ISSN 2304-0122 Ufa Mathematical Journal. Vol. 17. No 3 (2025). P. 62-78.

doi:10.13108/2025-17-3-62

ON ORBITS IN $\mathbb{C}^4$ OF 7–DIMENSIONAL LIE ALGEBRAS POSSESSING TWO ABELIAN SUBALGEBRAS

A.V. LOBODA, R.S. AKOPYAN

Abstract. The paper focuses on the problem on description of holomorphically homogeneous real hypersurfaces of multidimensional complex spaces based on the properties of the Lie algebras and their nilpotent and Abelian subalgebras corresponding to these manifolds. Using classifications of a large family of 7–dimensional solvable non–decomposable Lie algebras, earlier we studied the orbits of algebras with "strong" commutative properties. In particular, it was established that a 7–dimensional Lie algebra with an Abelian subalgebra of dimension 5 admits no Levi nondegenerate orbits in the space $\mathbb{C}^4$.

In the present paper we study all 82 types of solvable non–decomposable 7–dimensional Lie algebras, which have exactly two 4–dimensional Abelian subalgebras and a 6–dimensional nilradical. We prove that for 75 types of algebras, any 7–dimensional orbit in $\mathbb{C}^4$ is either Levi–degenerate or can be reduced to a tubular manifold by a holomorphic transformation. We provide all (up to local holomorphic coordinate transformations) realizations of 7 exceptional types of abstract Lie algebras as algebras of holomorphic vector fields in $\mathbb{C}^4$. For most of these realizations, we give coordinate descriptions of orbits, which are holomorphically homogeneous nondegenerate real hypersurfaces in this space.

Key words: Lie algebra, nilradical, Abelian ideal, homogeneous manifold, holomorphic transformation, vector field, orbit of algebra, tubular manifold, real hypersurface.

Mathematics Subject Classification: 22F30, 57M60, 53C30

1. Introduction

In connection with the problem on description of holomorphically homogeneous real hypersurfaces if the space $\mathbb{C}^4$, in this paper we study orbits of 7–dimensional Lie algebras in this space. We clarify that the case of a 4–dimensional complex space is the next one after obtaining the complete descriptions of locally holomorphically homogeneous hypersurfaces in the spaces $\mathbb{C}^2$ and $\mathbb{C}^3$, see [5], [11], [12]. And the dimension 7 is the minimal possible dimension of the Lie algebra of holomorphic vector fields on a homogeneous hypersurface in the space $\mathbb{C}^4$.

The family of abstract 7–dimensional Lie algebras is quite wide. A subfamily of such algebras satisfying the solvability and non–decomposability conditions is described in four papers [13], [14], [18], [19]; this subfamily contains 939 types of Lie algebras. In this paper we discuss the Lie algebras from [18], which contains a complete description of 7–dimensional (solvable non–decomposable) Lie algebras with 6–dimensional nilradicals. We note that this description counts 594 types of Lie algebras.

The family of holomorphically homogeneous hypersurfaces in the space $\mathbb{C}^4$ also seems to be quite wide, and therefore it is natural to consider individual subfamilies of such manifolds. We refer, for example, to the works [15] and [16], in which Levi–degenerate but k–nondegenerate

A.V. Loboda, R.S. Akopyan, On orbits in $\mathbb{C}^4$ of 7–dimensional Lie algebras possessing two Abelian subalgebras.

The research by A.V. Loboda is supported by the Russian Science Foundation, grant no. 23-21-00109, (https://rscf.ru/project/23-21-00109).

Submitted September 3, 2024.

(for $k = 2, 3$) homogeneous hypersurfaces in the complex spaces are studied, and, in particular, in the spaces $\mathbb{C}^4$.

In [7] the question on estimating the number of different holomorphically homogeneous Levi nondegenerate nontubular real hypersurfaces in this space was considered. The main result of [7] states that the 7–dimensional Lie algebras from the list in [18] associated with Levi nondegenerate non–tubular real hypersurfaces of $\mathbb{C}^4$ make a relatively small part of this list. The present paper concretizes this qualitative statement and it is essentially a continuation of [7]. The main goal of the paper is to select algebras admitting Levi nondegenerate orbits in $\mathbb{C}^4$ that are not reducible to tubes from the list of 7–dimensional Lie algebras in [18] containing exactly two 4–dimensional Abelian subalgebras.

Such formulation of the problem is explained by the results of [6] and [9]. By virtue of these results, for a 7–dimensional Lie algebra containing an Abelian subalgebra of dimension 5 or three Abelian subalgebras of dimension 4, any 7–dimensional orbit in $\mathbb{C}^4$ is either Levi degenerate or holomorphically equivalent to a tubular hypersurface.

In this paper, we specify the number of 7–dimensional Lie algebras possessing 6–dimensional nilradicals and containing exactly two 4–dimensional Abelian subalgebras. In total, there are 82 types of such algebras in [18].

The main result of this paper is the following statement.

Theorem 1.1. *At most 7 of 82 types of 7–dimensional Lie algebras possessing 6–dimensional nilradicals and two 4–dimensional Abelian subalgebras admit Levi nondegenerate 7–dimensional orbits in the space $\mathbb{C}^4$ not reducible to tubes.*

We note that the coordinate descriptions of most of these orbits were obtained in [8] and they are given in Section 3 of this paper. All these hypersurfaces are Levi nondegenerate, but the question of their possible reducibility (due to holomorphic transformations) to tubes is more complicated. The authors do not yet have answers to this question; this explains the clarification «at most» in the formulation of Theorem 1.1.

2. Nilradicals of 7–dimensional Lie algebras

The number of different 7–dimensional Lie algebras in [18], amounting to 594 types of such algebras, seems excessive for a practical work with a list of such objects. At the same time, all these almost 600 types are continuations of only 34 types of 6–dimensional nilpotent Lie algebras. In [18], these 6–dimensional algebras are presented by their tables of commutation relations in some (canonical) bases. The consideration of our article is even more simplified by the fact that only 10 of the 34 types of 6–dimensional nilpotent algebras have 7–dimensional continuations containing exactly two 4–dimensional Abelian subalgebras.

Remark 2.1. *The reliability of such quantitative descriptions of the considered lists of Lie algebras can be provided only by computer programs. Some of them have already been implemented (see, for example, [3]), and several more computer algorithms and schemes for their application to the discussed problem are being developed.*

The list of 6–dimensional nilpotent Lie algebras form ones having in [18] the following codes:

$$[6,1],\quad [6,9],\quad [6,13],\quad [6,18],\quad [6,21],\quad [6,22],\quad [6,23],\quad [6,24],\quad [6,26],\quad [6,29]. \qquad (2.1)$$

Below we provide non–trivial commutation relations in these algebras (in some canonical bases $e_1, \dots, e_6$):

Algebra $[6,1](7)$: $[e_3, e_6] = e_2$, $[e_4, e_5] = e_2$, $[e_4, e_6] = e_3$, $[e_5, e_6] = e_4$;

Algebra $[6,9](8)$: $[e_3, e_5] = e_2$, $[e_3, e_6] = e_1$, $[e_4, e_5] = -e_1$, $[e_4, e_6] = e_2$;

Algebra $[6,13](1)$: $[e_4, e_5] = e_1$, $[e_2, e_6] = e_1$, $[e_3, e_6] = e_2$, $[e_4, e_6] = e_3$, $[e_5, e_6] = e_4$;

Algebra $[6,18](7)$: $[e_2,e_5]=e_1,\quad [e_3,e_6]=e_1,\quad [e_4,e_6]=e_3,\quad [e_5,e_6]=e_4;$
Algebra $[6,21](7)$: $[e_2,e_4]=e_1,\quad [e_3,e_6]=e_1,\quad [e_4,e_6]=e_2,\quad [e_5,e_6]=e_3;$
Algebra $[6,22](7)$: $[e_4,e_5]=e_1,\quad [e_3,e_6]=e_2,\quad [e_4,e_6]=e_3,\quad [e_5,e_6]=e_4;$
Algebra $[6,23](10)$: $[e_3,e_5]=e_2,\quad [e_3,e_6]=e_1,\quad [e_4,e_6]=e_2,\quad [e_5,e_6]=e_4;$
Algebra $[6,24](8)$: $[e_3,e_5]=e_1,\quad [e_4,e_5]=e_1,\quad [e_4,e_6]=e_2,\quad [e_5,e_6]=e_4;$
Algebra $[6,26](9)$: $[e_3,e_5]=e_2,\quad [e_4,e_5]=e_1,\quad [e_4,e_6]=e_2,\quad [e_5,e_6]=e_4;$
Algebra $[6,29](18)$: $[e_2,e_3]=e_1,\quad [e_4,e_6]=e_1,\quad [e_5,e_6]=e_4.$

We clarify that the number in the brackets after the code of each of these 6–dimensional Lie algebras means the number of types of 7–dimensional Lie algebras possessing the given 6–dimensional algebra as a nilradical (the sum of all ten numbers in the brackets makes the 82 types mentioned above).

Remark 2.2. *The provided list differs from the similar one in* [7]. *For example, it does not contain the algebra* $[6,25]$, *the 7–dimensional continuations of which contain a single 4–dimensional Abelian subalgebra. The algebra* $[6,25]$ *appeared in the list in* [7] *due to an oversight, which has now been corrected by rechecking (using computer algorithms) the structure of the discussed algebras.*

Remark 2.3. *In the course of this rechecking, two 6–dimensional algebras* $[6,21]$ *and* $[6,29]$, *not discussed in* [7], *are added to the list of 6–dimensional nilpotent algebras we are interesting in this paper. Each of the 7–dimensional Lie algebras with the nilradical* $[6,21]$ *or* $[6,29]$, *contains exactly two 4–dimensional Abelian subalgebras. Some of these 7–dimensional algebras admit Levi nondegenerate 7–dimensional orbits in* $\mathbb{C}^4$. *However, all such hypersurfaces are reducible to tubes by the same quadratic change of variables.*

We clarify that the proof of the last statement for a large family of 18 types of 7–dimensional algebras possessing the 6–dimensional algebra $[6,29]$ as a nilradical is presented in [4].

Below we discuss the remaining 64 of the mentioned 82 types of 7–dimensional Lie algebras. In this case, we study in detail the realizations of a block of 7 types of algebras with the nilradical $[6,21]$, which were not included in the descriptions of [7]. A separate Section 5 of the paper is devoted to the second block of 8 types of algebras with the nilradical $[6,9]$. For the algebras in this block, the intersection of two 4–dimensional Abelian subalgebras is 2–dimensional, and not 3–dimensional (as for all other continuations of nilpotent algebras from the list (2.1)). The third block includes the description of nondegenerate orbits not reducible to tubes of the remaining 49 types of aforementioned 7–dimensional Lie algebras; this description was obtained in [8].

The description of such orbits for Lie algebras from the three named blocks, together with the assertion that they are absent for 18 types of continuations of the nilpotent algebra $[6,29]$, constitutes a complete solution to the main problem formulated in the Introduction.

For a visual representation of the conclusions about the algebra $[6,21]$ and 7 types of its 7–dimensional continuations, we need the corresponding tables of commutation relations corresponding to these algebras. Table 1 provides the nontrivial commutators $[e_k,e_j]$ (k is the row number of the table, j is the column number) of the elements of the canonical basis for the family (type) of Lie algebras $[7,[6,21],1,1]$ from [18].

The tables of commutation relations for the remaining six types of 7–dimensional Lie algebras with a nilradical isomorphic to the algebra $[6,1]$ differ from Table 1 only in the last column (and, accordingly, in the last row). Such columns for all 7–dimensional Lie algebras, which have the codes $[7,[6,21],1,k]$, $k=1,\dots,7$, in [18], are given in Table 2.

In view of Tables 1 and 2 we easily see that each of the discussed algebras $[7,[6,21],1,k]$ for $k=1,\dots,7$ possesses two Abelian subalgebras $I_4=<e_1,e_2,e_3,e_5>$ and $I_4'=<e_1,e_3,e_4,e_5>$.

Table 1. Table of commutation relations for the family of Lie algebras $[7,[6,21],1,1]$. The parameter of the family is an arbitrary real number m ($[e_4,e_7]=me_4$).

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1							$(2m+1)e_1$
e_2				e_1			$(m+1)e_2$
e_3						e_1	$2me_3$
e_4		$-e_1$			e_2	e_2	me_4
e_5				$-e_2$		e_3	$(2m-1)e_5$
e_6			$-e_1$	$-e_2$	$-e_3$		e_6
e_7	$-(2m+1)e_1$	$-(m+1)e_2$	$-2me_3$	$-me_4$	$-(2m-1)e_5$	$-e_6$	

Table 2. The commutators $[e_j,e_7]$ ($j=1,\dots,6$) for Lie algebras with the nilradical $[6,21]$ and codes $[7,[6,21],1,k]$ ($k=1,\dots,7$). The columns of table are headed by the last two symbols of the code of each type of algebra. The parameter m is an arbitrary real number, $\varepsilon=\pm1$.

Type of algebra	$[1,1]$	$[1,2]$	$[1,3]$	$[1,4]$	$[1,5]$	$[1,6]$	$[1,7]$
$[e_1,e_7]$	$(2m+1)e_1$	$2e_1$	$3e_1$	$2e_1$	e_1	$2e_1$	$3e_1$
$[e_2,e_7]$	$(m+1)e_2$	e_2	$2e_2$	e_2	e_2	e_2	$2e_2+e_3$
$[e_3,e_7]$	$2me_3$	$2e_3$	$2e_3$	$2e_3$		$2e_3$	$2e_3$
$[e_4,e_7]$	me_4	e_4	e_4	e_4	e_3	e_2+e_4	e_4+e_5
$[e_5,e_7]$	$(2m-1)e_5$	$2e_5$	e_5	εe_1+2e_5	$-e_5$	me_1+2e_5	e_5
$[e_6,e_7]$	e_6		εe_5+e_6		e_2+e_6		$-e_4+me_5+e_6$

Remark 2.4. *The first of them is the Abelian ideal in each of the discussed 7–dimensional algebras, while the second is just the Abelian subalgebra since* $[e_4,e_6]=e_2\notin I'_4$.

It is important to note that the intersection $I_4\cap I'_4$ of these subalgebras has the dimension 3.

Let us now briefly recall the scheme for describing nondegenerate orbits in $\mathbb{C}^4$ of 7–dimensional Lie algebras presented in [7], which uses the existence of 6–dimensional nilradicals and two 4–dimensional Abelian subalgebras.

1. Each of the discussed abstract 7–dimensional algebras is to be represented as an algebra of holomorphic vector fields. For this, it is sufficient to realize a basis of the 7–dimensional algebra by such fields. At the same time, all fields are to be tangent in the real sense to some Levi nondegenerate hypersurface, which is the orbit of this algebra.
2. We denote by g_7 and N_6, respectively, any of the discussed 7–dimensional Lie algebras and its 6–dimensional nilradical. It is convenient to begin the discussion of realization of algebra g_7 in the space $\mathbb{C}^4$ with the realization of basis N_6 since such a problem does not always have a solution.
3. If we succeed to realize the basis N_6 in the form of holomorphic vector fields in $\mathbb{C}^4$, then the discussion of representations of algebra g_7 is reduced to the study of system of 6 linear differential equations
$$[e_k,e_7]=\sum_{j=1}^{6}\alpha_{kj}e_j \tag{2.2}$$
for the unknown field e_7 with the known, from the table of commutation relations, coefficients α_{kj}.

4. The above described steps of the general scheme simplify significantly if the discussed 7–dimensional algebra (and, in fact, its subalgebra N_6) contains two 4–dimensional Abelian subalgebras. According to the main lemma in [7], [17], the search for only nondegenerate non–tubular orbits of 7–dimensional algebras in such a situation can be carried out by assuming that the quadruple of basis fields of the nilpotent ideal N_6 has a very simple form. In particular, the equations of system (2.2) turn out to be ordinary differential equations (and not partial differential equations).
5. At the final step the conditions
$$\text{Re}\,(X(\Phi)|_M) \equiv 0 \tag{2.3}$$
of the real tangency of sought hypersurface M by any field X from the discussed algebra g_7 are written as a system of 7 equations (according to the number of basis fields of the algebra g_7). This system is to be integrated.

In the framework of this scheme, in [7, Thms. 1, 2], there were written the realizations in $\mathbb{C}^4$ of the bases of nilpotent 6–dimensional Lie algebras from the aforementioned list, see Step 2 in the scheme. We mean the algebras different from [6, 21], [6, 29].

We specify that instead of the usual writing of holomorphic vector fields in $\mathbb{C}^4$ of type
$$X = a(z)\frac{\partial}{\partial z_1} + b(z)\frac{\partial}{\partial z_2} + c(z)\frac{\partial}{\partial z_3} + d(z)\frac{\partial}{\partial z_4},$$
hereafter we use its shortened form $X = (a(z), b(z), c(z), d(z))$. In view of this adoption, under the assumption of the existence of nondegenerate non–tubular orbits for 7–dimensional continuations of, for example, the Lie algebras [6, 1] and [6, 9], the bases of the realizations in $\mathbb{C}^4$ of these 6–dimensional algebras, up to local holomorphic transformations, read
$$\begin{aligned}
e_1 &= (0,0,0,1), & e_1 &= (0,0,0,1),\\
e_2 &= (0,0,1,0), & e_2 &= (0,0,1,0),\\
e_3 &= (0,0,-(z_1+C_3),D_3), & e_3 &= (0,1,0,0),\\
e_4 &= (0,1,0,0), & e_4 &= (0,i\varepsilon,i\varepsilon z_1+C_4,z_1+D_4),\\
e_5 &= (0,-z_1,z_2,-\tfrac{1}{2}D_3z_1^2+D_5), & e_5 &= (1,0,z_2,0),\\
e_6 &= (1,z_1(z_1+C_3),-z_2(z_1+C_3),D_3z_2), & e_6 &= (i\varepsilon,B_6,B_6z_1+C_6,z_2+D_6),
\end{aligned} \tag{2.4}$$
respectively, with some complex constants B_k, C_k, D_k.

It is easy to obtain similar statements for the algebras [6, 21], [6, 29].

Proposition 2.1. *Under the existence of nondegenerate non–tubular orbits for 7–dimensional continuations of the Lie algebras [6, 21] and [6, 29], the bases of realizations in $\mathbb{C}^4$ of these 6–dimensional algebras, up to local holomorphic transformations, read*
$$\begin{aligned}
e_1 &= (0,0,0,1), & e_1 &= (0,0,0,1),\\
e_2 &= (0,1,0,0), & e_2 &= (0,1,0,0),\\
e_3 &= (0,0,1,0), & e_3 &= (0,B_3,C_3,z_2+C_3z_1),\\
e_4 &= (0,-z_1+B_4,C_4,z_2+C_4z_1), & e_4 &= (0,0,1,0),\\
e_5 &= (0,0,-z_1+C_5,-\tfrac{1}{2}z_1^2+C_5z_1+D_5), & e_5 &= (0,0,-z_1,-\tfrac{1}{2}z_1^2+D_5),\\
e_6 &= (1,0,0,z_3), & e_6 &= (1,0,0,z_3),
\end{aligned} \tag{2.5}$$
respectively, where B_k, C_k, D_k are some complex constants.

Proof. We note that the basis for the realization in $\mathbb{C}^4$ of the 6–dimensional Lie algebra $[6,21]$ from the formulation of this proposition is constructed in [8, Proposition 2]. Here we present the derivation of formulas for the basis of the algebra $[6,29]$ realized in $\mathbb{C}^4$.

The description of this algebra by commutation relations at the beginning of the section shows that the 4-dimensional Abelian subalgebras of the nilpotent algebra $[6,29]$ are only $I_4 =< e_1, e_2, e_4, e_5 >$ and $I_4' =< e_1, e_3, e_4, e_5 >$. It can be verified that for each of the 18 continuations of $[6,29]$ to 7–dimensional Lie algebras described in [18], their 4–dimensional Abelian subalgebras are also only I_4 and I_4'. We consider an ordered set e_1, e_4, e_5, e_2 of pairwise commuting holomorphic vector fields tangent to a hypothetical nondegenerate hypersurface M of the space $\mathbb{C}^4$. Then we apply Lemma 1 from [7] (or Lemma 4.1 from [6]) on three variants of simplification of this set by holomorphic transformations. In two of these three variants of the realization of $[6,29]$, it is possible to rectify (transform into differentiations with respect to four independent variables z_k) the basis elements of the subalgebra I_4 (the first variant), or the subalgebra I_4' (the second variant). This implies the tubularity of hypersurface M.

Since we discuss homogeneous hypersurfaces not reducible to tubes, the only way in which such surfaces can exist is related with the simplification of the basis I_4 to

$$e_1 = (0,0,0,1), \qquad e_4 = (0,0,1,0), \qquad e_2 = (0,1,0,0), \qquad e_5 = (0,0,c_5(z_1), d_5(z_1))$$

by a holomorphic change of variables.

Now we consider the commutators of the fields e_3, e_6 with a triple of rectified fields e_1, e_2, e_4. Taking into account the non–trivial commutation relations $[e_2, e_3] = [e_4, e_6] = e_1$ in the algebra $[6,29]$, and the vanishing of the remaining commutators, we obtain the following simplified form of these fields

$$e_3 = (a_3(z_1), b_3(z_1), c_3(z_1), z_2 + d_3(z_1)), \qquad e_6 = (a_6(z_1), b_6(z_1), c_6(z_1), z_3 + d_6(z_1))$$

with some holomorphic functions $a_k(z_1)$, $b_k(z_1)$, $c_k(z_1)$, $d_k(z_1)$ of a single complex variable. The third component of the latter non–trivial relation $[e_5, e_6] = e_4$ implies the inequality $a_6(z_1) \neq 0$. The approaches of the works [1] and [10] then allow us to keep the rectified fields e_1, e_2, e_4 and to reduce the field e_6 by a holomorphic change to $e_6 = (1,0,0,z_3)$. After this, the simplification of the fields e_3 and e_5 to the form stated in Proposition 2.1 becomes in fact trivial. The proof is complete. □

The formulas of type (2.4) and (2.5) allow us to proceed to subsequent Steps 3 and 4 of the above scheme for describing Levi nondegenerate non–tubular orbits of the discussed 7–dimensional Lie algebras. We specify that in [7] only separate steps of this scheme were realized for some of the algebras we are interesting in. We shall discuss this issue systematically in the next sections.

3. Necessary conditions for existence of nondegenerate orbits

For 7–dimensional continuations of seven nilpotent algebras

$$[6,1], \quad [6,13], \quad [6,18], \quad [6,22], \quad [6,23], \quad [6,24], \quad [6,26], \tag{3.1}$$

being the main object of study in [7], the above scheme for investigating of orbits was completely realized in [8]. Using combined computer–manual algorithms, the system of differential equations (2.2) was investigated for 49 types of 7–dimensional Lie algebras possessing 6–dimensional algebras from the list (3.1) as nilradicals. However, the format of paper [8] and its focus primarily on the algorithmic side of the problem did not allow us to present (and discuss) important intermediate results. One of them is the following statement, which is of an independent mathematical interest.

Proposition 3.1. *Among the 49 types of Lie algebras possessing a 6–dimensional nilradical of one of the types (3.1) and two Abelian 4–dimensional subalgebras, at most 9 types can have nondegenerate non–tube–reducible orbits in $\mathbb{C}^4$. By holomorphic changes of coordinates, the bases of such algebras, realized as algebras of vector fields in $\mathbb{C}^4$, can be reduced to*

$[7,[6,1],1,1] \quad (\operatorname{Im} D_3 \neq 0):$
$e_1 = (0,0,0,1),$
$e_2 = (0,0,1,0),$
$e_3 = (0,0,-z_1,D_3),$
$e_4 = (0,1,0,0),$
$e_5 = (0,-z_1,z_2,-\frac{1}{2}D_3 z_1^2),$
$e_6 = (1,z_1^2,-z_1z_2,D_3z_2),$

$[7,[6,13],1,1]:$
$e_1 = (0,0,0,1),$
$e_2 = (0,0,1,0),$
$e_3 = (0,0,-z_1,-\frac{1}{2}z_1^2),$
$e_4 = (0,1,0,0),$
$e_5 = (0,-z_1,\frac{1}{3}z_1^3,z_2),$
$e_6 = (1,\frac{5}{6}z_1^3,-z_1z_2,z_3-\frac{1}{2}z_1^2z_2),$
$e_7 = (z_1,3z_2,5z_3,4z_4);$

$[7,[6,18],1,1] \quad (a=3,\ B_5 \in \mathbb{R}\setminus\{0\}):$
$e_1 = (0,0,0,1),$
$e_2 = (0,1,0,0),$
$e_3 = (0,0,1,0),$
$e_4 = (0,0,-z_1,-\frac{1}{2}z_1^2),$
$e_5 = (0,iB_5,\frac{1}{2}z_1^2,z_2+\frac{1}{3}z_1^3),$
$e_6 = (1,0,0,z_3),$
$e_7 = (z_1,3z_2,5z_3,6z_4);$

$[7,[6,22],1,1] \quad (a=1,\operatorname{Im} D_3 \neq 0):$
$e_1 = (0,0,0,1),$
$e_2 = (0,0,1,0),$
$e_3 = (0,0,-z_1,D_3),$
$e_4 = (0,1,0,0),$
$e_5 = (0,-z_1,\frac{1}{3}z_1^3,z_2),$
$e_6 = (1,-D_3z_1,-z_1z_2,D_3z_2),$
$e_7 = (z_1,2z_2,4z_3,3z_4);$

$[7,[6,23],1,1] \quad (a=2,\ D_4 \in \mathbb{R}\setminus\{0\}):$
$e_1 = (0,0,0,1),$
$e_2 = (0,0,1,0),$
$e_3 = (0,1,0,0),$
$e_4 = (0,0,-z_1,iD_4),$
$e_5 = (0,iD_4,z_2+\frac{1}{2}z_1^2,0),$
$e_6 = (1,0,0,z_2),$
$e_7 = (z_1,2z_2,4z_3,3z_4);$

$[7,[6,24],1,2] \quad (C_3 \in \mathbb{R}\setminus\{0\}):$
$e_1 = (0,0,0,1),$
$e_2 = (0,0,1,0),$
$e_3 = (0,0,iC_3,-z_1),$
$e_4 = (0,1,0,0),$
$e_5 = (1,0,0,z_2),$
$e_6 = (0,z_1,z_2,\frac{1}{2}z_1^2),$
$e_7 = (z_1,z_2,z_3,2z_4);$

$[7,[6,24],1,4] \quad (\varepsilon=\pm1, C_3 \in \mathbb{R}\setminus\{0\}):$
$e_1 = (0,0,0,1),$
$e_2 = (0,0,1,0),$
$e_3 = (0,0,iC_3,-z_1),$
$e_4 = (0,1,0,0),$
$e_5 = (1,0,0,z_2),$

$[7,[6,24],1,5] \quad (\operatorname{Im} C_3 \neq 0):$
$e_1 = (0,0,0,1),$
$e_2 = (0,0,1,0),$
$e_3 = (0,0,C_3,-z_1),$
$e_4 = (0,1,0,0),$
$e_5 = (1,0,0,z_2),$

$$e_6 = (0, z_1, z_2, \frac{1}{2}z_1^2),$$
$$e_7 = (z_1, z_2, z_3 + \varepsilon z_1, 2z_4);$$

$$e_6 = (0, z_1, z_2, \frac{1}{2}z_1^2),$$
$$e_7 = (z_1, z_2, z_3 + C_3 z_1, 2z_4 - \frac{1}{2}z_1^2);$$

$$\begin{aligned}
&[7, [6, 26], 1, 1] \quad (a = \frac{1}{2}, D_3 \in \mathbb{R} \setminus \{0\}) :\\
&e_1 = (0, 0, 0, 1),\\
&e_2 = (0, 0, 1, 0),\\
&e_3 = (0, 0, -z_1, iD_3),\\
&e_4 = (0, 1, 0, 0),\\
&e_5 = (1, 0, 0, z_2),\\
&e_6 = (0, z_1, z_2, \frac{1}{2}z_1^2),\\
&e_7 = (\frac{1}{2}z_1, \frac{3}{2}z_2, \frac{5}{2}z_3, 2z_4).
\end{aligned}$$

The significant reduction in the potential number of Levi nondegenerate non–tubular orbits fixed in Proposition 3.1 was illustrated in [8] by the example of continuations of the algebra $[6, 21]$. According to Tables 1 and 2 in Section 1, there are 7 types of such continuations $[7, [6, 21], 1, k]$, $k \in \{1, \dots, 7\}$. We present here the assertion from [8] about the orbits of these algebras.

Proposition 3.2. *The realizations of algebras in the family* $[7, [6, 21], 1, k]$ *in* $\mathbb{C}^4$ *that have nondegenerate non–tubular orbits are possible only for* $k \in \{1, 3, 7\}$*. In this case, in accordance with the formulas (2.5), the basis fields* e_1*,* e_2*,* e_3 *of the algebras under discussion have (in suitable coordinates) the form*

$$e_1 = (0, 0, 0, 1), \qquad e_2 = (0, 1, 0, 0), \qquad e_3 = (0, 0, 1, 0), \tag{3.2}$$

while the fields e_4*,* e_5*,* e_6*,* e_7 *are determined by the formulas*

$$\begin{aligned}
&[7, [6, 21], 1, 1] \quad (m, B_7, D_7 \in \mathbb{R}; C_4, C_7 \in \mathbb{C};\ m \cdot C_4 = 0) :\\
&e_4 = (0, -z_1, C_4, z_2 + C_4 z_1),\\
&e_5 = (0, 0, -z_1, -\frac{1}{2}z_1^2),\\
&e_6 = (1, 0, 0, z_3),\\
&e_7 = (z_1, (m + 1)z_2 + iB_7, 2mz_3 + C_7, (2m + 1)z_4 + C_7 z_1 + iD_7),
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
&[7, [6, 21], 1, 3] \quad (\varepsilon = \pm 1; D_6 \in \mathbb{C}) :\\
&e_4 = (0, -z_1, 0, z_2),\\
&e_5 = (0, 0, -z_1, -\frac{1}{2}z_1^2),\\
&e_6 = (1, 0, 0, z_3 + D_6),\\
&e_7 = (z_1, 2z_2, 2z_3 - \frac{\varepsilon}{2}z_1^2, 3z_4 - \frac{\varepsilon}{3}z_1^3 - 2D_6 z_1).
\end{aligned} \tag{3.4}$$

$$
\begin{aligned}
&[7,[6,21],1,7] \quad (m, D_4, D_6 \in \mathbb{R}; D_7 \in \mathbb{C}):\\
&e_4 = (0, -z_1, 0, z_2 + iD_4),\\
&e_5 = (0, 0, -z_1, -\frac{1}{2}z_1^2),\\
&e_6 = (1, 0, 0, z_3 + iD_6),\\
&e_7 = (z_1, 2z_2 + \frac{1}{2}z_1^2, z_2 + 2z_3 - \frac{m}{2}z_1^2, 3z_4 - \frac{m}{3}z_1^3 + D_7 z_1).
\end{aligned} \tag{3.5}
$$

Proof. All discussions related to this proposition naturally fall into two parts. In the first of them we derive the formulas (3.3), (3.4) and (3.5) obtained by routine calculations using the Maple package according to the scheme described in Section 2. The second part is related to obtaining contradictions that arise when trying to implement the same scheme for algebras of the discussed family for $k \in \{2, 4, 5, 6\}$.

For instance, for $k = 2$ according to the formulas of proposition 2.1, the first five basis fields of the algebra $[6, 21]$ have a zero first component. When calculating the preliminary form of the components of the field e_7 for this algebra, we use the form of the fields e_1, e_2, e_3, e_6 and the column of Table 2 describing the commutators $[e_j, e_7]$ in the algebra $[7, [6, 21], 1, 2]$. This yields the following description of the field

$$e_7 = (A_7, z_2 + B_7, 2z_3 + C_7, 2z_4 + C_7 z_1 + D_7) \tag{3.6}$$

with some complex constants A_7, B_7, C_7, D_7. The values of these constants and, first of all, the parameter A_7 should satisfy two more necessary conditions, namely, the commutation relations $[e_4, e_7] = e_4$ and $[e_5, e_7] = e_5$. In view of (3.6), the second component of the first of these relations reads

$$(-z_1 + B_4) + A_7 = (-z_1 + B_4).$$

This means that the first component of the field e_7 turns out to be zero for $k = 2$. The same reasoning (not affecting the commutator $[e_5, e_7]$) remains valid for $k = 4$. For $k = 6$ we similarly arrive at the conclusion $A_7 = 1$, so that for the basis field $e_7^* = e_7 - e_6$ of the realization of the algebra $[7, [6, 21], 1, 6]$ we again obtain a zero first component.

It is easy to see that the presence of six basis vector fields with zero z_1–component in the 7-dimensional Lie algebra in the space $\mathbb{C}^4$ leads, by the identity (2.2), to the independence of equation of any of its orbits of three complex variables z_2, z_3, z_4. Thus, the result of the procedure aimed at finding nondegenerate orbits for the 7–dimensional algebras $[7, [6, 21], 1, k]$ with $k = 2, 4, 6$ turns out to be only degenerate hypersurfaces.

Another algebra corresponding to the case $k = 5$ also does not admit realizations on nondegenerate non–tubular hypersurfaces of the space $\mathbb{C}^4$. Here the formula, similar to (3.6), has the form

$$e_7 = (z_1 + A_7, z_1 + z_2 + B_7, C_7, z_4 + C_7 z_1 + D_7).$$

In this case, according to Table 2, the relation $[e_4, e_7] = e_3$ should be satisfied for this algebra. However, the third components of both fields e_4 and e_7 are constants, and therefore this component is equal to zero for their commutator. This contradicts the form $e_3 = (0, 0, 1, 0)$ and, as a consequence, the existence of realizations of the algebra $[7, [6, 21], 1, 5]$ with the discussed properties. The proof is complete. □

4. Tubular orbits of discussed algebras

Propositions 3.1 and 3.2 justify the qualitative result of [7] on a «small» number of Lie algebras possessing the orbits with two properties

A) nondegeneracy,

B) irreducibility to tubes.
In this case, the reduction from 56 considered types of Lie algebras in two blocks to 9 + 3 written out specific types of bases is a significant advance in the discussed problem. However, in this section we shall show that the estimate of 12 is also incomplete and too overestimated.

Let us formulate precise statements that allow us to make such a conclusion.

Proposition 4.1 ([8]). *All nondegenerate orbits in $\mathbb{C}^4$ of 7–dimensional Lie algebras of four types*

$$[7,[6,18],1,1], \qquad [7,[6,24],1,2], \qquad [7,[6,24],1,4] \qquad [7,[6,24],1,5]$$

from Proposition 3.1 are holomorphically equivalent to tubular manifolds.

Proposition 4.2. *All regular 7–dimensional orbits of the algebras $[7,[6,21],1,k]$ in Proposition 3.2 are holomorphically equivalent to tubular manifolds.*

The proofs of these statements can be obtained by direct integration of systems of partial differential equations corresponding to the above written bases of realizations of all algebras from Propositions 3.1 and 3.2. The procedure of such integration gives visual descriptions of the orbits of these algebras, but it is quite tedious. Here we shall provide an arguing, which proves in a general way (without detailed calculations) the tubular character of the studied orbits of all three types of Lie algebras from Proposition 4.2.

Proof. Up to a translation of the variable $z_3^* = z_3 + D_6$, the bases of all three types, defined by formulas (3.2), (3.3), (3.4) and (3.5), can be represented in a unified generalized form

$$\begin{aligned} e_1 &=(0,0,0,1), & e_5 &= (0,0,-z_1,-\frac{1}{2}z_1^2),\\ e_2 &=(0,1,0,0), & e_6 &= (1,0,0,z_3),\\ e_3 &=(0,0,1,0), & e_7 &= (z_1,T_2(z),T_3(z),T_4(z)),\\ e_4 &=(0,-z_1,C_4,T_1(z)), \end{aligned} \tag{4.1}$$

where $T_k(z)$ are some holomorphic functions of $z = (z_1, z_2, z_3, z_4)$, the exact form of which is not essential for the subsequent discussions.

We clarify that the orbit in the space $\mathbb{C}^4$ of the Lie algebra with such a basis can, generally speaking, have a dimension less than 7. The existence of precisely the 7-dimensional orbit M passing through the point $Q \in \mathbb{C}^4$ is guaranteed (due to the Frobenius theorem [2]) by the condition of completeness of rank of the (7×8)–matrix formed by the real and imaginary parts of components of basis fields of algebra at the point Q.

We note that the presence of three rectified fields in this algebra means that the defining function of each orbit of the algebra is independent of the three real variables $x_k = \operatorname{Re} z_k$ $(k = 2, 3, 4)$. This allows us to simplify the check of completeness of the rank by removing from the mentioned (7×8)–matrix three rows (and, accordingly, three columns) corresponding to these fields. The resulting matrix of dimensions (4×5) for the basis described by the formulas (4.1) reads

$$\begin{pmatrix} 0 & 0 & -y_1 & C_{42} & t_1\\ 0 & 0 & 0 & y_1 & x_1y_1\\ 1 & 0 & 0 & 0 & y_3\\ x_1 & y_1 & t_2 & t_3 & t_4 \end{pmatrix}, \tag{4.2}$$

where $x_1 = \operatorname{Re} z_1$, $y_k = \operatorname{Im} z_k$, $t_k = \operatorname{Im} T_k$, $C_{42} = \operatorname{Im} C_4$.

It is clear that for $y_1 = 0$ the rows of matrix (4.2) are linearly dependent, so that the orbits of algebra (4.1) passing through the points of the space $\mathbb{C}^4$ with zero coordinate $\operatorname{Im} z_1$ are not regular hypersurfaces. And for $y_1 \neq 0$ the 4th–order minor of the matrix (4.1), obtained by discarding its last column and equal to $-y_1^3$, is nonzero. This ensures the full rank of matrix (4.2) and the set of fields (4.1).

The set of Equations (2.3) for 7 fields (4.1) is a system of linear equations with respect to 8 partial derivatives

$$\frac{\partial \Phi}{\partial x_k}, \qquad \frac{\partial \Phi}{\partial y_k}, \qquad k = 1, 2, 3, 4.$$

In the truncated version associated with the matrix (4.2) this system is linear with respect to five derivatives

$$\frac{\partial \Phi}{\partial x_1}, \qquad \frac{\partial \Phi}{\partial y_k}, \qquad k = 1, 2, 3, 4.$$

For the Lie algebra with a basis (4.1), we consider the value of the derivative $\dfrac{\partial \Phi}{\partial y_4}$ at some point of the regular orbit M in the space $\mathbb{C}^4$. This derivative cannot vanish because then at this point (due to the difference from zero of the minor considered above) all 8 partial derivatives would be zero. This contradicts the regularity of the discussed orbit.

Employing the inequality $\dfrac{\partial \Phi}{\partial y_4} \neq 0$, we can write the equation of the discussed orbit M (near a fixed point) in a form resolved with respect to y_4

$$y_4 = F(x_1, y_1, y_2, y_3). \tag{4.3}$$

Then for the defining function of M we have

$$\Phi = -y_4 + F(x_1, y_1, y_2, y_3) \quad \text{and} \quad \frac{\partial \Phi}{\partial y_4} = -1.$$

For a point $Q \in \mathbb{C}^4$ with a nonzero coordinate y_1 and the 7–dimensional orbit of the algebra (4.1) defined by Equation (4.3) and passing through this point, we consider the relations (2.3) corresponding to the second and third rows of the matrix (4.2). They read

$$\frac{\partial F(x_1, y_1, y_2, y_3)}{\partial y_3} = x_1, \qquad \frac{\partial F(x_1, y_1, y_2, y_3)}{\partial x_1} = y_3.$$

This means that any 7–dimensional regular orbit in the space $\mathbb{C}^4$ for algebras with a basis of the form (4.1) can be described by the equation

$$y_4 = x_1 y_3 + G(y_1, y_2) \tag{4.4}$$

with some analytical function of two variables G. Adding and subtracting the expression $x_3 y_1$ to the right hand side of the resulting equation and taking into consideration the identity

$$\operatorname{Im}(z_1 z_3) = x_1 y_3 + x_3 y_1,$$

after the holomorphic change of variables $z_3^* = -iz_3$, $z_4^* = z_4 - z_1 z_3$ we obtain a tubular equation of the surface (4.4)

$$y_4^* = y_1 y_3^* + G(y_1, y_2).$$

The proof is complete. □

5. Orbits of algebras with nilradical [6, 9]

We proceed to the presentation of a result similar to Propositions 3.1 and 3.2 and related to the algebra [6, 9]. First of all, we supplement the list of commutation relations of this algebra, given in Section 1, with commutators of the basis fields of this 6–dimensional algebra with the field e_7 within each of its eight 7–dimensional continuations. We present this information in the form of the following tables.

TABLE 3. Table of commutation relations for the Lie algebras $[7,[6,9],1,k]$ ($k = 1,2,3,4$). The parameters of individual algebras are arbitrary real numbers m, n.

Type of algebra	$[7,[6,9],1,1]$	$[7,[6,9],1,2]$	$[7,[6,9],1,3]$	$[7,[6,9],1,4]$
$[e_1,e_7]$	$2e_1-2me_2$	$-2e_2$		$2e_1$
$[e_2,e_7]$	$2me_1+2e_2$	$2e_1$		$2e_2$
$[e_3,e_7]$	$(n+1)e_3-me_4$	me_3-e_4	e_3	
$[e_4,e_7]$	$me_3+(n+1)e_4$	e_3+me_4	e_4	
$[e_5,e_7]$	$(1-n)e_5+me_6$	$-me_5+e_6$	$-e_5$	$2e_5$
$[e_6,e_7]$	$-me_5+(1-n)e_6$	e_5+me_6	$-e_6$	e_1+2e_6

TABLE 4. Table of commutation relations for the Lie algebras $[7,[6,9],2,k]$ ($k = 1,2$) and $[7,[6,9],3,k]$ ($k=1,2$). The parameters are $m,n,p\in\mathbb{R}$, $\varepsilon=\pm1$.

Type of algebra	$[7,[6,9],2,1]$	$[7,[6,9],2,2]$	$[7,[6,9],3,1]$	$[7,[6,9],3,2]$
$[e_1,e_7]$	$2e_1-2me_2$	$-2e_2$	$2me_1-2ne_2$	$2me_1+2e_2$
$[e_2,e_7]$	$2me_1+2e_2$	$2e_1$	$2ne_1+2me_2$	$-2e_1+2me_2$
$[e_3,e_7]$	e_3-me_4	$-e_4$	$(m+p)e_3-(n+\varepsilon)e_4$	$4e_4$
$[e_4,e_7]$	me_3+e_4	e_3	$(n+\varepsilon)e_3+(m+p)e_4$	$-4e_3$
$[e_5,e_7]$	$\varepsilon e_3+e_5+me_6$	εe_3+e_6	$(m-p)e_5+(n-\varepsilon)e_6$	$2me_5+2e_6$
$[e_6,e_7]$	$-\varepsilon e_4+me_5+e_6$	$-\varepsilon e_4-e_5$	$-(n-\varepsilon)e_5+(m-p)e_6$	$e_1-2e_5+2me_6$

Proposition 5.1. *Among the 8 types of 7–dimensional Lie algebras in the list in [18], which possess 6–dimensional nilradical $[6,9]$, at most 3 types can have nondegenerate non–tube–reducible orbits in $\mathbb{C}^4$. For all realizations of these three types, the basis fields $e_1,\dots,e_5$ have, up to holomorphic changes of coordinates, the following general form ($\varepsilon=\pm1$):*

$$e_1=(0,0,0,1),\quad e_2=(0,0,1,0),\quad e_3=(0,1,0,0),\quad e_4=(0,i\varepsilon,i\varepsilon z_1,z_1),\quad e_5=(1,0,z_2,0).$$

The two remaining fields of each algebra are described by the following formulas:

$$\begin{aligned}&[7,[6,9],1,1]\quad (m\in\mathbb{R}, n\geqslant 0; B_k, C_k, D_k, \hat{B}_7\in\mathbb{C};\ n\cdot B_6=0):\\ &e_6=(i\varepsilon, B_6, B_6z_1+C_6, z_2+D_6),\qquad e_7=(a_7(z),b_7(z),c_7(z),d_7(z)),\end{aligned}\tag{5.1}$$

where

$$\begin{aligned}&a_7(z)=(i\varepsilon m+1-n)z_1,\quad b_7(z)=(n+1-i\varepsilon m)z_2+mB_6z_1+B_7,\\ &c_7(z)=-i\varepsilon mz_1z_2+2z_3-2mz_4+mB_6z_1^2+\hat{B}_7z_1+C_7,\\ &d_7(z)=-mz_1z_2+2mz_3+2z_4+mD_6z_1+D_7;\end{aligned}$$

and

$$\begin{aligned}&[7,[6,9],1,2]\quad (m\geqslant 0; B_k, C_k, D_k, \hat{B}_7\in\mathbb{C};\ m\cdot B_6=0):\\ &e_6=(i\varepsilon, B_6, B_6z_1+C_6, z_2),\quad e_7=(a_7(z),b_7(z),c_7(z),d_7(z)),\end{aligned}\tag{5.2}$$

where

$$\begin{aligned}&a_7(z)=-(m-i\varepsilon)z_1,\qquad b_7(z)=(m-i\varepsilon)z_2+B_6z_1+2C_6,\\ &c_7(z)=-2z_4-i\varepsilon z_1z_2+B_6z_1^2+im\varepsilon C_6z_1,\qquad d_7(z)=2z_3-z_1z_2;\end{aligned}$$

and

$$\begin{aligned}&[7,[6,9],1,3]\quad (D_6\in\mathbb{C}, C_7, D_7\in\mathbb{R}):\\ &e_6=(i\varepsilon,0,-i\varepsilon D_6, z_2+D_6),\qquad e_7=(-z_1, z_2+D_6, D_6z_1+C_7, D_7).\end{aligned}\tag{5.3}$$

Proof. As in the proof of Proposition 3.2, the arguing can be split into two parts.

In one of them, we obtain the formulas (5.1)–(5.3) for the basis fields e_6, e_7 of each of the three families admitting the realizations in $\mathbb{C}^4$, which satisfy the necessary conditions A) and B). This part of the proof is based on the formulas (2.4) common to all algebras with nilradical [6,9].

For instance, for the algebra $[7,[6,9],1,3]$, the consideration of four commutation relations $[e_j, e_7]$, $j = 1,2,3,5$, leads us to the preliminary formula

$$e_7 = (-(z_1 + A_7), z_2 + B_7, B_7 z_1 + C_7, D_7) \tag{5.4}$$

with some complex constants A_7, B_7, C_7, D_7.

It remains to write out the components of two relations $[e_4, e_7] = e_4$ and $[e_6, e_7] = -e_6$ to clarify the possible values of these constants as well as some of the parameters in the formulas (2.4) for the fields e_4, e_6. For instance, calculating the commutator $[e_4, e_7]$, we find

$$[e_4, e_7] = i\varepsilon(0,1,0,0) + (z_1 + A_7)(0,0,i\varepsilon,1).$$

Equating the components of this vector field to the components of e_4, we obtain the identities

$$A_7 = D_4, \qquad C_4 = i\varepsilon D_4. \tag{5.5}$$

In the same way by the relation $[e_6, e_7] = -e_6$ we obtain the following necessary conditions for its validity

$$B_6 = 0, \qquad C_6 = -i\varepsilon B_7, \qquad B_7 = D_6. \tag{5.6}$$

Substituting the formulas (5.5) and (5.6) into (2.4) and (5.4), in view of the possibility to replace the field e_7 by $e_7 - (\operatorname{Re} C_7)e_2 - (\operatorname{Re} D_7)e_1)$, we get the desired statement for the algebra $[7,[6,9],1,3]$. Similar procedure for the families of algebras $[7,[6,9],1,1]$ and $[7,[6,9],1,2]$ leads us, after much more extensive calculations, to the stated formulas for the basis fields of these algebras.

The second part of the proof is related to the remaining Lie algebras having the nilradical $[6,9]$ and to obtaining conclusions for them, which contradict the discussed necessary conditions A) and B) of nondegeneracy and irreducibility to tubes (see Section 4). For illustration, here we provide a discussion for one of these algebras, namely, for $[7,[6,9],1,4]$.

For this algebra, the commutators of the four fields e_1, e_2, e_3, e_5 with the field e_7 are described by the last column of Table 3 (the components of these fields are given in the formulas (2.4)). Due to these clarifications, for the field e_7 we obtain the following preliminary form

$$e_7 = (2z_1 + A_7, B_7, 2z_3 + B_7 z_1 + C_7, 2z_4 + D_7) \tag{5.7}$$

with some complex constants A_7, B_7, C_7, D_7.

Now we consider the commutation relation $[e_6, e_7] = e_1 + 2e_6$. In view of (5.7) and the formula $e_6 = (i\varepsilon, B_6, B_6 z_1 + C_6, z_2 + D_6)$ in (2.4) the left hand side of this relation in the vector form reads

$$i\varepsilon \begin{pmatrix} 2 \\ 0 \\ B_7 \\ 0 \end{pmatrix} + (B_6 z_1 + C_6) \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} + (z_2 + D_6) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} - (2z_1 + A_7) \begin{pmatrix} 0 \\ 0 \\ B_6 \\ 0 \end{pmatrix} - B_7 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Then, in component notation, the considered relationship becomes the system of four scalar equations

$$\begin{aligned} &2i\varepsilon = 2i\varepsilon, \qquad 0 = 2B_6, \\ &i\varepsilon B_7 + 2(B_6 z_1 + C_6) - B_6(2z_1 + A_7) = 2(B_6 z_1 + C_6), \\ &2(z_2 + D_6) - B_7 = 2(z_2 + D_6) + 1. \end{aligned}$$

By the second equation in the last two equalities, we arrive at contradicting conditions

$$B_7 = 0 \quad \text{and} \quad B_7 = -1.$$

This means that it is impossible to realize the algebra $[7, [6, 9], 1, 4]$ in $\mathbb{C}^4$ satisfying the necessary conditions for the existence of at least one nondegenerate non–tubular orbit in this realization.

The algebras of the families $[7, [6, 9], 2, k]$ $(k = 1, 2)$ and $[7, [6, 9], 3, k]$ $(k = 1, 2)$ are discussed similarly. The proof is complete. □

Proposition 5.2. *For any values of the parameters in the formulas (5.3), all real hypersurfaces of the space* $\mathbb{C}^4$*, which are orbits of holomorphic realizations of the algebra* $[7, [6, 9], 1, 3]$*, turn out to be Levi–degenerate.*

Proof. The presence of a triple of rectified fields in the basis (5.3) means that the defining function of the orbit is independent of the variables $x_k = \operatorname{Re} z_k$ $(k = 1, 2, 3)$. The system of 4 equations for the defining function $\Phi(x_1, y_1, y_2, e_3, y_4)$ of any of the sought orbits corresponding to the remaining basis fields reads

$$\begin{aligned}
e_4 :& \quad \varepsilon \frac{\partial \Phi}{\partial y_2} + \varepsilon x_1 \frac{\partial \Phi}{\partial y_3} + y_1 \frac{\partial \Phi}{\partial y_4} = 0, \\
e_5 :& \quad \frac{\partial \Phi}{\partial x_1} + (y_2 - D_{62}) \frac{\partial \Phi}{\partial y_3} = 0, \\
e_6 :& \quad \varepsilon \frac{\partial \Phi}{\partial y_1} - \varepsilon D_{61} \frac{\partial \Phi}{\partial y_3} + y_2 \frac{\partial \Phi}{\partial y_4} = 0, \\
e_7 :& \quad -x_1 \frac{\partial \Phi}{\partial x_1} - y_1 \frac{\partial \Phi}{\partial y_1} + y_2 \frac{\partial \Phi}{\partial y_2} + (C_{72} + D_{61} y_1 + D_{62} x_1) \frac{\partial \Phi}{\partial y_3} + D_{72} \frac{\partial \Phi}{\partial y_4} = 0.
\end{aligned} \tag{5.8}$$

The solution to the second, shortest of these equations is the function

$$\Phi = F(y_3 - x_1(y_2 - D_{62}), y_1, y_2, y_4). \tag{5.9}$$

Denoting the arguments of this function, respectively, by t_1, t_2, t_3, t_4, we substitute (5.9) into the three remaining equations of the system (5.8). In the new variables, the equation corresponding to the field e_4 becomes

$$\varepsilon \frac{\partial F}{\partial t_3} + y_1 \frac{\partial F}{\partial t_4} = 0,$$

and its solution is $F = G(t_1, t_2, t_4 - \varepsilon t_2 t_3)$. In the new variables $s_1 = t_1$, $s_2 = t_2$, $s_3 = t_4 - \varepsilon t_2 t_3$ two latter equations of the studied system are written as

$$\varepsilon \left(\frac{\partial G}{\partial s_2} - D_{61} \frac{\partial G}{\partial s_1} \right) = 0, \qquad (C_{72} + D_{61}) \frac{\partial G}{\partial s_1} - s_2 \frac{\partial G}{\partial s_2} + D_{72} \frac{\partial G}{\partial s_3} = 0.$$

In the variables $\xi_1 = s_1 + D_{61} s_2$, $\xi_2 = s_3$ the solution to the system of these two equations reads

$$G = \varphi(D_{72}\xi_1 - C_{72}\xi_2)$$

with an arbitrary analytic function of one variable. This means that the solution of initial system of four equations is described by the formula

$$\Phi = D_{72}\xi_1 - C_{72}\xi_2 = A = const.$$

Returning back to the original variables, we get the general solution to the system (5.8)

$$y_2(\varepsilon C_{72} y_1 - D_{72} x_1) + (\alpha x_1 + \beta y_1 + \gamma y_3 + \delta y_4) - A = 0 \tag{5.10}$$

with some real coefficients in the linear part. If in this formula at least one of the pair of coefficients (γ, δ) vanishes, then the final equation (5.10) of the hypersurface in $\mathbb{C}^4$ turns out to be independent of one of complex variables. This means that the obtained surface can be represented as a direct product of the complex plane $\mathbb{C}$ and some 5–dimensional manifold, and hence, holomorphic degeneracy and, as a consequence, Levi–degeneracy. But even if this pair has a nonzero coefficient, the linear combination $\gamma z_3+\delta z_4$, considered as a new complex variable z_3^*, allows us to state the independence of equation (5.2) of the variable $z_4^* = z_4$. Thus, in this case too, the surface (5.2) is Levi–degenerate. The proof is complete. □

In the same way it is possible to integrate two types of algebras $[7, [6, 9], 1, 1]$ and $[7, [6, 9], 1, 2]$ for all values of the involved parameters. However, the resulting formulas (both intermediate and final), describing the orbits of these algebras, are excessively cumbersome. In this regard, here we give only two examples of homogeneous hypersurfaces corresponding to these algebras and obtained by integrating them for some particular values of the parameters.

Example 5.1. *The orbits of algebras* $[7, [6, 9], 1, 1]$ *defined by the formulas (5.1) for*

$$B_6 = C_6 = D_6 = B_7 = \hat{B}_7 = C_7 = D_7 = 0, \quad n = 0, \quad m \neq 0,$$

are reduced by holomorphic transformations to the surfaces with the equations

$$(y_3 - x_1y_2)^2 + (y_4 - \varepsilon y_1y_2)^2 = \exp\left(\frac{2}{m}\arctan\left(\frac{y_4 - \varepsilon y_1y_2}{y_3 - x_1y_2}\right)\right).$$

Example 5.2. *The orbits of algebras* $[7, [6, 9], 1, 2]$ *defined by the formulas (5.2) for* $B_6 = C_6 = 0$, *are reduced by holomorphic transformations to Levi nondegenerate hypersurface with the equation*

$$(y_3 - x_1y_2)^2 + (y_4 - \varepsilon y_1y_2)^2 = 1$$

independently of the admissible values of m.

It is easy to verify, for example, by using symbolic computations, that such generalizations of planar logarithmic spirals (in Example 5.1) and circles (Example 5.2) are Levi nondegenerate hypersurfaces of the space $\mathbb{C}^4$ at general points.

6. Final conclusions

The main conclusion of the article is the main result, Theorem 1.1 formulated in the Introduction. As specific additions to this theorem, we specify that the 7 mentioned types are the five types of Lie algebras from Proposition 3.1 and the two types of algebras in Proposition 5.1. For the last two types, in Section 5 we give examples of orbits corresponding to some particular values of the parameters included in the descriptions of the algebras; the algebras of these types contain the nilradical $[6, 9]$.

The first five types of 7–dimensional Lie algebras associated with the nilradicals $[6, 1]$, $[6, 13]$, $[6, 22]$, $[6, 23]$, $[6, 26]$, are integrated in [8] for all possible values of the parameters, which describe the realizations of these algebras in $\mathbb{C}^4$. Here we provide a description of all such Levi nondegenerate (not reducible to tubes) integral manifolds that are holomorphically homogeneous real hypersurfaces of the 4–dimensional complex space.

Theorem 6.1 ([8])**.** *Each nondegenerate non–tubular orbit of a 7–dimensional algebra that has a 6–dimensional nilradical of one of the types (3.1) is described up to holomorphic changes of coordinates by an equation of one of the following 5 types (A, B, C, D, Q, R are some real constants):*

$$[7, [6, 1], 1, 1]: \quad y_4 = 2y_1y_2 + (x_1^4 - 2x_1^2y_1^2) + \frac{2}{y_1}(2y_3 - x_1^2y_2) + \frac{2}{y_1^2} + Q\cdot\left(x_1y_2 - \frac{2}{3}x_1^3y_1\right) + R\cdot y_1^4;$$

$$[7, [6, 13], 1, 1]: \quad y_4 = x_1y_3 + \frac{1}{3}x_1y_2(3x_1^2 - y_1^2) + \frac{5}{36}x_1^2y_1(-3x_1^4 + 3x_1^2y_1^2 - y_1^4) - By_1^7 - \frac{1}{2}\frac{y_2^2}{y_1};$$

$$[7,[6,22],1,1]: \quad y_4 = \frac{B}{3}y_1y_2 + \frac{AB}{3}x_1(x_1^2 - y_1^2) + \frac{B^2}{6}x_1^2y_1 - \frac{1}{y_1}\Big(\frac{B^2}{4}x_1^4 + Bx_1^2y_2 + By_3 + \frac{1}{2}y_2^2\Big) + Cy_1^3;$$

$$[7,[6,23],1,1]: \quad y_3 = By_1^4 - 2x_1y_1y_2 + 2y_1y_4 - y_2^2;$$

$$[7,[6,26],1,1]: \quad y_3 = y_1y_4 + x_1y_1y_2 \pm \frac{y_2^2}{y_1} + By_1^5.$$

Completing the discussion of 64 considered types of 7–dimensional Lie algebras we clarify that the question of whether the property B) is satisfied or not for the hypersurfaces from the 7 families obtained remains open and it requires an additional study.

In conclusion, we give another version of a brief formulation of the results of paper: *among 10 nilpotent 6–dimensional Lie algebras containing two 4–dimensional Abelian subalgebras, at most six algebras have 7-dimensional (non–decomposable solvable) continuations with Levi nondegenerate 7–dimensional orbits in the space* $\mathbb{C}^4$*, which are not reducible to tubes.*

BIBLIOGRAPHY

1. A.V. Atanov, A.V. Loboda. *Decomposable five–dimensional Lie algebras in the problem on holomorphic homogeneity in* $\mathbb{C}^3$ // J. Math. Sci., New York **268**:1, 84–113 (2022). https://doi.org/10.1007/s10958-022-06182-3
2. R.L. Bishop, R.J. Crittenden. *Geometry of manifolds.* Academic Press, New York (1964).
3. V.V. Krutskikh, A.V. Loboda. *Computer data processing in one multidimensional mathematical problem* // in *Informatics: problems, methods, technologies*, “Velborn” Ltd., Voronezh, 411–419 (2021). (in Russian).
4. V.V. Krutskikh, A.V. Loboda. *Application of systems analysis and computer algorithms in studying orbits of 7–dimensional Lie algebras* // Matem. Fizika Kompyut. Modelir. **27**:3, 38–59 (2024). (in Russian). https://doi.org/10.15688/mpcm.jvolsu.2024.3.4
5. A.V. Loboda. *Holomorphically homogeneous real hypersurfaces in* $\mathbb{C}^3$ // Trans. Mosc. Math. Soc. **2020**, 169–228 (2020). https://doi.org/10.1090/mosc/309
6. A.V. Loboda, V.K. Kaverina. *On degeneracy of orbits of nilpotent Lie algebras* // Ufa Math. J. **14**:1, 52–76 (2022). https://doi.org/10.13108/2022-14-1-52
7. A.V. Loboda. *On 7–dimensional Lie algebras admitting Levi nondegenerate orbits in* $\mathbb{C}^4$ // Trudy Mosk. Mat. Obsch. **84**:2, 205–230 (2023). (in Russian). https://www.mathnet.ru/eng/mmo/v84/i2/p205
8. A.V. Loboda, A.V. Atanov, P.E. Albutkina. *On algorithms for describing homogeneous submanifolds of multidimensional spaces* // In: *Informatics: problems, methodology, technology.* Voronezh State Univ., Voronezh 380–391 (2024). (in Russian).
9. A.V. Atanov, A.V. Loboda. *On degenerate orbits of real Lie algebras in multidimensional complex spaces* // Russ. J. Math. Phys. **30**:4, 432–442 (2023). https://doi.org/10.1134/S1061920823040027
10. V.K. Beloshapka, I.G. Kossovskiy. *Homogeneous hypersurfaces in* $\mathbb{C}^3$*, associated with a model CR-cubic* // J. Geom. Anal. **20**:3, 538–564 (2010). https://doi.org/10.1007/s12220-010-9117-4
11. E. Cartan. *Sur la géométrie pseudoconforme des hypersurfaces de l'espace de deux variables complexes* // Ann. Math. Pura Appl. **11**, 17–90 (1933). https://doi.org/10.1007/BF02417822
12. B. Doubrov, J. Merker, D. The. *Classification of simply-transitive Levi non-degenerate hypersurfaces in* $\mathbb{C}^3$ // Int. Math. Res. Not. **2022**:19, 15421–15473 (2022). https://doi.org/10.1093/imrn/rnab147
13. M.P. Gong. *Classification of nilpotent Lie algebras of dimension 7 (over algebraically closed fields and* $\mathbb{R}$*).* PhD Thesis, Univ. Waterloo, Waterloo (1998).
14. F. Hindeleh, G. Thompson. *Seven dimensional Lie algebras with a four-dimensional nilradical* // Algebras Groups Geom. **25**:3, 243–265 (2008).

15. M. Kolář, I. Kossovskiy, D. Sykes. *New examples of 2-nondegenerate real hypersurfaces in* $\mathbb{C}^N$ *with arbitrary nilpotent symbols* // J. Lond. Math. Soc., II. Ser. **110**:2, e12962 (2024). https://doi.org/10.1112/jlms.12962
16. B. Kruglikov, A. Santi. *On 3–nondegenerate CR manifolds in dimension 7. I: The transitive case.* // J. Reine Angew. Math. **2025**:820, 167–211 (2025). https://doi.org/10.1515/crelle-2024-0093
17. A.V. Loboda, R.S. Akopyan, V.V. Krutskikh. *On the orbits of nilpotent 7-dimensional Lie algebras in 4-dimensional complex space* // J. Sib. Fed. Univ., Math. Phys. **13**:3, 360–372 (2020). https://doi.org/10.17516/1997-1397-2020-13-3-360-372
18. A.R. Parry. *Classification of Real Indecomposable Solvable Lie Algebras of Small Dimension with Codimension One Nilradicals.* Master's Thesis, Logan, Utah (2007).
19. A.L. Vu, T.A. Nguyen, T.T.C. Nguyen, T.T.M. Nguyen, T.N. Vo. *Classification of 7–dimensional solvable Lie algebras having 5–dimensional nilradicals* // Commun. Algebra **51**:5, 1866–1885 (2023). https://doi.org/10.1080/00927872.2022.2145300

Alexander Vasilievich Loboda,
Voronezh State Technical University,
Moskovsky av. 14,
394026, Voronezh, Russia

Lomonosov Moscow State University,
Leninskie Gory 1,
119991, Moscow, Russia
E-mail: `lobvgasu@yandex.ru`

Ripsime Sergoevna Akopyan,
MIREA — Russian Technological University,
Vernadsky av. 78,
119454, Moscow, Russia
E-mail: `akrim111@yandex.ru`