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ON RECOVERING PROBLEM FOR STURM — LIOUVILLE OPERATOR WITH TWO FROZEN ARGUMENTS

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Abstract. Inverse spectral problems consist in recovering operators by their spectral characteristics. The problem of recovering the Sturm— Liouville operator with one frozen argument by one spectrum was considered earlier in works by various authors. In this paper, we study the uniqueness of recovering the operator with two frozen arguments and different coefficients p, q by the spectra of two boundary value problems. This case is significantly more difficult than the case of one frozen argument since the operator is no longer a one–dimensional perturbation. We prove that the operator with two frozen arguments can not be recovered by two spectra in the general case. For the unique recovery, one has to impose some conditions on the coefficients. We assume that the coefficients p and q are zero on some interval and prove the uniqueness theorem. We also obtain formulas for regularized traces of two spectra. The result is formulated in terms of the convergence of a certain series, which allows us to avoid smoothness conditions for the coefficients.

Keywords: inverse spectral problem, frozen argument, nonlocal operator, Sturm — Liouville operator, regularized trace formula, uniqueness theorem.

Mathematics Subject Classification: 34K29, 34A55

1. Introduction

We consider the inverse spectral problem for the Sturm — Liouville operator with two frozen arguments

$$\ell y = -y''(x) + p(x)y(a) + q(x)y(b), \quad x \in (0, \pi),$$

where $p, q \in L_2(0, \pi)$ are complex–valued, and the parameters $a, b \in (0, \pi)$ are fixed and called the frozen arguments. In contrast to pure differential operators studied in the framework of classical theory of inverse spectral problems [3], [5], [14], [21], [34], the operator ℓy is non–local. Non–local operators possess specific spectral properties, see [9], [10], [12], [18], [28], [31], [33], and require to develop methods different from ones in the classical theory of inverse spectral problems.

In previous works, there were studied the Sturm — Liouville operators with one frozen argument, that is, in the case $q = 0$, and with various boundary conditions, see [2], [9], [11], [13], [15]–[17], [19], [25]–[28], [30], [32]. There was considered the inverse problem, which consisted in recovering the coefficient p by the spectrum of the operator. The most complete results were obtained for the boundary conditions

$$y^{(\alpha)}(0) = y^{(\beta)}(\pi) = 0,$$

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where $\alpha, \beta \in \{0, 1\}$ denote the order of derivatives. In the rational case $\frac{a}{\pi} \in \mathbb{Q}$ a part of the spectrum can degenerate (since it is independent of p) and for the unique recovering of p we need additional data apart of the spectrum, see [13], [15], [16]. In the irrational case $\frac{a}{\pi} \notin \mathbb{Q}$ the degeneration phenomenon does not appear, and p is uniquely recovered by the spectrum, see [32]. Thus, for almost all $a \in (0, \pi)$ we have the unique recovering of one coefficient by one spectrum.

A general approach to both cases was developed in [25] and [26], and later it was generalized for the operators

$$\tilde{\ell}y = -y''(x) + p(x)y(a) + r(x)y(x),$$

see [27]. It was observed in [19] that the Sturm — Liouville operator with one frozen argument is a one–dimensional perturbation of the differential operator $\ell_0 y = -y''$, and there was studied the inverse problem for the corresponding class of one–dimensional perturbations. Recently there arose an interest to the operators with several frozen arguments taken with the same coefficients [29]:

$$\ell_1 y = -y''(x) + \sum_{k=1}^{m} y(a_k)p(x), \quad m \in \mathbb{N}.$$

They are one–dimensional perturbations of the type, which was studied in [19]. These operators do not lead to a situation, which differs essentially from the case of a single frozen argument, and the inverse spectral problems can be studied by the methods of the works [19], [25]–[27]. At the same time, the operator ℓy is not a one–dimensional perturbation, and for this operator there are no known methods in the theory of inverse spectral problems.

We introduce the boundary value problem $\mathcal{L}_j(p, q)$ with the index $j = 0, 1$

$$\ell y = \lambda y(x), \tag{1.1}$$

$$y^{(j)}(0) = y(\pi) = 0, \tag{1.2}$$

and by $\{\lambda_{nj}\}_{n\geqslant 1}$ we denote its spectrum. We consider the following inverse problem.

Inverse problem 1.1. *Given the spectra $\{\lambda_{n0}\}_{n\geqslant 1}$ and $\{\lambda_{n1}\}_{n\geqslant 1}$, find p and q.*

First of all, we are interesting in the uniqueness of solution to this inverse problem, that is, whether different pairs (p, q) always produce different pairs of spectra $(\{\lambda_{n0}\}_{n\geqslant 1}, \{\lambda_{n1}\}_{n\geqslant 1})$. We immediately note that for $a = b$ only the sum of p and q makes sense, and in this case the solution to Inverse problem 1.1 is not unique. To exclude this situation, we impose the restriction

$$0 < a < b < \pi. \tag{1.3}$$

We shall show that for all a and b obeying (1.3), the solution to Inverse problem 1.1 is not unique. For the unique determination of p and q we have to specify the formulation of the inverse problem by prescribing an additional information. We suppose that the coefficients p and q simultaneously vanish on $[0, b]$ or $[a, \pi]$. Under this condition we shall prove a theorem on the unique recovering of p and q by two spectra.

We shall also obtain the formula for the regularized traces of the spectra $\{\lambda_{nj}\}_{n\geqslant 1}$, $j = 0, 1$. By the regularized trace we mean the series of differences between the eigenvalues of two operators, one being a perturbation of the other. From the physical point of view, this notion reflects the measure of energy defect of a quantum system, see [4]. The basic results of theory of regularized traces were provided in the survey [7].

The formulas for regularized traces of operators with a single frozen argument were obtained in [20], [22]–[24] for an absolutely continuous coefficient p. In [22], under the conditions $p \in$

$W_2^1[0,\pi]$ and $q \equiv 0$ it was proved that

$$\sum_{n=1}^{\infty}\left(\lambda_{nj} - \left(n - \frac{j}{2}\right)^2\right) = p(a), \quad j = 0, 1.$$

Here we obtain the formulas for the regularized traces under more general conditions $p, q \in L_1[0,\pi]$:

$$\sum_{n=1}^{\infty}\left(\lambda_{nj} - \left(n - \frac{j}{2}\right)^2\right) = \sum_{n=1}^{\infty} s_{nj}, \quad j = 0, 1, \tag{1.4}$$

where s_{nj} are expressed in terms of the Fourier coefficients of the functions p and q over the system of eigenfunctions of unperturbed operator $-y''$. The formula (1.4) is treated so that either both series diverge or they converge to the same number. If p is absolutely continuous in the vicinity of the point a, and q is absolutely continuous in the vicinity of the point b, then we have the convergence to the number $p(a) + q(b)$. The formulas for the regularized traces of the operators with two frozen arguments can be also obtained from the results of works [6] and [8], but this requires stronger restrictions for the coefficients.

The paper is organized as follows. In Section 2 we construct characteristic functions and obtain asymptotic formulas for the spectra, see Theorem 2.1. In Section 3 we construct an example of distinct pairs (p, q), which lead to the same pair of spectra, see Theorem 3.1. In Section 4 we specify the formulation of inverse problem and prove the uniqueness theorem, see Theorem 4.1. In Section 5 we obtain the formulas for regularized traces of the spectra $\{\lambda_{nj}\}_{n\geqslant 1}$, see Theorem 5.1. In Appendix we provide the details of proof of formula (2.3) for the characteristic functions.

2. Characteristic functions

Let us obtain the characteristic functions for the boundary value problems $\mathcal{L}_j(p, q)$ with $j = 0, 1$. It is known that for $f \in L_2(0,\pi)$ each solution of the equation $-y''(x) + f(x) = \lambda y(x)$ can be represented as

$$y(x) = x_1 \frac{\sin\rho x}{\rho} + x_2 \cos\rho x + \int_0^x \frac{\sin\rho(x-t)}{\rho} f(t)\, dt, \quad \rho^2 = \lambda, \quad x_1,\, x_2 \in \mathbb{C}.$$

Letting $f(t) = x_3 p(t) + x_4 q(t)$, we obtain

$$y(x) = x_1 \frac{\sin\rho x}{\rho} + x_2 \cos\rho x + x_3 \int_0^x \frac{\sin\rho(x-t)}{\rho} p(t)\, dt + x_4 \int_0^x \frac{\sin\rho(x-t)}{\rho} q(t)\, dt. \tag{2.1}$$

This function is a solution to Equation (1.1) if and only if $y(a) = x_3$ and $y(b) = x_4$. This solution is non–trivial if and only if the vector $(x_j)_{j=1}^4$ is non–zero.

Substituting the expression (2.1) into the conditions (1.2) with $j = 0, 1$ and the identities $y(a) = x_3$, $y(b) = x_4$, we arrive at the system of linear equations

$$\begin{aligned}
&x_{2-j} = 0,\\
&x_1 \frac{\sin \rho\pi}{\rho} + x_2 \cos \rho\pi + x_3 \int_0^\pi \frac{\sin \rho(\pi - t)}{\rho} p(t)\,dt + x_4 \int_0^\pi \frac{\sin \rho(\pi - t)}{\rho} q(t)\,dt = 0,\\
&x_1 \frac{\sin \rho a}{\rho} + x_2 \cos \rho a + x_3 \int_0^a \frac{\sin \rho(a - t)}{\rho} p(t)\,dt + x_4 \int_0^a \frac{\sin \rho(a - t)}{\rho} q(t)\,dt = x_3,\\
&x_1 \frac{\sin \rho b}{\rho} + x_2 \cos \rho b + x_3 \int_0^b \frac{\sin \rho(b - t)}{\rho} p(t)\,dt + x_4 \int_0^b \frac{\sin \rho(b - t)}{\rho} q(t)\,dt = x_4.
\end{aligned} \tag{2.2}$$

This system possesses a non–zero solution $(x_k)_{k=1}^4$ if and only if

$$\Delta_j(\lambda) := \begin{vmatrix} \varphi_j(\rho, \pi) & \int_0^\pi \frac{\sin \rho(\pi - t)}{\rho} p(t)\,dt & \int_0^\pi \frac{\sin \rho(\pi - t)}{\rho} q(t)\,dt \\ \varphi_j(\rho, a) & \int_0^a \frac{\sin \rho(a - t)}{\rho} p(t)\,dt - 1 & \int_0^a \frac{\sin \rho(a - t)}{\rho} q(t)\,dt \\ \varphi_j(\rho, b) & \int_0^b \frac{\sin \rho(b - t)}{\rho} p(t)\,dt & \int_0^b \frac{\sin \rho(b - t)}{\rho} q(t)\,dt - 1 \end{vmatrix} = 0,$$

where we have employed the notation

$$\varphi_j(\rho, z) = \begin{cases} \rho^{-1} \sin \rho z, & j = 0, \\ \cos \rho z, & j = 1. \end{cases}$$

In what follows we indicate the dependence on p and q by the variables after the colon; thus, $\Delta_j(\lambda) = \Delta_j(\lambda; p, q)$. This dependence can be omitted if we do not focus on particular values of p and q.

For $j = 0, 1$ the function $\Delta_j(\lambda)$ is the characteristic function of the boundary value problem $\mathcal{L}_j(p, q)$: the zeroes of this function coincide with the spectrum of the boundary value problem. Since the Taylor expansions of entire functions $\rho^{-1} \sin \rho z$ and $\cos \rho z$ involve only even powers of ρ, the functions Δ_0 and Δ_1 are entire in λ.

Expanding the determinants, after calculations we obtain the representations

$$\Delta_j(\lambda) = \varphi_j(\rho, \pi) + A_{j0}(\lambda) + A_{j1}(\lambda) + B_j(\lambda), \quad j = 0, 1, \tag{2.3}$$

where

$$\begin{aligned}
A_{j0}(\lambda) = A_{j0}(\lambda; p) = \varphi_j(\rho, a) \int_a^\pi \frac{\sin \rho(\pi - t)}{\rho} p(t)\,dt + \frac{\sin \rho(\pi - a)}{\rho} \int_0^a \varphi_j(\rho, t)\, p(t)\,dt,\\
A_{j1}(\lambda) = A_{j1}(\lambda; q) = \varphi_j(\rho, b) \int_b^\pi \frac{\sin \rho(\pi - t)}{\rho} q(t)\,dt + \frac{\sin \rho(\pi - b)}{\rho} \int_0^b \varphi_j(\rho, t)\, q(t)\,dt,
\end{aligned} \tag{2.4}$$

and $B_j(\lambda)$ is defined as

$$\begin{aligned}
B_j(\lambda) = B_j(\lambda;p,q) = &\frac{\sin\rho(\pi-b)}{\rho}\bigg(\int_0^a \varphi_j(\rho,t)\,p(t)\,dt\int_a^b \frac{\sin\rho(\xi-a)}{\rho}q(\xi)\,d\xi \\
&- \int_0^a \varphi_j(\rho,\xi)\,q(\xi)\,d\xi\int_a^b \frac{\sin\rho(t-a)}{\rho}p(t)\,dt\bigg) \\
&+ \frac{\sin\rho(b-a)}{\rho}\bigg(\int_0^a \varphi_j(\rho,t)\,p(t)\,dt\int_b^\pi \frac{\sin\rho(\pi-\xi)}{\rho}q(\xi)\,d\xi \\
&- \int_0^a \varphi_j(\rho,\xi)\,q(\xi)\,d\xi\int_b^\pi \frac{\sin\rho(\pi-t)}{\rho}p(t)\,dt\bigg) \\
&+ \varphi_j(\rho,a)\bigg(\int_b^\pi \frac{\sin\rho(\pi-\xi)}{\rho}q(\xi)\,d\xi\int_a^b \frac{\sin\rho(b-t)}{\rho}p(t)\,dt \\
&- \int_b^\pi \frac{\sin\rho(\pi-t)}{\rho}p(t)\,dt\int_a^b \frac{\sin\rho(b-\xi)}{\rho}q(\xi)\,d\xi\bigg) \\
&+ \varphi_j(\rho,a)\frac{\sin\rho(\pi-b)}{\rho}\int_a^b\int_a^b \frac{\sin\rho(\xi-t)}{\rho}q(\xi)\,p(t)\,dt\,d\xi;
\end{aligned} \tag{2.5}$$

for more detail see the Appendix. The representation (2.3) shows that $\Delta_j(\lambda)$ are entire functions of order $\frac{1}{2}$ and type π. By means of the standard method based on applying the Rouché theorem, see [21], we prove the following theorem.

Theorem 2.1. *For $j=0,1$ the asymptotic formulas hold*

$$\lambda_{nj} = \Big(n-\frac{j}{2}\Big)^2 + \varkappa_{nj}, \quad \{\varkappa_{nj}\}_{n\geqslant 1}\in \ell_2.$$

In what follows some properties of the terms in the representation (2.3) play an important role. The terms $A_{j0}(\lambda;p)$ and $A_{j1}(\lambda;q)$ depend linearly on p and q, respectively. The term $B_j(\lambda;p,q)$ depends bilinearly in p and q. The formula (2.5) shows that this dependence is antisymmetric: $B_j(\lambda;p,q) = -B_j(\lambda;q,p)$; the pairs of antisymmetric terms are united in brackets in (2.5). Because of this, if $p=q$, then $B_j(\lambda;p,q)=0$. By the bilinearity this implies

$$B_j(\lambda;q,\alpha q) = 0, \quad \alpha\in\mathbb{C}. \tag{2.6}$$

By (2.5) we also see that $B_j(\lambda;p,q)=0$ if

$$\left[\begin{array}{ll} p(x)\equiv 0,\ q(x)\equiv 0, & x\in[a,\pi], \\ p(x)\equiv 0,\ q(x)\equiv 0, & x\in[0,b]. \end{array}\right. \tag{2.7}$$

The construction of characteristic function for the boundary value problem $\mathcal{L}_0$ was considered in [29] in the particular case $p=q$. Under this condition, $B_j(\lambda)=0$ and our representation (2.3) is consistent with the obtained there formula. One more case when $B_j(\lambda)=0$ is $q=0$. Under this condition we also have $A_{j1}(\lambda)=0$, and the formula (2.3) gives the representation for the characterstic function of the operator with one frozen argument, which is consistent with the one obtained earlier, see [13], [15], [16].

3. Non–uniqueness of solution to inverse problem 1.1

In this section we construct different pairs of coefficients (p, q), which produce the same pairs of spectra $\Big(\{\lambda_{n0}\}_{n\geqslant 1}, \{\lambda_{n1}\}_{n\geqslant 1}\Big)$. This will prove the non–uniqueness of solution of Inverse problem 1.1.

We continue the functions p and q to $\mathbb{R}\setminus[0,\pi]$ by zero. Following the lines of proof of Lemma 1 in [16], we obtain the representations

$$\begin{aligned}
A_0(\lambda;p,q) &:= A_{00}(\lambda)+A_{01}(\lambda)=\frac{1}{2}\int\limits_0^\pi \frac{\cos\rho t}{\rho^2}W_0(t)dt,\\
A_1(\lambda;p,q) &:= A_{10}(\lambda)+A_{11}(\lambda)=\frac{1}{2}\int\limits_0^\pi \frac{\sin\rho t}{\rho}W_1(t)dt,
\end{aligned}\tag{3.1}$$

where

$$\begin{aligned}
W_j(t;p,q) =&(-1)^{j+1}p(t+a-\pi)+(-1)^{j+1}p(\pi-t+a)\\
&+(-1)^j p(\pi-a+t)+p(\pi-a-t)\\
&+(-1)^{j+1}q(t+b-\pi)+(-1)^{j+1}q(\pi-t+b)\\
&+(-1)^j q(\pi-b+t)+q(\pi-b-t), \qquad t\in[0,\pi], \qquad j=0,1.
\end{aligned}$$

Lemma 3.1. *We let $T=\min\{a, b-a, \pi-b\}$. Let $G(t)\in L_2(\mathbb{R})$ be an arbitrary even non–trivial function, which vanishes identically outside the segment $[-T,T]$. Then for the functions*

$$s(t)=G(b-t), \qquad r(t)=-G(a-t) \tag{3.2}$$

we have $A_j(\lambda;s,r)=0$, $j=0,1$.

Proof. For $t\in[0,\pi]$ we introduce the functions

$$\begin{aligned}
u_0(t;p,q) &:= \frac{W_0(t)+W_1(t)}{2} = p(\pi-a-t)+q(\pi-b-t),\\
u_1(t;p,q) &:= \frac{W_0(t)-W_1(t)}{2} = -p(t+a-\pi)-p(\pi-t+a)+p(\pi-a+t)\\
&\qquad -q(t+b-\pi)-q(\pi-t+b)+q(\pi-b+t).
\end{aligned}\tag{3.3}$$

We note that s and r vanish identically outside $[0,\pi]$, and this is why in (3.3) we can formally replace p by s and q by r. We are going to prove that $u_0(t;s,r)=u_1(t;s,r)=0$.

Indeed,

$$u_0(t;s,r)=s(\pi-a-t)+r(\pi-b-t)\overset{(3.2)}{=}G(b-\pi+a+t)-G(a-\pi+b+t)=0.$$

In $u_1(t;s,r)$ we group the terms as follows

$$\begin{aligned}
u_1(t;s,r)=&-\big(s(t+a-\pi)+r(\pi-t+b)\big)\\
&-\big(s(\pi-t+a)+r(t+b-\pi)\big)+\big(s(\pi-a+t)+r(\pi-b+t)\big).
\end{aligned}$$

Applying (3.2) to each pair of terms and using the parity of the function $G(t)$, we arrive at the identity $u_1(t;s,r)=0$.

The identities $u_0(t;s,r)=u_1(t;s,r)=0$ imply $W_0(t;s,r)=0$ and $W_1(t;s,r)=0$. By (3.1) we obtain $A_j(\lambda;s,r)=0$ for $j=0,1$. □

Theorem 3.1. *Different pairs of coefficients*

$$(p,q)=(-r,s+r) \quad \textit{and} \quad (p,q)=(-s-r,s)$$

produce the same pairs of spectra $\Big(\{\lambda_{n0}\}_{n\geqslant1},\{\lambda_{n1}\}_{n\geqslant1}\Big)$. *Thus, Inverse problem 1.1 can not have a unique solution.*

Proof. Since G in Lemma 3.1 is a non–trivial function, the same is true for s and r, and this is why $(-r,s+r)\neq(-s-r,s)$. Let $j=0,1$. By the linearity of $A_{j0}(\lambda;p)$ and $A_{j1}(\lambda;q)$ with respect to p and q we have

$$\begin{aligned}A_j(\lambda;-r,s+r)-A_j(\lambda;-s-r,s)&=A_{j0}(\lambda;-r)+A_{j1}(\lambda;s+r)-A_{j0}(\lambda;-s-r)-A_{j1}(\lambda;s)\\&=A_{j0}(\lambda;s)+A_{j1}(\lambda;r)=A_j(\lambda;s,r)=0,\end{aligned}$$

where the latter identity holds due to Lemma 3.1. Thus,

$$A_j(\lambda;-r,s+r)=A_j(\lambda;-s-r,s).$$

Using the property (2.6) and the bilinearity of $B_j(\lambda;p,q)$ with respect to p and q, we obtain

$$B_j(\lambda;-r,s+r)=B_j(\lambda;-r,s)=B_j(\lambda;-s-r,s).$$

By the formula (2.3) we have $\Delta_j(\lambda;-r,s+r)=\Delta_j(\lambda;-s-r,s)$. This means that the pairs $(p,q)=(-r,s+r)$ and $(p,q)=(-s-r,s)$ give the same spectrum $\{\lambda_{nj}\}_{n\geqslant1}$. □

If we choose some function G obeying the assumptions of Lemma 3.1, then in Theorem 3.1 we obtain particular pairs of the coefficients (p,q).

Example 3.1. Let $a=\frac{\pi}{4}$ and $b=\frac{\pi}{2}$. Then the function

$$G(t)=\chi_{[-T,T]}(t),\quad T=\frac{\pi}{4},\quad \chi_S(t):=\begin{cases}1, & t\in S,\\ 0, & t\notin S,\end{cases}$$

obeys the assumptions of Lemma 3.1. We find $s(t)=\chi_{[\frac{\pi}{4};\frac{3\pi}{4}]}(t)$ and $r(t)=-\chi_{[0;\frac{\pi}{2}]}(t)$. By Theorem 3.1 the following pairs of coefficients (p,q) produce the same pair of spectra

$$\begin{aligned}&p(t)=\chi_{[0;\frac{\pi}{2}]}(t), \qquad q(t)=\chi_{[\frac{\pi}{4};\frac{3\pi}{4}]}(t)-\chi_{[0;\frac{\pi}{2}]}(t);\\&p(t)=-\chi_{[\frac{\pi}{4};\frac{3\pi}{4}]}(t)+\chi_{[0;\frac{\pi}{2}]}(t), \qquad q(t)=\chi_{[\frac{\pi}{4};\frac{3\pi}{4}]}(t).\end{aligned}$$

The direct substitution of each pair (p,q) into the formulas (2.3)–(2.5) justifies that the characteristic functions coincide

$$\begin{aligned}\Delta_0(\lambda;p,q)=&\frac{1}{\rho}\sin\rho\pi+\frac{1}{2\rho^3}\Big(-3\sin\rho\pi+5\sin\frac{3\rho\pi}{4}-2\sin\frac{\rho\pi}{2}+\sin\frac{\rho\pi}{4}\Big)\\&+\frac{1}{\rho^5}\left(\sin\frac{\rho\pi}{2}\Big[\cos\frac{\rho\pi}{4}-1\Big]^2+2\sin\frac{\rho\pi}{4}\Big[\cos\frac{\rho\pi}{4}-1\Big]\Big[\cos\frac{\rho\pi}{2}-\cos\frac{\rho\pi}{4}\Big]\right),\\ \Delta_1(\lambda;p,q)=&\cos\rho\pi+\frac{1}{2\rho^2}\Big(-3\cos\rho\pi+3\cos\frac{3\rho\pi}{4}+\cos\frac{\rho\pi}{4}-1\Big)\\&+\frac{1}{\rho^4}\left(\sin\frac{\rho\pi}{2}\sin\frac{\rho\pi}{4}\Big[1-\cos\frac{\rho\pi}{4}\Big]+\sin^2\frac{\rho\pi}{4}\Big[\cos\frac{\rho\pi}{4}-\cos\frac{\rho\pi}{2}\Big]\right.\\&\left.+\cos\frac{\rho\pi}{4}\Big[\cos\frac{\rho\pi}{4}-\cos\frac{\rho\pi}{2}\Big]\Big[1-\cos\frac{\rho\pi}{4}\Big]\right).\end{aligned}$$

4. Inverse problem with additional conditions

We consider Inverse problem 1.1 under additional conditions for p and q.

Inverse problem 4.1. *It is known that the pair of functions (p,q) satisfies the conditons (2.7). Given the spectra $\{\lambda_{n0}\}_{n\geqslant 1}$ and $\{\lambda_{n1}\}_{n\geqslant 1}$, recover p and q.*

We are going to prove the uniqueness of solution to Inverse problem 4.1. Apart of the boundary value problems $\mathcal{L}_0(p,q)$ and $\mathcal{L}_1(p,q)$, we consider the boundary value problems $\mathcal{L}_0(\tilde{p},\tilde{q})$ and $\mathcal{L}_1(\tilde{p},\tilde{q})$ with some coefficients $\tilde{p},\tilde{q}\in L_2(0,\pi)$. For $j=0,1$ by $\{\tilde{\lambda}_{nj}\}_{n\geqslant 1}$ we denote the spectrum of the boundary value problem $\mathcal{L}_j(\tilde{p},\tilde{q})$.

Theorem 4.1. *Let the functions p, q, $\tilde{p}$ and $\tilde{q}$ obey one of the following two conditions:*

1. *Each function vanishes on $[a,\pi]$;*
2. *Each function vanishes on $[0,b]$.*

Then the identities $\{\lambda_{n0}\}_{n\geqslant 1}=\{\tilde{\lambda}_{n0}\}_{n\geqslant 1}$ and $\{\lambda_{n1}\}_{n\geqslant 1}=\{\tilde{\lambda}_{n1}\}_{n\geqslant 1}$ imply that $p=\tilde{p}$ and $q=\tilde{q}$.

We shall need the next lemma.

Lemma 4.1. *The characteristic functions are uniquely recovered by the spectra*

$$\Delta_j(\lambda)=\pi^{1-j}\prod_{k=1}^{\infty}\frac{\lambda_{nj}-\lambda}{(n-\frac{j}{2})^2},\quad j=0,1.$$

The proof of the lemma is standard, see the proof of Theorem 1.1.4 in [21]. In the proof, the asymptotic formulas of Theorem 2.1 are applied as well as the formulas

$$\Delta_0(\lambda)=\frac{\sin\rho\pi}{\rho}+O\Big(\frac{e^{|\tau|\pi}}{\rho^2}\Big),\qquad \Delta_1(\lambda)=\cos\rho\pi+O\Big(\frac{e^{|\tau|\pi}}{\rho}\Big),\qquad \tau=\operatorname{Im}\,\rho,$$

which are implied by (2.3)–(2.5).

Proof of Theorem 4.1. It follows from Lemma 4.1 that

$$\Delta_j(\lambda;p,q)=\Delta_j(\lambda;\tilde{p},\tilde{q}),\quad j=0,1.$$

Since $B_j(\lambda;p,q)=B_j(\lambda;\tilde{p},\tilde{q})=0$, in the representation (2.3) we have $A_j(\lambda;p,q)=A_j(\lambda;\tilde{p},\tilde{q})$, and in (3.1) we obtain $W_j(t;p,q)=W_j(t;\tilde{p},\tilde{q})$, $j=0,1$. This yields

$$u_j(t;p,q)=u_j(t;\tilde{p},\tilde{q}),\quad j=0,1. \tag{4.1}$$

We denote $\hat{p}=p-\tilde{p}$ and $\hat{q}=q-\tilde{q}$. It follows from (3.3) and (4.1) that

$$\begin{gathered}\hat{p}(\pi-a-t)+\hat{q}(\pi-b-t)=0,\\ -\hat{p}(t+a-\pi)-\hat{p}(\pi-t+a)+\hat{p}(\pi-a+t)\\ -\hat{q}(t+b-\pi)-\hat{q}(\pi-t+b)+\hat{q}(\pi-b+t)=0,\end{gathered}\tag{4.2}$$

where $t\in[0,\pi]$. Considering the first identity in (4.2) for $t\in[0,\pi-a]$, after the change of variable $z=\pi-a-t$ we get

$$\hat{p}(z)+\hat{q}(z+a-b)=0,\quad z\in[0,\pi-a]. \tag{4.3}$$

For the sake of definiteness we suppose that p, q, $\tilde{p}$ and $\tilde{q}$ are zero on $[0,b]$. Considering the second identity in (4.2) for $t\in[0,a]$, in view of the identity $\hat{p}=\hat{q}=0$ on $[0,b]$ we obtain $\hat{p}(\pi-a+t)+\hat{q}(\pi-b+t)=0$. The change of variable $z=\pi-a-t$ gives the identity (4.3) for $z\in[\pi-a,\pi]$. Thus, we arrive at the formula

$$\hat{p}(z)+\hat{q}(z+a-b)=0,\quad z\in[0,\pi]. \tag{4.4}$$

Similarly, the second identity in (4.2) for $t \in [b,\pi]$ gives $\hat p(\pi - t + a) + \hat q(\pi - t + b) = 0$, and after the change of variable we obtain

$$\hat p(z + a - b) + \hat q(z) = 0, \quad z \in [b,\pi]. \tag{4.5}$$

Since $\hat p = \hat q = 0$ on $[0,b]$, considering $b \leqslant z \leqslant \min(2b - a, \pi)$ in the formulas (4.4) and (4.5), we arrive at the identities $\hat p(z) = \hat q(z) = 0$. Repeating this step, by the induction we prove

$$\hat p = \hat q = 0, \quad b + (k-1)(b-a) \leqslant z \leqslant \min(b + k(b-a), \pi), \quad k = 1, \dots, n,$$

where $n \in \mathbb{N}$ is the smallest number such that $b + n(b-a) \geqslant \pi$. These identities mean that $\hat p = \hat q = 0$ on $[b,\pi]$, and this proves the theorem. The case, when p, q, $\tilde p$ and $\tilde q$ vanish identically on $[a,\pi]$, is considered in the same way. The proof is complete. □

5. Formulas for regularized traces

For $n \geqslant 1$ we let

$$a_{n0} = \frac{2}{\pi}\sin na \int_0^\pi \sin nt\, p(t)\, dt, \qquad a_{n1} = \frac{2}{\pi}\cos\Big(n - \frac12\Big)a \int_0^\pi \cos\Big(n-\frac12\Big)t\, p(t)\, dt,$$

$$b_{n0} = \frac{2}{\pi}\sin nb \int_0^\pi \sin nt\, q(t)\, dt, \qquad b_{n1} = \frac{2}{\pi}\cos\Big(n - \frac12\Big)b \int_0^\pi \cos\Big(n-\frac12\Big)t\, q(t)\, dt$$

and introduce the numbers $s_{nj} = a_{nj} + b_{nj}$, $j = 0,1$.

Theorem 5.1. *Let $p, q \in L_1[0,\pi]$ and $j = 0,1$. The series $\sum_{n=1}^{\infty}\Big(\lambda_{nj} - \big(n - \frac{j}{2}\big)^2\Big)$ converges if and only if the series $\sum_{n=1}^{\infty} s_{nj}$ does. In the convergence case the formula (1.4) holds.*

To prove this theorem, we shall need the following lemma.

Lemma 5.1. *Let $c < d$ and $f \in L_1[c,d]$. Then*

$$\int_c^d e^{i\rho(t-c)} f(t)\, dt = o\big(e^{|\tau|(d-c)}\big), \qquad \int_c^d e^{i\rho(d-t)} f(t)\, dt = o\big(e^{|\tau|(d-c)}\big), \quad \rho \to \infty, \tag{5.1}$$

where $\tau = \operatorname{Im}\rho$.

Proof. By the change of variable both identities in (5.1) are reduced to the asymptotic formula

$$I(\rho) := \int_0^z e^{i\rho t} g(t)\, dt = o\big(e^{|\tau| z}\big), \quad \rho \to \infty, \tag{5.2}$$

where $z = d - c > 0$ and $g \in L_1[0,z]$. To prove (5.2), we consider an arbitrary $\varepsilon > 0$. There exists a continuously differentiable function $\tilde g \in C^{(1)}[0,z]$ such that

$$\int_0^z |\tilde g(t) - g(t)|\, dt < \frac{\varepsilon}{2}.$$

Denoting

$$I_1(\rho) = \int_0^z e^{i\rho t}\tilde g(t)\, dt,$$

we find

$$\left|\int_0^z e^{i\rho t} g(t)\,dt\right| \leqslant \int_0^z |e^{i\rho t}|\,|\tilde g(t)-g(t)|\,dt + |I_1(\rho)| \leqslant e^{|\tau|z}\frac{\varepsilon}{2} + |I_1(\rho)|. \tag{5.3}$$

Integrating by parts in $I_1(\rho)$, we obtain

$$|I_1(\rho)| \leqslant |\rho|^{-1} M_\varepsilon e^{|\tau|z}, \qquad M_\varepsilon = 2\sup_{t\in[0,z]} |\tilde g(t)| + \int_0^{\pi-a} |\tilde g'(t)|\,dt.$$

For sufficiently large $|\rho| > 2\varepsilon^{-1}M_\varepsilon$ we have $|I_1(\rho)| \leqslant e^{|\tau|z}\frac{\varepsilon}{2}$. Applying this estimate to the right hand side in (5.3), for an arbitrary $\varepsilon > 0$ we obtain the inequality

$$|I(\rho)| \leqslant \varepsilon e^{|\tau|z}, \qquad |\rho| > 2\varepsilon^{-1}M_\varepsilon,$$

which implies (5.2). The proof is complete. □

Proof of Theorem 5.1. For the sake of definiteness we consider the case $j = 0$; the arguing for $j = 1$ is similar. We denote

$$S(\lambda) = \frac{\sin\rho\pi}{\rho}, \qquad \Gamma_N = \left\{\lambda\in\mathbb{C}\colon |\lambda| = \left(N+\frac{1}{2}\right)^2\right\}, \quad N\in\mathbb{N}.$$

Then

$$I_N := \sum_{n=1}^{N} (\lambda_{n0} - n^2) = \frac{1}{2\pi i}\int_{\Gamma_N} \lambda\left(\ln\frac{\Delta_0(\lambda)}{S(\lambda)}\right)' d\lambda. \tag{5.4}$$

For $\lambda\in\Gamma_N$ we estimate

$$\frac{\Delta_0(\lambda)}{S(\lambda)} = 1 + f(\lambda), \qquad f(\lambda) := \frac{A_{00}(\lambda) + A_{01}(\lambda) + B_0(\lambda)}{S(\lambda)}.$$

In the standard way, see [21], it can be proved that

$$|S(\lambda)| \geqslant C\frac{e^{|\tau|\pi}}{|\rho|}, \qquad \lambda\in\Gamma_N. \tag{5.5}$$

Since

$$\sin\rho\xi = \frac{e^{i\rho\xi} - e^{-i\rho\xi}}{2i}, \qquad \cos\rho\xi = \frac{e^{i\rho\xi} + e^{-i\rho\xi}}{2},$$

we can apply Lemma 5.1 to each integral in (2.4) and (2.5). This leads us to the asymptotics

$$A_{00}(\lambda) = o\Big(\frac{e^{|\tau|\pi}}{\rho^2}\Big), \qquad A_{01}(\lambda) = o\Big(\frac{e^{|\tau|\pi}}{\rho^2}\Big), \qquad B_0(\lambda) = o\Big(\frac{e^{|\tau|\pi}}{\rho^3}\Big). \tag{5.6}$$

Thus, by (5.5) and (5.6) we have $f(\lambda) = o(\rho^{-1})$, $\lambda\in\Gamma_N$, and for large N the increment of the argument of $\Delta_0(\lambda)/S(\lambda)$ on the contour Γ_N is equal to 0. Integrating by parts in (5.4), we arrive at the formula

$$I_N = -\frac{1}{2\pi i}\int_{\Gamma_N} \ln\Big(1+f(\lambda)\Big)\,d\lambda.$$

Applying the Taylor expansion to $\ln(1+f(\lambda))$, in view of the identities

$$f(\lambda) = o(\rho^{-1}) \quad\text{and}\quad \frac{B_0(\lambda)}{S(\lambda)} = o(\rho^{-2})$$

we obtain

$$I_N = -\frac{1}{2\pi i}\int\limits_{\Gamma_N}\left(\frac{A_{00}(\lambda)}{S(\lambda)} + \frac{A_{01}(\lambda)}{S(\lambda)} + o\Big(\frac{1}{\rho^2}\Big)\right)d\lambda = -\sum_{n=1}^{N}\operatorname*{Res}_{\lambda=n^2}\frac{A_{00}(\lambda)}{S(\lambda)} - \sum_{n=1}^{N}\operatorname*{Res}_{\lambda=n^2}\frac{A_{01}(\lambda)}{S(\lambda)} + o(1).$$

Calculating

$$\operatorname*{Res}_{\lambda=n^2}\frac{A_{00}(\lambda)}{S(\lambda)} = -a_{n0}, \qquad \operatorname*{Res}_{\lambda=n^2}\frac{A_{01}(\lambda)}{S(\lambda)} = -b_{n0},$$

we arrive at the formula

$$I_N = \sum_{n=1}^{N} s_{n0} + o(1).$$

Sending N to ∞, we complete the proof. □

The trigonometric systems of functions $\{\sqrt{\frac{\pi}{2}}\sin nt\}_{n\geqslant 1}$ and $\{\sqrt{\frac{\pi}{2}}\cos(n-\frac{1}{2})t\}_{n\geqslant 1}$ are orthonormalized bases in $L_2(0,\pi)$ since these are the systems of eigenfunctions of the unperturbed operator $-y''$ with the boundary conditions (1.2) for $j=0$ and $j=1$, respectively. The series $\sum\limits_{n=1}^{\infty} a_{nj}$ is the Fourier series of function p at the point a, the series $\sum\limits_{n=1}^{\infty} b_{nj}$ is the Fourier series of function q at the point b. There are several convergence tests at a point for the Fourier series of function f, see [1]. In particular, it is sufficient to claim the absolute continuity of f in the vicinity of a point to ensure that the series converges to the value of f at this point. If $p\in AC[a-\varepsilon,a+\varepsilon]$ and $q\in AC[b-\varepsilon,b+\varepsilon]$ for some $\varepsilon>0$, then

$$\sum_{n=1}^{\infty} a_{nj} = p(a), \qquad \sum_{n=1}^{\infty} b_{nj} = q(b),$$

and

$$\sum_{n=1}^{\infty}\left(\lambda_{nj} - \Big(n-\frac{j}{2}\Big)^2\right) = p(a)+q(b),$$

which agrees with the results of the previous works [20], [22]. We note that the convergence of series in (1.4) can hold even if the series $\sum\limits_{n=1}^{\infty} a_{nj}$ and $\sum\limits_{n=1}^{\infty} b_{nj}$ diverge.

Example 5.1. Let $j=0$, $a=\frac{\pi}{3}$ and $b=\pi-a$. We take functions $p,q\in L_2(0,\pi)$ such that

$$\int\limits_0^{\pi} p(t)\sin nt\,dt = \frac{\pi}{2n}\operatorname{sgn}\big(\sin na\big), \quad n\geqslant 1, \qquad q(t) = -p(\pi-t).$$

Then $a_{n0}=\frac{\sqrt{3}}{2n}$ for n, which is not a multiple of 3, and $a_{n0}=0$ otherwise. At the same time,

$$\sum_{n=1}^{\infty} a_{n0} = \frac{\sqrt{3}}{2}\sum_{k=1}^{\infty}\left(\frac{1}{3k-2}+\frac{1}{3k-1}\right) > \frac{\sqrt{3}}{2}\sum_{k=1}^{\infty}\frac{1}{3k},$$

and the series diverges. On the other hand, using the properties $b=\pi-a$ and $q(t)=-p(\pi-t)$, we obtain $b_{n0}=-a_{n0}$ and $s_{n0}=0$, $n\geqslant 1$. Thus, the series $\sum\limits_{n=1}^{\infty} a_{n0}$ and $\sum\limits_{n=1}^{\infty} b_{n0}$ diverge and $\sum\limits_{n=1}^{\infty} s_{n0}=0$. By Theorem 5.1 we arrive at the formula

$$\sum_{n=1}^{\infty}(\lambda_{n0}-n^2) = 0.$$

The latter identity holds once $b=\pi-a$ and $q(t)=-p(\pi-t)$.

Appendix: proof of formula (2.3)

For the sake of definiteness we consider the case $j=0$. Expanding the determinant along the third column and then along the second column, we obtain

$$\Delta_0(\lambda) = D_{00} + D_{01} + D_{10} + D_{11},$$

where

$$D_{00} := \begin{vmatrix} \frac{\sin\rho\pi}{\rho} & 0 & 0 \\ \frac{\sin\rho a}{\rho} & -1 & 0 \\ \frac{\sin\rho b}{\rho} & 0 & -1 \end{vmatrix}, \quad D_{11} := \begin{vmatrix} \frac{\sin\rho\pi}{\rho} & \int\limits_0^\pi \frac{\sin\rho(\pi-t)}{\rho}p(t)\,dt & \int\limits_0^\pi \frac{\sin\rho(\pi-t)}{\rho}q(t)\,dt \\ \frac{\sin\rho a}{\rho} & \int\limits_0^a \frac{\sin\rho(a-t)}{\rho}p(t)\,dt & \int\limits_0^a \frac{\sin\rho(a-t)}{\rho}q(t)\,dt \\ \frac{\sin\rho b}{\rho} & \int\limits_0^b \frac{\sin\rho(b-t)}{\rho}p(t)\,dt & \int\limits_0^b \frac{\sin\rho(b-t)}{\rho}q(t)\,dt \end{vmatrix},$$

$$D_{01} := \begin{vmatrix} \frac{\sin\rho\pi}{\rho} & \int\limits_0^\pi \frac{\sin\rho(\pi-t)}{\rho}p(t)\,dt & 0 \\ \frac{\sin\rho a}{\rho} & \int\limits_0^a \frac{\sin\rho(a-t)}{\rho}p(t)\,dt & 0 \\ \frac{\sin\rho b}{\rho} & \int\limits_0^b \frac{\sin\rho(b-t)}{\rho}p(t)\,dt & -1 \end{vmatrix}, \quad D_{10} := \begin{vmatrix} \frac{\sin\rho\pi}{\rho} & 0 & \int\limits_0^\pi \frac{\sin\rho(\pi-t)}{\rho}q(t)\,dt \\ \frac{\sin\rho a}{\rho} & -1 & \int\limits_0^a \frac{\sin\rho(a-t)}{\rho}q(t)\,dt \\ \frac{\sin\rho b}{\rho} & 0 & \int\limits_0^b \frac{\sin\rho(b-t)}{\rho}q(t)\,dt \end{vmatrix}.$$

It is clear that $D_{00} = \frac{\sin\rho\pi}{\rho}$, which gives the first term in (2.3). We consider D_{01} :

$$\begin{aligned} D_{01} &= \frac{\sin\rho a}{\rho}\int\limits_0^\pi \frac{\sin\rho(\pi-t)}{\rho}p(t)\,dt - \frac{\sin\rho\pi}{\rho}\int\limits_0^a \frac{\sin\rho(a-t)}{\rho}p(t)\,dt \\ &= \frac{\sin\rho a}{\rho}\int\limits_a^\pi \frac{\sin\rho(\pi-t)}{\rho}p(t)\,dt + \frac{1}{\rho^2}\int\limits_0^a \big[\sin\rho a\sin\rho(\pi-t) - \sin\rho\pi\sin\rho(a-t)\big]p(t)\,dt. \end{aligned}$$

In what follows we shall need the trigonometric formulas

$$\begin{aligned} \sin\alpha\sin(\beta-\gamma) - \sin\beta\sin(\alpha-\gamma) &= \sin\gamma\sin(\beta-\alpha), \\ \sin\alpha\cos(\beta-\gamma) - \cos\beta\sin(\alpha-\gamma) &= \sin\gamma\cos(\beta-\alpha). \end{aligned} \tag{5.7}$$

Applying the first formula from (5.7), we obtain

$$\sin\rho a\sin\rho(\pi-t) - \sin\rho\pi\sin\rho(a-t) = \sin\rho t\sin\rho(\pi-a),$$

and we arrive at the identity $D_{01} = A_{00}(\lambda)$. In the same way we find $D_{10} = A_{01}(\lambda)$.

We let $B_0(\lambda) = D_{11}$ and reduce $B_0(\lambda)$ to the needed form (2.5). We expand the determinant D_{11} along the second and third columns by splitting the integrals into the sums

$$\int\limits_0^\pi = \int\limits_0^a + \int\limits_a^b + \int\limits_b^\pi, \qquad \int\limits_0^b = \int\limits_0^a + \int\limits_a^b.$$

We write out the term, which contains the integrals of the functions p and q only on the segments $[0, a]$:

$$\begin{vmatrix} \dfrac{\sin \rho\pi}{\rho} & \displaystyle\int_0^a \frac{\sin \rho(\pi - t)}{\rho} p(t)\, dt & \displaystyle\int_0^a \frac{\sin \rho(\pi - t)}{\rho} q(t)\, dt \\ \dfrac{\sin \rho a}{\rho} & \displaystyle\int_0^a \frac{\sin \rho(a - t)}{\rho} p(t)\, dt & \displaystyle\int_0^a \frac{\sin \rho(a - t)}{\rho} q(t)\, dt \\ \dfrac{\sin \rho b}{\rho} & \displaystyle\int_0^a \frac{\sin \rho(b - t)}{\rho} p(t)\, dt & \displaystyle\int_0^a \frac{\sin \rho(b - t)}{\rho} q(t)\, dt \end{vmatrix}. \tag{5.8}$$

From the first row we deduct the second one multiplied by $\cos \rho(\pi - a)$, and from the third row we deduct the second one multiplied by $\cos \rho(b - a)$. Denoting

$$x_1 = \frac{\sin \rho(\pi - a)}{\rho}, \qquad x_3 = \frac{\sin \rho(b - a)}{\rho},$$

we obtain

$$\begin{vmatrix} x_1 \cos \rho a & x_1 \displaystyle\int_0^a \cos \rho(a - t) p(t)\, dt & x_1 \displaystyle\int_0^a \cos \rho(a - t) q(t)\, dt \\ \dfrac{\sin \rho a}{\rho} & \displaystyle\int_0^a \frac{\sin \rho(a - t)}{\rho} p(t)\, dt & \displaystyle\int_0^a \frac{\sin \rho(a - t)}{\rho} q(t)\, dt \\ x_3 \cos \rho a & x_3 \displaystyle\int_0^a \cos \rho(a - t) p(t)\, dt & x_3 \displaystyle\int_0^a \cos \rho(a - t) q(t)\, dt \end{vmatrix} = 0,$$

since the first and third rows are linearly dependent. Thus, the determinant (5.8) is equal to zero and in (2.5) there is no term involving as factors the integral of the function p over $[0, a]$ and the integral of the function q over $[0, a]$.

We write out the term, which involves as factors the integral of p over $[0, a]$ and the integral of q over $[a, b]$:

$$\begin{vmatrix} \dfrac{\sin \rho\pi}{\rho} & \displaystyle\int_0^a \frac{\sin \rho(\pi - t)}{\rho} p(t)\, dt & \displaystyle\int_a^b \frac{\sin \rho(\pi - t)}{\rho} q(t)\, dt \\ \dfrac{\sin \rho a}{\rho} & \displaystyle\int_0^a \frac{\sin \rho(a - t)}{\rho} p(t)\, dt & 0 \\ \dfrac{\sin \rho b}{\rho} & \displaystyle\int_0^a \frac{\sin \rho(b - t)}{\rho} p(t)\, dt & \displaystyle\int_a^b \frac{\sin \rho(b - t)}{\rho} q(t)\, dt \end{vmatrix}.$$

We multiply the third row by $\cos\rho(\pi-b)$ and deduct it from the first one, then we multiply the second row by $\cos\rho(b-a)$ and deduct it from the third row. We obtain

$$\frac{\sin\rho(\pi-b)}{\rho^3}\begin{vmatrix} \cos\rho b & \int\limits_0^a \cos\rho(b-t)p(t)\,dt & \int\limits_a^b \cos\rho(b-t)q(t)\,dt \\ \sin\rho a & \int\limits_0^a \sin\rho(a-t)p(t)\,dt & 0 \\ \sin\rho(b-a)\cos\rho a & \sin\rho(b-a)\int\limits_0^a \cos\rho(a-t)p(t)\,dt & \int\limits_a^b \sin\rho(b-t)q(t)\,dt \end{vmatrix}$$

$$=\frac{\sin\rho(\pi-b)}{\rho^3}\Bigg(\int\limits_a^b \cos\rho(b-\xi)q(\xi)\,d\xi\int\limits_0^a\big[\sin\rho a\sin\rho(b-t)-\sin\rho b\sin\rho(a-t)\big]p(t)\,dt$$

$$-\int\limits_a^b \sin\rho(b-\xi)q(\xi)\,d\xi\int\limits_0^a\big[\sin\rho a\cos\rho(b-t)-\cos\rho b\sin\rho(a-t)\big]p(t)\,dt\Bigg).$$

Applying the formulas (5.7) to the terms in square brackets, we arrive at the expression

$$\frac{\sin\rho(\pi-b)}{\rho^3}\int\limits_0^a \sin\rho t\,p(t)\,dt\int\limits_a^b\big(\sin\rho(b-a)\cos\rho(b-\xi)-\cos\rho(b-a)\sin\rho(b-\xi)\big)q(\xi)\,d\xi$$

$$=\frac{\sin\rho(\pi-b)}{\rho^3}\int\limits_0^a \sin\rho t\,p(t)\,dt\int\limits_a^b \sin\rho(\xi-a)\,q(\xi)\,d\xi$$

that gives the first term in the first brackets in (2.5). The other terms in the expansion of the determinant D_{11} can be considered in the same way.

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