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ON HOMOCLINIC POINTS AND TOPOLOGICAL ENTROPY OF CONTINUOUS MAPS ON ONE–DIMENSIONAL RAMIFIED CONTINUA

E.N. MAKHROVA

Abstract. Let X be a dendroid, $f : X \to X$ be a continuous map, p be a periodic point of f and let x be a homoclinic point in X to the periodic point p. We study the properties of the homoclinic point x and the unstable manifold of the point p. We investigate the local structure of X under which the existence of a homoclinic point implies the positive topological entropy of f. We also present differences in the properties of homoclinic points and the unstable manifolds of periodic points for continuous maps defined on dendroids, dendrites and finite trees.

Keywords: dendroid, dendrite, finite tree, continuous map, unstable manifold, homoclinic point, topological entropy.

Mathematics Subject Classification: 37B40, 37B45, 37E25, 54F50

1. Introduction

By *continuum* we mean a non–empty compact connected metric space. Let X be a continuum, p be an arbitrary point from X. If the connected components of boundaries of an arbitrary neighbourhood $U(p)$ of the point p in X are singleton sets, then X is called the *one–dimensional continuum*, see, for example, [3, Ch. 2, Sect. 25, I].

At present, there is an increasing interest in dynamical systems on one–dimensional ramified continua with a complex topological structure. This is due to the fact that these continua appear, for example, as Julia sets in complicated dynamical systems [24], as limit sets of dynamical systems with phase spaces of dimension not less than two [9], [10], as global attractors of skew products and integrable mappings [15], [16], in problems of mathematical physics [7], [13], etc.

In this paper we study homoclinic points, first discovered by H. Poincaré in problems of celestial mechanics [6, Ch. XXXIII], for continuous mappings on one–dimensional ramified continua such as dendroids; dendrites and finite trees are their special cases, see Definition 2.2. The existence of homoclinic points for continuous mappings of a closed interval or finite trees is equivalent to the positivity of topological entropy [2], [11]. At the same time, the presence of homoclinic points in continuous mappings of dendrites, which are not finite trees, does not imply that the topological entropy is positive or vanishes [2], [5], [14], [19], [21]. Moreover, the complexity of the structure of one–dimensional ramified continua leads to the fact that even homeomorphisms and monotone mappings defined on them exhibit properties not specific for continuous mappings of a closed interval, see, for example, [21], [22]. This is why, an effective approach to studying the dynamics of mappings on these continua is one that establishes such

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features of their topological structure that the mappings under consideration exhibit properties similar to the properties of continuous mappings of closed intervals or finite trees. A similar approach was first implemented for continuous mappings of dendrites in [12], as well as in [21], [23].

Since the dendroids have a more complicated topological structure than dendrites, in this paper we study conditions on the local structure of dendroids under which the existence of a homoclinic point implies the positivity of topological entropy of mappings defined on them. In addition, we study the properties of unstable manifold of a periodic point, which also depend on the structure of the dendroid. We note that the properties of unstable manifold of continuous mappings on finite trees were studied in [17].

2. Preliminaries and main results

We denote by $\mathbb{N}$ the set of natural numbers, by $\mathbb{C}$ the set of complex numbers and by $\mathbf{i}$ the imaginary unit.

Let X be a one–dimensional continuum with metric d, and A be a subset of X. We denote by $\operatorname{diam} A$ the diameter of set A, by $\operatorname{card} A$ the cardinality of set A; by $\partial(A)$ the boundary of set A. By $A^{(1)}$ we denote the derived set, which is the set of all limit points of set A.

Following [2], a connected subset of the continuum X whose closure is homeomorphic to the segment $[0;1]$ on the real line $\mathbb{R}^1$ is called the *arc*.

The symbol $[x;y]$ denotes the arc with endpoints at the points x and y, which contains these points; we let $(x;y] = [x;y] \setminus \{x\}$, $[x;y) = [x;y] \setminus \{y\}$, $(x;y) = [x;y]\setminus\{x;y\}$.

We shall employ the definition of the order of a point in the sense of Menger — Urysohn [4, Ch. 6, Sect. 51].

Definition 2.1 ([4, Ch. 6, Sect. 51]). *Let X be a one-dimensional continuum, z be a point in X.*

1) *We say that the order of z is finite if there exists $\mathfrak{n} \in \mathbb{N}$ such that for each neighborhood $U(z)$ of z in X there exists a neighborhood $U_1(z) \subset U(z)$ such that* $\operatorname{card} \partial(U_1(z)) = \mathfrak{n}$, *and there is no subneighborhood $U_2(z) \subset U_1(z)$ such that* $\operatorname{card} \partial(U_2(z)) < \mathfrak{n}$[1]. *In this case, we say that the order of z is equal to $\mathfrak{n}$ (*$\operatorname{ord} z = \mathfrak{n}$*).*
2) *The order of a point z is called infinite if for each number $\mathfrak{n} \in \mathbb{N}$ there exists a neighborhood $U(z)$ of the point z in X such that for each neighborhood $U_1(z) \subset U(z)$ we have* $\operatorname{card} \partial(U_1(z)) > \mathfrak{n}$.

Points of finite order exceeding 2 and points of infinite order are called *ramification points* of the continuum X. Points of order 1 are called *endpoints* of the continuum X. By $R(X)$ ($E(X)$) we denote the set of ramification points (endpoints) of the continuum X.

A continuum X is called *unicoherent* if for each subcontinua A and B in X satisfying the condition $A \cup B = X$, the intersection $A \cap B$ is connected.

We note that each segment on the real line $\mathbb{R}^1$ is unicoherent, and a circle is not a unicoherent set.

A continuum X is *hereditarily unicoherent* if each subcontinuum Y in X is unicoherent.

A continuum is called *arcwise connected* if any pair of its points can be connected by an arc.

Definition 2.2. *A continuum X is called a dendroid if X is arcwise connected and hereditarily unicoherent.*

A locally connected dendroid is called a dendrite.

A dendrite with a finite set of endpoints is called a finite tree.

We mention the following properties of dendroids.

[1]Concerning the order relation on the set of cardinal numbers, see, for example, [1, Ch. 3]

Lemma 2.1 ([20]). *Let X be a dendroid. Then*

1) *X is a one–dimensional continuum;*
2) *X contains no subsets homeomorphic to a circle;*
3) *each two distinct points x, y in X can be connected by a single arc $[x; y]$;*
4) *each subcontinuum of a dendroid is a dendroid.*

Let p be an arbitrary point in a dendroid X. Then $X \setminus \{p\}$ consists of one or more connected components, which are called *components of* p.

It follows from the definition 2.1 that if the order of a point p in a dendroid X is finite, then the number of components of p is finite. The converse is not true for dendroids that are not dendrites (see, e.g., [20]). For instance, in a dendroid

$$X = [0; \mathbf{i}] \cup [0; 1] \cup \bigcup_{n=0}^{\infty} \left[\frac{1}{2^n}; \frac{1}{2^n} + \mathbf{i}\right] [1],$$

which is not a dendrite, each point p from $[0; \mathbf{i}]$ is by Definition 2.1 a ramification point of infinite order, but the point p has either one component (if $x = \mathbf{i}$) or two components (if $p \in [0; \mathbf{i})$). But if a dendroid X is a dendrite, and the number of components of a point p in X is finite, then it coincides with the order of the point p, see [4, Ch.6, Sect. 51, VI].

Let $f : X \to X$ be a continuous mapping of X. A point p in X is called a periodic point of f if there exists a natural number $m \geqslant 1$ such that $f^m(p) = p$; the smallest m satisfying this condition is called the period of periodic point p. If $m = 1$, then p is called a fixed point of f.

The set of periodic (fixed) points of f is denoted by $\mathrm{Per}(f)$ $(\mathrm{Fix}(f))$.

Definition 2.3 ([11]). *An unstable manifold of a periodic point p of period m of a continuous mapping $f : X \to X$ is a set of points $W^u(p, f^m)$ of a dendroid X such that for each point $z \in W^u(p, f^m)$ and an arbitrary neighborhood $U(p)$ of the point p in X, which does not contain the point z, there exists a natural number $i \geqslant 1$ such that $z \in f^{im}(U(p))$.*

Definition 2.4 ([11]). *A stable manifold of a periodic point p of period m of a continuous mapping $f : X \to X$ is a set of points $W^s(p, f^m)$ of a dendroid X such that $\omega(z, f^m) = \{p\}$ for each point $z \in W^s(p, f^m)$, where $\omega(z, f^m)$ is the ω–limit set of trajectory of point z with respect to the mapping f^m.*

Definition 2.5 ([6]). *A point $z \in X$ is called a homoclinic point of the mapping $f : X \to X$ if there exists a periodic point $p \in X$ of period m such that $p \neq z$, and $z \in W^u(p, f^m) \bigcap W^s(p, f^m)$.*

We recall the definition of topological entropy introduced first in [8].

Let X be a compact topological space, $f : X \to X$ be a continuous mapping, and U be an open cover of X. Since X is compact, there exists a finite subcover of X. Denote by $N(U)$ the cardinality of the smallest subcover extracted from U. For any two covers U, V of X, we let

$$U \vee V = \{A \cap B, \quad \text{where} \quad A \in U,\, B \in V\}.$$

Definition 2.6 ([8]). *The topological entropy $h(f, U)$ of a mapping f with respect to a cover U is*

$$\lim_{n\to\infty} \frac{\log N(U \vee f^{-1}(U) \vee \ldots \vee f^{-n+1}(U))}{n},$$

where $f^{-j}(\cdot)$ stands for the jth complete preimage of a set $(\cdot)$.

The topological entropy $h(f)$ of mapping f is $\sup\limits_U h(f, U)$.

[1]The notation of a segment coincides with the notation of an arc since each segment on a plane is an arc according to the previously introduced definition of the arc.

We proceed to the main results of the work. The first theorem is a generalization of the known properties of unstable manifold of a periodic point of continuous mappings defined on a closed interval and a finite tree [11], [17].

Theorem 2.1. *Let $f : X \to X$ be a continuous mapping of a dendroid X, $p \in \mathrm{Per}(f)$, m be a period of the point p. Then*

1) *$W^u(p, f^m)$ is invariant under f^m, that is, $f^m(W^u(p, f^m)) \subseteq W^u(p, f^m)$;*
2) *if in addition $\operatorname{ord} p$ is finite and $p \notin R^{(1)}(X)$, then $W^u(p, f^m)$ is arcwise connected.*

The next statement was proved in [17].

Lemma 2.2 ([17]). *Let $f : X \to X$ be a continuous mapping of a finite tree X, point $p \in \mathrm{Fix}(f)$. If $\overline{W^u(p, f)} \setminus W^u(p, f) \neq \emptyset$, then $\overline{W^u(p, f)} \setminus W^u(p, f) \subset \mathrm{Per}(f)$.*

In this paper we show that Lemma 2.2 fails for continuous mappings defined on a dendroid, see Example 3.2.

In the next theorem we obtain local conditions on the structure of a dendroid, which ensure an analogue of the assertion proved for continuous mappings of a closed interval [11].

Theorem 2.2. *Let $f : X \to X$ be a continuous mapping of a dendroid X and let a fixed point $p \in X$ satisfy the following conditions:*

1) *$p \notin R^{(1)}(X)$;*
2) *$\operatorname{ord} p$ is finite.*

Then for each point $x \in W^u(p, f)$, $x \neq p$, and for each neighborhood $U(p)$ of the point p in X there exist a point $y \in U(p) \cap W^u(p, f)$ and a natural number $n \geqslant 1$ such that $f^n(y) = x$.

In the paper we also show that the violation of at least one of Conditions 1) or 2) of Theorem 2.2 destroys the statement of the theorem, see Examples 4.1 and 4.2.

Since each point of a finite tree satisfies Conditions 1), 2) of Theorem 2.2, we obtain the following statement.

Corollary 2.1. *Let $f : X \to X$ be a continuous mapping of a finite tree X, $p \in \mathrm{Fix}(f)$. Then for each point $x \in W^u(p, f)$, $x \neq p$, and for each neighborhood $U(p)$ of p in X there exist a point $y \in U(p) \cap W^u(p, f)$ and a natural number n such that $f^n(y) = x$.*

Since each homoclinic point to a periodic point p of the mapping f belongs to the unstable manifold of the point p, by Theorem 2.2 we obtain the following statement.

Corollary 2.2. *Let $f : X \to X$ be a continuous mapping of a dendroid X, f have a homoclinic point $x \in X$ to a periodic point $p \in X$ of period m, and the following conditions hold:*

1) *$p \notin R^{(1)}(X)$;*
2) *$\operatorname{ord} p$ is finite.*

Then for each neighborhood $U(p)$ of the point p there exist a homoclinic point $y \in U(p)$ to the periodic point p and a natural number $n \geqslant 1$ such that $f^{mn}(y) = x$.

Thus, if the assumptions of Corollary 2.2 are satisfied, in an arbitrary neighborhood of a periodic point p there exist an infinite number of homoclinic points of the mapping f, which is not true if at least one of the conditions 1) – 2) of this corollary is violated, see Examples 4.1 and 4.2 below.

In the next theorem we obtain local conditions on the structure of a dendroid, under which the existence of a homoclinic point of the mapping f implies the positivity of the topological entropy.

Theorem 2.3. *Let $f : X \to X$ be a continuous mapping of a dendroid X, and there exists a homoclinic point $x \in X$ to a periodic point $p \in X$, where the point p satisfies the following conditions:*
1) $p \notin R^{(1)}(X)$;
2) $\operatorname{ord} p$ *is finite.*

Then the topological entropy of the mapping f is positive.

3. Proof of Theorem 2.1

To prove the theorem 2.1 we need the concept of an n–od, where $n \geqslant 3$. The set of points in the complex plane, the n–th power of which belongs to the segment $[0; 1]$ is called the n–od. We note that the n–od has a single ramification point 0.

Lemma 3.1. *Let $f : X \to X$ be a continuous mapping of the dendroid X, $p \in \operatorname{Fix}(f)$. Then $f(W^u(p, f)) \subseteq W^u(p, f)$.*

Proof. Let $x \in W^u(p, f)$ be a point. By Definition 2.3, for each neighborhood $U(p)$ of p there exists a natural number $n \geqslant 1$ such that $x \in f^n(U(p))$. Then $f(x) \in f^{n+1}(U(p))$, that is, $f(x) \in W^u(p, f)$. The proof is complete. □

Lemma 3.2. *Let X be a dendroid and a point p in X satisfy the conditions*
1) $p \notin R^{(1)}(X)$;
2) $\operatorname{ord} p = n$, *where* $n \in \mathbb{N}$.

Then for each neighborhood $U(p)$ of a point p in X there exists a neighborhood $U_1(p) \subset U(p)$ such that $\overline{U_1(p)}$ is either an arc (for $n = 1$ or 2) or is homeomorphic to the n–od (for $n \geqslant 3$).

Proof. Let $U(p)$ be an arbitrary neighborhood of p in X. Since the order of p is finite, it follows from Definition 2.1 that the number of components of p is n. We denote by $X_i(p)$ the components of p, where $1 \leqslant i \leqslant n$. Since $p \notin R^{(1)}(X)$, for every $1 \leqslant i \leqslant n$ there exists a point $\alpha_i \in X_i(p) \cap U(p)$ such that
$$(p; \alpha_i] \cap R(X) = \emptyset.$$
We let $Y = \bigcup_{i=1}^{n} [p; \alpha_i)$. Then $\overline{Y}$ is either an arc (for $n = 1$ or 2) or is homeomorphic to the n–od (for $n \geqslant 3$). If $Y \subseteq U(p)$, then we let $U_1(p) = Y$, and this case we arrive at the desired statement.

We consider the case $Y \nsubseteq U(p)$. Since $\overline{Y}$ is an arc or is homeomorphic to the n–od, then $\overline{Y}$ is a locally connected continuum. Therefore, there exists a connected neighborhood $U_1(p) \subset Y \cap U(p)$. Then $\overline{U_1(p)}$ is either an arc (for $n = 1$ or 2) or is homeomorphic to the n–od (for $n \geqslant 3$). The proof is complete. □

Lemma 3.3. *Let $f : X \to X$ be a continuous mapping of a dendroid X, $p \in \operatorname{Fix}(f)$, $\operatorname{ord} p$ be finite, and $p \notin R^{(1)}(X)$. Then $W^u(p, f)$ is arcwise connected.*

Proof. Since $p \in W^u(p, f)$, it is sufficient to show that for each point x in $W^u(p, f) \setminus \{p\}$ there exists an arc $[p; x]$ such that $[p; x] \subseteq W^u(p, f)$.

Since X is a dendroid, by Property 3) of Lemma 2.1 there exists a unique arc $[p; x] \subset X$. Let us show that each point z in $(p; x)$ belongs to $W^u(p, f)$.

Let $U(p)$ be an arbitrary neighborhood of p in X that does not contain z. Since $\operatorname{ord} p$ is finite and $p \notin R^{(1)}(X)$, by Lemma 3.2 there exists a neighborhood $U_1(p) \subset U(p)$ such that $\overline{U_1(p)}$ is an arc or homeomorphic to the n-od.

Since $x \in W^u(p, f)$, by Definition 2.3 for a given neighborhood $U_1(p)$ of p there exists a natural number $k \geqslant 1$ such that $x \in f^k(U_1(p))$. Then there exists a point $y \in U_1(p)$ such that

$f^k(y) = x$. Since $y \in U_1(p)$, by the choice of the neighborhood $U_1(p)$ we have $[p; y] \subset U_1(p)$. On the other hand, it follows from the continuity of f that the set $f^k([p; y])$ is connected and contains the points p and x. Therefore, $[p; x] \subseteq f^k([p; y])$. Since $z \in (p; x)$, we obtain

$$z \in f^k([p; y]) \subset f^k(U_1(p)).$$

Since $U_1(p) \subset U(p)$, and $U(p)$ is an arbitrary neighborhood of the point p, we have $z \in W^u(p, f)$. The proof is complete. □

Theorem 2.1 is implied by Lemmas 3.1 and 3.3. If $\operatorname{ord} p$ is not finite, then Lemma 3.3 fails. The idea of the next example belongs to the referee.

Example 3.1. *Let* $I_n = \left[\frac{1}{2^n}; \frac{1}{2^n} + \mathbf{i}\right]$, *where* $n \geqslant 0$. *On the dendroid*

$$X = [0; \mathbf{i}] \cup [0; 1] \cup \bigcup_{n=0}^{\infty} I_n$$

we define the mapping $f : X \to X$ *as follows*

1) $f(x) = x$ *if* $x \in [0; \mathbf{i}] \cup I_0$;
2) $f(x) = 2x$ *if* $x \in \left[0; \frac{1}{2}\right]$;
3) $f(x) = 1$ *if* $x \in \left[\frac{1}{2}; 1\right]$;
4) $f : I_{n+1} \to I_n$ *is a linear homeomorphism with the following property: for each point* x *in* I_{n+1}, *the identity* $\operatorname{Im} x = \operatorname{Im} f(x)$ *holds, where* $\operatorname{Im}(\cdot)$ *is the imaginary part of* $(\cdot)$. *Then* $f(I_{n+1}) = I_n$ *for every integer* $n \geqslant 0$.

The constructed mapping f *is continuous, the unstable manifold of any fixed point* x *from* $(0, \mathbf{i}]$ *is not connected. We note that according to Definition 2.1 the order of each point* $x \in (0; \mathbf{i}]$ *is infinite.*

In conclusion of this section, we show that Lemma 2.2 fails for a continuous mapping defined on a dendroid.

Example 3.2. *Let*

$$X = [0; 1] \cup \bigcup_{j=0}^{+\infty} I_j, \quad \text{where} \quad I_j = \left[0; \exp\left(\frac{\pi}{2^j}\mathbf{i}\right)\right], \quad j \geqslant 0.$$

At each point x *of* $(0; 1]$, *the continuum* X *is not locally connected. Therefore,* X *is a dendroid, which is not a dendrite.*

To define the mapping $f : X \to X$, *we need two auxiliary mappings. The first of them,* $h_j : I_j \to [0; 1]$ $(j \geqslant 1)$, *is defined by the formula*

$$h_j(x) = x \cdot \exp\left(-\frac{\pi}{2^j}\mathbf{i}\right) \quad \text{for any point} \quad x \in I_j, \quad j \geqslant 1.$$

We note that h_j *is a homeomorphism, and* $h_j(I_j) = [0; 1]$.

We define the second auxiliary mapping $g_j : I_j \to I_j$ $(j \geqslant 1)$ *by letting*

$$g_j(x) = h_j^{-1}(\sqrt{h_j(x)}) \quad \text{for any point} \quad x \in I_j, \quad j \geqslant 1.$$

We note that

$$\operatorname{Per}(g_j) = \operatorname{Fix}(g_j) = \left\{0; \exp\left(\frac{\pi}{2^j}\mathbf{i}\right)\right\}$$

for each number $j \geqslant 1$ *and for any point* x *in* $I_j \setminus \left\{0, \exp\left(\frac{\pi}{2^j}\mathbf{i}\right)\right\}$ *the* ω*–limit set* $\omega(x, g_j)$ *of trajectory of point* x *with respect to the mapping* g_j *is the fixed point* $\exp\left(\frac{\pi}{2^j}\mathbf{i}\right)$.

We define the mapping $f : X \to X$ *as follows:*

1) $f(x)=x$ *if* $x\in[-1;0]$;
2) $f(x)=g_j(x)\cdot\exp\left(\frac{\pi}{2^j}\mathbf{i}\right)$ *if* $x\in I_j$ *for all* $j\geqslant 1$;
3) $f(x)=\sqrt{x}$ *if* $x\in[0;1]$.

The constructed mapping f is continuous and $\operatorname{Per}(f)=\operatorname{Fix}(f)=[-1;0]\cup\{1\}$.

We are going to show that each point y in $I_j\setminus\left\{0,\exp\left(\frac{\pi}{2^j}\mathbf{i}\right)\right\}$ $(j\geqslant 1)$ *belongs to* $W^u(0,f)$. *Let $U(0)$ be a neighborhood of 0, which does not contain y, and let x be an arbitrary point in* $U(0)\cap(I_j\setminus\{0\})$. *Since* $\omega(x,g_j)=\left\{\exp\left(\frac{\pi}{2^j}\mathbf{i}\right)\right\}$, *there exists a natural number* $n\geqslant 1$ *such that* $g_j^n(x)\in\left(y;\exp\left(\frac{\pi}{2^j}\mathbf{i}\right)\right)$. *Then*

$$y\in g_j^n(U(0)\cap I_j).$$

Item 2) of the construction of mapping f implies that $y\in f^n(U(0)\cap I_{j+n})$. *Therefore,* $y\in f^n(U(0))$. *Thus,*

$$X\setminus\bigcup_{j=1}^{\infty}\left\{\exp\left(\frac{\pi}{2^j}\mathbf{i}\right)\right\}\subset W^u(0,f).$$

Since $f(I_1)=I_0$, *by Lemma 3.1* $(-1;0]\subset W^u(0,f)$. *Since for each neighborhood $U(0)$ of the point 0 obeying* $U(0)\cap E(X)=\emptyset$ *and any natural number* $n\geqslant 1$ *the condition*

$$f^n(U(0))\cap E(X)=\emptyset$$

is satisfied, we find $W^u(0,f)\cap E(X)=\emptyset$. *Summarizing the above facts, we obtain the identity*

$$W^u(0,f)=X\setminus E(X).$$

Then $\overline{W^u(0,f)}\setminus W^u(0,f)=E(X)$, *and each point* $\exp\left(\frac{\pi}{2^j}\mathbf{i}\right)$, $j\geqslant 1$, *in $E(X)$ is not a periodic point of the mapping f.*

4. Proof of Theorem 2.2

Suppose that the statement of Theorem 2.2 is false. Then there exist a point $x\in W^u(p,f)$ and a neighborhood $U(p)$ of p in X that does not contain x satisfying the condition

$$f^n(U(p)\cap W^u(p,f))\cap\{x\}=\emptyset \quad \text{for each} \quad n\in\mathbb{N}. \tag{4.1}$$

Let $r=\operatorname{ord}p$, where $r\in\mathbb{N}$. Since the order of p is finite and $p\notin R^{(1)}(X)$, by Lemma 3.2 there exists a neighborhood $U_1(p)\subset U(p)$ such that $\overline{U_1(p)}$ is either an arc (if $r=1$ or 2) or is homeomorphic to the r–od (if $r\geqslant 3$). Put $Y=\overline{U_1(p)}$.

Since $p,x\in W^u(p,f)$, by Lemma 3.3 we have $[p;x]\subseteq W^u(p,f)$. Therefore,

$$[p;x]\cap Y\subseteq W^u(p,f). \tag{4.2}$$

Hence, there exists at least one component of the point p in Y that contains points from $W^u(p,f)$.

On the other hand, since $x\in W^u(p,f)$, it follows from (4.1), the condition $Y\subset U(p)$ and the connectivity of $W^u(p,f)$ that there exists at least one component of p in Y that does not contain points from $W^u(p,f)$. Taking into consideration (4.2), we obtain that the order of p is greater than 1, that is, $r\geqslant 2$.

Let s be the number of components of p that do not contain points from $W^u(p,f)$. Then $1\leqslant s\leqslant r-1$. Denote by $Y_{i_1}(p),\dots,Y_{i_s}(p)$ the components of p in Y, for which

$$Y_{i_j}(p)\cap W^u(p,f)=\emptyset, \qquad 1\leqslant j\leqslant s, \tag{4.3}$$

and $Y_{i_{s+1}}(p), \dots, Y_{i_r}(p)$ are the components of point p, for which

$$Y_{i_j}(p) \cap W^u(p, f) \neq \emptyset, \qquad s+1 \leqslant j \leqslant r.$$

Since $W^u(p, f)$ is connected, $Y_{i_j}(p) \cap W^u(p, f)$ is also connected $(s+1 \leqslant j \leqslant r)$. This is why without loss of generality we suppose that

$$Y_{i_j}(p) \subseteq W^u(p, f), \qquad s+1 \leqslant j \leqslant r. \tag{4.4}$$

Otherwise, using the local connectivity of Y and connectivity of $W^u(p, f)$, we choose a connected subneighborhood $U_0(p) \subset U_1(p)$ such that all components of the point p in $\overline{U_0(p)}$ containing points from $W^u(p, f)$ belong entirely to $W^u(p, f)$.

By continuity of f and the local connectivity of the continuum Y, there exists a connected neighborhood $U_2(p)$ of the point p such that $U_2(p) \subset U_1(p)$, and

$$f(U_2(p)) \subset U_1(p). \tag{4.5}$$

Since $U_2(p)$ is a connected neighborhood and $U_2(p) \subset U_1(p)$, then $\overline{U_2(p)}$ is either an arc (if $r = 1$ or 2) or is homeomorphic to the r–od (if $r \geqslant 3$). Therefore, $Y_{i_j}(p) \cap \partial(U_2(p))$ a singleton set for each $1 \leqslant j \leqslant r$. We let $\alpha_j = Y_{i_j}(p) \cap \partial(U_2(p))$ for $1 \leqslant j \leqslant s$. By (4.3), $\alpha_j \notin W^u(p, f)$, $1 \leqslant j \leqslant s$. Therefore, there exists a neighborhood $U_3(p)$ of the point p such that $U_3(p) \subset U_2(p)$, and

$$f^j(U_3(p)) \cap \{\alpha_1, \dots, \alpha_s\} = \emptyset, \qquad j \geqslant 1. \tag{4.6}$$

We are going to show that $f^n(U_3(p)) \cap \{x\} = \emptyset$ for each number $n \geqslant 1$. We suppose the contrary, then there exist a point $y \in U_3(p)$ and $n_0 \in \mathbb{N}$ such that $f^{n_0}(y) = x$. Since $x \notin U(p)$, and $U_3(p) \subset U_2(p) \subset U_1(p) \subset U(p)$, we have $x \notin U_3(p)$. Thus, $y \in U_3(p)$, and $x \notin U_3(p)$. Therefore, there exists a natural number $k \geqslant 1$ such that

$$\{y, f(y), \dots, f^{k-1}(y)\} \subset U_3(p), \qquad f^k(y) \notin U_3(p).$$

Then by the embedding $U_3(p) \subset U_2(p)$ and (4.5) we obtain $f^k(y) \in U_1(p)$. Hence, taking into consideration that $U_1(p) \subset U(p)$, $x \notin U(p)$, we get $k < n_0$.

Thus,

$$f^k(y) \in U_1(p) \setminus U_3(p) \subset \bigcup_{j=1}^{r} Y_{i_j}(p) \setminus U_3(p).$$

Then, in view of the conditions: $p \in \mathrm{Fix}(f)$, and f is continuous mapping, we obtain

$$f^k([p;\, y]) \cap \partial U_3(p) \neq \emptyset.$$

By (4.6),

$$f^k([p;\, y]) \cap \{\alpha_1, \dots, \alpha_s\} = \emptyset.$$

Therefore,

$$f^k(y) \in \bigcup_{j=s+1}^{r} Y_{i_j}(p) \setminus U_3(p).$$

Then, taking into account (4.4), we obtain that $f^k(y) \in W^u(p, f)$. Hence,

$$x \in f^{n_0-k}(U_3(p) \cap W^u(p, f)),$$

which contradicts (4.1) since $U_3(p) \subset U(p)$. Thus, $f^n(U_3(p)) \cap \{x\} = \emptyset$ for each $n \geqslant 1$. The latter contradicts the condition $x \in W^u(p, f)$. The proof of Theorem 2.2 is complete.

The following two examples show that the statement of Theorem 2.2 fails if at least one of Conditions 1), 2) is not satisfied. In Example 4.1 we construct a continuous mapping on the continuum X_1, on which Condition 1) is not satisfied, and in Example 4.2 we do a continuous mapping on the continuum X_2, on which Condition 2) is violated.

Example 4.1. *1. Construction of the continuum X_1.*

For each integer $k \geqslant 0$, let $I_k = \left[\frac{1}{2^k}; \frac{1}{2^k} + \frac{\mathbf{i}}{2^k}\right]$ be vertical segments, whose lengths tend to 0 as $k \to +\infty$. We define the dendroid

$$Z = [0;1] \cup \bigcup_{k=0}^{\infty} I_k,$$

which is a dendrite. The dendrite Z has a countable number of ramification points, the order of each ramification point is 3, and $0 \in R^{(1)}(Z)$. The segment $[0;1]$ is called the base of dendrite Z.

For each $j \in \mathbb{N}$, we denote by Z_j the dendrite obtained from dendrite Z by contraction in all directions by $\frac{1}{2^{j+1}}$ times and shifting along the segment $[0;1]$ by $\frac{1}{2^j}$ to the right. As a result, the base of dendrite Z_j coincides with the segment $\left[\frac{1}{2^j}; \frac{3}{2^{j+1}}\right]$ $(j \geqslant 1)$. Each vertical segment of dendrite Z_j obtained by the contraction and shift of segment I_k is denoted by $I_k^{(j)}$, where $j \geqslant 1$, $k \geqslant 0$. We observe

$$\lim_{j\to\infty} \operatorname{diam} Z_j = 0. \tag{4.7}$$

Let

$$J_k = \left[1; 1 + \frac{1}{2^k}\exp\left(\frac{\pi}{2^k}\mathbf{i}\right)\right],$$

where $k \geqslant 1$, and $Y = \bigcup_{k=1}^{\infty} J_k$. The continuum Y is a dendrite and has a single ramification point 1 of infinite order.

We define the continuum

$$X_1 = [0;1] \cup \bigcup_{j=1}^{\infty} Z_j \cup Y,$$

see Figure 1. By (4.7), the continuum X_1 is locally connected, therefore X_1 is a dendrite. We note that the dendrite X_1 has a countable number of ramification points, and $0 \in R^{(1)}(X_1)$.

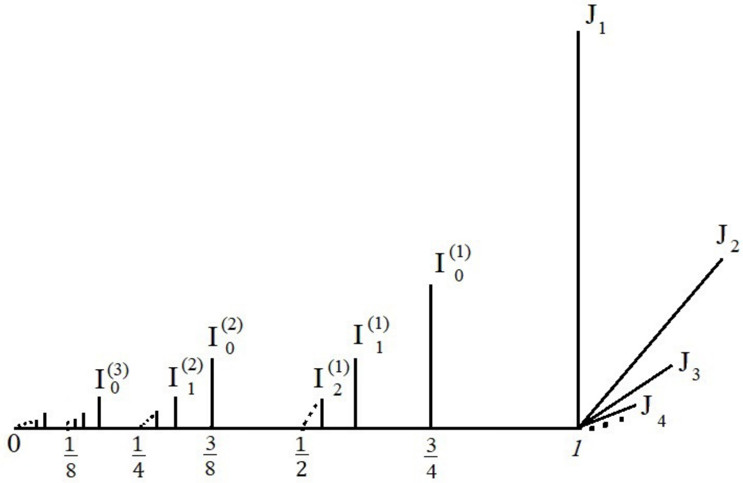

FIGURE 1. Dendrite X_1

2. Define the mapping $f : X_1 \to X_1$ as follows:

1) $f(0) = 0$.

2) *For each* $j \geqslant 2$, *we let* $f(x) = x + \dfrac{1}{2^j}$ *for* $x \in Z_j$. *As a result, each dendrite of* Z_j $(j \geqslant 2)$ *is shift by* $\dfrac{1}{2^j}$ *units to the right along the interval* $[0;1]$, *and* $f(I_k^{(j)}) = I_{k+1}^{(j-1)}$ *for all* $k \geqslant 0$. *Thus,* $f(Z_j) \subset Z_{j-1}$ *for each* $j \geqslant 2$. *We note that each segment*

$$\left[\frac{1}{2^j}; \frac{3}{2^{j+1}}\right],$$

which is the base of dendrite Z_j, *is mapped onto the segment* $\left[\dfrac{1}{2^{j-1}}; \dfrac{5}{2^{j+1}}\right]$ $(j \geqslant 2)$.

3) *For each number* $j \geqslant 1$ *we define the linear homeomorphism*

$$f : \left[\frac{3}{2^{j+2}}; \frac{1}{2^j}\right] \to \left[\frac{5}{2^{j+2}}; \frac{1}{2^{j-1}}\right] \quad \text{so that} \quad f\left(\frac{3}{2^{j+2}}\right) = \frac{5}{2^{j+2}}, f\left(\frac{1}{2^j}\right) = \frac{1}{2^{j-1}};$$

4) $f(x) = 1$ *if* $x \in \left[\dfrac{3}{4}; 1\right]$.

5) *We define* $f : Z_1 \to Y$ *such that*

$$f\left(\left[\frac{1}{2}; \frac{3}{4}\right]\right) = 1,$$

$f : I_k^{(1)} \to J_{k+1}$ *is a linear homeomorphism with the property:* $f(I_k^{(1)}) = J_{k+1}$ *for each* $k \geqslant 0$. *Then* $f(Z_1) = Y$.

6) *For each number* $k \geqslant 2$, *we define a linear homeomorphism* $f : J_k \to J_{k-1}$ *such that*

$$f\left(1 + \frac{1}{2^k}\exp\left(\frac{\pi}{2^k}\mathbf{i}\right)\right) = 1 + \frac{1}{2^{k-1}}\exp\left(\frac{\pi}{2^{k-1}}\mathbf{i}\right), \qquad f(0) = 0.$$

Then $f(J_k) = J_{k-1}$.

7) *Let* $f : J_1 \to [0;1]$ *be a linear homomorphism such that*

$$f(1) = 1, \qquad f\left(1 + \frac{1}{2}\exp\left(\frac{\pi}{2}\mathbf{i}\right)\right) = 0.$$

The mapping f *is continuous,* $\mathrm{Per}(f) = \mathrm{Fix}(f) = \{0; 1\}$. *It follows from Item 2) of constructing the mapping* f *that*

$$f^{j-1}(I_k^{(j)}) = I_{k+j-1}^{(1)}, \qquad j \geqslant 2 \quad \text{and} \quad k \geqslant 0.$$

Then, taking into account Item 5) of constructing the mapping f, *we obtain the identities*

$$f^j(I_k^{(j)}) = f(f^{j-1}(I_k^{(j)})) = f(I_{k+j-1}^{(1)}) = J_{k+j}, \qquad \text{where} \quad j \geqslant 1, \quad k \geqslant 0.$$

They yield

$$f^j(Z_j) = \bigcup_{i=j}^{\infty} J_i, \qquad j \geqslant 1. \tag{4.8}$$

We are going to show that the unstable manifold $W^u(0, f)$ *of the point* 0 *is the set* $[0;1] \cup Y$. *It follows from Items 2) – 3) of the construction of mapping* f *that each point* $x \in (0;1]$ *belongs to* $W^u(0, f)$. *Let us show that each point* e *from* $Y \setminus \{1\}$ *belongs to* $W^u(0, f)$. *Let* $U(0)$ *be an arbitrary neighborhood of the point* 0 *in* X_1 *that does not contain the point* 1. *By* (4.7) *there exists a natural number* $j_0 \geqslant 1$ *such that* $Z_{j_0} \subset U(0)$. *According to the construction of the dendrite* Y, *there exists a natural number* $k_0 \geqslant 1$ *such that* $e \in J_{k_0}$. *If* $k_0 \geqslant j_0$, *then, by* (4.8),

$e \in f^{j_0}(U(0))$. If $k_0 < j_0$, then by the choice of j_0, (4.8) and Item 6) of the construction of mapping f we obtain

$$f^{2j_0-k_0}(U(0)) \supset f^{j_0-k_0}(f^{j_0}(Z_{j_0})) \supset f^{j_0-k_0}(\bigcup_{i=j_0}^{\infty} J_i) \supset f^{j_0-k_0}(J_{j_0}) = J_{k_0} \ni e.$$

Thus, $e \in W^u(0, f)$. Therefore,

$$[0;1] \cup Y \subseteq W^u(0, f). \tag{4.9}$$

It follows from Item 2) of constructing the mapping f that $f^{-k-1}(I_k^{(j)}) = \emptyset$ for $j \geqslant 1$, $k \geqslant 0$. Hence, by (4.9) we obtain

$$W^u(0, f) = [0;1] \cup Y.$$

Thus, for each neighborhood $U(0)$ of the point 0 in X that does not contain the point 1 we have $U(0) \cap W^u(0, f) \subset [0;1)$. But for each point $x \in Y \setminus \{1\}$ and each natural number $n \geqslant 1$ we have $f^{-n}(x) \cap [0;1) = \emptyset$. Hence, for an arbitrary point $x \in Y \setminus \{1\}$ we have

$$\bigcup_{n=1}^{+\infty} f^{-n}(x) \cap (U(0) \cap W^u(0, f)) = \emptyset.$$

Thus, if Condition 1) of Theorem 2.2 is violated, then the statement of Theorem 2.2 fails.

In conclusion, we note that by Item 7) of the construction of the mapping f, each endpoint $1+\dfrac{1}{2^k}\exp\left(\dfrac{\pi}{2^k}\mathbf{i}\right)$, $k \geqslant 1$, of the continuum Y is a homoclinic point of the mapping f to the fixed point 0. But none of them has homoclinic points among their preimages in the neighborhood $U(0)$ of the point 0. Thus, Corollary 2.2 fails if its Condition 1) is violated.

Example 4.2. *Let us construct the continuum X_2. For each natural $k \geqslant 1$ we let*

$$I_k = \left[0; \frac{1}{2^{k-1}} \exp\left(\frac{\pi}{2^k}\mathbf{i}\right)\right].$$

As in Example 4.1 we define

$$J_k = \left[1; 1 + \frac{1}{2^k} \exp\left(\frac{\pi}{2^k}\mathbf{i}\right)\right],$$

where $k \geqslant 1$. We observe that

$$\lim_{k\to\infty} \operatorname{diam} I_k = \lim_{k\to\infty} \operatorname{diam} J_k = 0. \tag{4.10}$$

We let

$$X_2 = [0;1] \cup \bigcup_{k=1}^{\infty} (I_k \cup J_k),$$

see Figure 2.

By (4.10) the continuum X_2 is locally connected. Therefore, X_2 is a dendrite with only two ramification points 0 and 1 of infinite order.

We denote $Y = \bigcup_{k\geqslant 1} J_k$, and let $e_k = 1 + \dfrac{1}{2^k}\exp\left(\dfrac{\pi}{2^k}\mathbf{i}\right)$ be the endpoints of the continuum Y, $k \geqslant 1$.

We proceed to constructing the mapping $f : X_2 \to X_2$.

1) *Let $f(z) = z$ if $z \in [0;1]$.*
2) *Let $f : \left[0; \dfrac{\mathbf{i}}{2}\right] \to [0;\, 1]$ be a linear homeomorphism such that*

$$f(0) = 0, \qquad f\left(\frac{\mathbf{i}}{2}\right) = 1.$$

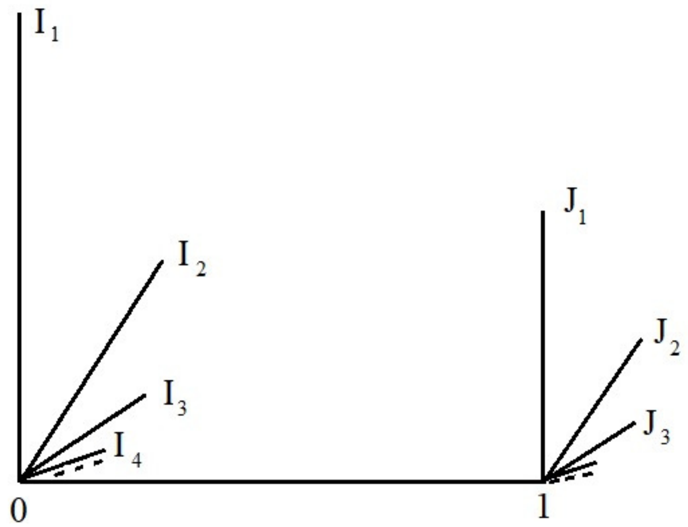

FIGURE 2. Dendrite X_2

3) *To define the mapping* $f : \left[\frac{\mathbf{i}}{2}; \mathbf{i}\right] \to Y$, *we denote* $z_k = \left(\frac{1}{2} + \frac{1}{2^k}\right)\mathbf{i}$, $k \geqslant 1$. *We note that* $\lim_{k\to\infty} z_k = \frac{\mathbf{i}}{2}$.
For all $k \geqslant 1$, *we define a linear homeomorphism* $f : [z_k; z_{k+1}] \to [e_k; e_{k+1}]$ *such that* $f(z_k) = e_k$, $f(z_{k+1}) = e_{k+1}$. *Then* $f([z_k; z_{k+1}]) = J_k \cup J_{k+1}$, $(k \geqslant 1)$. *Therefore,* $f\left(\left[\frac{\mathbf{i}}{2}; \mathbf{i}\right]\right) = Y$.
4) *To define the mapping* $f : I_{k+1} \to I_k$ $(k \geqslant 1)$, *we divide each segment* I_k *into 2 equal parts and denote by* $I_k^{(1)}$ *the half that contains the point* 0, *and by* $I_k^{(2)}$ *the other half. We let*

$$f(z) = 2z \cdot \exp\left(\frac{\pi}{2^{k+1}}\mathbf{i}\right), \quad \text{if} \quad z \in I_{k+1}^{(1)} \quad \text{for all} \quad k \geqslant 1. \tag{4.11}$$

Then $f(I_{k+1}^{(1)}) = I_k^{(1)}$ *for* $k \geqslant 1$. *On each segment*

$$I_{k+1}^{(2)} = \left[\frac{1}{2^{k+1}} \exp\left(\frac{\pi}{2^{k+1}}\mathbf{i}\right); \frac{1}{2^k} \exp\left(\frac{\pi}{2^{k+1}}\mathbf{i}\right)\right] \quad (k \geqslant 1)$$

we define the linear homeomorphism

$$f : I_{k+1}^{(2)} \to \left[\frac{1}{2^k} \exp\left(\frac{\pi}{2^k}\mathbf{i}\right); \frac{3}{2^{k+1}} \exp\left(\frac{\pi}{2^k}\mathbf{i}\right)\right]$$

such that

$$f\left(\frac{1}{2^{k+1}} \exp\left(\frac{\pi}{2^{k+1}}\mathbf{i}\right)\right) = \frac{1}{2^k} \exp\left(\frac{\pi}{2^k}\mathbf{i}\right), \qquad f\left(\frac{1}{2^k} \exp\left(\frac{\pi}{2^{k+1}}\mathbf{i}\right)\right) = \frac{3}{2^{k+1}} \exp\left(\frac{\pi}{2^k}\mathbf{i}\right),$$

and the distance between any two points from $I_{k+1}^{(2)}$ *is preserved, that is, for all points* x, y *in* $I_{k+1}^{(2)}$ *the identity* $d(x, y) = d(f(x), f(y))$ *holds. Then* $f(I_{k+1}^{(2)}) \subset I_k^{(2)}$, *or, more precisely, the segment* $I_{k+1}^{(2)}$ *is mapped only into the half of segment* $I_k^{(2)}$. *Then for each endpoint* $\frac{1}{2^{k-1}} \exp\left(\frac{\pi}{2^k}\mathbf{i}\right)$ *belonging to* I_k *the condition is satisfied:*

$$f^{k-1}\left(\frac{1}{2^{k-1}} \exp\left(\frac{\pi}{2^k}\mathbf{i}\right)\right) = z_k, \qquad k \geqslant 2. \tag{4.12}$$

Thus, by (4.11) and (4.12) we obtain the identity

$$f^{k-1}(I_k) = [0; z_k], \qquad k \geqslant 2. \tag{4.13}$$

5) *For each $k \geqslant 1$, we define a linear homeomorphism $f : J_{k+1} \to J_k$ such that $f(e_{k+1}) = e_k$, $f(1) = 1$.*
6) *Let $f : J_1 \to [0; 1]$ be a linear homeomorphism such that $f(e_1) = 0$, $f(1) = 1$.*

The constructed mapping f is continuous, $\mathrm{Per}(f) = \mathrm{Fix}(f) = [0; 1]$.
We are going to show that

$$W^u(0, f) = [0; 1] \cup Y \cup \bigcup_{k=1}^{\infty} I_k^{(1)}.$$

First let us show that each point e of the continuum Y belongs to $W^u(0, f)$. Let $e \in J_k$ $(k \geqslant 1)$, and $U(0)$ be an arbitrary neighborhood of 0 that does not contain 1. By (4.10) there exists a natural number $j_0 \geqslant k$ such that $I_{j_0} \subset U(0)$. Applying (4.13), we obtain the identity

$$f^{j_0-1}(I_{j_0}) = [0;\ z_{j_0}].$$

Then, by virtue of points 2) and 3) of the definition of the mapping f, we have

$$f^{j_0}(I_{j_0}) = f(f^{j_0-1}(I_{j_0})) = f([0;\ z_{j_0}]) = f\left(\left[0;\ \frac{\mathbf{i}}{2}\right] \cup \left[\frac{\mathbf{i}}{2};\ z_{j_0}\right]\right) = [0; 1] \cup \bigcup_{k=j_0}^{\infty} J_k \supset J_{j_0},$$

and by Item 5)

$$f^{2j_0-k}(I_{j_0}) = f^{j_0-k}(f^{j_0}(I_{j_0})) \supset f^{j_0-k}(I_{j_0}) = J_k.$$

Therefore, $f^{2j_0-k}(U(0)) \cap \{e\} \neq \emptyset$. The above reasoning yield

$$[0; 1] \cup Y \cup \bigcup_{k=1}^{\infty} I_k^{(1)} \subset W^u(0, f).$$

Let us show that $\left(\frac{\mathbf{i}}{2}; \mathbf{i}\right] \cap W^u(0, f) = \emptyset$. Let x be any point from $\left(\frac{\mathbf{i}}{2}; \mathbf{i}\right]$. Then there exists a natural number $k_0 \geqslant 1$ such that $x \in (z_{k_0+1};\ z_{k_0}]$. We choose a neighborhood $U(0)$ of the point 0 such that

$$U(0) \cap I_{k_0+1}^{(2)} = \emptyset. \tag{4.14}$$

By (4.13) and Item 3) of the definition of mapping f

$$f^j(I_k) \cap (z_{k_0+1};\ z_{k_0}] = \emptyset, \qquad k \geqslant k_0 + 1, \qquad j \geqslant 1. \tag{4.15}$$

It follows from the condition (4.14) that $U(0) \cap I_k^{(2)} = \emptyset$ for $1 \leqslant k \leqslant k_0$. Then, by (4.11),

$$f^{k-1}(U(0) \cap I_k) \subset \left[0;\ \frac{\mathbf{i}}{2}\right] \qquad \textit{for} \qquad 2 \leqslant k \leqslant k_0.$$

Hence, by Item 2) of the definition of mapping f we obtain

$$f^j(U(0) \cap I_k) \cap (z_{k_0+1}; z_{k_0}] = \emptyset, \qquad j \geqslant 1, \qquad 2 \leqslant k \leqslant k_0. \tag{4.16}$$

Since $U(0) \cap I_1 \subset [0; \mathbf{i}]$, according to Items 1) and 2) of the definition of mapping f

$$f^j(U(0) \cap I_1) \cap (z_{k_0+1};\ z_{k_0}] = \emptyset, \qquad j \geqslant 1. \tag{4.17}$$

It follows from (4.15), (4.16) *and* (4.17) *that* $f^j(U(0)) \cap \{x\} = \emptyset$ *for each point* $x \in \left(\frac{\mathbf{i}}{2}; \mathbf{i}\right]$. *Thus,* $\left(\frac{\mathbf{i}}{2}; \mathbf{i}\right] \cap W^u(0, f) = \emptyset$. *Therefore, for all* $j \geqslant 1$

$$f^{-j}\left(\left(\frac{\mathbf{i}}{2}; \mathbf{i}\right]\right) \cap W^u(0, f) = \emptyset.$$

Hence, by Item 3) of the construction of mapping f*, for each point* $e \in Y \setminus \{1\}$ *and an arbitrary neighborhood* $U(0)$ *of the point* 0 *that does not contain* 1*, the following condition is satisfied:*

$$\bigcup_{j=1}^{+\infty} f^{-j}(e) \cap (W^u(0, f) \cap U(0)) = \emptyset.$$

Thus, if Condition 2) *of Theorem* 2.2 *is not satisfied, the statement of this theorem fails.*

We also note that each endpoint e *of the continuum* Y *is a homoclinic point to the fixed point* 0*, but among the preimages of the point* e *there are no homoclinic points to the fixed point* 0*, that is, Corollary* 2.2 *also fails if its Condition* 2) *is violated.*

5. Proof of Theorem 2.3

To prove Theorem 2.3, we need the notion of a horseshoe adapted to the considered case, which goes back to Smale [25], as well as auxiliary statements.

Definition 5.1. *We say that the mapping* $f : X \to X$ *of a continuum* X *has a horseshoe if there exist disjoint subcontinua* $A, B \subset X$ *such that*

$$f(A) \cap f(B) \supset A \cup B.$$

The following theorem is a direct consequence of Definitions 2.6 and 5.1, see, for example, [2].

Theorem 5.1 ([2]). *Let* $f : X \to X$ *be a continuous mapping of* X*, and let* f *be a horseshoe. Then the topological entropy of* f *is positive.*

We shall need the following property of topological entropy.

Lemma 5.1 ([8]). *For each continuous mapping* $f : X \to X$ *of a compact topological space* X *and each natural number* $n \geqslant 1$ *the identity*

$$h(f^n) = n \cdot h(f)$$

holds, where $h(\cdot)$ *is the topological entropy of the mapping* $(\cdot)$.

Proof of Theorem 2.3. To prove the positivity of topological entropy of f, we use Theorem 5.1 and Lemma 5.1. Let us show the existence of a horseshoe for some iteration of f.

Let x be a homoclinic point to a periodic point p of period m, $U(p)$ be an arbitrary neighborhood of p in X that does not contain x. Denote by $r = \operatorname{ord} p$, where $r \in \mathbb{N}$. Since $\operatorname{ord} p$ is finite, $p \notin R^{(1)}(X)$, by Lemma 3.2 there exists a neighborhood $U_1(p)$ of p in X such that $U_1(p) \subset U(p)$, and $\overline{U_1(p)}$ is either an arc (for $r = 1$ or 2) or is homeomorphic to r–od (for $r \geqslant 3$).

By Corollary 2.2 there exist a homoclinic point $x_1 \in U_1(p)$ to a periodic point p and a natural number n_1 such that $f^{mn_1}(x_1) = x$. Since $\overline{U_1(p)}$ is a locally connected continuum, there exists a connected neighborhood $U_2(p) \subset U_1(p)$ of p such that $x_1 \notin U_2(p)$. Again by Corollary 2.2 there exists a homoclinic point $x_2 \in U_2(p)$ to a periodic point p and a natural number n_2 such that $f^{mn_2}(x_2) = x_1$. We repeat the above reasoning $(r + 1)$ times. As a result, we construct connected neighborhoods $U_k(p)$ of the point p $(1 \leqslant k \leqslant r+1)$, homoclinic points $\{x_1, \dots, x_{r+1}\}$ and natural numbers $\{n_1, \dots, n_{r+1}\}$ with the following properties

1) $U_{k+1}(p) \subset U_k(p)$ for each $1 \leqslant k \leqslant r$;
2) $x_{k+1} \in U_{k+1}(p)$, but $x_k \notin U_{k+1}(p)$ for all $1 \leqslant k \leqslant r$;
3) $f^{mn_{k+1}}(x_{k+1}) = x_k$ for $1 \leqslant k \leqslant r$.

Since the order of p is r, there exist distinct homoclinic points $x_i, x_j \in \{x_1, \dots, x_{r+1}\}$ that lie on the same component of p. Let $i < j$ for the sake of definiteness. Since $U_i(p)$ and $U_j(p)$ are connected neighborhoods, and $U_j(p) \subset U_i(p) \subset U_1(p)$, then $\overline{U_i(p)}$ and $\overline{U_j(p)}$ are either arcs or homeomorphic to r–od. Then by Properties 1) – 3) we obtain

$$x_j \in (p; x_i), \quad \text{and} \quad f^{m(n_{i+1}+\ldots+n_k+\ldots+n_j)}(x_j) = x_i, \quad \text{where} \quad i+1 \leqslant k \leqslant j.$$

Let $g = f^{m(n_{i+1}+\ldots+n_k+\ldots+n_j)}$, $y = x_j$. Then $g(p) = p$, $y \in (p; g(y))$. Therefore, by the continuity of the mapping g

$$g([p; y]) \supseteq [g(p); g(y)] = [p; g(y)] \supset [p; y]. \tag{5.1}$$

Since y is a homoclinic point to a fixed point p, by definitions 2.4 and 2.5 $\omega(y, g) = \{p\}$. Therefore, for any neighborhood $U(p)$ of p that does not contain y, there exists a natural number $s \geqslant 2$ such that $g^s(y) \in U(p)$. Then

$$y \in (g^s(y); g(y)). \tag{5.2}$$

We denote by V the component of y that contains $g(y)$. Then it follows from (5.2) that

$$g(y) \in V, \qquad g^s(y) \notin V.$$

Therefore, there exists a natural number $2 \leqslant j_0 \leqslant s$ such that

$$g^j(y) \in V \quad \text{for any} \quad 1 \leqslant j \leqslant j_0 - 1, \qquad g^{j_0}(y) \notin V.$$

Then the points $g^{j_0}(y)$ and $g^{j_0-1}(y)$ lie on different components of y, that is,

$$y \in (g^{j_0}(y); g^{j_0-1}(y)). \tag{5.3}$$

It follows from the continuity of the mapping g that

$$g^{j_0-1}([y; g(y)]) \supseteq [g^{j_0-1}(y); g^{j_0-1}(g(y))] = [g^{j_0-1}(y); g^{j_0}(y)].$$

Taking into account (5.3), we obtain that

$$g^{j_0-1}([y; g(y)]) \supset [g^{j_0-1}(y); y].$$

This inclusion yields

$$g(g^{j_0-1}([y; g(y)]) \supset g([g^{j_0-1}(y); y]),$$

that is,

$$g^{j_0}([y; g(y)]) \supset [g^{j_0}(y); g(y)].$$

Hence, by (5.3),

$$g^{j_0}([y; g(y)]) \supset [g^{j_0}(y); g(y)] \supset [y; g(y)].$$

Let $h = g^{j_0}$. Then $h([y; g(y)]) \supset [y; g(y)]$. But then

$$h^j([y; g(y)]) \supset [y; g(y)], \qquad j \geqslant 1. \tag{5.4}$$

It follows from (5.1) that there exists a point $y_1 \in (p; y)$ such that $g(y_1) = y$. Then

$$g([p; y_1]) \supseteq [p; y] \supset [p; y_1].$$

Therefore, there exists a point $y_2 \in (p; y_1)$ such that $g(y_2) = y_1$. We repeat the above reasoning j_0 times. As a result, we construct a set of points $\{y_1, \dots, y_{j_0}\}$ with the following properties:

$$g(y_1) = y, g(y_j) = y_{j-1}, \quad \text{and} \quad y_j \in (p; y_{j-1}) \quad \text{for} \quad 2 \leqslant j \leqslant j_0.$$

We let $u = y_{j_0}$, $v = y_{j_0-1}$. Then $[u; v] \cap [y; g(y)] = \emptyset$ and

$$h([u; v]) = g^{j_0}([u; v]) \supseteq [g^{j_0}(u); g^{j_0}(v)] = [y; g(y)], \tag{5.5}$$

Since y is a homoclinic point of f, by Definition 2.4, $\omega(y, f^m) = \{p\}$. Since $p \in \mathrm{Fix}(f^m)$, we have $\omega(y, h) = \{p\}$. Therefore, for each neighborhood $U(p)$ of p satisfying the condition $U(p) \cap \{u\} = \emptyset$, there exists a natural number j_1 such that for each $j \geqslant j_1$ the condition $h^j(y) \in U(p)$ is satisfied. Therefore,

$$u \in (h^j(y); y), \qquad j \geqslant j_1.$$

Then, taking into consideration (5.4), we get

$$h^j([y; g(y)]) \supset [u; g(y)] \supset [u; v] \cup [y; g(y)], \qquad j \geqslant j_1. \tag{5.6}$$

Therefore, by (5.5) we obtain

$$h^{j_1+1}([u; v]) \supset h^{j_1}([y; g(y)]) \supset [u; v] \cup [y; g(y)]. \tag{5.7}$$

It follows from (5.6) and (5.7) that the mapping h^{j_1+1} has a horseshoe $([u; v], [y; g(y)])$.

According to Theorem 5.1, the mapping h has a positive topological entropy. By Lemma 5.1 the topological entropy of f is also positive. The proof is complete. $\square$

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BIBLIOGRAPHY

1. P.S. Aleksandrov. *Introduction to Set Theory and General Topology.* Nauka, Moscow (1977). (in Russian).
2. L.S. Efremova and E.N. Makhrova. *One–dimensional dynamical systems* // Russ. Math. Surv. **76**:5, 821–881 (2021). https://doi.org/10.1070/RM9998
3. K. Kuratowski. *Topology. I.* Academic Press, New York; PWN–Polish Scientific Publishers, Warszawa (1966).
4. K. Kuratowski. *Topology. II.* Academic Press, New York; PWN–Polish Scientific Publishers, Warszawa (1968).
5. E.N. Makhrova. *Homoclinic points and topological entropy of a continuous mapping of a dendrite* // J. Math. Sci., New York **158**:2, 241–248 (2009). https://doi.org/10.1007/s10958-009-9392-4
6. H. Poincaré. *New methods in celestial mechanics. 1. Periodic and asymptotic solutions. 2. Approximations by series. 3. Integral invariants and asymptotic properties of certain solutions* American Institute of Physics, Bristol (1993).
7. V.Zh. Sakbaev, O.G. Smolyanov. *Diffusion and quantum dynamics on graphs* // Dokl. Math. **88**:1, 404–408 (2013). https://doi.org/10.1134/S1064562413040108
8. R.L. Adler, A.G. Konheim, M.H. McAndrew. *Topological entropy* // Trans. Am. Math. Soc. **114**, 309–319 (1965). https://doi.org/10.2307/1994177
9. S.J. Agronsky, J.G. Ceder. *What sets can be ω–limit sets in E^n?* // Real Anal. Exch. **17(1991/92)**:1, 97–109 (1992).
10. F. Balibrea, G.–G. Juan Luis. *Continua with empty interior as ω–limit sets* // Appl. Gen. Topol. **6**:2, 195–205 (2005).
11. L. Block. *Homoclinic points of mappings of the interval* // Proc. Am. Math. Soc. **72**:3, 576–580 (1978). https://doi.org/10.1090/S0002-9939-1978-0509258-X
12. J. Charatonik, A. Illanes. *Mappings on dendrites* // Topology Appl. **144**:1–3, 109–132 (2004). https://doi.org/10.1016/j.topol.2004.04.004
13. V. Chernyshev, A. Tolchennikov, A. Shafarevich. *Behavior of quasi–particles on hybrid spaces. Relations to the geometry of geodesics and to the problems of analytic number theory* // Regul. Chaotic Dyn. **21**:5, 531–537 (2016). https://doi.org/10.1134/S156035471605004X

14. L.S. Efremova, E.N. Makhrova. *On homoclinic points of piecewice monotone mappings of dendrites* // in Proc. Int. Conf. “Progress in Nonlinear Science”, **1**, 225–228 (2002).
15. L.S. Efremova. *Example of the smooth skew product in the plane with the one-dimensional ramified continuum as the global attractor* // ESAIM, Proc. **36**, 15–25 (2012). https://doi.org/10.1051/proc/201236002
16. L.S. Efremova. *Ramified continua as global attractors of C^1-smooth self-maps of a cylinder close to skew products* // J. Difference Equ. Appl. **29**:9–12, 1244–1274 (2023). https://doi.org/10.1080/10236198.2023.2204144
17. H. Hosaka, H. Kato. *Continuous maps on trees and nonwandering sets* // Topology Appl. **81**:1, 35–46 (1997). https://doi.org/10.1016/S0166-8641(97)00015-1
18. Z. Kočan, V. Kornecká–Kurková, M. Málek. *Entropy, horseshoes and homoclinic trajectories on trees, graphs and dendrites* // Ergodic Theory Dyn. Syst. **31**:1, 165–175 (2011). https://doi.org/10.1017/S0143385710000325
19. Z. Kočan. *Chaos on one–dimensional compact metric spaces* // Int. J. Bifurcation Chaos Appl. Sci. Eng. **22**:10, 1250259 (2012). https://doi.org/10.1142/S0218127412502598
20. A. Lelek. *On plane dendroids and their end points in the classical sense* // Fundam. Math. **49**, 301–319 (1960).
21. E.N. Makhrova. *Monotone maps on dendrites* // Discontin. Nonlinearity Complex. **9**:4, 541–552 (2020). https://doi.org/10.5890/DNC.2020.12.007
22. E.N. Makhrova. *On limit sets of monotone maps on dendroids* // Appl. Math. Nonlinear Sci. **5**:2, 311–316 (2020). https://doi.org/10.2478/amns.2020.2.00056
23. E.N. Makhrova. *Remarks on minimal sets on dendrites and finite graphs* // J. Difference Equ. Appl. **29**:9–12, 1313–1322 (2023). https://doi.org/10.1080/10236198.2023.2220417
24. H.–O. Peitgen, P. H. Richter, *The Beauty of Fractals: Images of Complex Dynamical Systems.* Springer–Verlag, Berlin (1986).
25. S. Smale, *Differentiable dynamical systems* // Bull. Am. Math. Soc. **73**, 747–817 (1967). https://doi.org/10.1090/S0002-9904-1967-11798-1

Elena Nikolaevna Makhrova,
Lobachevsky State University
of Nizhny Novgorod
Gagarin av. 23,
603022, Nizhny Novgorod, Russia
E-mail: elena_makhrova@inbox.ru