}i, \qquad d_{0,2,0}=-\frac{1}{2}, \qquad d_{1,2,0}=1, \qquad d_{2,2,0}=-\frac{1}{2}.$$

The polynomials $u_{\mu,0}$, $\mu=0,1,2$, read

$$u_{0,0}(t)=1, \qquad u_{1,0}(t)=\frac{3}{2}i-2ie^{it}+\frac{1}{2}ie^{i2t}, \qquad u_{2,0}(t)=-\frac{1}{2}+e^{it}-\frac{1}{2}e^{i2t},$$

and the polynomial $x_{2,0}$ reads

$$x_{2,0}(t)=x(t_0)+x'(t_0)\left(\frac{3}{2}i-2ie^{it}+\frac{1}{2}ie^{i2t}\right)-x''(t_0)\left(\frac{1}{2}-e^{it}+\frac{1}{2}e^{i2t}\right), \qquad t\in\mathbb{R}.$$

Now we take $m=3$. Let us find the values of parameters in the formulas (5.4). The coefficients $\beta_{\mu,0}$, $\mu=0,1,2,3$, are equal to

$$\beta_{0,0}=0, \qquad \beta_{1,0}=i, \qquad \beta_{2,0}=2i, \qquad \beta_{3,0}=3i,$$

thus,

$$\omega_3'(\beta_{0,0})=6i, \qquad \omega_3'(\beta_{1,0})=-2i, \qquad \omega_3'(\beta_{2,0})=2i, \qquad \omega_3'(\beta_{3,0})=-6i.$$

Let us calculate the values of the functions $B_{\mu,l}(\beta_{0,0},\beta_{1,0},\beta_{2,0},\beta_{3,0})$, $\mu,l=0,1,2,3$. For the sake of brevity, we shall omit the vector $(\beta_{0,0},\beta_{1,0},\beta_{2,0},\beta_{3,0})$ in the variables of the function $B_{\mu,l}$

$$\begin{array}{llll} B_{0,0}=-6i, & B_{0,1}=0, & B_{0,2}=0, & B_{0,3}=0,\\ B_{1,0}=-11, & B_{1,1}=-6, & B_{1,2}=-3, & B_{1,3}=-2,\\ B_{2,0}=-6i, & B_{2,1}=5i, & B_{2,2}=4i, & B_{2,3}=3i,\\ B_{3,0}=1, & B_{3,1}=1, & B_{3,2}=1, & B_{3,3}=1. \end{array}$$

This allows us to calculate the coefficients $d_{l,\mu,0}$, $\mu,l=0,1,2,3$,

$$\begin{array}{llll} d_{0,0,0}=1, & d_{1,0,0}=0, & d_{2,0,0}=0, & d_{3,0,0}=0,\\ d_{0,1,0}=\dfrac{11}{6}i, & d_{1,1,0}=-3i, & d_{2,1,0}=\dfrac{3}{2}i, & d_{3,1,0}=-\dfrac{1}{3}i,\\ d_{0,2,0}=-1, & d_{1,2,0}=\dfrac{5}{2}, & d_{2,2,0}=-2, & d_{3,2,0}=\dfrac{1}{2},\\ d_{0,3,0}=-\dfrac{1}{6}i, & d_{1,3,0}=\dfrac{1}{2}i, & d_{2,3,0}=-\dfrac{1}{2}i, & d_{3,3,0}=\dfrac{1}{6}i. \end{array}$$

The fundamental polynomials $u_{\mu,0}$, $\mu=0,1,2,3$, read

$$u_{0,0}(t)=1, \qquad u_{1,0}(t)=\frac{11}{6}i-3ie^{it}+\frac{3}{2}ie^{i2t}-\frac{1}{3}ie^{i3t},$$
$$u_{2,0}(t)=-1+\frac{5}{2}e^{it}-2e^{i2t}+\frac{1}{2}e^{i3t}, \qquad u_{3,0}(t)=-\frac{1}{6}i+\frac{1}{2}ie^{i}t-\frac{1}{2}ie^{i2t}+\frac{1}{6}ie^{i3t},$$

and the polynomial $x_{3,0}$ reads

$$x_{3,0}(t)=x(t_0)+x'(t_0)\left(\frac{11}{6}i-3ie^{it}+\frac{3}{2}ie^{i2t}-\frac{1}{3}ie^{i3t}\right)-x''(t_0)(1-\frac{5}{2}e^{it}+2e^{i2t}-\frac{1}{2}e^{i3t})$$
$$-x'''(t_0)\left(\frac{1}{6}i-\frac{1}{2}e^{it}+\frac{1}{2}ie^{i2t}-\frac{1}{6}ie^{i3t}\right), \qquad t\in\mathbb{R}.$$

The given examples show that in contrast to the non–periodic polynomials of the usual Maclaurin series, in the periodic series of the Maclaurin type, the addition of one more term requires the recalculation of all previous terms.

We proceed to the case $m>0$, $n>0$. In [10], the author has already used the interpolation polynomials $x_{1,n}$, $n=0,1,\dots,$ of the first multiplicity as an approximation aggregate for the exact solving of singular integro–differential equations with the Hilbert kernel. Here, we give an example of a Hermite — Fejér type polynomial of order $m=2$. By Hermite — Fejér polynomials, we mean trigonometric polynomials with multiple nodes, the derivatives of which at the interpolation nodes may, unlike traditional Hermite — Fej'er polynomials, be non–zero.

We fix $n\in\mathbb{N}_0$ and $m=2$ and construct the polynomial $x_{2,n}$. Using the results of constructing such polynomials for the cases $m=0$, $n\geqslant 0$ and $n=0$, $m\geqslant 0$, one can immediately write out the explicit form of such polynomials:

$$x_{2,n}=\frac{1}{(2n+1)^3}\sum_{k=0}^{2n}\left(x(t_k)+x'(t_k)\left(\frac{3}{2}i-2ie^{i(t-t_k)}+\frac{1}{2}ie^{i2(t-t_k)}\right)\right.$$
$$\left.-x''(t_k)\left(\frac{1}{2}-e^{i(t-t_k)}+\frac{1}{2}e^{i2(t-t_k)}\right)\right)\left(\frac{1-e^{i(2n+1)(t-t_k)}}{(1-e^{i(t-t_k)})}\right)^3.$$

The results of this paper can be developed in different directions. First of all, it is natural to generalize the construction of multiple interpolation polynomials to the case of fractional order derivatives. For this purpose, we can use fractional derivatives defined by the author in [9]. The results of constructing trigonometric polynomials of multiple interpolation over unequal nodes is also of interest. It is important to establish the approximative properties of such polynomials in different functional spaces.

It was shown in [3] that in the case of multiple interpolation of periodic real–valued functions, the interpolation polynomials converge to the interpolated function in the space C^m for odd m and do not converge for even m. This property is likely preserved for the complex–valued functions considered in this paper. Finally, it is interesting to consider the problem of multiple interpolation over an even number of nodes. Due to the multiplicity of interpolation polynomials of this type for the same function, we expect interesting results in this case.

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