ISSN 2304-0122 Ufa Mathematical Journal. Vol. 17. No 3 (2025). P. 9-16.

doi:10.13108/2025-17-3-9

ON DEGENERATE SOLUTIONS OF SECOND ORDER ELLIPTIC EQUATIONS IN PLANE

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Abstract. In the work we study conditions, under which a solution to a second order partial differential equation in the unit disk on the plane degenerates. We prove that each degenerate solution is either a polynomial of degree at most 2 or a linear combination of a constant and the logarithm of a fractional–rational expression. In proof of the main result we use the Taylor series expansion of the degenerate solution of the equation at an arbitrary point and study the dependence of coefficients of resulting series on the coefficients at the lower powers of the same series.

Keywords: elliptic equation, degenerate function, Jacobian.

Mathematics Subject Classification: 35C99, 35J15

1. Introduction

Let

$$L = \overline{\partial}\partial_\beta, \tag{1.1}$$

where $\overline{\partial}$ is the Cauchy — Riemann operator and $\partial_\beta = \frac{\partial}{\partial x} + \beta i \frac{\partial}{\partial y}$, while $\beta \neq \pm 1$ is a non–zero real number. In the present work we study the conditions, under which a solution u to the equation

$$Lu = 0 \tag{1.2}$$

degenerates, that is, the set of its values in the complex plane has no internal points.

If $\beta = -1$, Equation (1.2) is the Laplace equation, and the functions satisfying this equation are harmonic. There exist rather many degenerate harmonic functions.

In particular, the real part of each complex–valued harmonic function is a real–valued harmonic function. The latter property is not extended to the case $\beta \neq -1$: in this case, a solution of Equation (1.2) can be real–valued in some domain only if it is a polynomial of degree at most 2 in the variables x and y.

If $\beta = 1$, then Equation (1.2) reads $\overline{\partial}^2 u = 0$, and the functions solving this equation are called bi–analytic. If a bi–analytic function degenerates, then it is either equal to $A(e^{i\alpha}z + \overline{z}) + A_1$, or to $A(z-c)^\gamma(\overline{z}-\overline{c}) + A_1$, where A, A_1, c, α, γ are constants $\operatorname{Im}\alpha = 0$, $|\gamma| = 1$ [1, Thm. 1.11].

It follows from [2, Thm. 5] that if $\beta > 0$, $\beta \neq 1$ and a solution $u(z)$ to Equation (1.2) is a polynomial of degree exceeding 2, then the function $u(z)$ can not degenerate.

2. Main results and proofs

We introduce the following notation. Let

$$z_\beta = x + \frac{i}{\beta}y = \frac{\beta+1}{2\beta}z + \frac{\beta-1}{2\beta}\overline{z}.$$

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Submitted August 22, 2024.

We define a linear mapping T_β of the plane $\mathbb{C}$ by the formula $T_\beta(z) = z_\beta$. Let a function $u(z)$ satisfy Equation (1.2) in the domain D. Then

$$u(z) = f(z) - f_\beta(z_\beta), \tag{2.1}$$

where the functions f and f_β are holomorphic in the domains D and $T_\beta(D)$, respectively. We denote by $I_u(z)$ the Jacobian of the mapping $u(z)$ at a point $z \in D$. It is clear that

$$I_u(z) = \left|\frac{\partial u}{\partial z}\right|^2 - \left|\frac{\partial u}{\partial \bar{z}}\right|^2.$$

If $I_u(z) = 0$ and the identity (2.1) holds, then

$$\left|\frac{f'_\beta(z_\beta)}{f'(z)} - \frac{1+\beta}{2}\right| = \left|\frac{1-\beta}{2}\right|; \tag{2.2}$$

see [2, Sect. 2], the corresponding arguing in the proof of Theorem 1 remains true also for $\beta > 0$, $\beta \neq 1$.

Now we are in position to formulate the main result of work.

Theorem 2.1. *Let a function $u(z)$ satisfy Equation (1.2) in some domain D, $\beta \in \mathbb{R} \setminus \{0; \pm 1\}$, and degenerate. Then either $u(z)$ is polynomial of degree at most 2 or*

$$u(z) = c_1 + c_2 \ln \frac{z - c}{z_\beta - T_\beta c},$$

where c, c_1, c_2 are some complex numbers.

To prove this theorem, we shall need the following lemmas.

Lemma 2.1. *In the vicinity of an arbitrary point $c \in \mathbb{C}$ there exists at most one degenerate solution to Equation (1.2) with prescribed coefficients at the monomials of degree up to 3 in the Taylor expansions of this solution about the point c under the condition that the gradient of this solution at the point c is non–zero.*

Lemma 2.2. *Let a function $u(z)$ satisfies the conditions*

1. *$u(z)$ is a solution to Equation (1.2) in the vicinity of the point 0;*
2. *$u(z) = f(z) - f_\beta(z_\beta)$, where*

$$f(z) = c_0 + \sum_{k=1}^{\infty} c_k z^k, \qquad f_\beta(z_\beta) = \sum_{k=1}^{\infty} c_{\beta,k} z_\beta^k, \qquad c_1 \neq 0, \qquad c_{\beta,1} \neq 0.$$

Then the coefficients at the polynomials of degree at most $n-1$ in the Taylor expansion of function $\ln \frac{f'_\beta(z_\beta)}{f'(z)}$ about the point 0 are uniquely determined by the coefficients c_k and $c_{\beta,k}$, $k = 1, 2, \dots, n$. And vice versa, the coefficients c_k, $k = 2, \dots, n$ and $c_{\beta,k}$, $k = 1, 2, \dots, n$, are uniquely determined by the coefficients at the monomials of degree at most $n-1$ in the Taylor expansion of the function $\ln \frac{f'_\beta(z_\beta)}{f'(z)}$ in the vicinity of the point 0 and by the coefficient c_1.

Lemma 2.3. *Let $u(z) = f(z) - f_\beta(z_\beta)$ be a degenerate solution to Equation (1.2) in the vicinity of point 0, and the gradient of function $u(z)$ at the point 0 is non–zero. Then there exists at most one function $\ln \frac{f'_\beta(z_\beta)}{f'(z)}$ with prescribed coefficients at monomials of degree at most 2 in the Taylor expansion of this function about the point 0.*

Proof of Lemma 2.2. In the vicinity of the point 0 we have $f'(z)=\sum\limits_{k=1}^{\infty}kc_kz^{k-1}$, therefore,

$$\begin{aligned}\ln f'(z) &= \ln c_1 + \ln\left(1+\sum_{k=2}^{\infty}k\frac{c_k}{c_1}z^{k-1}\right)\\ &= \ln c_1 + \sum_{l=1}^{\infty}\frac{(-1)^{l-1}}{l}\left(\sum_{k=1}^{\infty}(k+1)\frac{c_{k+1}}{c_1}z^k\right)^l = \ln c_1 + \sum_{l=1}^{\infty}d_lz^l,\end{aligned}$$

where

$$d_1=\frac{2c_2}{c_1},\qquad d_l=\frac{(l+1)c_{l+1}}{c_1}+\varepsilon_l,\qquad l=2,3,\dots,\tag{2.3}$$

the constants ε_l depends only on the coefficients $c_1,\dots,c_l$, $l=2,3,\dots$.

Using similar arguing, we obtain that

$$\ln f'_\beta(z_\beta)=\ln c_{\beta,1}+\sum_{l=1}^{\infty}d_{\beta,l}z_\beta^l,$$

where

$$d_{\beta,1}=\frac{2c_{\beta,2}}{c_{\beta,1}},\qquad d_{\beta,l}=\frac{(l+1)c_{\beta,l+1}}{c_{\beta,1}}+\varepsilon_{\beta,l},\qquad l=2,3,\dots,\tag{2.4}$$

the constants $\varepsilon_{\beta,l}$ depend only on the coefficients $c_{\beta,1},\dots,c_{\beta,l}$, $l=2,3,\dots$.

Thus, in the vicinity of the point 0 we have

$$\ln\frac{f'_\beta(z_\beta)}{f'(z)}=d_0-\left(\sum_{l=1}^{\infty}d_lz^l\right)+\sum_{l=1}^{\infty}d_{\beta,l}z_\beta^l,$$

where $d_0=\ln\frac{c_{\beta,1}}{c_1}$. Together with (2.3) and (2.4) this imply that the coefficients d_0, d_k and $d_{\beta,k}$, $k=1,2,\dots,n-1$, are determined uniquely by the coefficients c_k and $c_{\beta,k}$, $k=1,2,\dots,n$, and vice versa, the coefficients c_k, $k=2,\dots,n$, and $c_{\beta,k}$, $k=1,2,\dots,n$, are uniquely determined by the coefficients d_0, c_1, d_k and $d_{\beta,k}$, $k=1,2,\dots,n-1$. The proof is complete. □

Proof of Lemma 2.1. Let $u(z)$ be a degenerate solution of Equation (1.2) in the vicinity of the point c and the gradient of $u(z)$ at the point c do not vanish. Without loss of generality we suppose that $c=0$ and that in the vicinity of 0 the Taylor expansion of $u(z)$ reads

$$u(z)=a_0+x+\alpha y+\sum_{n=2}^{\infty}(a_nz^n+b_nz_\beta^n),$$

where α is some real number.

By means of elementary calculations we obtain that the sum of monomials of third degree in the Taylor series of function $I_u(z)$ in the vicinity of the point 0 is equal to

$$\begin{aligned}&4\left(\operatorname{Re}a_4+\frac{\operatorname{Re}b_4}{\beta}-\alpha(\operatorname{Im}a_4+\operatorname{Im}b_4)+l_{30}\right)x^3\\ &-12\left(\operatorname{Im}a_4+\frac{\operatorname{Im}b_4}{\beta^2}+\alpha\left(\operatorname{Re}a_4+\frac{\operatorname{Re}b_4}{\beta}\right)+l_{21}\right)x^2y\\ &-12\left(\operatorname{Re}a_4+\frac{\operatorname{Re}b_4}{\beta^3}-\alpha(\operatorname{Im}a_4+\frac{\operatorname{Im}b_4}{\beta^2})+l_{12}\right)xy^2\\ &+4\left(\operatorname{Im}a_4+\frac{\operatorname{Im}b_4}{\beta^4}+\alpha\left(\operatorname{Re}a_4+\frac{\operatorname{Re}b_4}{\beta^3}\right)+l_{03}\right)y^3,\end{aligned}$$

where the numbers l_{30}, l_{21}, l_{12}, l_{03} depend only on the coefficients α, a_2, a_3, b_2, b_3.

If $u(z)$ is a degenerate function, then $I_u(z) = 0$ in the vicinity of the point 0. We thus have

$$\begin{cases} \operatorname{Re} a_4 + \dfrac{\operatorname{Re} b_4}{\beta} - \alpha\left(\operatorname{Im} a_4 + \operatorname{Im} b_4\right) + l_{30} = 0, \\ \alpha\left(\operatorname{Re} a_4 + \dfrac{\operatorname{Re} b_4}{\beta}\right) + \operatorname{Im} a_4 + \dfrac{\operatorname{Im} b_4}{\beta^2} + l_{21} = 0, \\ \operatorname{Re} a_4 + \dfrac{\operatorname{Re} b_4}{\beta^3} - \alpha\left(\operatorname{Im} a_4 + \dfrac{\operatorname{Im} b_4}{\beta^2}\right) + l_{12} = 0, \\ \alpha\left(\operatorname{Re} a_4 + \dfrac{\operatorname{Re} b_4}{\beta^3}\right) + \operatorname{Im} a_4 + \dfrac{\operatorname{Im} b_4}{\beta^4} + l_{03} = 0. \end{cases}$$

We have obtained a system of linear equations with the unknowns $\operatorname{Re} a_4$, $\operatorname{Re} b_4$, $\operatorname{Im} a_4$, $\operatorname{Im} b_4$. Its determinant is equal to

$$\begin{vmatrix} 1 & \frac{1}{\beta} & -\alpha & -\alpha \\ \alpha & \frac{\alpha}{\beta} & 1 & \frac{1}{\beta^2} \\ 1 & \frac{1}{\beta^3} & -\alpha & \frac{-\alpha}{\beta^2} \\ \alpha & \frac{\alpha}{\beta^3} & 1 & \frac{1}{\beta^4} \end{vmatrix} = \left(\frac{1}{\beta^3} - \frac{1}{\beta}\right)(1+\alpha^2)\left(\frac{1}{\beta^2} + \alpha^2 - \frac{1}{\beta^4} - \frac{\alpha^2}{\beta^2}\right)$$

$$= \left(\frac{1}{\beta^3} - \frac{1}{\beta}\right)(1+\alpha^2)\left(\frac{1}{\beta^2} + \alpha^2)(1 - \frac{1}{\beta^2}\right) \neq 0$$

since $\beta \neq \pm 1$. This is why the system is uniquely solvable with respect to $\operatorname{Re} a_4$, $\operatorname{Re} b_4$, $\operatorname{Im} a_4$, $\operatorname{Im} b_4$, that is, the coefficients a_4 and b_4 are uniquely determined for the given coefficients α, a_2, a_3, b_2, b_3.

We proceed by induction. Suppose that all coefficients a_i and b_i, $i = 4, \dots, n-1$, $n > 4$, are uniquely determined by the coefficients α, a_1, a_2, a_3, b_1, b_2, b_3. We denote

$$f(z) = a_0 + \frac{1+\alpha\beta i}{1-\beta} z + \sum_{k=2}^{\infty} a_k z^k, \qquad f_\beta(z_\beta) = \frac{\beta(1+\alpha i)}{1-\beta} z_\beta - \sum_{k=2}^{\infty} b_k z_\beta^k.$$

Then $u(z) = f(z) - f_\beta(z_\beta)$. Since $I_u(z) = 0$ in the vicinity of the point 0, we have

$$\left|\frac{f'_\beta(z_\beta)}{f'(z)} - \frac{1+\beta}{2}\right| = \left|\frac{1-\beta}{2}\right|$$

for all neighbourhoods of the point 0. Then the function $\ln \frac{f'_\beta(z_\beta)}{f'(z)}$, defined and satisfying Equation (1.2) in the vicinity of the point 0, also degenerates. By Lemma 2.2 all terms of the Taylor series of function $\ln \frac{f'_\beta(z_\beta)}{f'(z)}$ in the vicinity of the point 0 are determined up to the third power. Since the function $\ln \frac{f'_\beta(z_\beta)}{f'(z)}$ is also a degenerate solution to Equation (1.2), by the induction assumption we uniquely determine the coefficients at the monomials of degree at most $n-1$ in the Taylor expansion of this function. Applying once again Lemma 2.2, we uniquely determine the coefficients a_n and b_n. Thus, $u(z)$ is uniquely determined by the coefficients a_0, α, a_2, a_3, b_2, b_3. The proof is complete. □

Proof of Lemma 2.3. Suppose that we are given the coefficients at the monomials of degree at most 2 in the Taylor expansion of the function $\ln \frac{f'_\beta(z_\beta)}{f'(z)}$ about the point 0. It follows from Lemma 2.2 that in this case the coefficients at the monomials of degree at most 3 in the Taylor expansion of function $u(z)$ about the point 0 are determined uniquely if $u(0) = 0$, $f'(0) = 1$. By Lemma 2.1 this yields the uniqueness of function $u(z)$, and hence, of $\ln \frac{f'_\beta(z_\beta)}{f'(z)}$. The proof is complete. □

Proof of Theorem 2.1. Without loss of generality we suppose that

$$u(z) = f(z) - f_\beta(z_\beta)$$

is a degenerate solution of Equation (1.2) in the vicinity of the point 0, and the gradient of the function $u(z)$ at the point 0 is non–zero. It has been established in the proof Lemma 2.1 that in this case the function $\ln \frac{f'_\beta(z_\beta)}{f'(z)}$ is also a degenerate solution of this equation in the vicinity of the point 0. We denote $g(z) = \ln f'(z)$, $g_\beta(z_\beta) = \ln f'(z)$. Then

$$\ln \frac{f'_\beta(z_\beta)}{f'(z)} = g_\beta(z_\beta) - g(z).$$

If the function $\ln \frac{f'_\beta(z_\beta)}{f'(z)}$ is constant, then the functions $f'(z)$ and $f'_\beta(z_\beta)$ are also constant and the function $u(z)$ is linear.

If the function $\ln \frac{f'_\beta(z_\beta)}{f'(z)}$ is non–constant, without loss of generality we can suppose that $g'(0) \neq 0$, $g'_\beta(0) \neq 0$. Applying Lemma 2.3 and arguing as in the proof of this lemma, we obtain that the function $\ln \frac{g'_\beta(z_\beta)}{g'(z)}$ is determined uniquely by the linear part of its Taylor expansion about the point 0.

Let

$$\ln \frac{g'_\beta(z_\beta)}{g'(z)} = \sigma_0 + \sigma_1 z - \sigma_2 z_\beta + \ldots$$

in the vicinity of the point 0. If the function $\ln \frac{g'_\beta(z_\beta)}{g'(z)}$ is constant, then the functions $g'(z)$ and $g'_\beta(z_\beta)$ are also constants and therefore, the function $\ln \frac{f'_\beta(z_\beta)}{f'(z)}$ is also linear. Since this function degenerates, the set of its values lies on some straight line. Then the set of values of the function $\frac{f'_\beta(z_\beta)}{f'(z)}$ lies either on a straight line or on a circle centered at the point 0 or on a spiral. At the same time, the set of values of this function lies on a circle centered at the point $\frac{1+\beta}{2} \neq 0$. Thus, the function $\frac{f'_\beta(z_\beta)}{f'(z)}$ can take only a discrete set of values and hence, it is constant. Then the functions $f'(z)$ and $f'_\beta(z_\beta)$ are also constant and therefore, the function $u(z)$ is linear.

If the function $\ln \frac{g'_\beta(z_\beta)}{g'(z)}$ is non–constant, without loss of generality we suppose that $\sigma_1 \neq 0, \sigma_2 \neq 0$. Since the function $\ln \frac{f'_\beta(z_\beta)}{f'(z)}$ degenerates, the function $\ln \frac{g'_\beta(z_\beta)}{g'(z)}$ does, too. But the Jacobian of the latter function vanishes identically in the vicinity of the point 0 and hence,

$$\left| \frac{\sigma_2}{\sigma_1} - \frac{1+\beta}{2} \right| = \left| \frac{1-\beta}{2} \right|.$$

This is why $\sigma_1 = \frac{\gamma}{c}, \sigma_2 = \frac{\gamma}{T_\beta c}$, where γ and c are some complex constants. Since the function

$$\ln \frac{z-c}{z_\beta - T_\beta c} = \ln \frac{c}{T_\beta c} - \frac{z}{c} + \frac{z_\beta}{T_\beta c} + \ldots$$

degenerates, we can suppose that

$$\ln \frac{g'_\beta(z_\beta)}{g'(z)} = \gamma_0 + \gamma_1 \ln \frac{z-c}{z_\beta - T_\beta c},$$

where γ_0 and γ_1 are some complex constants, and we can prove that for some γ_0 and γ_1 there exist corresponding functions f and f_β, and hence, a function $u(z)$ being a degenerate solution to Equation (1.2). Since the parameters c, γ_0 and γ_1 determine completely the linear part of Taylor expansion of the function $\ln \frac{g'_\beta(z_\beta)}{g'(z)}$, the above proven facts imply that in this case there are no other degenerate solutions of this equation in the vicinity of the point 0.

We have
$$\frac{g'_\beta(z_\beta)}{g'(z)} = e^{\gamma_0}\left(\frac{z-c}{z_\beta - T_\beta c}\right)^{\gamma_1}.$$
The left hand side of the latter identity satisfies the condition
$$\left|\frac{g'_\beta(z_\beta)}{g'(z)} - \frac{1+\beta}{2}\right| = \left|\frac{1-\beta}{2}\right|.$$
At the same time,
$$\left|\frac{z-c}{z_\beta - T_\beta c} - \frac{1+\beta}{2}\right| = \left|\frac{1-\beta}{2}\right|.$$
Since $\frac{1+\beta}{2} \neq 0$, the set of points $\left(\frac{z-c}{z_\beta - T_\beta c}\right)^{\gamma_1}$ lies on some circumference if and only if $\gamma_1 = \pm 1$.

Thus, either
$$\frac{g'_\beta(z_\beta)}{g'(z)} = \beta\frac{z_\beta - T_\beta c}{z-c} \qquad (\gamma_1 = -1, \gamma_0 = \ln\beta),$$
or
$$\frac{g'_\beta(z_\beta)}{g'(z)} = \frac{z-c}{z_\beta - T_\beta c} \qquad (\gamma_1 = 1, \gamma_0 = 0).$$
In the first case we have
$$g'(z) = \gamma_2(z-c), \qquad g'_\beta(z_\beta) = \beta\gamma_2(z_\beta - T_\beta c),$$
where γ_2 is some non–zero complex constant. Therefore,
$$\ln\frac{f'_\beta(z_\beta)}{f'(z)} = g_\beta(z_\beta) - g(z) = \gamma_3 + \frac{\gamma_2}{2}(\beta(z_\beta - T_\beta c)^2 - (z-c)^2),$$
where γ_3 is some non–zero complex constant. Since the polynomial
$$\beta(z_\beta - T_\beta c)^2 - (z-c)^2$$
takes only real values, the set of values of function $\ln\frac{f'_\beta(z_\beta)}{f'(z)}$ lies on some straight line. But, as it has been shown above, in this case the function $u(z)$ is linear.

If
$$\frac{g'_\beta(z_\beta)}{g'(z)} = \frac{z-c}{z_\beta - T_\beta c},$$
then
$$g'(z) = \frac{\gamma_4}{z-c}, \qquad g'_\beta(z_\beta) = \frac{\gamma_4}{z_\beta - T_\beta c},$$
where γ_4 is some non–zero complex constant. Therefore,
$$\ln\frac{f'_\beta(z_\beta)}{f'(z)} = g_\beta(z_\beta) - g(z) = \gamma_5 + \gamma_4\ln\frac{z_\beta - T_\beta c}{z-c},$$
where γ_5 is some non–zero complex constant. Reproducing the above arguing with $g(z)$ replaced by $f(z)$ and $g_\beta(z)$ replaced by $f_\beta(z)$, we obtain that either
$$u(z) = \gamma_6 + \gamma_7((z-c)^2 - \beta(z_\beta - T_\beta c)^2),$$
where γ_6, γ_7 are some complex constants, and then $u(z)$ is a polynomial of the second degree, or
$$u(z) = c_1 + c_2\ln\frac{z-c}{z_\beta - T_\beta c},$$
where c_1, c_2 are some complex constants. The proof is complete. □

Corollary 2.1. *Let a function $u(z)$ satisfies Equation (1.2) in some domain D, $\beta \in \mathbb{R} \setminus \{0; \pm 1\}$, and degenerates. Then the set of values of function $u(z)$ is either constant, or a straight line, or, in the case $\beta < 0$, the curve*

$$z = c_1 + c_2 \left(\ln \left(\frac{1+\beta}{2} \cos\varphi + \sqrt{\left(\frac{1-\beta}{2}\right)^2 - \left(\frac{1+\beta}{2}\right)^2 \sin^2\varphi} \right) + i\varphi \right), \qquad 0 < \varphi < 4\pi,$$

while in the case $\beta > 0$ it is the curve

$$z = c_1 + c_2 \left(\ln \left(\frac{1+\beta}{2} \cos\varphi \pm \sqrt{\left(\frac{1-\beta}{2}\right)^2 - \left(\frac{1+\beta}{2}\right)^2 \sin^2\varphi} \right) + i\varphi \right),$$

$$\pi - \arcsin \left| \frac{1-\beta}{1+\beta} \right| \leqslant \varphi \leqslant \pi + \arcsin \left| \frac{1-\beta}{1+\beta} \right|,$$

where c, c_1, c_2 are some complex constants.

Proof. If $u(z)$ is a degenerate polynomial of the first degree, then it reads

$$u(z) = c(\alpha_1 x + \alpha_2 y), \quad c \in \mathbb{C}, \quad \alpha_1, \alpha_2 \in \mathbb{R},$$

and hence, the set of its values is a straight line.

If $u(z)$ is a polynomial of the second degree, then by means of the parallel translation it is reduced to the form

$$u(z) = \gamma_1 + \gamma_2(z^2 + b z_\beta^2),$$

where γ_1 and γ_2 are some complex constants. At the same time, if $u(z)$ is a degenerate polynomial, then $I_u(z) = 0$ for all $z \in \mathbb{C}$. In this case the identity (2.2) becomes

$$\left| b\frac{z_\beta}{z} + \frac{1+\beta}{2} \right| = \left| \frac{1-\beta}{2} \right|$$

for all $z \in \mathbb{C}$.

Since

$$\left| \frac{z_\beta}{z} - \frac{1+\beta}{2\beta} \right| = \left| \frac{1-\beta}{2\beta} \right|$$

for all $z \in \mathbb{C}$, we obtain $b = -\beta$, and hence,

$$u(z) = \gamma_1 + \gamma_2(z^2 - \beta z_\beta^2) = \gamma_1 + \gamma_2 \left((1-\beta)x^2 + \left(\frac{1}{\beta} - 1 \right) y^2 \right).$$

Thus, the set of values of $u(z)$ is a straight line.

Now let

$$u(z) = c_1 + c_2 \ln \frac{z - c}{z_\beta - T_\beta c},$$

where c_1, c_2 are some complex constant. The set of values of function $\frac{z-c}{z_\beta - T_\beta c}$ is a circumference of radius $\left|\frac{1-\beta}{2}\right|$ centered at the point $\frac{1+\beta}{2}$. For $\beta < 0$, in the polar coordinates, this circumference is expressed by the equation

$$r = \frac{1+\beta}{2} \cos\varphi + \sqrt{\left(\frac{1-\beta}{2}\right)^2 - \left(\frac{1+\beta}{2}\right)^2 \sin^2\varphi}, \qquad 0 \leqslant \varphi \leqslant 2\pi,$$

the function $u(z)$ is defined and satisfies Equation (1.2) in the domain $\mathbb{C} \setminus \Gamma$, where

$$\Gamma = \{y = \operatorname{Im} c, \ x \geqslant 0\}.$$

The increment of the argument of expression $\frac{z-c}{z_\beta - T_\beta c}$ under the total passage in the positive direction of the circumference centered at the point c is equal to 4π, and hence, the set of values of this function is the curve

$$z = c_1 + c_2 \left(\ln \left(\frac{1+\beta}{2} \cos\varphi + \sqrt{\left(\frac{1-\beta}{2}\right)^2 - \left(\frac{1+\beta}{2}\right)^2 \sin^2\varphi} \right) + i\varphi \right), \qquad 0 < \varphi < 4\pi.$$

Similarly, for $\beta > 0$, in the polar coordinates this circumference is expressed by the equation

$$r = \frac{1+\beta}{2} \cos\varphi \pm \sqrt{\left(\frac{1-\beta}{2}\right)^2 - \left(\frac{1+\beta}{2}\right)^2 \sin^2\varphi},$$
$$\pi - \arcsin\left|\frac{1-\beta}{1+\beta}\right| \leqslant \varphi \leqslant \pi + \arcsin\left|\frac{1-\beta}{1+\beta}\right|,$$

the function $u(z)$ is defined and satisfies Equation (1.2) in the domain $\mathbb{C} \setminus \{c\}$, and hence, the set of values of this function is the curve

$$z = c_1 + c_2 \left(\ln \left(\frac{1+\beta}{2} \cos\varphi \pm \sqrt{\left(\frac{1-\beta}{2}\right)^2 - \left(\frac{1+\beta}{2}\right)^2 \sin^2\varphi} \right) + i\varphi \right),$$
$$\pi - \arcsin\left|\frac{1-\beta}{1+\beta}\right| \leqslant \varphi \leqslant \pi + \arcsin\left|\frac{1-\beta}{1+\beta}\right|.$$

The proof is complete. □

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