ISSN 2304-0122 Ufa Mathematical Journal. Vol. 17. No 3 (2025). P. 17-45.

doi:10.13108/2025-17-3-17

SEMIANALYTIC APPROXIMATION OF NORMAL DERIVATIVE OF DOUBLE LAYER POTENTIAL NEAR AND AT BOUNDARY OF TWO–DIMENSIONAL DOMAIN

D.Yu. IVANOV

Abstract. The normal derivatives (ND) of the double layer potential (DLP) are defined on a boundary of a domain by hyper–singular integrals. This is why, it is impossible to calculate ND DLP with a satisfactory accuracy either on the boundary or in its vicinity using traditional quadrature formulas, which allow one to calculate ND DLP with a good accuracy at a sufficient distance from the boundary. In the present paper, we obtain semi–analytical approximations of ND DLP for the two–dimensional Laplace equation, which uniformly converge with an almost cubic velocity in a closed near–boundary domain that includes the boundary. For this purpose, we use exact integration over the smooth component of the distance function near the observation point, an additive–multiplicative method for extracting a singularity, and a piecewise quadratic interpolation of slowly varying functions. We provide the results of calculating ND DLP in a closed near–boundary domain of a unit circle, which confirm the uniform almost cubic convergence of the proposed approximations.

Keywords: quadratic formulas, analytic integration, double layer potential, uniform convergence.

Mathematics Subject Classification: 3108, 65E05

1. Introduction

The boundary element method (BEM), along with the finite element method (FEM) and the finite difference method (FDM), is one of the main methods for the approximate solving of boundary value problems [3, Sect. 2.5]. In the BEM, the discretization is performed only on the boundary $\partial\Omega$ of a domain Ω, in contrast to the FEM and FDM, where it is necessary to discretize the entire domain Ω. The BEM is based on the analytical method of boundary integral equations (BIE) [16, Sect. 1.1]. In the framework of the BIE method, the solution to the boundary value problem at any point $x \in \Omega$ is sought in the form of the so–called potential, which, using the integral operator, is expressed through the unknown density defined on the boundary $\partial\Omega$ as a solution to the BIE. In two–dimensional problems, the potentials are curvilinear integrals over the arc length s. The BIE operator is also expressed in terms of the potentials and their derivatives defined on the boundary $\partial\Omega$. For example, solutions of the internal and external Dirichlet problems are expressed as a double layer potential (DLP), where the density is a solution of the corresponding BIE of the second kind, the integral operator of which also has the form of a DLP [10, Sect. 3.4]. Therefore, to implement the BEM, an approximation of the potentials and their derivatives is required. For this purpose, within the framework of the two–dimensional BEM, the curve $\partial\Omega$ is divided into arcs, the so–called boundary elements (BE). On each BE, the density is approximated by a polynomial coinciding with the density at the given nodes, that is, the piecewise polynomial interpolation is

D.Yu. Ivanov, Semianalytic approximation of normal derivative of double layer potential near and at boundary of two–dimensional domain.

Submitted July 14, 2024.

performed. The integrals arising on the BE cannot be calculated exactly in the general case. To calculate them, quadrature formulas are used, usually these are the Gauss formulas [2, Ch. 3, Sect. 5, Subsect. 3]. Gauss formulas allow one to attain a satisfactory accuracy if the potential is calculated at a sufficient distance from the boundary $\partial\Omega$. If the observation point x, at which the potential is calculated, is located sufficiently close to the nodes of the Gauss formula, the accuracy, as a rule, becomes unsatisfactory. This phenomenon, called the boundary layer effect [32], is due to the fact that the kernel of the integral operator $K(x, x')$ has a singularity at $x = x' \in \partial\Omega$, and it is impossible to calculate the potential at the nodes of the Gauss formula. The integrals on the GE are called singular (SI) if the observation point x belongs to the BE, and almost singular (ASI) if the point x is sufficiently close to the BE [28].

The need for calculations near the boundary of the domain arises when solving problems in thin–walled and multilayer structures, thin coatings, films, at the ends of cracks [23], [25], [31]. To calculate the SI and ASI, special methods are used, including semi–analytical ones [11]–[14], [17], [23], [24], [27], [29], [30], [31]. One of such methods, outlined in [24], [27], [31] and proposed for approximating two–dimensional potentials and their derivatives near and at the boundary $\partial\Omega$, is based on the passage from the integration variable s to the variable $\rho = (r^2 - d^2)^{\frac{1}{2}}$, where r and d are the distances from the observation point x to the integration point $x' \in \partial\Omega$ and to the curve $\partial\Omega$, respectively. The integrals over ρ are calculated exactly. For this purpose, a multiplicative method of extracting a singularity is used: the integrand is split into two factors, one of which, including the Jacobian, is a slowly varying function and is therefore approximated by polynomials, while the other factor, singular or close to singular, is taken as a weight function. This approach has two advantages. First, as it was shown in [4], it can be implemented for any sufficiently smooth analytically defined curve $\partial\Omega$ since the integrals over the variable ρ are calculated exactly in many known cases and depend on the curve $\partial\Omega$ only parametrically, that is, the coordinate functions of the curve $\partial\Omega$ cannot fundamentally change the form of the integrand. Second, there are no visible reasons preventing unbounded growth of approximation order in the framework of this method. Indeed, the approximation order is determined by the order of the polynomial in the numerator, therefore, the complexity of the integral does not increase essentially while increasing the order of polynomial. We note that in the framework of semi–analytical methods based on the approximation of integrals over the arc length s, the unbounded growth of the approximation order is prevented by the fact that the powers of the distance function r^{2n} ($n \in \mathbb{N}$) are in the denominator of the integrand. Indeed, in order to make the exact integration possible, the function r^2 is replaced by a polynomial in powers of s. Since the polynomial is in the denominator, its degree cannot exceed two, otherwise obtaining an analytical expression for the antiderivative becomes difficult. Therefore, linear approximation of coordinate functions is mainly used [11]–[14], [17], [23], [29]. If the quadratic approximation is employed, then the polynomial obtained instead of r^2 is still truncated to a quadratic one [30], and within this approach, usage of higher–degree polynomials to approximate coordinate functions makes no sense.

In works of the author, by using exact integration with respect to the variable ρ, there were obtained uniformly converging approximations of the thermal potential of a simple layer potential (SLP) near the boundary of a two–dimensional domain [4], approximations of the thermal DLP [5], approximations of the DLP for the Laplace equation [6], approximations of the normal derivative of the thermal SLP [7], approximations of the DLP for the dissipative Helmholtz equation [8], approximations of the ND SLP for the Laplace equation [9]. In the present work, on the base of exact integration with respect to the variable ρ, we obtain uniformly converging approximations of the ND DLP for the two–dimensional Laplace equation near and at the boundary. The need to calculate the values of the ND DLP on the boundary $\partial\Omega$ arises when solving various BIEs [10, Sects. 3.5–3.7, 3.9]. If the solution to the Dirichlet problem is sought in the form of a DLP, then the ND DLP for the Laplace equation can be used to calculate the steady–state heat flux, the fluid velocity during steady flow, and the strength of an electrostatic field. We note that the solution to the Dirichlet problem in the form of a DLP has the advantage that stable BIEs of the second kind are used, whereas unstable BIEs of the first kind are used when obtaining the solution to the Dirichlet problem using the SLP.

It is known that DLP with a continuous density generally speaking possesses no derivatives at the boundary, but if the density is smooth enough, the ND DLP can be defined at the boundary to a continuous function in the entire space [15, Sect. 2.5]. By the ND DLP on the boundary we mean the values obtained by the continuous continuation of ND DLP from the domain Ω. They can be calculated by using hyper–singular integrals, which exist only in the sense of a finite Hadamard value, the integrand in which has the singularity r^{-2} [15, Sect. 1.6]. In [15, Sect. 13.6] the ND DLP on the boundary is approximately calculated on the base of a semi–analytical approximation of such integrals at the support points, which are the midpoints of the BE: at each BE the density is replaced by a constant function with a node at the support point, and the integrals arising after this in the case of the Laplace equation are calculated analytically by the discrete vortex method. It was shown in [21] that the ND DLP on the boundary can be expressed in terms of less singular integrals that exist in the sense of the Cauchy principal value. In [1], [22], the ND DLP, expressed in terms of such integrals, was approximated at the support points being the midpoints of the BE on the base of the quadrature formula of mean rectangles. The approximations of the ND DLP on the boundary of a two–dimensional domain Ω obtained in [15, Sect. 13.6] and [1], [22] have errors $O(h\ln h)$ and $O(h^{\frac{1}{2}})$, respectively, where h is the length of the BE.

In the present work we construct the approximations for ND DLP in a closed near–boundary domain, which includes the boundary and have the semi–width equalling to the third of radius of Lyapunov circle. The integrals obtained as the result of continuation by continuity and existing only in the sense of Hadamard finite value are approximated on the boundary. At the first step, the boundary is divided into BEs of equal length and, for fixed d, we calculate the approximations for the ND DLP at support points, the projections of which onto the curve $\partial\Omega$ coincide with the midpoints of the BEs. We employ the piecewise quadratic interpolation (PQI) with three nodes at the ends and in the middle of the BE to approximate the density. Exact integration with respect to ρ is performed in some neighborhood of the projection of each support point, where the Jacobian remains positive and bounded (not on the SI and ASI, as in [24], [27], [31], but on arcs of fixed length). The integral with respect to ρ is represented as a sum of three integrals of different degrees of singularity, each of which is approximated on the base of the above described multiplicative method of extracting a singularity. The PQI is also used to approximate slowly changing factors in the integrands. To achieve the highest approximation order, an additional discretization with respect to ρ is performed in such a way that the BEs with respect to ρ are located symmetrically with respect to each support point. Such additional discretization requires no additional values of the density. Outside the neighborhoods where exact integration with respect to ρ is performed, Gauss formulas of at least fifth order are used to approximate the integrals arising after the PQI of the density. At the second step, in order to double the number of reference points used without involving new values of the density, all BEs are shifted by the half of BE length, and then, just as at the first step, the approximations of the ND DLP are calculated at the reference points corresponding to the midpoints of the new BEs. At the third step, for a fixed d, the PQI is performed with respect to the variable s with nodes at the reference points. This allows us to calculate approximate values of the NP DLP at any point of the closed near–boundary domain described by the curvilinear coordinates s and d. We prove that the approximations of ND DLP obtained in this way converge uniformly in the closed near–boundary domain at the rate $O(h^3\ln h)$. The final section presents the results of calculating ND DLP in a closed near–boundary domain of a unit circle, which confirm the almost cubic rate of uniform convergence of semi–analytical approximations, and also demonstrate the impossibility of using exclusively Gaussian formulas to approximate integrals arising under the interpolation of density.

2. Formulation of problem

Let Ω_+ be a two–dimensional simply connected domain with a boundary $\partial\Omega$. In the Cartesian coordinates (x_1, x_2) we define parametric equations of the curve $\partial\Omega$: $x_1 = \tilde{x}_1(s)$, $x_2 = \tilde{x}_2(s)$. The absolute value of the parameter s is equal to the length of the arc beginning at some fixed point and ending at the point $\tilde{x}(s) := (\tilde{x}_1(s), \tilde{x}_2(s))$, and it increases as the domain Ω_+ is to the left while passing the boundary $\partial\Omega$. The functions $\tilde{x}_1(s)$, $\tilde{x}_2(s)$ $(s \in \mathbb{R})$ are periodic with the period $2S$, where S is the half length of $\partial\Omega$, and make a one–to–one correspondence between the sets $[-S, S)$ and $\partial\Omega$. In what follows we write $\partial\Omega \in C^n$ if there exists continuous on the closed set $I_S := [-S, S]$ derivatives $\tilde{x}_i^{(l)}(s)$ $(l = \overline{0, n},\ i = 1, 2)$, and $\tilde{x}_i^{(l)}(-S + 0) = \tilde{x}_i^{(l)}(S - 0)$. We suppose that $\partial\Omega \in C^2$ unless otherwise is said.

By $C(\partial\Omega)$ we denote the Banach space of $2S$–periodic continuous on the entire real axis $\mathbb{R}$ real functions $f(s)$ with the norm

$$\|f\|_{C(\partial\Omega)} := \sup_{s \in I_S} |f(s)|\,.$$

By $C^n(\partial\Omega)$ $(n \in \mathbb{Z}_+)$ we denote the Banach spaces of the functions $f \in C(\partial\Omega)$ having continuous on the set I_S derivatives $f^{(l)}(s)$ $(l = \overline{1, n})$ with the norm

$$\|f\|_{C^n(\partial\Omega)} := \sum_{l=0}^{n} \left\|f^{(l)}\right\|_{C(\partial\Omega)} \quad (C^0(\partial\Omega) = C(\partial\Omega)).$$

By $\vec{e}(s)$ we denote the unit vector directed along the tangential line to $\partial\Omega$ at the point $\tilde{x}(s)$ to the side of increasing of the parameter s, while $\vec{n}(s)$ stands for the unit normal to the curve $\partial\Omega$ at the point $\tilde{x}(s)$ directed inside the domain Ω_+. The vectors $\vec{e}(s)$, $\vec{n}(s)$ form the right system and their coordinates (x_1, x_2) are calculated by means of the formulas $\vec{e}(s) = (\tilde{x}_1'(s), \tilde{x}_2'(s))$, $\vec{n}(s) = (-\tilde{x}_2'(s), \tilde{x}_1'(s))$.

We introduce local Cartesian coordinates (ξ_s, η_s) with the origins at the points $\tilde{x}(s)$ and the abscissa and ordinate axes directed along the vectors $\vec{e}(s)$ and $\vec{n}(s)$, respectively. The points $\tilde{x}_d(s)$ $(d \in \mathbb{R})$ with the local coordinates $(\xi_s, \eta_s) = (0, d)$ form the normal to the curve $\partial\Omega$ passing through $\tilde{x}(s)$ $(\tilde{x}_0(s) := \tilde{x}(s))$. There exists a set I_D of form $[-D_1, 0) \cup (0, D_2]$ $(D_1, D_2 > 0)$ such that for $(s, d) \in \mathrm{T} := I_S \times I_D$ the correspondence between the points $\tilde{x}(s)$ and $\tilde{x}_d(s)$ is one–to–one. For instance, $I_D = [-D, 0) \cup (0, D]$, where D is the third of the radius of the Lyapunov circle [18, Sect. 102]. Since $\partial\Omega \in C^2$, for a fixed $d \in I_D$ the points $\tilde{x}_d(s)$ form a closed curve $\partial\Omega_d \in C^1$, and the normals $\tilde{x}(s)\tilde{x}_d(s)$ to the curves $\partial\Omega$ are also the normals to the curve $\partial\Omega_d$. The curves $\partial\Omega_d$ are called the curves parallel to the curve $\partial\Omega$.

On the set T we define the function u

$$u(s, d) := \int\limits_{I_S} g\, v(s + \sigma)\, d\sigma. \tag{2.1}$$

Here

$$g(s, \sigma, d) := \partial_{\vec{n}(s)} \partial_{\vec{n}(s+\sigma)} \ln r^{-1} = -2^{-1} \partial_{\vec{n}(s)} \left[\frac{\left(\partial_{\vec{n}(s+\sigma)} r^2\right)}{r^2} \right],$$

$$v \in C(\partial\Omega), \qquad r(s, \sigma, d) := |\vec{r}|\,, \qquad \vec{r}(s, \sigma, d) := \overrightarrow{\tilde{x}_d(s)\tilde{x}(s + \sigma)};$$

the differentiations $\partial_{\vec{n}(s)}$ and $\partial_{\vec{n}(s+\sigma)}$ are made at the points $\tilde{x}_d(s)$ and $\tilde{x}(s + \sigma)$ in the directions $\vec{n}(s)$ and $\vec{n}(s + \sigma)$, respectively. We adopt that sometimes we do not write the variables of a function if they are the same as in the definition of function. The function $U(\tilde{x}_d(s)) := (2\pi)^{-1} u$

is ND DLP with the density v in the near–boundary domain Ω_D formed by the points $\tilde{x}_d(s)$ as $(s,d) \in \mathrm{T}$.

By the known theorems [10, Sect. 2.5, Thms. 2.21, 2.23] the function $U(\tilde{x}_d(s))$ can be defined by the continuity at $d=0$, $s \in I_S$. Here we obtain the approximations $\breve{U}(\tilde{x}_d(s))$ of the function $U(\tilde{x}_d(s))$ uniformly converging on the set $\overline{\mathrm{T}}$ with the rate $O(h^3 \ln h)$, that is, they satisfy the estimates

$$\left|\breve{U} - U\right| \leqslant ch^3 \left|\ln h\right| \quad ((s,d) \in \overline{\mathrm{T}}), \tag{2.2}$$

where c is a positive constants independent of h and (s,d).

3. Some facts about quadratic interpolation

We consider a quadratic interpolation, which will be used, for some real function $f(z)$. Let $h > 0$, $z_0 \in \mathbb{R}$, $z_{\pm 1} := z_0 \pm h$. By $C^k(I_h)$ $(k = 0,1,\ldots)$ we denote the Banach spaces of continuous on the segment $I_h \equiv [z_{-1}, z_1]$ functions $f(z)$ having continuous on I_h derivatives $f^{(l)}(z)$ $(l = \overline{1,k})$ with the norm

$$\|f\|_{C^k(I_h)} \equiv \sum_{l=0}^{k} \sup_{z \in I_h} \left|f^{(l)}(z)\right|.$$

Let $f \in C^0(I_h)$. We define the Lagrange interpolation polynomial $\hat{f}(z)$ with the equidistant nodes z_{-1}, z_0, z_1 [2, Ch. 2, Sect. 5]

$$\hat{f}(z) := f(z_{-1}) + (z - z_{-1}) f_1(z_{-1}, z_1) + (z - z_{-1})(z - z_1) f_2(z_{-1}, z_1, z_0).$$

Here $f_1(z_{-1}, z_1)$, $f_2(z_{-1}, z_1, z_0)$ are separated distance of 1st and 2nd orders with non–coinciding values of the variable [2, Ch. 2, Sect. 5, Subsect. 1]

$$f_1(z_{-1}, z_1) := \frac{f(z_1) - f(z_{-1})}{z_1 - z_{-1}},$$
$$f_2(z_{-1}, z_1, z_0) := \frac{f_1(z_1, z_0) - f_1(z_{-1}, z_1)}{z_0 - z_{-1}}.$$

The separated differences are symmetric functions, that is, their values do not change under each permutation of the variables [2, Ch. 2, Sect. 5, Subsect. 1, Prop. 3]. At the nodes, the identities $\tilde{f}(z_m) = f(z_m)$ $(m = \overline{-1,1})$ hold. For $z \in (z_{-1}, z_0) \cup (z_0, z_1)$ the following two formulas hold

$$f(z) = \hat{f}(z) + (z - z_{-1})(z - z_1)(z - z_0) f_3(z, z_{-1}, z_1, z_0) \qquad (f \in C^0(I_h)), \tag{3.1}$$

$$\begin{aligned} f(z) = \hat{f}(z) &+ (z - z_{-1})(z - z_1)(z - z_0) f_3(z_{-1}, z_1, z_0, z_0) \\ &+ (z - z_{-1})(z - z_1)(z - z_0)^2 f_4(z, z_{-1}, z_1, z_0, z_0) \qquad (f \in C^1(I_h)), \end{aligned} \tag{3.2}$$

see the formulas for the residual term in the Newton interpolation formula [2, Ch. 2, Sect. 5, Subsect. 3], [2, Ch. 2, Sect. 11, Subsect. 5]). Here f_3, f_4 are separated differences of 3rd and 4th orders, the variables of which coincide or can coincide

$$f_3(z, z_{-1}, z_1, z_0) := \frac{f_2(z_{-1}, z_1, z_0) - f_2(z, z_{-1}, z_1)}{z_0 - z} \qquad (z \in (z_{-1}, z_0) \cup (z_0, z_1),\ f \in C^0(I_h)),$$

$$f_3(z_m, z_{-1}, z_1, z_0) := \lim_{z' \to z_m} f_3(z', z_{-1}, z_1, z_0) \qquad (m = \overline{-1,1},\ f \in C^1(I_h));$$

$$f_4(z, z_{-1}, z_1, z_0, z_0) := \frac{f_3(z_{-1}, z_1, z_0, z_0) - f_3(z, z_{-1}, z_1, z_0)}{z_0 - z} \quad (z \in (z_{-1}, z_0) \cup (z_0, z_1),\ f \in C^1(I_h)),$$

$$f_4(z_m, z_{-1}, z_1, z_0, z_0) := \lim_{z' \to z_m} f_4(z', z_{-1}, z_1, z_0, z_0) \qquad (m = \overline{-1,1},\ f \in C^2(I_h)).$$

For each $z \in [z_{-1}, z_1]$ there exist $\zeta_1(z), \zeta_2(z) \in [z_{-1}, z_1]$ such that

$$f_3(z, z_{-1}, z_1, z_0) = 6^{-1} f^{(3)}(\zeta_1) \qquad (f \in C^3(I_h)), \tag{3.3}$$

$$f_4(z, z_{-1}, z_1, z_0, z_0) = 24^{-1} f^{(4)}(\zeta_2) \qquad (f \in C^4(I_h)), \tag{3.4}$$

see [2, Ch. 2, Sect. 5, Subsect. 3], [2, Ch. 2, Sect. 11, Subsect. 5]. By the formulas (3.1), (3.3), we have the majorant estimate for interpolation error on the segment $[z_{-1}, z_1]$:

$$\left|\hat{f}(z) - f(z)\right| \leqslant c_\lambda \sup_{z \in [z_{-1}, z_1]} \left|f^{(3)}(z)\right| h^3 \quad \left(z \in [z_{-1}, z_1],\ f \in C^3(I_h),\ c_\lambda := \frac{\sqrt{3}}{27}\right). \tag{3.5}$$

The interpolation polynomial $\hat{f}(z)$ can be also written as the sum

$$\hat{f}(z) = \sum_{m=-1}^{1} f(z_m) \Lambda_m^{[z_{-1}, z_1]}(z) \qquad (z \in [z_{-1}, z_1]), \tag{3.6}$$

where

$$\Lambda_m^{[z_{-1}, z_1]}(z) := \prod_{j=-1\ (j \neq m)}^{1} \frac{z - z_j}{z_m - z_j} \qquad (m = \overline{-1, 1}),$$

see [2, Ch. 2, Sect. 2, Subsect. 1]. On the base of the formula (3.6) and the inequalities

$$\left|\Lambda_m^{[z_{-1}, z_1]}(z)\right| \leqslant 1 \qquad (z \in [z_{-1}, z_1], \quad m = \overline{0, 2}),$$

by means of the mean value theorem for $z \in [z_{-1}, z_1]$ we obtain the estimates

$$\left|\hat{f}(z)\right| \leqslant c_{\Lambda,0} \max_{m=\overline{-1,1}} |f(z_m)| \qquad (f \in C^0(I_h),\ c_{\Lambda,0} := 3), \tag{3.7}$$

$$\left|\hat{f}(z) - f(z)\right| \leqslant c_{\lambda,0} \sup_{z \in [z_{-1}, z_1]} \left|f^{(1)}(z)\right| h \qquad (f \in C^1(I_h),\ c_{\lambda,0} := 3). \tag{3.8}$$

For $z \in [z_{-1}, z_1]$, for the derivatives $\hat{f}^{(j)}(z)$ $(j = 1, 2)$ we have the formulas

$$\begin{aligned}
\hat{f}^{(1)}(z) &= \frac{(2z - z_0 - z_1) \int_{z_0}^{z_{-1}} f^{(1)}(\zeta) d\zeta}{(z_{-1} - z_0)(z_{-1} - z_1)} + \frac{(2z - z_0 - z_{-1}) \int_{z_0}^{z_1} f^{(1)}(\zeta) d\zeta}{(z_1 - z_0)(z_1 - z_{-1})} \qquad (f \in C^1(I_h)), \\
\hat{f}^{(2)}(z) &= \frac{2 \int_{z_0}^{z_{-1}} f^{(2)}(\zeta)(z_{-1} - \zeta) d\zeta}{(z_{-1} - z_0)(z_{-1} - z_1)} + \frac{2 \int_{z_0}^{z_1} f^{(2)}(\zeta)(z_1 - \zeta) d\zeta}{(z_1 - z_0)(z_1 - z_{-1})} \qquad (f \in C^2(I_h))
\end{aligned} \tag{3.9}$$

obtained by the representation (3.6) and the Taylor form with the residual term in the integral form (TFETIF) [19, Sect. 318]. By the mean value theorem the formulas (3.9) allow us to obtain the estimates for $z \in [z_{-1}, z_1]$

$$\left|\hat{f}^{(j)}(z)\right| \leqslant c_{\Lambda,j} \sup_{z \in [z_{-1}, z_1]} \left|f^{(j)}(z)\right| \qquad (f \in C^j([z_{-1}, z_1]),\ j = 1, 2), \tag{3.10}$$

$$\left|\hat{f}^{(j)}(z) - f^{(j)}(z)\right| \leqslant c_{\lambda,j} \sup_{z \in [z_{-1}, z_1]} \left|f^{(j+1)}(z)\right| h \qquad (f \in C^{j+1}([z_{-1}, z_1]),\ j = 1, 2), \tag{3.11}$$

where $c_{\Lambda,1} := 3$, $c_{\Lambda,2} := 2^{-1}$, $c_{\lambda,1} := 4$, $c_{\lambda,2} := 2$.

4. Additive– multiplicative selection of singularity

Here we consider the possibility of passage from the integration variable σ to the integration variable ρ in the integrals (2.1), which will be used for the semi–analytic approximation of ND DLP. Let $\vec{r}_0(s,\sigma) := \overrightarrow{\tilde{x}(s)\tilde{x}(s+\sigma)}$, $r_0(s,\sigma) := |\vec{r}_0|$. On the set $\Theta := I_S \times I_S$ we define the functions $\psi_i(s,\sigma)$ $(i=\overline{0,4})$: for $\sigma \neq 0$ we let

$$\psi_i := \frac{\varphi_i}{\sigma^2} \quad (i=\overline{0,2}), \qquad \psi_i := \frac{\varphi_i}{\sigma} \quad (i=3,4),$$

where

$$\begin{aligned}
\varphi_0(s,\sigma) &:= r_0^2 = [\tilde{x}_1(s+\sigma) - \tilde{x}_1(s)]^2 + [\tilde{x}_2(s+\sigma) - \tilde{x}_2(s)]^2, \\
\varphi_1(s,\sigma) &:= 2^{-1}\partial_{\vec{n}(s+\sigma)} r_0^2 = -\tilde{x}_2'(s+\sigma)\,[\tilde{x}_1(s+\sigma) - \tilde{x}_1(s)] + \tilde{x}_1'(s+\sigma)\,[\tilde{x}_2(s+\sigma) - \tilde{x}_2(s)] \\
&= (\vec{n}(s+\sigma)\,, \vec{r}_0)_{\mathbb{R}^2}, \\
\varphi_2(s,\sigma) &:= 2^{-1}\partial_{\vec{n}(s)} r_0^2 = -\tilde{x}_2'(s)\,[\tilde{x}_1(s) - \tilde{x}_1(s+\sigma)] + \tilde{x}_1'(s)\,[\tilde{x}_2(s) - \tilde{x}_2(s+\sigma)] \\
&= -\,(\vec{n}(s), \vec{r}_0)_{\mathbb{R}^2}, \\
\varphi_3(s,\sigma) &:= 2^{-1}\partial_\sigma \varphi_0 = \tilde{x}_1'(s+\sigma)\,[\tilde{x}_1(s+\sigma) - \tilde{x}_1(s)] + \tilde{x}_2'(s+\sigma)\,[\tilde{x}_2(s+\sigma) - \tilde{x}_2(s)] \\
&= (\vec{e}(s+\sigma)\,, \vec{r}_0)_{\mathbb{R}^2}, \\
\varphi_4(s,\sigma) &:= \partial_\sigma \varphi_2 = \tilde{x}_2'(s)\,\tilde{x}_1'(s+\sigma) - \tilde{x}_1'(s)\,\tilde{x}_2'(s+\sigma) = -\,(\vec{n}(s), \vec{e}(s+\sigma))_{\mathbb{R}^2},
\end{aligned}$$

$(\cdot,\cdot)_{\mathbb{R}^2}$ is the scalar product in the Euclidean space $\mathbb{R}^2$), while for $\sigma = 0$ we let

$$\psi_0 = \psi_3 := 1, \qquad \psi_1 = \psi_2 = 2^{-1}\psi_4 := 2^{-1}\left(\tilde{x}_2'(s)\,\tilde{x}_1''(s) - \tilde{x}_1'(s)\,\tilde{x}_2''(s)\right) = -2^{-1}K(s).$$

Here $K(s)$ is the signed curvature [20, Subsect. 250] of the curve $\partial\Omega$ at the point $\tilde{x}(s)$. Similarly to [8, Thm. 2.1] one can show that under the condition $\partial\Omega \in C^{n+2}$ $(n \in \mathbb{Z}_+)$ there exist continuous on the set Θ derivative $\partial_s^k \partial_\sigma^l \psi_i$, $k = \overline{0, n-l}$, $l = \overline{0,n}$ for $i = \overline{0,2}, 4$; $k = \overline{0, n+1-l}$, $l = \overline{0, n+1}$ for $i = 3$.

The local coordinates (ξ_s, η_s) of the points $\tilde{x}_d(s)$ and $\tilde{x}(s+\sigma)$ are equal to $(0,d)$ and $\left((\vec{e}(s), \vec{r}_0)_{\mathbb{R}^2}, (\vec{n}(s), \vec{r}_0)_{\mathbb{R}^2}\right)$, respectively, and this is why

$$r^2 = \left|\overrightarrow{\tilde{x}_d(s)\tilde{x}(s+\sigma)}\right|^2 = r_0^2 - 2d\,(\vec{n}(s), \vec{r}_0)_{\mathbb{R}^2} + d^2.$$

On the set $\Upsilon := \Theta \times \overline{I}_D = I_S \times \overline{\mathrm{T}}$ we define the functions $\varphi_0'(s,\sigma,d)$, $\psi_0'(s,\sigma,d)$, $\psi_1'(s,\sigma,d)$:

$$\varphi_0' := r^2 - d^2 = \varphi_0 + 2d\varphi_2, \qquad \psi_0' := \psi_0 + 2d\psi_2, \qquad \psi_1' := \psi_3 + d\psi_4.$$

Since $\varphi_0' \geqslant 0$ for $(s,\sigma,d) \in \Upsilon$ ($\varphi_0' > 0$ for $\sigma \neq 0$, $\varphi_0' = 0$ for $\sigma = 0$) [8, Sect. 2], on the set Υ we can define the function $\rho'(s,\sigma,d)$:

$$\rho' := \sqrt{\varphi_0'}, \quad \text{if} \quad \sigma \geqslant 0; \qquad \rho' := -\sqrt{\varphi_0'}, \quad \text{if} \quad \sigma < 0.$$

Since

$$r^2 = \varphi_0 + 2d\,\varphi_2 + d^2,$$

we have

$$2^{-1}\partial_{\vec{n}(s+\sigma)} r^2 = \varphi_1 + d\varphi_5,$$

where

$$\varphi_5(s,\sigma) := -\tilde{x}_1'(s)\tilde{x}_1'(s+\sigma) - \tilde{x}_2'(s)\tilde{x}_2'(s+\sigma) \quad \text{and} \quad \partial_{\vec{n}(s+\sigma)} \ln r^{-1} = -\frac{\varphi_1 + d\,\varphi_5}{\varphi_0 + 2d\,\varphi_2 + d^2}.$$

This is why for $(s,\sigma,d) \in \Upsilon$ (except for $d = \sigma = 0$), the function $g = \partial_d \partial_{\vec{n}(s+\sigma)} \ln r^{-1}$ can be written as

$$g = a_1(\rho', d)\;\delta_1 + a_2(\rho', d)\,\delta_2 + a_3(\rho', d)\,\delta_3, \tag{4.1}$$

where

$$a_1(\rho,d) := \left(-\rho^2+d^2\right)A^{-2},\quad a_2(\rho,d) := 2d\rho^2A^{-2},\quad a_3(\rho,d) := \rho^4A^{-2},\quad A(\rho,d) := \rho^2+d^2,$$

$$\delta_1(s,\sigma) := \varphi_5,\quad \delta_2(s,\sigma,d) := \frac{\psi_1+\psi_2\varphi_5}{\psi_0'},\quad \delta_3(s,\sigma,d) := \frac{2\psi_1\psi_2}{(\psi_0')^2}.$$

In what follows we suppose that the set $I_D = [-D_1,0)\cup(0,D_2]$ $(D_1,D_2>0)$ is defined by the conditions

(i) for $(s,d)\in\mathrm{T}$, the correspondence between the points $\tilde{x}(s)$ and $\tilde{x}_d(s)$ is one–to–one;
(ii) for $(s,d)\in\mathrm{T}$ the inequalities $1-dK(s)>0$ hold.

For instance, $I_D = [-D,0)\cup(0,D]$, since at the same time $1-dK(s)\geqslant\dfrac{2}{3}$ $((s,d)\in\mathrm{T})$ [8, Sect. 2]. We can ensure an arbitrary smoothness for the functions δ_i in (s,σ) by an appropriate smoothness of the boundary $\partial\Omega$. Indeed, since

$$\psi_0(s,0)=1,\qquad \psi_2(s,0)=-2^{-1}K(s),$$

we have

$$\psi_0'(s,0,d)=1-dK(s)>0\quad\text{for}\quad (s,d)\in\overline{\mathrm{T}}.$$

Moreover, $\psi_0' = \dfrac{\varphi_0'}{\sigma^2}>0$ for $(s,\sigma,d)\in\Upsilon$, $\sigma\neq0$. This is why $\psi_0'>0$ on the set Υ, under the condition $\partial\Omega\in C^{n+2}$ $(n\in\mathbb{Z}_+)$ there exist continuous on the set Υ derivatives $\partial_s^k\partial_\sigma^l\delta_i$ $(k=\overline{0,n-l},\ l=\overline{0,n},\ i=\overline{1,3})$.

We also note that $\psi_3(s,0)=1$, $\psi_4(s,0)=-K(s)$. Therefore,

$$\psi_1'(s,0,d)=1-dK(s)>0\quad\text{for}\quad (s,d)\in\overline{\mathrm{T}},$$

and the next statement is true.

Theorem 4.1 (cf. [4, Thm. 5])**.** *Let $\partial\Omega\in C^2$. Then there exists a sufficiently small number $\Sigma_0>0$ such that on the set $\Upsilon' := I_S\times\Xi\times\overline{I}_D$, where $\Xi := [-\Sigma_0,\Sigma_0]$, the function*

$$\delta_0(s,\sigma,d) := (\partial_\sigma\rho')^{-1} = \frac{\sqrt{\psi_0'}}{\psi_1'}$$

is defined everywhere and positive.

It is obvious that the number Σ_0 is not unique. In particular, we can obtain the number Σ_0 as follows. For a fixed $s\in I_S$ by E_s we denote a closed arc of the curve $\partial\Omega$ bounded by two parallel straight lines located at the distance D from the straight line $\tilde{x}_{-D}(s)\tilde{x}_D(s)$ and $\tilde{x}(s)\in\mathrm{E}_s$. The set of values σ, for which $\tilde{x}(s+\sigma)\in\mathrm{E}_s$, we denote by Ξ_s, the boundaries of Ξ_s are denoted Σ_s', Σ_s'', and then $\Xi_s=[\Sigma_s',\Sigma_s'']$, $\Sigma_s'\leqslant-D$, $\Sigma_s''\geqslant D$; $\tilde{x}(s-\Sigma_s')$, $\tilde{x}(s+\Sigma_s'')$ is the boundaries of the arc E_s, see Figure. 1. By [4, Thm. 5] as the number Σ_0 we can take the smallest of the values $\inf\limits_{s\in I_S}|\Sigma_s'|$, $\inf\limits_{s\in I_S}\Sigma_s''$.

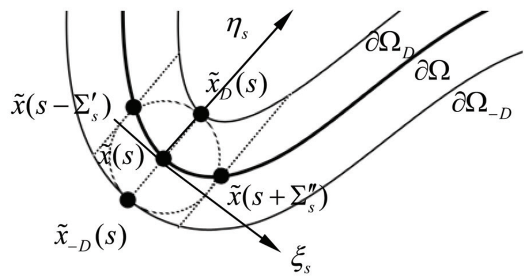

FIGURE 1. Construction of the arc E_s.

The function δ_0,which is the Jacobian of passage from the integration variable σ to the integration variable $\rho = \rho'$ can be also made arbitrarily smooth in (s, σ) by an appropriate smoothness of the curve $\partial\Omega$: under the condition $\partial\Omega \in C^{n+2}$ $(n \in \mathbb{Z}_+)$ there exist continuous on the set Υ derivatives $\partial_s^k \partial_\sigma^l \delta_0$ $(k = \overline{0, n-l},\ l = \overline{0, n})$.

Corollary 4.1. *Let* $\partial\Omega \in C^{n+2}$, $n \in \mathbb{Z}_+$. *Then for each fixed* $(s, d) \in \overline{\mathrm{T}}$ *the functions* $\tilde{\rho}_{s,d}(\sigma) := \rho'$ *diffeomorphically, with the smoothness* C^{n+1}, *map the set* Ξ *onto the sets* $\tilde{\Xi}_{s,d} := \tilde{\rho}_{s,d}(\Xi)$. *The functions* $\sigma'(s, \rho, d) := \tilde{\sigma}_{s,d}(\rho)$ $(\tilde{\sigma}_{s,d}(\rho)$ *is the inverse function for* $\tilde{\rho}_{s,d}(\sigma))$, $\delta_0'(s, \rho, d) := \delta_0\left(s, \tilde{\sigma}_{s,d}(\rho), d\right)$, $\delta_i'(s, \rho, d) := \delta_i\left(s, \tilde{\sigma}_{s,d}(\rho), d\right)\delta_0'$ $(i = \overline{1,3})$ *have continuous on the set* $\tilde{\Upsilon}' := \left\{(s, \rho, d) : s \in I_S, \rho \in \tilde{\Xi}_{s,d}, d \in \overline{I}_D\right\}$ *derivatives* $\partial_\rho^k \partial_s^l \sigma'$ $(k = \overline{0, n-l+1},\ l = \overline{0, n})$, $\partial_s^k \partial_\rho^l \delta_i'$ $(k = \overline{0, n-l},\ l = \overline{0, n},\ i = \overline{0, 3})$.

By the identities (4.1) and Corollary 4.1, on the set T the function u can be represented as the sum $u = \sum_{i=1}^{4} u_i$, where

$$u_i(s, d) := \int_{\tilde{\Xi}_{s,d}} a_i w_i'\, d\rho \quad (i = \overline{1,3}), \qquad u_4(s, d) := \int_{I_S \setminus \Xi} w_4 d\sigma,$$

$w_i'(s, \rho, d) := \delta_i' v(s + \tilde{\sigma}_{s,d}(\rho))$ $(i = \overline{1,3})$, $w_4(s, \sigma, d) := g\, v(s + \sigma)$. Here the integrands a_i $(i = \overline{1,3})$ have a singularity at $d = \rho = 0$, while for the functions w_i' $(i = \overline{1,3})$ we can ensure an arrbitrary smoothness in (s, ρ) by an appropriate smoothness of the boundary curve $\partial\Omega$ and the density v: under the conditions $\partial\Omega \in C^{n+2}$, $v \in C^n(\partial\Omega)$ $(n \in \mathbb{Z}_+)$ there exist continuous on the set $\tilde{\Upsilon}'$ derivatives $\partial_s^k \partial_\rho^l w_i'$ $(k = \overline{0, n-l},\ l = \overline{0, n},\ i = \overline{1,3})$. The integrand w_4 can be also made arbitrarily smooth in (s, σ) by an appropiate smoothness of $\partial\Omega$ and v: since $r \geqslant \rho' > 0$ for $\sigma \neq 0$, $(s, d) \in \overline{\mathrm{T}}$, under the conditions $\partial\Omega \in C^{n+2}$, $v \in C^n(\partial\Omega)$ $(n \in \mathbb{Z}_+)$ there exist continuous derivatives $\partial_s^k \partial_\sigma^l w_4$ $(k = \overline{0, n-l},\ l = \overline{0, n})$ for $\sigma \in \overline{I_S \setminus [-\Sigma_0, \Sigma_0]}$, $(s, d) \in \overline{\mathrm{T}}$.

5. Description of semi–analytical approximations for ND DLP near and at boundary of domain

Since $\tilde{\rho}_{s,d}(0) = 0$ for $(s, d) \in \overline{\mathrm{T}}$, by Corollary 4.1 there exist the numbers

$$\mathrm{P} := \min\{-\mathrm{P}_-, \mathrm{P}_+\}, \quad \mathrm{P}_- := \sup_{(s,d) \in \overline{\mathrm{T}}} \tilde{\rho}_{s,d}(-\Sigma_0) < 0, \quad \mathrm{P}_+ := \inf_{(s,d) \in \overline{\mathrm{T}}} \tilde{\rho}_{s,d}(\Sigma_0) > 0.$$

Since $\tilde{\sigma}_{s,d}(0) = 0$ for $(s, d) \in \overline{\mathrm{T}}$, by Corollary 4.1 there exists a number

$$\Sigma_1 := \min\left\{-\Sigma_-, \Sigma_+\right\},$$

where

$$\Sigma_- := \sup_{(s,d) \in \overline{\mathrm{T}}} \tilde{\sigma}_{s,d}(-\mathrm{P}) < 0, \qquad \Sigma_+ := \inf_{(s,d) \in \overline{\mathrm{T}}} \tilde{\sigma}_{s,d}(\mathrm{P}) > 0.$$

By definition,

$$\rho_{s,d}\left([-\Sigma_1, \Sigma_1]\right) \subseteq [-\mathrm{P}, \mathrm{P}] \subseteq \tilde{\Xi}_{s,d} \qquad ((s, d) \in \overline{\mathrm{T}}).$$

Let $L \in \mathbb{N}$,

$$h(L) := \frac{S}{2L+1}, \qquad s_l := lh \quad (l \in \mathbb{Z}).$$

We note that $s_l \in I_S$ for $l = \overline{-2L-1, 2L+1}$ and $\tilde{x}(s_{l+4L+2}) = \tilde{x}(s_l)$ for all $l \in \mathbb{Z}$, in particular, $\tilde{x}(s_{-2L-1}) = \tilde{x}(s_{2L+1})$. In introduce the sets I_L consisting of the points s_{2l} for $l = \overline{-L, L}$ and the sets $\mathrm{T}_L := I_L \times I_D$. We take a sufficiently large number $L_0 \in \mathbb{N}$ such that $h \in \left(0, \dfrac{\Sigma_1}{3}\right]$ for $L \geqslant L_0$.

Let $L \geqslant L_0$. On the set $\overline{\mathrm{T}_L}$ we define the functions

$$\mathrm{P}'(s,d) := \max\left\{-\mathrm{P}'_-, \mathrm{P}'_+\right\}, \qquad \mathrm{P}'_-(s,d) := \tilde{\rho}_{s,d}(s_{-2L'-1}), \qquad \mathrm{P}'_+(s,d) := \tilde{\rho}_{s,d}(s_{2L'+1}),$$
$$\Sigma'_-(s,d) := \tilde{\sigma}_{s,d}(-\mathrm{P}'), \qquad \Sigma'_+(s,d) := \tilde{\sigma}_{s,d}(\mathrm{P}').$$

The values of function $L'(s,d) \in \mathbb{N}$ are such that $s_{2L'+1} \leqslant \Sigma_1$, $s_{2L'+3} > \Sigma_1$. By definition,

$$[-\mathrm{P}', \mathrm{P}'] \subseteq [-\mathrm{P}, \mathrm{P}] \subseteq \tilde{\Xi}_{s,d} \qquad ((s,d) \in \overline{\mathrm{T}}_L).$$

On Figure 2 the segment $[-\mathrm{P}', \mathrm{P}']$ is obtained under the assumptions

$$-\mathrm{P}_- > \mathrm{P}_+, \qquad -\Sigma_- < \Sigma_+, \qquad -\mathrm{P}'_- < \mathrm{P}'_+.$$

Moreover, by the definition,

$$\left[\Sigma'_-, \Sigma'_+\right] \supseteq \left[s_{-2L'-1}, s_{2L'+1}\right] \supseteq \left[-\frac{3\Sigma_1}{5}, \frac{3\Sigma_1}{5}\right] \qquad ((s,d) \in \overline{\mathrm{T}}_L),$$

since

$$s_{2L'+3} = (2L'+3)h > \Sigma_1, \quad \text{and} \quad s_{2L'+1} = (2L'+1)h > \frac{(2L'+1)\Sigma_1}{2L'+3} \geqslant \frac{3\Sigma_1}{5} \quad \text{for} \quad L' \in \mathbb{N}.$$

Thus, for all $(s,d) \in \overline{\mathrm{T}}_L$, $L \geqslant L_0$ the segment $[-\mathrm{P}', \mathrm{P}']$ is always in the domain, in which we can pass from the integration variable σ to the integration variable $\rho = \rho'$, and the boundaries of the set $\overline{I_S \backslash \left[\Sigma'_-, \Sigma'_+\right]}$ do not approach the zero point closer than by the fixed distance $\dfrac{3\Sigma_1}{5}$.

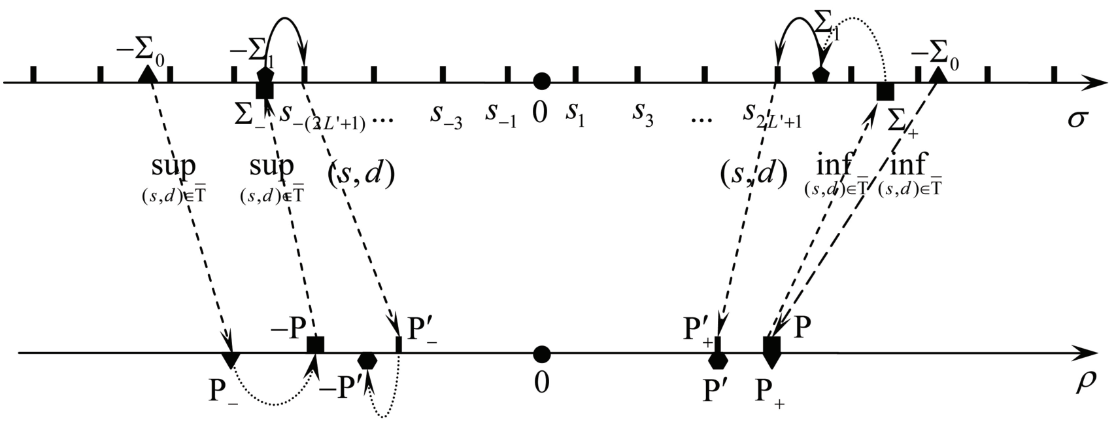

FIGURE 2. The scheme of obtaining the segments $[-\mathrm{P}', \mathrm{P}']$

On the sets $[-\mathrm{P}', \mathrm{P}']$ $((s,d) \in \overline{\mathrm{T}}_L)$ we define the sets of points $\{\rho_{s,d,2l+1}\}_{l=-2L'-2}^{2L'+1}$, which consist of the points $\tilde{\rho}_{s,d}(s_{2l+1})$ $(l = \overline{-L'-1, L'})$ and their mirror symmetries with respect to the point $\rho = 0$:

$$\{\rho_{s,d,2l+1}\}_{l=-2L'-2}^{2L'+1} := \{\pm\tilde{\rho}_{s,d}(s_{2l+1})\}_{l=-L'-1}^{L'}, \qquad \rho_{s,d,2l-1} \leqslant \rho_{s,d,2l+1}$$

for $l = \overline{-2L'-1, 2L'+1}$, $(s,d) \in \overline{\mathrm{T}}_L$, see Figure 3. We also define the sets of points $\{\rho_{s,d,2l}\}_{l=-2L'-1}^{2L'+1}$:

$$\rho_{s,d,2l} := 2^{-1}\left(\rho_{s,d,2l-1} + \rho_{s,d,2l+1}\right),$$

and we denote by $h'_{s,d,l}$ the lengths of segments $[\rho_{s,d,-2l-1}, \rho_{s,d,-2l+1}]$ $(l = \overline{-2L'-1, 2L'+1})$. By definition,

$$\bigcup_{l=-2L'-1}^{2L'+1} [\rho_{s,d,2l-1}, \rho_{s,d,2l+1}] = [-\mathrm{P}', \mathrm{P}'];$$

the segments $[\rho_{s,d,-2l-1}, \rho_{s,d,-2l+1}]$ and $[\rho_{s,d,2l-1}, \rho_{s,d,2l+1}]$ are symmetric with respect to the point $\rho_{s,d,0} = 0$: $\rho_{s,d,-(2l+1)} = -\rho_{s,d,2l+1}$, $h'_{s,d,-l} = h'_{s,d,l}$ $(l = \overline{0, 2L'+1}$, $(s,d) \in \overline{\mathrm{T}}_L)$.

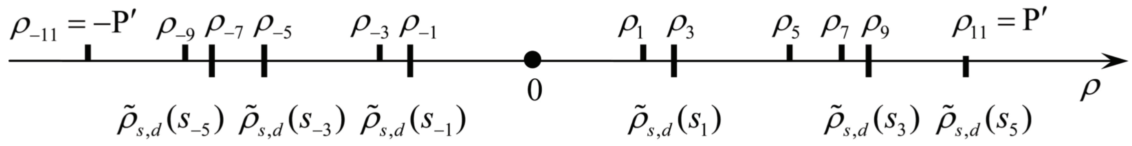

FIGURE 3. Sets of points $\{\rho_{s,d,2l+1}\}_{l=-2L'-2}^{2L'+1}$ and $\{\tilde{\rho}_{s,d}(s_{2l+1})\}_{l=-L'-1}^{L'}$ for $L' = 2$

We define PQA $\tilde{v}$ of the density $v \in C(\partial\Omega)$ in the variable s on the segments $[s_{2l-1}, s_{2l+1}]$ with the nodes at points s_{2l-1}, s_{2l}, s_{2l+1}:

$$\tilde{v}(s) := \sum_{m=-1}^{1} v(s_{2l+m})\Lambda_m^{[s_{2l-1},s_{2l+1}]}(s) \quad (s \in [s_{2l-1}, s_{2l+1}], \quad l = \overline{-L, L}).$$

We define the functions $\tilde{w}'_i(s,\rho,d) := \delta'_i \tilde{v}(s + \tilde{\sigma}_{s,d}(\rho))$ $(i = \overline{1,3})$, $\tilde{w}_4(s,\sigma,d) := g\,\tilde{v}(s+\sigma)$. We denote by $\hat{w}'_i(s,\rho,d)$ the PQA of functions $\tilde{w}'_i$ $(i = \overline{1,3})$ in the variable ρ on the segments $[\rho_{s,d,2l-1}, \rho_{s,d,2l+1}]$ with nodes at points $\rho_{s,d,2l-1}$, $\rho_{s,d,2l}$, $\rho_{s,d,2l+1}$:

$$\hat{w}'_i(s,\rho,d) := \hat{w}'_{i,l} \quad \left(\rho \in [\rho_{s,d,2l-1}, \rho_{s,d,2l+1}], \quad l = \overline{-2L'-1, 2L'+1}, \quad (s,d) \in \overline{\mathrm{T}}_L\right),$$

$$\hat{w}'_{i,l}(s,\rho,d) := \begin{cases} \sum\limits_{m=-1}^{1} \tilde{w}'_i(s, \rho_{s,d,2l+m}, d)\, \Lambda_m^{[\rho_{s,d,2l-1},\rho_{s,d,2l+1}]}(\rho) & (\rho_{s,d,2l-1} < \rho_{s,d,2l+1}), \\ \tilde{w}'_i(s, \rho_{s,d,2l}, d) & (\rho_{s,d,2l-1} = \rho_{s,d,2l+1}). \end{cases}$$

We denote by $\hat{w}_{4,l}$ the approximations of integrals $\int\limits_{\beta_{s,d,2l-1}}^{\beta_{s,d,2l+1}} \tilde{w}_4(s,\sigma,d)d\sigma$ on the base of the Gauss formulas with γ nodes

$$\hat{w}_{4,l}(s,d) := h''_{s,d,l} \sum_{j=1}^{\gamma} \eta_j \tilde{w}_4(s, \beta_{s,d,l,j}, d) \quad (l = \overline{-L, L}, \quad (s,d) \in \overline{\mathrm{T}}_L).$$

Here

$$\begin{aligned} &h''_{s,d,l} := 2^{-1}\left(\beta_{s,d,2l+1} - \beta_{s,d,2l-1}\right), \qquad \beta_{s,d,l,j} := \bar{\beta}_{s,d,l} + h''_{s,d,l} z_j, \\ &\bar{\beta}_{s,d,l} := 2^{-1}(\beta_{s,d,2l-1} + \beta_{s,d,2l+1}) \quad (l = \overline{-L, L}); \\ &\beta_{s,d,2l-1} := \min\left\{s_{2l-1}, \Sigma'_-\right\}, \quad \text{for} \quad l = \overline{-L, 0}, \\ &\beta_{s,d,2l+1} := \max\left\{s_{2l+1}, \Sigma'_+\right\} \quad \text{for} \quad l = \overline{0, L} \quad ((s,d) \in \overline{\mathrm{T}}_L); \end{aligned}$$

z_j are the roots of the polynomial $\left(\dfrac{d^\gamma}{dz^\gamma}\right)\left(z^2-1\right)^\gamma$ on the interval $(-1,1)$; the weight coefficients η_j obey the conditions $\sum\limits_{j=1}^{\gamma} \eta_j = 2$, $\eta_j > 0$ [2, Ch. 3, Sect. 5]. By definition,

$$\bigcup_{l=-L}^{L} [\beta_{s,2l-1}, \beta_{s,2l+1}] = \overline{I_S \setminus [\Sigma'_-, \Sigma'_+]}, \qquad (s,d) \in \overline{\mathrm{T}}_L.$$

The semi–analytic approximations $\hat{u}(s,d)$ of function $u(s,d)$ are defined on the sets T_L by means of sums $\hat{u} := \sum_{i=1}^{4} \hat{u}_i$, where

$$\hat{u}_i(s,d) := \int\limits_{[-\mathrm{P}',\mathrm{P}']} a_i \hat{w}_i' d\rho \qquad ((s,d) \in \mathrm{T}_L, \quad i = \overline{1,3}), \tag{5.1}$$

$$\hat{u}_4(s,d) := \sum_{l=-L}^{L} \hat{w}_{4,l} \quad ((s,d) \in \mathrm{T}_L). \tag{5.2}$$

Theorem 5.1. *Let $\partial\Omega \in C^2$, $v \in C(\partial\Omega)$, $L \in \mathbb{N}$, $L \geqslant L_0$. Then the function $\hat{u}(s,d)$ is continuous on the set T_L and can be defined by continuity for $d = 0$ that makes it continuous on the set $\overline{\mathrm{T}}_L$.*

Proof. The observation point can differ from the nodes in the Gauss formula since $r > 0$ for $\sigma \in \overline{I_S \setminus [s_{-2L'-1}, s_{2L'+1}]}$, $(s,d) \in \overline{\mathrm{T}}_L$, and this is why the function $\hat{u}_4$ can be defined by continuity for $d = 0$ to a continuous function on the set $\overline{\mathrm{T}}_L$ by means of the formula (5.2).

The functions $\hat{u}_i$ $(i = \overline{1,3})$ can be represented as the sums

$$\hat{u}_i := \sum_{l=-2L'-1}^{2L'+1} J_{i,l}, \qquad \text{where} \quad J_{i,l}(s,d) := \int\limits_{\rho_{s,d,2l-1}}^{\rho_{s,d,2l+1}} a_i \hat{w}_{i,l}' d\rho.$$

In their turn, the integrals $J_{i,l}$ for $(s,d) \in \mathrm{T}_L$, $\rho_{s,d,2l-1} < \rho_{s,d,2l+1}$ can be represented as the sums

$$J_{i,l} = \sum_{j=0}^{2} \tilde{A}_{i,j,l} \gamma_{i,j,l}, \quad \text{where} \quad \tilde{A}_{i,j,l}(d) := \int\limits_{\rho_{s,d,2l-1}}^{\rho_{s,d,2l+1}} a_i \rho^j d\rho;$$

$\gamma_{i,j,l}(s,d)$ are the coefficients of the Lagrange polynomilas $\hat{w}_{i,l}' = \gamma_{i,0,l} + \gamma_{i,1,l}\rho + \gamma_{i,2,l}\rho^2$, which are continuous on the set $\overline{\mathrm{T}}_L$ functions. The functions $\tilde{A}_{i,j,l}$ for $(s,d) \in \mathrm{T}_L$ are calculated by the identities

$$\tilde{A}_{i,j,l}(d) = A_{i,j}(\rho_{s,d,2l+1}, d) - A_{i,j}(\rho_{s,d,2l-1}, d) \quad (j = \overline{0,2},\ l = \overline{-2L'-1, 2L'+1},\ i = \overline{1,3}), \tag{5.3}$$

where

$$\begin{aligned}
&A_{1,0}(\rho,d) := \rho A^{-1},\\
&A_{1,1}(\rho,d) := -d^2 A^{-1} - 2^{-1} \ln A,\\
&A_{1,2}(\rho,d) := -\rho - d^2 \rho A^{-1} + 2d \arctan\left(\frac{\rho}{d}\right);\\
&A_{2,0}(\rho,d) := -d\rho A^{-1} + \arctan\left(\frac{\rho}{d}\right),\\
&A_{2,1}(\rho,d) := d^3 A^{-1} + d \ln A,\\
&A_{2,2}(\rho,d) := 2d\rho + d^3 \rho A^{-1} - 3d^2 \arctan\left(\frac{\rho}{d}\right);\\
&A_{3,0}(\rho,d) := 2\rho + d^2 \rho A^{-1} - 3d \arctan\left(\frac{\rho}{d}\right),\\
&A_{3,1}(\rho,d) := \rho^2 - d^4 A^{-1} - 2d^2 \ln A,\\
&A_{3,2}(\rho,d) := \frac{2\rho^3}{3} - 4d^2\rho - d^4 \rho A^{-1} + 5d^3 \arctan\left(\frac{\rho}{d}\right).
\end{aligned}$$

Therefore, by means of the identities (5.3) and $\gamma_{2,0,0} = 0$ the integrals $J_{i,l}$ can be defined for $d = 0$ to continuous on the set $\overline{\mathrm{T}}_L$ functions, and then by the formulas (5.1) the functions

$\hat{u}_i$ ($i = \overline{1,3}$) can be defined to continuous on the set $\overline{\mathrm{T}}_L$ functions. At the same time the integral $J_{1,0}(s,0)$ exists only in the sense of Hadamard finite value [25, Frm. (1.6.6)], the integral $J_{1,1}(s,0)$ does only in the sense of Cauchy principal value; the other integrals $J_{i,l}(s,0)$ are converging improper integrals. The proof is complete. □

We introduce the PQA $\tilde{v}'$ of the density $v \in C(\partial\Omega)$ in the variable s on the segments $[s_{2l}, s_{2l+2}]$ with the nodes at the points s_{2l}, s_{2l+1}, s_{2l+2}:

$$\tilde{v}'(s) := \sum_{m=-1}^{1} v(s_{2l+1+m})\Lambda_m^{[s_{2l},s_{2l+2}]}(s) \qquad \left(s \in [s_{2l}, s_{2l+2}], \quad l = \overline{-L, L}\right).$$

We introduce the sets I'_L consisting of the points s_{2l+1} for $l = \overline{-L, L}$ and the set $\mathrm{T}'_L := I'_L \times I_D$. By means of the functions $\tilde{v}'$, on the sets T'_L the approximations $\hat{u}'(s,d)$ of function u in the same way how on the sets T_L the approximations $\hat{u}(s,d)$ were defined by means of the functions $\tilde{v}$. Similar to Theorem 5.1 we define the functions $\hat{u}'$ at $d = 0$ to continuous ones on the sets $\overline{\mathrm{T}}'_L$.

The semi–analytic approximations $\breve{U}\left(\tilde{x}_d(s)\right)$ of ND DLP $U\left(\tilde{x}_d(s)\right)$, which can be calculated at each point $\tilde{x}_d(s)$ of a closed near–boundary domain $\overline{\Omega_D}$ under the condition $v \in C(\partial\Omega)$, are defined by the formula $\breve{U} := (2\pi)^{-1}\breve{u}$, where

$$\breve{u}(s,d) := \hat{u}'(s_{2l-1}, d)\Lambda_{-1}^{[s_{2l-1},s_{2l+1}]}(s) + \hat{u}(s_{2l}, d)\Lambda_0^{[s_{2l-1},s_{2l+1}]}(s) + \hat{u}'(s_{2l+1}, d)\Lambda_1^{[s_{2l-1},s_{2l+1}]}(s), \quad (5.4)$$

$d \in \overline{I_D}$, $s \in [s_{2l-1}, s_{2l+1}]$, $l = \overline{-L, L}$. The values of functions $\hat{u}$, $\hat{u}'$ for $d = 0$, which are used here, can be obtained via the continuous extension in accordance with Theorem 5.1.

6. Justification of possibility of tangential interpolation of ND DLP near and at boundary of domain

In this section we prove the uniform convergence of the interpolants of the ND DLP with respect to the variable s in the closed near–boundary domain $\overline{\Omega}_D$. In order to do this, we first obtain conditions for the existence and continuity of the tangent derivatives of the ND DLP in the domain $\overline{\Omega}_D$.

Let $\rho_- < 0$, $\rho_+ > 0$ be constants such that $[\rho_-, \rho_+] \subseteq [\mathrm{P}_-, \mathrm{P}_+]$. By σ_-, σ_+ we denote the functions $\sigma_-(s,d) := \tilde{\sigma}_{s,d}(\rho_-)$, $\sigma_+(s,d) := \tilde{\sigma}_{s,d}(\rho_+)$. By Corollary 4.1 the functions $-\sigma_-$, σ_+ are positive on the set $\overline{\mathrm{T}}$. According to the identities (4.1), for $(s,d) \in \mathrm{T}$ the function u can be represented as the sum $u = \sum_{i=1}^{4} u_i$, where

$$u_i(s,d) := \int_{\rho_-}^{\rho_+} a_i w'_i d\rho \quad (i = \overline{1,3}), \qquad u_4(s,d) := \int_{I_S \setminus [\sigma_-, \sigma_+]} w_4 d\sigma. \qquad (6.1)$$

Theorem 6.1. *Let $\partial\Omega \in C^{n+4}$, $v \in C^{n+2}(\partial\Omega)$, $n \in \mathbb{Z}_+$. Then the function $u(s,d)$ is continuous on the set T and can be defined at $d = 0$ to a continuous on the set $\overline{\mathrm{T}}$ function. On the set $\overline{\mathrm{T}}$, the function $u(s,d)$ defined to a continuous one, possesses the derivatives $\partial_s^l u$ ($l = \overline{1,n}$), which are continuous on the sets T and I_S for $d = 0$. There exists uniformly in $s \in I_S$ converging limits*

$$\lim_{d \to \pm 0} \partial_s^l u(s,d) = \pm\pi \left(\partial_s^l w'_2\right)(s,0,0) + \partial_s^l u(s,0) \quad (l = \overline{1,n}, \quad s \in I_S), \qquad (6.2)$$

by means of which the derivatives $\partial_s^l u$ ($l = \overline{1,n}$) can be defined to continuous either on the set $I_S \times [0, D_2]$ or on the set $I_S \times [-D_1, 0]$ functions.

Proof. Let $\partial\Omega \in C^{n+2}$, $v \in C^n(\partial\Omega)$, $n \in \mathbb{Z}_+$. Since $r > 0$ for $\sigma \in \overline{I_S \setminus [\sigma_-, \sigma_+]}$, $(s,d) \in \overline{\mathrm{T}}$, there exist continuous derivatives $\partial_s^l w_4$ $(l = \overline{0,n})$ for $\sigma \in \overline{I_S \setminus [\sigma_-, \sigma_+]}$, $(s,d) \in \overline{\mathrm{T}}$ and according to Corollary 4.1 the derivatives $\partial_s^l \sigma_-$, $\partial_s^l \sigma_+$ $(l = \overline{0,n})$ are continuous on the set $\overline{\mathrm{T}}$. This is why the function u_4 can be defined for $d = 0$ to a continuous on $\overline{\mathrm{T}}$ function means of corresponding integrals (6.1), and then there exist continuous on the set $\overline{\mathrm{T}}$ derivatives $\partial_s^l u_4$ $(l = \overline{1,n})$, which can be represented as the linear combinations of the integrals

$$\int\limits_{I_S \setminus [\sigma_-, \sigma_+]} \partial_s^l w_4 d\sigma$$

and the expressions of form

$$\partial_s^{l-k} w_4(s, \sigma_\pm, d) \partial_s^k \sigma_\pm \qquad (k = \overline{1,l}, l = \overline{1,n}).$$

The functions a_i $(i = \overline{1,3})$ are continuous for $d \in I_D$, $\rho \in \mathbb{R}$. By Corollary 4.1 the derivatives $\partial_s^l w_i'$ $(l = \overline{0,n},\ i = \overline{1,3})$ are continuous on the set $\tilde{\Upsilon}'$. This is why the functions $u_i(s,d)$ $(i = \overline{1,3})$ defined by the corresponding integrals (6.1) are continuous on the set T, and there exist continuous on T derivatives

$$\partial_s^l u_i = \int\limits_{\rho_-}^{\rho_+} a_i \partial_s^l w_i' \, d\rho \qquad (l = \overline{1,n}, i = \overline{1,3}).$$

For $d \in \overline{I}_D$, $\rho \in \mathbb{R}$ (except for $d = \rho = 0$) the function a_3 is continuous and bounded: $|a_3| \leqslant 1$, this is why the function u_3 can be defined for $d = 0$ to a continuous on $\overline{\mathrm{T}}$ function by the corresponding integrals (6.1), and then the derivatives $\partial_s^l u_3$ $(l = \overline{0,n})$ are continuous on the set $\overline{\mathrm{T}}$ and are calculated by the integrals

$$\partial_s^l u_3 = \int\limits_{\rho_-}^{\rho_+} a_3 \partial_s^l w_3' \, d\rho.$$

For each fixed $\varepsilon \in (0, \rho_+] \cap (0, -\rho_-]$ we have uniformly in $s \in I_S$ converging limits

$$\lim_{d\to 0} \int\limits_{\rho_-}^{-\varepsilon} a_2 \partial_s^l w_2' d\rho = \lim_{d\to 0} \int\limits_{\varepsilon}^{\rho_+} a_2 \partial_s^l w_2' d\rho = 0 \qquad (l = \overline{0,n}). \tag{6.3}$$

We also have uniformly in $s \in I_S$ converging limits

$$\begin{aligned} \lim_{\varepsilon\to+0} \left(\lim_{d\to\pm 0} \int\limits_{-\varepsilon}^{\varepsilon} a_2 \partial_s^l w_2' d\rho \right)(s) &= 2 \lim_{\varepsilon\to+0} \left(\lim_{d\to\pm 0} \left(\partial_s^l w_2'\right)(s, \bar\rho_{s,d,\varepsilon}, d)\, A_{2,0}(\varepsilon, d) \right) \\ &= \pm\pi \left(\partial_s^l w_2'\right)(s,0,0) \qquad (l = \overline{0,n}), \end{aligned} \tag{6.4}$$

where $\bar\rho_{s,d,\varepsilon}$ is some point in the segment $[-\varepsilon, \varepsilon]$, the exact location of which depends $(s,d) \in \mathrm{T}$. By the identities (6.3), (6.4) there exist uniformly in $s \in I_S$ converging limits

$$\lim_{d\to\pm 0} \partial_s^l u_2(s,d) = \pm\pi \left(\partial_s^l w_2'\right)(s,0,0) \qquad (l = \overline{0,n}). \tag{6.5}$$

In view of the identities (6.5) and $w_2'(s,0,0) = \delta_2'(s,0,0) = 0$ $(s \in I_S)$ the function u_2 becomes continuous on the set $\overline{\mathrm{T}}$ if we define being zero for $d = 0$, $s \in I_S$. This agrees with the formula (6.1) since $a_2(\rho, 0) = 0$ for $\rho \neq 0$.

Let $\partial\Omega \in C^{n+4}$, $v \in C^{n+2}(\partial\Omega)$, $n \in \mathbb{Z}_+$. Using TFETIF, we represent the function w_1' as the sum

$$w_1' = w_{1,0}' + w_{1,1}'\rho + w_{1,2}'\rho^2 \quad ((s,d) \in \overline{\mathrm{T}},\ \rho \in [\rho_-, \rho_+]), \tag{6.6}$$

where

$$w'_{1,0}(s,d) := w'_1(s,0,d), \qquad w'_{1,1}(s,d) := \left(\partial_\rho w'_1\right)(s,0,d),$$

$$w'_{1,2}(s,\rho,d) := \rho^{-2}\int\limits_0^\rho \left(\partial_\rho^2 w'_1\right)(s,\varsigma,d)\,(\rho-\varsigma)\,d\varsigma \quad (\rho\neq 0),$$

$$w'_{1,2}(s,0,d) := 2^{-1}\left(\partial_\rho^2 w'_1\right)(s,0,d).$$

The derivatives $\partial_s^l w'_{1,j}$ ($l=\overline{0,n}$, $j=\overline{0,2}$) are continuous by Corollary 4.1: $\partial_s^l w'_{1,0}$, $\partial_s^l w'_{1,1}$ on the set $\overline{\mathrm{T}}$, $\partial_s^l w'_{1,2}$ on the set $\tilde{\Upsilon}'$. In accordance with the formulas (6.1), (6.6), for $(s,d)\in\mathrm{T}$ we represent the function u_1 as the sum

$$u_1 = \sum_{j=0}^{2} u_{1,j},$$

where

$$u_{1,j}(s,d) := w'_{1,j}\tilde{A}_{1,j}, \qquad \tilde{A}_{1,j}(d) := \int\limits_{\rho_-}^{\rho_+} a_1\rho^j\,d\rho \quad (j=0,1), \qquad u_{1,2}(s,d) := \int\limits_{\rho_-}^{\rho_+} a_1\rho^2 w'_{1,2}\,d\rho.$$

The functions $\tilde{A}_{1,0}(d)$, $\tilde{A}_{1,1}(d)$ can be defined for $d=0$ to continuous on the set $\overline{I}_D$ functions since $-\rho_-,\rho_+>0$ and $\tilde{A}_{1,j}(d) = A_{1,j}(\rho_+,d) - A_{1,j}(\rho_-,d)$ ($j=0,1$). We note that the obtained in this way values $\tilde{A}_{1,j}(0)$ ($j=0,1$) coincide respectively with the integral $\tilde{A}_{1,0}(0)$ in the sense of Hadamard finite value, see the definion in [15, Frm. (1.6.5)] and with the integral $\tilde{A}_{1,1}(0)$ in the sense of the Cauchy principle value. Let

$$u_{1,j}(s,0) := \tilde{A}_{1,j}(0)\,w'_{1,j}(s,0) \qquad (s\in I_S,\ j=0,1).$$

Then the derivatives $\partial_s^l u_{1,j}$ are continuous on $\overline{\mathrm{T}}$ since

$$\partial_s^l u_{1,j} = \tilde{A}_{1,j}\partial_s^l w'_{1,j} \qquad (l=\overline{0,n},\ j=0,1).$$

We consider the representation of form

$$u_{1,2} = u_{1,2,1} + u_{1,2,2},$$

where

$$u_{1,2,k}(s,d) := \int\limits_{\rho_-}^{\rho_+} a_{1,k} w'_{1,2}\,d\rho \qquad \left((s,d)\in\mathrm{T},\ k=1,2\right),$$

$$a_{1,1}(\rho,d) := -\rho^4 A^{-2}, \qquad a_{1,2}(\rho,d) := d^2\rho^2 A^{-2}.$$

Taking into consideration that $a_{1,1}(\rho,0)=-1$, $a_{1,2}(\rho,0)=0$ for $\rho\neq 0$, we define the functions $u_{1,2,k}$ for $d=0$:

$$u_{1,2,1}(s,0) := -\int\limits_{\rho_-}^{\rho_+} w'_{1,2}\big|_{d=0}\,d\rho, \qquad u_{1,2,2}(s,0) := 0 \quad (s\in I_S).$$

Then the derivatives

$$\partial_s^l u_{1,2,1} = \int\limits_{\rho_-}^{\rho_+} a_{1,1}\partial_s^l w'_{1,2}\,d\rho \qquad (l=\overline{0,n})$$

exist and continuous on the set $\overline{\mathrm{T}}$ since $|a_{1,1}|\leqslant 1$ for $d\in\overline{I}_D$, $\rho\in\mathbb{R}$ except for $d=\rho=0$. For $d\neq 0$, $s\in I_S$ the function $a_{1,2}$ does not change the sign on the integration segments $[\rho_-,\rho_+]$, and this is why by the mean value theorem $\partial_s^l u_{1,2,2} = 2^{-1} d\tilde{A}_2 \partial_s^l w'(s, \bar{\rho}_{s,d,l}, d)$. Here $\bar{\rho}_{s,d,l}$ is some point in $[\rho_-, \rho_+]$, the locatio of which depends on d, s, l; $\tilde{A}_2(d) := A_{2,0}(\rho_+, d) - A_{2,0}(\rho_-, d)$ is a uniformly bounded in $d \in I_D$ function. Therefore, as $d \to 0$, the values

$$\partial_s^l u_{1,2,2} = \int\limits_{\rho_-}^{\rho_+} a_{1,2} \partial_s^l w'_{1,2}\, d\rho$$

tend to the values $\partial_s^l u_{1,2,2}(s,0) = 0$ $(l = \overline{0,n})$ uniformly in $s \in I_S$, that is, the derivatives $\partial_s^l u_{1,2,2}$ exist and continuous on the set $\overline{\mathrm{T}}$.

Thus, let

$$u_1(s,0) := \sum_{j=0}^{1} \tilde{A}_{1,j}(0)\, w'_{1,j}(s,0) + u_{1,2,1}(s,0) \qquad (s \in I_S).$$

Then the function u_1 and derivatives $\partial_s^l u_1$ $(l = \overline{1,n})$ are continuous on the set $\overline{\mathrm{T}}$. Finally, let

$$u(s,0) := u_1(s,0) + u_3(s,0) + u_4(s,0) \qquad (s \in I_S),$$

where $u_3(s,0)$, $u_4(s,0)$ are calculated by the corresponding integrals (6.1). Then by the above proven facts the derivatives $\partial_s^l u(s,0)$ $(l = \overline{1,n})$ are continuous in $s \in I_S$, the function u is continuous on the set $\overline{\mathrm{T}}$, and due uniformly in $s \in I_S$ converging limits (6.5) there exist uniformly in $s \in I_S$ converging limits (6.2). These limits allow us to define the derivatives $\partial_s^l u$ $(l = \overline{1,n})$ to continuous either on the set $I_S \times [0, D_2]$ or on the set $I_S \times [-D_1, 0]$ functions. The proof is complete. □

We note that the possibility of defining ND DLP on the boundary of domain by continuity was obtained in the known theorems [10, Sect. 2.5, Thms. 2.21, 2.23].

Let $\partial\Omega \in C^4$, $v \in C^2(\partial\Omega)$. On the set $\overline{\mathrm{T}}$ we define PQA $\ddot{u}$ of the function u in the variable s on the segments $[s_{2l-1}, s_{2l+1}]$ with the nodes at the points s_{2l-1}, s_{2l}, s_{2l+1}:

$$\ddot{u}(s,d) := \sum_{m=-1}^{1} u(s_{2l+m}, d) \Lambda_m^{[s_{2l-1}, s_{2l+1}]}(s) \quad \left(d \in \bar{I}_D,\ s \in [s_{2l-1}, s_{2l+1}],\ l = \overline{-L, L}\right). \tag{6.7}$$

The used values of function $u(s,d)$ for $d = 0$, $s \in I_S$ are obtained by the continuation by continuity.

Theorem 6.2. *Let $L \in \mathbb{N}$, $\partial\Omega \in C^7$, $v \in C^5(\partial\Omega)$. Then, as $L \to \infty$, the functions $\ddot{u}(s,d)$ converge to the functions $u(s,d)$ defined for $d = 0$ by continuity, uniformly in $(s,d) \in \overline{\mathrm{T}}$ with at least cubic rate.*

Proof. By the estimate (3.5) and Theorem 6.1 the inequalities hold

$$|\ddot{u} - u| \leqslant c_\lambda h^3 \sup_{(s,d)\in\overline{\mathrm{T}}} \left|\partial_s^3 u\right| \qquad ((s,d) \in \overline{\mathrm{T}}). \tag{6.8}$$

The proof is complete. □

7. Proof of uniform convergence of semi–analytic approximations of ND DLP near and at boundary of domain

On the sets T_L we define the approximations $\tilde{u}$ of function u

$$\tilde{u}(s,d) := \int\limits_{I_S} g\tilde{v}(s+\sigma)\, d\sigma.$$

Theorem 7.1. *Let $\partial\Omega \in C^4$, $v \in C(\partial\Omega)$, $L \in \mathbb{N}$, $L \geqslant L_0$. Then the function $\tilde{u}(s,d)$ is continuous on the set T_L and can be defined for $d = 0$ to a continuous on the set $\overline{\mathrm{T}}_L$ function.*

Proof. The proof follows the lines of the proof of Theorem 6.1. We point out special moments. Since the continuity is to be proved only on the set $\overline{\mathrm{T}}_L$, as ρ_-, ρ_+ we take the quantities

$$\sup_{(s,d)\in\overline{\mathrm{T}}} \tilde{\rho}_{s,d}(-h) < 0, \qquad \inf_{(s,d)\in\overline{\mathrm{T}}} \tilde{\rho}_{s,d}(h) > 0.$$

We represent the function $\tilde{u}$ as the sum $\tilde{u} = \sum_{i=1}^{4} \tilde{u}_i$, where the functions $\tilde{u}_i$ $(i = \overline{1,4})$ are similar to the functions u_i $(i = \overline{1,4})$:

$$\tilde{u}_i(s,d) := \int_{\rho_-}^{\rho_+} a_i \tilde{w}_i'\, d\rho \quad (i = \overline{1,3}), \qquad \tilde{u}_4(s,d) := \int_{I_S\setminus[\sigma_-,\sigma_+]} \tilde{w}_4 d\sigma \quad ((s,d)\in \mathrm{T}_L).$$

The end points of segments $[s_{2l-1}, s_{2l+1}]$ are the nodes of PQA and this is why the function $\tilde{v}(s)$, as well as $v(s)$, is continuous on the set I_S. The identity

$$\lim_{d\to\pm 0} \tilde{u}_2(s,d) = 0,$$

which is similar to (6.5) for $l = 0$, holds on the set I_L since

$$\tilde{w}_2'(s,0,0) = \delta_2'(s,0,0) = 0 \quad \text{for} \quad s \in I_L.$$

An identity similar to (6.6) holds for $\rho \in [\rho_-, \rho_+]$, $(s,d) \in \overline{\mathrm{T}}_L$ for the function $\tilde{w}_1'$ under the condition $v \in C(\partial\Omega)$, which is weaker than the condition $v \in C^2(\partial\Omega)$ for the function w_1' since for all $v \in C(\partial\Omega)$, $s \in I_L$ the function $\tilde{v}(s+\sigma)$ is quadratic in the variable $\sigma \in [-h,h]$. The proof is complete. $\square$

Theorem 7.2. *Let $L \in \mathbb{N}$, $L \geqslant L_0$, $\partial\Omega \in C^4$. Then the functions $u(s,d)$, $\tilde{u}(s,d)$ defined by continuity for $d = 0$, obey the estimates for $(s,d) \in \overline{\mathrm{T}}_L$*

$$|u - \tilde{u}| \leqslant C_1 \|v\|_{C^4(\partial\Omega)} h^3 \ln(2L+1) \quad ((s,d) \in \overline{\mathrm{T}}_L), \tag{7.1}$$

where C_1 is a positive constant independent of L and (s,d).

Proof. Let $v \in C^4(\partial\Omega)$. According to the identities (4.1), for $(s,d) \in \mathrm{T}_L$ the functions u, $\tilde{u}$ can be represented as the sums

$$u = \sum_{i=1}^{3} u_i, \qquad \tilde{u} = \sum_{i=1}^{3} \tilde{u}_i \quad (i = \overline{1,3}),$$

where

$$u_i(s,d) := \int_{I_s} a_i(\rho',d) w_i\, d\sigma, \qquad \tilde{u}_i(s,d) := \int_{I_s} a_i(\rho',d)\tilde{w}_i\, d\sigma, \tag{7.2}$$

$$w_i(s,\sigma,d) := \delta_i v(s+\sigma), \qquad \tilde{w}_i(s,\sigma,d) := \delta_i \tilde{v}(s+\sigma) \qquad (i = \overline{1,3}).$$

The function $a_3(\rho',d)$ is bounded for $(s,d) \in \mathrm{T}$, $\sigma \in I_S$: $|a_3(\rho',d)| \leqslant 1$. This is why in view of the inequalities (3.5) we have the estimates

$$|\tilde{u}_3 - u_3| \leqslant c_3 \|v\|_{C^3(\partial\Omega)} h^3 \sup_{(s,\sigma,d)\in\Upsilon'} |\delta_3| \qquad ((s,d)\in \mathrm{T}_L), \qquad c_3 := 2Sc_\lambda. \tag{7.3}$$

There exist positive constants

$$c_\rho' := \inf_{(s,\sigma,d)\in\Upsilon} \sqrt{\psi_0'}, \qquad c_\rho'' := \sup_{(s,\sigma,d)\in\Upsilon} \sqrt{\psi_0'}, \qquad c_\rho := \frac{c_\rho''}{c_\rho'}.$$

For $(s,d)\in\mathrm{T}$ the inequalities hold

$$\int\limits_{I_S}|a_2(\rho',d)|\,d\sigma\leqslant\int\limits_{I_S}\frac{2\,|d|\left(\sigma c''_\rho\right)^2}{\left[\left(c'_\rho\sigma\right)^2+d^2\right]^2}d\sigma\leqslant\frac{2c_\rho^2}{c'_\rho}\sup_{d\in I_D}\left|A_{2,0}(c'_\rho S,|d|)\right|.$$

In view of the inequalities (3.5), we obtain the estimates

$$|\tilde u_2-u_2|\leqslant c_2\,\|v\|_{C^3(\partial\Omega)}\,h^3\quad((s,d)\in\mathrm{T}_L),\tag{7.4}$$

where

$$c_2:=\frac{2c_\lambda c_\rho^2}{c'_\rho}\sup_{d\in\overline{I}_D}\left|A_{2,0}(c'_\rho S,|d|)\right|\sup_{(s,\sigma,d)\in\Upsilon'}|\delta_2|\,.$$

Letting

$$z=s+s_{2l}+t,\qquad f(z)=v(s+s_{2l}+t),\qquad z_{-1}=s+s_{2l-1},$$
$$z_1=s+s_{2l+1},\qquad z_0=s+s_{2l},\qquad \hat f(z)=\tilde v(s+s_{2l}+t)$$

in the formula (3.1), we arrive at the identities

$$v(s+s_{2l}+t)-\tilde v(s+s_{2l}+t)=\tilde v_{3,l}t\,\omega,$$

where

$$\omega(t):=(t+h)\,(t-h)\,,\quad \tilde v_{3,l}(s,t):=f_3(z,z_{-1},z_1,z_0),\quad s\in I_L,\quad t\in[-h,h],\quad l=\overline{-L,L}.$$

We use TFETIF to get

$$\delta_1-\delta_{1,0}=\delta_{1,1}\sigma,$$

where

$$\delta_{1,0}(s):=\delta_1(s,0)=-1,\qquad \delta_{1,1}(s,\sigma):=\sigma^{-1}\int\limits_0^\sigma\left(\partial_\sigma\delta_1\right)(s,\varsigma)d\varsigma$$

for $\sigma\neq0$, $\delta_{1,1}(s,0):=\left(\partial_\sigma\delta_1\right)(s,0)$. We can write the deviation $u_1-\tilde u_1$ for $(s,d)\in\mathrm{T}_L$ as

$$u_1-\tilde u_1=\sum_{l=-L}^{L}\int\limits_{-h}^{h}\left(g_{1,l}\delta_1(s,s_{2l}+t)+g_{2,l}\delta_{1,1}(s,s_{2l}+t)-g_{3,l}\,t\right)\tilde v_{3,l}\omega\,dt,\tag{7.5}$$

where

$$g_{1,l}(s,t,d):=\left[\left(a_1(\rho',d)-a_1(\sigma,d)\right)t\right]_{\sigma=s_{2l}+t},$$
$$g_{2,l}(t,d):=\left[a_1(\sigma,d)\sigma t\right]_{\sigma=s_{2l}+t},\quad g_{3,l}(t,d):=\left[a_1(\sigma,d)\right]_{\sigma=s_{2l}+t}.$$

On the base of TFETIF

$$\psi'_0-\psi_{0,0}=\psi_{0,1}\sigma+2d\psi_2,$$

where

$$\psi_{0,0}(s):=\psi_0(s,0)=1,\qquad \psi_{0,1}(s,\sigma):=\sigma^{-1}\int\limits_0^\sigma\left(\partial_\sigma\psi_0\right)(s,\varsigma)d\varsigma$$

for $\sigma\neq0$,

$$\psi_{0,1}(s,0):=\left(\partial_\sigma\psi_0\right)(s,0),$$

and the identity $\left(\rho'\right)^2=\sigma^2\psi'_0$ for $(s,d)\in\mathrm{T}$, $\sigma\in I_S$ we have the identities

$$a_1(\rho',d)-a_1(\sigma,d)=\left(\psi_{0,1}\sigma+2d\,\psi_2\right)b,\quad b(s,\sigma,d):=\frac{\psi'_0\sigma^6-\left(\psi'_0+1\right)d^2\sigma^4-3d^4\sigma^2}{\left(\sigma^2+d^2\right)^2\left(\sigma^2\psi'_0+d^2\right)^2}.$$

Moreover,

$$\left|\frac{t}{\sigma}\right| \leqslant (2l-1)^{-1}$$

for $\sigma = s_{\pm 2l} + t$, $t \in [-h,h]$, $l = \overline{1,L}$; for $(s,d) \in \mathrm{T}$, $\sigma \in I_S$ the absolute value of function $(\psi_{0,1}\sigma + 2d\,\psi_2)\,b\sigma$ is bounded by some positive constant c_1' and $|a_1(\sigma,d)\sigma^2| \leqslant 1$ for $(\sigma,d) \in \mathrm{T}$. This is why for $(s,d) \in \mathrm{T}_L$, $t \in [-h,h]$ we have the inequalities

$$|g_{1,0}| \leqslant c_1', \quad |g_{2,0}| \leqslant 1; \quad |g_{1,l}| \leqslant (2\,|l|-1)^{-1}\,c_1', \quad |g_{2,l}| \leqslant (2\,|l|-1)^{-1} \quad (\pm l = \overline{1,L}). \tag{7.6}$$

We also have the relations

$$\sum_{l=1}^{L} 2(2l-1)^{-1} \leqslant \ln(2L+1)\,, \qquad \int_{-h}^{h} |\omega|\;dt = \frac{4}{3}h^3. \tag{7.7}$$

There exist positive constants $\tilde{c}_0 := \sup\limits_{(s,\sigma)\in\Theta} |\delta_1|$, $\tilde{c}_1 := \sup\limits_{(s,\sigma)\in\Theta} |\delta_{1,1}|$. By the formulas (7.6), (7.7), (3.3) we obtain the estimates

$$\sum_{l=-L}^{L}\left|\int_{-h}^{h}(g_{1,l}\delta_1 + g_{2,l}\delta_{1,1})\,\tilde{v}_{3,l}\omega\;dt\right| \leqslant c_{1,1}\,\|v\|_{C^3(\partial\Omega)}\ln(2L+1)\,h^3 \quad ((s,d)\in\mathrm{T}_L), \tag{7.8}$$

where $c_{1,1} := \frac{4}{9}\,(c_1'\tilde{c}_0 + \tilde{c}_1)$.

Since $\omega(-t) = \omega(t)$ and $g_{3,-l}(-t,d) = g_{3,l}(t,d)$ $(t \in [-h,h],\ l = \overline{1,L})$, and the signs of the functions $g_{3,l}$, ω do not change as $t \in [-h,h]$, in view of the formula (3.3) and the mean value theorem for $(s,d) \in \mathrm{T}_L$, $l = \overline{1,L}$ we have the identities

$$\begin{aligned}\int_{-h}^{h} g_{3,-l}\tilde{v}_{3,-l}t\omega\,dt + \int_{-h}^{h} g_{3,l}\tilde{v}_{3,l}t\omega\,dt &= \int_{-h}^{h}(\tilde{v}_{3,l}(s,t) - \tilde{v}_{3,-l}(s,-t))\,tg_{3,l}\omega\,dt\\ &= 6^{-1}\left(v^{(3)}(s+s_{2l}+\zeta_l) - v^{(3)}(s-s_{2l}-\zeta_l)\right)\zeta_l\int_{-h}^{h} g_{3,l}\omega\,dt,\end{aligned} \tag{7.9}$$

where $\zeta_l(s,d) \in [-h,h]$. For $(s,d) \in \mathrm{T}_L$, $l = \overline{1,L}$ the inequalities hold

$$\begin{aligned}&\left|v^{(3)}(s+s_{2l}+\zeta_l) - v^{(3)}(s-s_{2l}-\zeta_l)\right| \leqslant \|v\|_{C^4(\partial\Omega)}\,2(2l+1)h,\\ &|g_{3,l}| \leqslant (2l-1)^{-2}h^{-2}, \qquad |\zeta_l| \leqslant h, \qquad \frac{(2l+1)}{(2l-1)} \leqslant 3.\end{aligned} \tag{7.10}$$

By the relations (7.7), (7.9), (7.10) we get the estimates

$$\left|\sum_{l=\overline{-L,-1},\ \overline{1,L}}\int_{-h}^{h} g_{3,l}t\tilde{v}_{3,l}\omega\,dt\right| \leqslant c_{1,2}\,\|v\|_{C^4(\partial\Omega)}\ln(2L+1)\,h^3 \quad ((s,d)\in\mathrm{T}_L), \quad c_{1,2} := \frac{2}{3}. \tag{7.11}$$

Comparing the Newton formulas (3.1), (3.2) and letting $z = s+t$, $z_{-1} = s-h$, $z_1 = s+h$, $z_0 = s$, we can represent the function $\tilde{v}_{3,0}$ as the sum $\tilde{v}_{3,0} = \tilde{v}_3 + \tilde{v}_4 t$, where

$$\tilde{v}_3(s) := f_3(z_{-1},z_1,z_0,z_0), \quad \tilde{v}_4(s,t) := f_4(z,z_{-1},z_1,z_0,z_0), \quad s \in I_L, \quad t \in [-h,h].$$

Taking into consideration that

$$\int_{-h}^{h} g_{3,0}t\omega\,dt = 0 \quad (d \in I_D) \quad \text{and} \quad \left|g_{3,0}(t,d)t^2\right| \leqslant 1 \quad (d \in I_D,\ t \in [-h,h]),$$

by the formulas (3.4) and $\int\limits_{-h}^{h} |\omega|\, dt = \frac{4}{3}h^3$ we obtain the estimates

$$\left|\int\limits_{-h}^{h} g_{3,0} t \tilde{v}_{3,0}\omega\, dt\right| = \left|\int\limits_{-h}^{h} g_{3,0} t^2 \tilde{v}_4\, \omega\, dt\right| \leqslant c_{1,3} \|v\|_{C^4(\partial\Omega)} h^3 \quad ((s,d) \in \mathrm{T}_L), \quad c_{1,3} := 18^{-1}. \tag{7.12}$$

By the formula (7.5) and estimates (7.8), (7.11), (7.12) we obtain

$$|u_1 - \tilde{u}_1| \leqslant c_1 \|v\|_{C^4(\partial\Omega)} \ln(2L+1)\, h^3 \quad ((s,d) \in \mathrm{T}_L), \quad c_1 := c_{1,1} + c_{1,2} + c_{1,3}. \tag{7.13}$$

In view of the formulas $u = \sum\limits_{i=1}^{3} u_i$, $\tilde{u} = \sum\limits_{i=1}^{3} \tilde{u}_i$ $(i = \overline{1,3})$ and the estimates (7.3), (7.4), (7.13) for $(s,d) \in \mathrm{T}_L$ we arrive at the estimates (7.1), where $C_1 := c_1 + c_2 + c_3$. By Theorem 6.1, 7.1 the inequalities (7.1) can be extended by continuity to the entire set $\overline{\mathrm{T}}_L$. The proof is complete. □

Let $L \in \mathbb{N}$, $L \geqslant L_0$, $v \in C(\partial\Omega)$. On the set T_L we represent the function $\tilde{u}$ as the sum $\tilde{u} = \sum\limits_{i=1}^{4} \tilde{u}_i$, where

$$\tilde{u}_i(s,d) := \int\limits_{[-\mathrm{P}',\mathrm{P}']} a_i \tilde{w}_i' d\rho \quad (i = \overline{1,3}), \qquad \tilde{u}_4(s,d) \equiv \int\limits_{I_S \setminus [\Sigma_-', \Sigma_+']} \tilde{w}_4 d\sigma \quad ((s,d) \in \mathrm{T}_L).$$

Theorem 7.3. *Let $\gamma, L \in \mathbb{N}$, $L \geqslant L_0$, $\gamma \geqslant 2$, $\partial\Omega \in C^{2\gamma+2}$. Then, for $(s,d) \in \overline{\mathrm{T}}_L$, the functions $\tilde{u}(s,d)$, $\hat{u}(s,d)$, defined by continuity for $d = 0$, satisfy the estimates*

$$|\hat{u} - \tilde{u}| \leqslant C_2 \|v\|_{C^3(\partial\Omega)} h^3 \ln(2L+1), \tag{7.14}$$

where C_2 is a positive constant independent of L and (s,d).

Proof. Let $v \in C^3(\partial\Omega)$. We define the set

$$\Upsilon'' := I_S \times \left[-\frac{3\Sigma_1}{5}, \frac{3\Sigma_1}{5}\right] \times \overline{I_D}.$$

Since $r > 0$ for $(s,\sigma,d) \in \overline{\Upsilon \setminus \Upsilon''}$, for $\partial\Omega \in C^{n+1}$, $n \in \mathbb{Z}_+$ we can define the positive constants

$$\hat{c}_{4,j} := \sup_{(s,\sigma,d) \in \overline{\Upsilon \setminus \Upsilon''}} \left|\partial_\sigma^j g\right| \qquad (j = \overline{0,n}).$$

For $L \geqslant L_0$, $(s,d) \in \overline{\mathrm{T}}_L$ we have the embeddings $\left[-\frac{3\Sigma_1}{5}, \frac{3\Sigma_1}{5}\right] \subseteq [s_{-2L'-1}, s_{2L'+1}]$, and this is why for $(s,d) \in \overline{\mathrm{T}}_L$ and almost all $\sigma \in \overline{I_S \setminus [s_{-2L'-1}, s_{2L'+1}]}$ the derivative $\partial_\sigma^{2\gamma} \tilde{w}_4$ exists and by the inequalities (3.7), (3.10) we have

$$\left|\partial_\sigma^{2\gamma} \tilde{w}_4\right| \leqslant \tilde{c}_4 \|v\|_{C^2(\partial\Omega)} \quad ((s,d) \in \overline{\mathrm{T}}_L), \quad \tilde{c}_4 := \hat{c}_{4,2\gamma} c_{\Lambda,0} + 2\gamma \hat{c}_{4,2\gamma-1} c_{\Lambda,1} + \gamma(2\gamma-1)\hat{c}_{4,2\gamma-2} c_{\Lambda,2}.$$

Using the estimate for the error term in the Gauss formulas [2, Ch. 3, Sect. 5, Subsect. 2], we obtain the estimate for the error of approximation of function $\tilde{u}_4$:

$$|\hat{u}_4 - \tilde{u}_4| \leqslant c_4 \|v\|_{C^2(\partial\Omega)} h^{2\gamma} \quad ((s,d) \in \mathrm{T}_L), \qquad c_4 := 2S(\gamma!)^4 \left[(2\gamma)!\right]^{-3} (2\gamma+1)^{-1} \tilde{c}_4. \tag{7.15}$$

By Corollary 4.1, for $\partial\Omega \in C^{n+2}$, $n \in \mathbb{Z}_+$ we can define the positive constants

$$\hat{c}_{i,j} := \sup_{(s,\rho,d) \in \tilde{\Upsilon}'} \left|\partial_\rho^j \delta_i'\right| \qquad (j = \overline{0,n}, i = \overline{0,3}).$$

For almost all $(s,\rho,d) \in \tilde{\Upsilon}'$ the derivatives $\partial_\rho^3 \tilde{w}_i'$ $(i = \overline{1,3})$ exist and by the inequalities (3.7), (3.10) we have

$$\left|\partial_\rho^3 \tilde{w}_i'\right| \leqslant \tilde{c}_i \|v\|_{C^2(\partial\Omega)},$$

where

$$\tilde{c}_i := \hat{c}_{i,3}c_{\Lambda,0} + (3\hat{c}_{i,2}\hat{c}_{0,0} + 3\hat{c}_{i,1}\hat{c}_{0,1} + \hat{c}_{i,0}\hat{c}_{0,2})\, c_{\Lambda,1} + 3\left(\hat{c}_{i,1}\hat{c}_{0,0}^2 + \hat{c}_{i,0}\hat{c}_{0,1}\hat{c}_{0,0}\right) c_{\Lambda,2} \quad (i = \overline{1,3}).$$

Using the inequalities

$$h'_{s,d,l} \leqslant c''_\rho h \qquad ((s,d) \in \overline{\mathrm{T}}_L,\ l = \overline{-2L'-1, 2L'+1}),$$

similar to the estimates (7.3), (7.4) we obtain the estimates for the errors of approximations of functions $\tilde{u}_3$, $\tilde{u}_2$:

$$|\hat{u}_i - \tilde{u}_i| \leqslant c_i \left\|v\right\|_{C^2(\partial\Omega)} h^3 \quad ((s,d) \in \mathrm{T}_L, \quad i = 2, 3). \tag{7.16}$$

Here

$$c_2 := 2c_\lambda \tilde{c}_2 (c''_\rho)^3 \sup_{d\in I_D} \left|A_{2,0}(\mathrm{P}, |d|)\right|, \qquad c_3 := 2Sc_\lambda \tilde{c}_3 (c''_\rho)^3.$$

Letting

$$\begin{aligned} &z = \rho_{s,d,2l} + t, \qquad f(z) = \tilde{w}'_1(s, \rho_{s,d,2l} + t, d), \qquad z_{-1} = \rho_{s,d,2l-1},\\ &z_1 = \rho_{s,d,2l+1}, \qquad z_0 = \rho_{s,d,2l}, \qquad \hat{f}(z) = \hat{w}'_1(s, \rho_{s,d,2l} + t, d),\\ &t \in \left[-h'_{s,d,l}, h'_{s,d,l}\right], \qquad (s,d) \in \overline{\mathrm{T}}_L, \qquad l = \overline{-2L'-1, 2L'+1}, \end{aligned}$$

in the formula (3.1), we arrive at the identities

$$\tilde{w}'_1(s, \rho_{s,d,2l} + t, d) - \hat{w}'_1(s, \rho_{s,d,2l} + t, d) = \hat{w}'_{\mathrm{III},l} t\, \omega_{s,d,l},$$

where

$$\omega_{s,d,l}(t) := \left(t + h'_{s,d,l}\right)\left(t - h'_{s,d,l}\right), \qquad \hat{w}'_{\mathrm{III},l}(s,t,d) := f_3(z, z_{-1}, z_1, z_0).$$

For $(s,d) \in \mathrm{T}_L$ the deviation $\tilde{u}_1 - \hat{u}_1$ can be written as

$$\tilde{u}_1 - \hat{u}_1 = \sum_{l=-2L'-1}^{2L'+1} \int_{-h'_{s,d,l}}^{h'_{s,d,l}} g_l \hat{w}'_{\mathrm{III},l} t\omega_{s,d,l} dt, \qquad g_l(s,t,d) := \left[a_1(\rho,d)\right]_{\rho=\rho_{s,d,2l}+t}. \tag{7.17}$$

Let us estimate the sum (7.17) without the term for $l = 0$. Since

$$h'_{s,d,-l} = h'_{s,d,l}, \qquad \omega_{s,d,-l}(-t) = \omega_{s,d,l}(t), \qquad g_{-l}(s,-t,d) = g_l(s,t,d)$$

for $t \in \left[-h'_{s,d,l}, h'_{s,d,l}\right]$, $(s,d) \in \overline{\mathrm{T}}_L$, $l = \overline{1, 2L'+1}$, and the sign of function g_l, $\omega_{s,d,l}$ does not change for $t \in \left[-h'_{s,d,l}, h'_{s,d,l}\right]$, in view of the formula (3.3) and the mean value theorem for $(s,d) \in \mathrm{T}_L$, $l = \overline{1, 2L'+1}$ we have the identities

$$\begin{aligned} &\int_{-h'_{s,d,-l}}^{h'_{s,d,-l}} g_{-l} \hat{w}'_{\mathrm{III},-l} t\omega_{s,d,-l}\, dt + \int_{-h'_{s,d,l}}^{h'_{s,d,l}} g_l \hat{w}'_{\mathrm{III},l} t\omega_{s,d,l}\, dt\\ &\qquad = 6^{-1}\left(\left[\partial^3_\rho \tilde{w}'_1\right]_{\rho=\zeta_l^+} - \left[\partial^3_\rho \tilde{w}'_1\right]_{\rho=\zeta_l^-}\right) \zeta_l \int_{-h'_{s,d,l}}^{h'_{s,d,l}} g_l \omega_{s,d,l}\, dt, \end{aligned} \tag{7.18}$$

where

$$\zeta_l^\pm(s,d) := \pm\rho_{s,d,2l} \pm \zeta_l, \qquad \zeta_l(s,d) \in \left[-h'_{s,d,l}, h'_{s,d,l}\right], \qquad l = \overline{1, 2L'+1}.$$

We also have

$$
\begin{aligned}
\left[\partial_\rho^3 \tilde{w}_1'\right]_{\rho=\zeta_l^+} - \left[\partial_\rho^3 \tilde{w}_1'\right]_{\rho=\zeta_l^-} &= \sum_{k=0}^{2} \left(\left[\partial_\rho^{3-k}\delta_1'\right]_{\rho=\zeta_l^+} - \left[\partial_\rho^{3-k}\delta_1'\right]_{\rho=\zeta_l^-} \right) \left[\partial_\rho^k \tilde{v}'\right]_{\rho=\zeta_l^+} \\
&\quad + \left[\partial_\rho^{3-k}\delta_1'\right]_{\rho=\zeta_l^-} \left(\left[\partial_\rho^k \tilde{v}'\right]_{\rho=\zeta_l^+} - \left[\partial_\rho^k v'\right]_{\rho=\zeta_l^+} \right) \\
&\quad + \left[\partial_\rho^{3-k}\delta_1'\right]_{\rho=\zeta_l^-} \left(\left[\partial_\rho^k v'\right]_{\rho=\zeta_l^+} - \left[\partial_\rho^k v'\right]_{\rho=\zeta_l^-} \right) \\
&\quad + \left[\partial_\rho^{3-k}\delta_1'\right]_{\rho=\zeta_l^-} \left(\left[\partial_\rho^k v'\right]_{\rho=\zeta_l^-} - \left[\partial_\rho^k \tilde{v}'\right]_{\rho=\zeta_l^-} \right),
\end{aligned}
\tag{7.19}
$$

where

$$
v'(s,\rho,d) := v(s+\tilde{\sigma}_{s,d}(\rho)), \qquad \tilde{v}'(s,\rho,d) := \tilde{v}(s+\tilde{\sigma}_{s,d}(\rho)).
$$

We define the positive constants

$$
\begin{aligned}
&\hat{c}_{1,0}' := \hat{c}_{1,0}, && \hat{c}_{1,j}' := \max\left\{\hat{c}_{1,j}, \hat{c}_{i,j-1}'\right\} \quad (j=\overline{1,n}); \\
&c_\Lambda' := \max\left\{c_{\Lambda,0}, c_{\Lambda,1}, c_{\Lambda,2}\right\}, && c_\lambda' := \max\left\{c_{\lambda,0}, c_{\lambda,1}, c_{\lambda,2}\right\}.
\end{aligned}
$$

For $v \in C^3(\partial\Omega)$, $(s,\rho,d) \in \tilde{\Upsilon}'$ we have the estimates

$$
\left|\partial_\rho^j v'\right| \leqslant c_{v,j} \left\|v\right\|_{C^j(\partial\Omega)} \quad (j=\overline{0,3}),
$$

where

$$
\begin{aligned}
&c_{v,0} := 1, \qquad c_{v,1} := \max\left\{\hat{c}_{0,0}, c_{v,0}\right\}, \\
&c_{v,2} := \max\left\{\hat{c}_{0,0}^2 + \hat{c}_{0,1}, c_{v,1}\right\}, \\
&c_{v,3} := \max\left\{\hat{c}_{0,3}^3 + 3\hat{c}_{0,0}\hat{c}_{0,1} + \hat{c}_{0,2}, c_{v,2}\right\}.
\end{aligned}
$$

Moreover, $\rho \leqslant \tilde{\rho}_{s,d}(s_{2l+1}) \leqslant c_\rho'' s_{2l+1}$ if $\rho \in [\rho_{s,d,2l-1}, \rho_{s,d,2l+1}]$, $l = \overline{1, 2L'+1}$. Taking into consideration the inequalities (3.7), (3.8), (3.10), (3.11), for $t \in [-h_{s,d,l}', h_{s,d,l}']$, $l = \overline{1, 2L'+1}$, $(s,d) \in \overline{\mathrm{T}}_L$ we get the estimates

$$
\begin{aligned}
&\left|\left[\partial_\rho^j \delta_1'\right]_{\rho=\zeta_l^\pm}\right| \leqslant c_\delta \qquad (j=\overline{0,3}), && c_\delta := \hat{c}_{1,3}'; \\
&\left|\left[\partial_\rho^j \delta_1'\right]_{\rho=\zeta_l^+} - \left[\partial_\rho^j \delta_1'\right]_{\rho=\zeta_l^-}\right| \leqslant c_\delta' \left(2l+1\right) h \qquad (j=\overline{0,3}), && c_\delta' := 2\hat{c}_{1,4}' c_\rho''; \\
&\left|\left[\partial_\rho^j v'\right]_{\rho=\zeta_l^+} - \left[\partial_\rho^j v'\right]_{\rho=\zeta_l^-}\right| \leqslant c_v' \left\|v\right\|_{C^3(\partial\Omega)} \left(2l+1\right) h \qquad (j=\overline{0,2}), && c_v' := 2c_{v,3} c_\rho''; \\
&\left|\left[\partial_\rho^j \tilde{v}'\right]_{\rho=\zeta_l^\pm}\right| \leqslant \tilde{c}_v \left\|v\right\|_{C^2(\partial\Omega)} \qquad (j=\overline{0,2}), && \tilde{c}_v := c_\Lambda' c_{v,2}; \\
&\left|\left[\partial_\rho^j \tilde{v}' - \partial_\rho^j v'\right]_{\rho=\zeta_l^\pm}\right| \leqslant \tilde{c}_v' \left\|v\right\|_{C^3(\partial\Omega)} h \qquad (j=\overline{0,2}), && \tilde{c}_v' := c_\lambda' c_{v,3} c_\rho''.
\end{aligned}
\tag{7.20}
$$

By the estimates (7.20) and identities (7.19), for $t \in [-h_{s,d,l}', h_{s,d,l}']$, $l = \overline{1, 2L'+1}$, $(s,d) \in \overline{\mathrm{T}}_L$ the estimates hold

$$
\left|\left[\partial_\rho^3 \tilde{w}_1'\right]_{\rho=\zeta_l^+} - \left[\partial_\rho^3 \tilde{w}_1'\right]_{\rho=\zeta_l^-}\right| \leqslant \tilde{c}_{1,1} \left\|v\right\|_{C^3(\partial\Omega)} \left(2l+1\right) h, \quad \tilde{c}_{1,1} := 3\left(c_\delta' \tilde{c}_v + c_\delta \left(2\tilde{c}_v' + c_v'\right)\right). \tag{7.21}
$$

Let

$$
\rho_{s,d,2l-1}' := \min\left\{-\tilde{\rho}_{s,d}(s_{-(2l-1)}), \tilde{\rho}_{s,d}(s_{2l-1})\right\}, \qquad l = \overline{1, L'}, \qquad (s,d) \in \overline{\mathrm{T}}_L.
$$

We observe that $\rho \geqslant \tilde{\rho}_{s,d,1}' \geqslant c_\rho' s_1$ if $\rho \in [\rho_{s,d,1}, \rho_{s,d,3}]$ and by the Dirichlet principle $\rho \geqslant \tilde{\rho}_{s,d,2l'-1}' \geqslant c_\rho' s_{2l'-1}$ if $\rho \in [\rho_{s,d,2l-1}, \rho_{s,d,2l+1}]$, $l = 2l' + l''$, $l' = \overline{1, L'}$, $l'' = 0, 1$. This is why the inequalities hold

$$
|g_1| \leqslant \left(c_\rho' h\right)^{-2} \quad \text{for} \quad t \in [-h_{s,d,1}, h_{s,d,1}],
$$

and

$$|g_l| \leqslant \left(c'_\rho h\right)^{-2} \left(2\left|l'\right|-1\right)^{-2} \quad \text{for} \quad t \in [-h'_{s,d,l}, h'_{s,d,l}],$$

$l = 2l' + l''$, $l' = \overline{1, L'}$, $l'' = 0, 1$ ($(s,d) \in \mathrm{T}_L$). Taking into consideration the inequalities

$$|\zeta_l| \leqslant c''_\rho h \qquad (l = \overline{1, 2L'+1}),$$

$$\frac{(2l+1)}{(2l'-1)} \leqslant 7 \qquad (l = 2l' + l'',\ l' = \overline{1, L'},\ l'' = 0, 1)$$

and

$$\int\limits_{-h'_{s,d,l}}^{h'_{s,d,l}} |\omega_{s,d,l}|\, dt \leqslant \frac{4}{3} \left(c''_\rho h\right)^3 \quad (l = \overline{1, 2L'+1},\ (s,d) \in \overline{\mathrm{T}}_L),$$

the identities (7.18) and estimates (7.21), similar to the inequalities (7.11) we obtain the estimates

$$\left| \sum_{l = \overline{-2L'-1,-1},\ \overline{1,2L'+1}} \int\limits_{-h'_{s,d,l}}^{h'_{s,d,l}} g_l \hat{w}'_{\mathrm{III},l} t \omega_{s,d,l}\, dt \right| \leqslant c_{1,1} \|v\|_{C^3(\partial\Omega)} \ln(2L'+1)\, h^3 \quad ((s,d) \in \mathrm{T}_L). \quad (7.22)$$

Here

$$c_{1,1} := \frac{14}{9} \tilde{c}_{1,1} \left(c_\rho c''_\rho\right)^2.$$

We proceed to estimate the term in the sum (7.17) for $l = 0$. Comparing the Newton formulas (3.1), (3.2) and letting $z = t$, $z_{-1} = -h_{s,d,0}$, $z_1 = h_{s,d,0}$, $z_0 = 0$, for $t \in [-h_{s,d,0}, h_{s,d,0}]$, $(s,d) \in \overline{\mathrm{T}}_L$ we can represent the function $\hat{w}'_{\mathrm{III},0}$as the sum

$$\hat{w}'_{\mathrm{III},0} = \hat{w}'_{\mathrm{III}} + \hat{w}'_{\mathrm{IV}} t,$$

where

$$\hat{w}'_{\mathrm{III}}(s,d) := f_3(z_{-1}, z_1, z_0, z_0), \qquad \hat{w}'_{\mathrm{IV}}(s,t,d) := f_4(z, z_{-1}, z_1, z_0, z_0).$$

For almost all $(s,\rho,d) \in \tilde{\Upsilon}'$ the derivative $\partial_\rho^4 \tilde{w}'_1$ exists and by the formula (3.4) and inequalities (3.7), (3.10) we have

$$|\hat{w}'_{\mathrm{IV}}| \leqslant 24^{-1} \sup_{(s,\rho,d)\in\tilde{\Upsilon}'} \left|\partial_\rho^4 \tilde{w}'_1\right| \leqslant 24^{-1} \tilde{c}_{1,2} \|v\|_{C^2(\partial\Omega)},$$

where

$$\begin{aligned} \tilde{c}_{1,2} :=& \hat{c}_{1,4} c_{\Lambda,0} + \left(4\hat{c}_{1,3}\hat{c}_{0,0} + 6\hat{c}_{1,2}\hat{c}_{0,1} + 4\hat{c}_{1,1}\hat{c}_{0,2} + \hat{c}_{1,0}\hat{c}_{0,3}\right) c_{\Lambda,1} \\ &+ \left(6\hat{c}_{1,2}\hat{c}_{0,0}^2 + 12\hat{c}_{1,1}\hat{c}_{0,0}\hat{c}_{0,1} + 3\hat{c}_{1,0}\hat{c}_{0,1}^2 + 4\hat{c}_{1,0}\hat{c}_{0,0}\hat{c}_{0,2}\right) c_{\Lambda,2}. \end{aligned}$$

Taking into consideration that

$$\int\limits_{-h_{s,d,0}}^{h_{s,d,0}} g_0 t \omega_{s,d,0}\, dt = 0, \qquad \int\limits_{-h_{s,d,0}}^{h_{s,d,0}} |\omega_{s,d,0}|\, dt \leqslant \frac{4}{3} \left(c''_\rho h\right)^3 \quad \text{for} \quad (s,d) \in \mathrm{T}_L$$

and

$$\left|g_0 t^2\right| \leqslant 1 \quad \text{for} \quad (s,d) \in \mathrm{T}_L, \quad t \in [-h_{s,d,0}, h_{s,d,0}],$$

we obtain the estimates

$$\left| \int\limits_{-h_{s,d,0}}^{h_{s,d,0}} g_0 \hat{w}'_{\mathrm{III},0} t \omega_{s,d,0} dt \right| = \left| \int\limits_{-h_{s,d,0}}^{h_{s,d,0}} g_0 t^2 \hat{w}'_{\mathrm{IV}} \omega_{s,d,0} dt \right| \leqslant c_{1,2} \|v\|_{C^2(\partial\Omega)} h^3 \quad ((s,d) \in \mathrm{T}_L). \quad (7.23)$$

Here $c_{1,2} := 18^{-1} \tilde{c}_{1,2} (c''_\rho)^3$.

By the formula (7.17) and estimates (7.22), (7.23) we obtain

$$|\hat{u}_1 - \tilde{u}_1| \leqslant c_1 \|v\|_{C^3(\partial\Omega)} \ln(2L'+1)\, h^3 \quad ((s,d) \in \mathrm{T}_L), \quad c_1 := c_{1,1} + c_{1,2}. \tag{7.24}$$

In view of the identities $\tilde{u} = \sum_{i=1}^{4} \tilde{u}_i$, $\hat{u} \equiv \sum_{i=1}^{4} \hat{u}_i$ and estimates (7.15), (7.16), (7.24), for $(s,d) \in \mathrm{T}_L$ we obtain the estimates (7.14), where $C_2 := c_1 + c_2 + c_3 + c_4$. By Theorems 5.1, 7.1, the inequalities (7.14) can be continued to the entire set $\overline{\mathrm{T}}_L$. The proof is complete. □

By the inequalities (7.1), (7.14) under the conditions $\gamma, L \in \mathbb{N}$, $L \geqslant L_0$, $\gamma \geqslant 2$, $\partial\Omega \in C^{2\gamma+2}$, $v \in C^4(\partial\Omega)$ we obtain the estimates

$$|\hat{u} - u| \leqslant C_0 \|v\|_{C^4(\partial\Omega)} h^3 \ln(2L+1) \quad ((s,d) \in \overline{\mathrm{T}}_L), \quad C_0 := C_1 + C_2. \tag{7.25}$$

Similarly, there exists a positive constant C_0' such that under the conditions $\gamma, L \in \mathbb{N}$, $L \geqslant L_0$, $\gamma \geqslant 2$, $\partial\Omega \in C^{2\gamma+2}$, $v \in C^4(\partial\Omega)$ the estimates hold

$$|\hat{u}' - u'| \leqslant C_0' \|v\|_{C^4(\partial\Omega)} h^3 \ln(2L+1) \quad ((s,d) \in \overline{\mathrm{T}}_L'). \tag{7.26}$$

According to the estimates (7.25), (7.26), the functions $\hat{u}$, $\hat{u}'$ approximate the function u with the error $O(h^3 \ln(2L+1))$ on the corresponding sets $\overline{\mathrm{T}}_L$, $\overline{\mathrm{T}}_L'$. Under the conditions $\gamma, L \in \mathbb{N}$, $L \geqslant L_0$, $\gamma \geqslant 2$, $\partial\Omega \in C^{2\gamma+2}$, $v \in C^4(\partial\Omega)$ we have the estimates

$$|\breve{u} - u| \leqslant C \|v\|_{C^4(\partial\Omega)} h^3 \ln(2L+1) \quad ((s,d) \in \overline{\mathrm{T}}_L \cup \overline{\mathrm{T}}_L'), \quad C := C_0 + C_0'. \tag{7.27}$$

According to the formulas (5.4), (6.7), for $(s,d) \in \overline{\mathrm{T}}$ the function $\breve{u} - \ddot{u}$ is calculated by the formula

$$\breve{u} - \ddot{u} = [\hat{u}' - u]_{s=s_{2l-1}} \Lambda_{-1}^{[s_{2l-1}, s_{2l+1}]} + [\hat{u} - u]_{s=s_{2l}} \Lambda_0^{[s_{2l-1}, s_{2l+1}]} + [\hat{u}' - u]_{s=s_{2l+1}} \Lambda_1^{[s_{2l-1}, s_{2l+1}]},$$

that is, the function $\breve{u} - \ddot{u}$ on the segments $[s_{2l-1}, s_{2l+1}]$ is described by the quadratic Lagrange polynomials determined by the given values of the function $\breve{u} - u$ at the nodes s_{2l-1}, s_{2l}, s_{2l+1}. This is why in view of the inequalities (3.7), (7.27) under the conditions $\gamma, L \in \mathbb{N}$, $L \geqslant L_0$, $\gamma \geqslant 2$, $\partial\Omega \in C^{2\gamma+2}$, $v \in C^4(\partial\Omega)$ we obtain the estimates

$$|\breve{u} - \ddot{u}| \leqslant c_{\Lambda,0} C \|v\|_{C^4(\partial\Omega)} h^3 \ln(2L+1) \quad ((s,d) \in \overline{\mathrm{T}}). \tag{7.28}$$

By the estimates (6.8), (7.28) under the conditions $\gamma, L \in \mathbb{N}$, $L \geqslant L_0$, $L \geqslant \dfrac{S^2-1}{2}$, $(\ln(2L+1) \leqslant 2|\ln h|)$, $\gamma \geqslant 2$, $\partial\Omega \in C^{2\gamma+3}$, $v \in C^5(\partial\Omega)$ we get the estimates (2.2), where

$$c := \pi^{-1} \left(c_{\Lambda,0} C \|v\|_{C^4(\partial\Omega)} + c_\lambda \sup_{(s,d)\in\overline{\mathrm{T}}} \left|\partial_s^3 u\right| \right).$$

We arrive at the main statement of the work.

Corollary 7.1. *Let $\gamma, L \in \mathbb{N}$, $L \geqslant L_0$, $\gamma \geqslant 2$, $\partial\Omega \in C^{2\gamma+3}$, $v \in C^5(\partial\Omega)$. Then, as $L \to \infty$, the functions $\breve{U}(\tilde{x}_d(s))$ converge to the function $U(\tilde{x}_d(s))$, defined for $d = 0$ by continuity, uniformly in $(s,d) \in \overline{\mathrm{T}}$ with the rate $O(h^3 \ln h)$.*

8. Numerical experiments

We consider ND DLP near and at the boundary of unit circle ($d \in [-0.01, 0.01]$) with the density $v(s) = \cos(s)$ ($s \in \overline{I}_S = [-\pi, \pi]$). The exact values of ND DLP $U(\tilde{x}_d(s))$ are calculated by the formulas

$$U = -2^{-1} \cos(s) \quad \text{for} \quad d \in [0,1],$$

$$U = -2^{-1}\frac{\cos(s)}{(1-d)^2} \quad \text{for} \quad d \in (-\infty, 0]\,.$$

For this geometry the third of the radius of Lyapunov circle is $D = \dfrac{2}{3\pi} \approx 0.21$. The derivative $\partial_\sigma \rho'$ is calculated by means of the expression $\sqrt{1-d}\cos\left(\dfrac{\sigma}{2}\right)$ for all $(s,\sigma,d) \in \Upsilon$, and this is why $\partial_\sigma\rho' > 0$ for $\sigma \in (-\pi,\pi)$, and as Σ_0 we can take any number $\Sigma_0 \in (0,\pi)$, and by the symmetry $\Sigma_1 = \Sigma_0$. We calculate the semi–analytic approximations of ND DLP $\breve{U}\left(\tilde{x}_d(s)\right)$ are calculated by means of the formula (5.4), and for calculating the function $\hat{u}_4$ we use the Gauss formula with $\gamma = 2$ nodes. The calculation of approximate and exact values of ND DLP is made with the double precision on the sets X_L formed by the points $\tilde{x}_{d_i}(s_{l+\frac{j}{4}})$ $(i = \overline{-5,5}$, $l = \overline{-2L-1, 2L}$, $j = \overline{0,3})$, where

$$s_{l+\frac{j}{4}} := \left(l + \frac{j}{4}\right)h, \qquad d_0 := 0, \qquad d_{\pm i} := \pm 10^{i+1}, \qquad i = \overline{1,4}, \qquad d_{\pm 5} := \pm 10^{-15}.$$

To determine experimentally the approximation accuracy we find the values ΔU, which are the maxima of absolute values of errors on the sets X_L:

$$\Delta U(h,\Sigma_0) := \max_{i=\overline{-5,5},\; l=\overline{-2L-1,2L},\; j=\overline{0,3}} \left|\breve{U}\left(\tilde{x}_{d_i}(s_{l+\frac{j}{4}})\right) - U\left(\tilde{x}_{d_i}(s_{l+\frac{j}{4}})\right)\right|.$$

In Table 1 in each cell we provide the values of ΔU calculated for given Σ_0 and h.

TABLE 1.

Σ_0/π	$h_1 = \pi/15$	$h_2 = \pi/31$	$h_3 = \pi/63$	$h_4 = \pi/127$	$h_5 = \pi/255$
0.006	$7.70 \cdot 10^{0}$	$6.94 \cdot 10^{0}$	$5.39 \cdot 10^{0}$	$2.30 \cdot 10^{0}$	$5.07 \cdot 10^{-2}$
0.01	$4.33 \cdot 10^{0}$	$3.56 \cdot 10^{0}$	$2.02 \cdot 10^{0}$	$8.17 \cdot 10^{-2}$	$6.91 \cdot 10^{-3}$
0.03	$9.60 \cdot 10^{-1}$	$1.88 \cdot 10^{-1}$	$3.22 \cdot 10^{-3}$	$4.41 \cdot 10^{-4}$	$4.70 \cdot 10^{-5}$
0.05	$2.87 \cdot 10^{-1}$	$5.63 \cdot 10^{-3}$	$1.07 \cdot 10^{-3}$	$6.39 \cdot 10^{-5}$	$6.08 \cdot 10^{-6}$
0.068	$3.06 \cdot 10^{-2}$	$1.01 \cdot 10^{-3}$	$1.63 \cdot 10^{-4}$	$1.87 \cdot 10^{-5}$	$1.18 \cdot 10^{-6}$
0.10	$3.27 \cdot 10^{-3}$	$6.27 \cdot 10^{-4}$	$3.48 \cdot 10^{-5}$	$3.47 \cdot 10^{-6}$	$2.17 \cdot 10^{-7}$
0.15	$4.25 \cdot 10^{-4}$	$9.74 \cdot 10^{-5}$	$8.85 \cdot 10^{-6}$	$1.00 \cdot 10^{-6}$	$8.35 \cdot 10^{-8}$
0.50	$2.96 \cdot 10^{-4}$	$3.78 \cdot 10^{-5}$	$5.14 \cdot 10^{-6}$	$6.60 \cdot 10^{-7}$	$6.82 \cdot 10^{-8}$
0.90	$4.66 \cdot 10^{-4}$	$5.35 \cdot 10^{-5}$	$6.83 \cdot 10^{-6}$	$7.68 \cdot 10^{-7}$	$7.43 \cdot 10^{-8}$
0.99	$1.29 \cdot 10^{-2}$	$2.00 \cdot 10^{-3}$	$2.21 \cdot 10^{-4}$	$3.68 \cdot 10^{-5}$	$4.79 \cdot 10^{-6}$
0.999	$1.79 \cdot 10^{-1}$	$4.06 \cdot 10^{-2}$	$9.10 \cdot 10^{-3}$	$1.91 \cdot 10^{-3}$	$3.44 \cdot 10^{-4}$
0.9999	$1.85 \cdot 10^{0}$	$4.34 \cdot 10^{-1}$	$1.04 \cdot 10^{-1}$	$2.53 \cdot 10^{-2}$	$6.02 \cdot 10^{-3}$
0.99999	$1.85 \cdot 10^{1}$	$4.37 \cdot 10^{0}$	$1.06 \cdot 10^{0}$	$2.60 \cdot 10^{-1}$	$6.37 \cdot 10^{-2}$

Calculating the degree of convergence rates

$$\upsilon := \ln\left(\frac{\Delta U(h_k)}{\Delta U(h_{k+1})}\right) / \ln\left(\frac{h_k}{h_{k+1}}\right)$$

while passing from the discretization step h_k to h_{k+1} $(k = \overline{1,4})$, we observe that in a wide range of possible values of Σ_0 the proposed semi–analytical approximation has a uniform almost cubic rate of convergence $(\upsilon \approx 3)$ in the closed near–boundary domain, which is in good agreement with Corollary 7.1. In particular, a pronounced cubic rate of convergence is observed for the value $\dfrac{\Sigma_0}{\pi} = \pi^{-1}\arcsin D \approx 0.068$, when the distance from the observation point to the nearest nodes in the Gauss formulas is approximately equal to the third of radius of Lyapunov circle (see the remark after Theorem 4.1). The convergence rate decreases with decreasing Σ_0, when the observation point $\tilde{x}_d(s)$ approaches the nodes in the Gauss formulas, and as $\Sigma_0 \to \pi$, when the

Jacobian $(\partial_\sigma \rho')^{-1}$, which makes the passage from the integration variable over the arc length σ to the variable ρ, increases unboundedly.

Thus, the proposed combined method, which uses the exact integration with respect to the variable ρ on a fixed length part of the boundary near the observation point and the Gauss formula on the rest of the boundary, ensures uniform almost cubic convergence of the approximations of the ND DLP in the near–boundary domain including the boundary. In this case, the part of the boundary on which the exact integration with respect to ρ is made should be made as large as possible to achieve the greatest accuracy, provided that the Jacobian of the transition to the variable ρ remains positive and does not take on too large values.

Now we consider the calculation of ND DLP with the density $v(s) = \cos(s)$ at some distance from the boundary of the unit circle ($d \geqslant 0.01$ and $d \leqslant -0.01$). Here for $d \in (-\infty, 1)$, $s \in I_S$ the correspondence between the points $\tilde{x}(s)$ and $\tilde{x}_d(s)$ is one–to–one $1 - dK(s) = 1 - d > 0$, and this is why as the set I_D we can take any segment $[-D_1, 0) \cup (0, D_2]$, where $D_1 > 0$, $D_2 \in (0, 1)$. The possibility to choose the number Σ_0 is independent of the choice of numbers D_1, D_2 since $\partial_\sigma \rho' = \sqrt{1-d}\cos\left(\frac{\sigma}{2}\right) > 0$ for all $d \in (-\infty, 1)$, $\sigma \in (-\pi, \pi)$. For fixed d we calculate the values of functions $\breve{U}\left(\tilde{x}_d(s)\right)$ (for $\gamma = 2$) and $U\left(\tilde{x}_d(s)\right)$ on the sets $X'_L(d)$ formed by the points $\tilde{x}_d(s_{l+\frac{j}{4}})$ ($l = \overline{-2L-1, 2L}$, $j = \overline{0,3}$). For the experimental determination of approximation accuracy we find the values $\Delta' U$, which are the absolute values of errors on the sets $X'_L(d)$ for fixed values d, Σ_0:

$$\Delta' U(d, h, \Sigma_0) := \max_{l=\overline{-2L-1,2L},\ j=\overline{0,3}} \left| \breve{U}\left(\tilde{x}_d(s_{l+\frac{j}{4}})\right) - \breve{U}\left(\tilde{x}_d(s_{l+\frac{j}{4}})\right) \right|.$$

In Table 2 in each cell we provide two series of values $\Delta' U$: for $\Sigma_0 = \frac{\pi}{2}$ and $\Sigma_0 = 0$ (from top to down) calculated for given values of d and h.

The data in Table 2 indicate that the use of exclusively Gauss formulas for calculating the integrals arising after PQA of the density ensures the cubic convergence of the approximations of the ND DLP only at some distance from the boundary, and it is impossible in a sufficiently narrow closed near–boundary domain. The semi–analytical approximations $\breve{U}\left(\tilde{x}_d(s)\right)$ converge with an almost cubic rate uniformly in $(s, d) \in \overline{\mathrm{T}}$ in a wide range of possible values of D_1, D_2. The convergence rate decreases as $d \to 1$ (the points $\tilde{x}_d(s)$ approach the center of the circle), when the Jacobian $(\partial_\sigma \rho')^{-1}$ increases unboundedly.

9. Conclusion

In this paper, we obtain semi–analytical approximations of ND DLP for the two–dimensional Laplace equation $\breve{U}\left(\tilde{x}_d(s)\right)$, which converge uniformly in a closed near–boundary domain $\overline{\Omega}_D$ including the boundary with the rate $O(h^3 \ln h)$ under the conditions $\partial\Omega \in C^7$, $v \in C^5(\partial\Omega)$. We use quadratic polynomials and the Gauss formulas of at least fifth order to approximate slowly varying functions. In the same way, for each $n \in \mathbb{N}$, one can obtain approximations converging uniformly with the rate $O(h^{2n+1} \ln h)$ if one uses polynomials of degree $2n$ and the Gauss formulas of order at least $O(h^{2n+3})$.

The disadvantages of the method include the rather high requirements imposed on the smoothness of the boundary curve $\partial\Omega$. In addition, the issue on approximation of the boundary in the framework of this method was not studied. The advantage of the method is the possibility of its implementation for any analytically defined boundary curves. Suppose that the boundary approximation is an independent problem and it is satisfactorily solved by using an analytically defined curve that has sufficient smoothness, that is, such a curve can be treated as the true boundary of the domain. Then the proposed method can always be implemented, regardless of the complexity of curve. To implement other semi–analytical methods based on exact integration over the arc length, additional approximation of the coordinate functions of the curve is required: piecewise constant, piecewise linear or piecewise quadratic, which leads

Table 2.

d	$h_1=\pi/15$	$h_2=\pi/31$	$h_3=\pi/63$	$h_4=\pi/127$	$h_5=\pi/255$
0.01	$2.96\cdot10^{-4}$ $5.92\cdot10^{0}$	$3.78\cdot10^{-5}$ $1.18\cdot10^{1}$	$5.14\cdot10^{-6}$ $1.96\cdot10^{1}$	$6.60\cdot10^{-7}$ $1.99\cdot10^{1}$	$6.82\cdot10^{-8}$ $7.23\cdot10^{0}$
0.1	$3.72\cdot10^{-4}$ $1.73\cdot10^{0}$	$3.47\cdot10^{-5}$ $4.01\cdot10^{-1}$	$3.87\cdot10^{-6}$ $2.24\cdot10^{-2}$	$4.73\cdot10^{-7}$ $5.27\cdot10^{-5}$	$5.84\cdot10^{-8}$ $5.84\cdot10^{-8}$
0.5	$2.98\cdot10^{-4}$ $4.24\cdot10^{-4}$	$3.28\cdot10^{-5}$ $3.26\cdot10^{-5}$	$3.88\cdot10^{-6}$ $3.88\cdot10^{-6}$	$4.73\cdot10^{-7}$ $4.73\cdot10^{-7}$	$5.84\cdot10^{-8}$ $5.84\cdot10^{-8}$
0.9	$3.10\cdot10^{-4}$ $2.89\cdot10^{-4}$	$3.32\cdot10^{-5}$ $3.26\cdot10^{-5}$	$3.90\cdot10^{-6}$ $3.88\cdot10^{-6}$	$4.74\cdot10^{-7}$ $4.73\cdot10^{-7}$	$6.01\cdot10^{-8}$ $5.84\cdot10^{-8}$
0.99	$3.08\cdot10^{-4}$ $2.89\cdot10^{-4}$	$3.32\cdot10^{-5}$ $3.26\cdot10^{-5}$	$3.90\cdot10^{-6}$ $3.88\cdot10^{-6}$	$4.74\cdot10^{-7}$ $4.73\cdot10^{-7}$	$2.65\cdot10^{-7}$ $5.84\cdot10^{-8}$
0.999	$3.08\cdot10^{-4}$ $2.89\cdot10^{-4}$	$3.29\cdot10^{-5}$ $3.26\cdot10^{-5}$	$5.46\cdot10^{-6}$ $3.88\cdot10^{-6}$	$6.74\cdot10^{-6}$ $4.73\cdot10^{-7}$	$5.50\cdot10^{-4}$ $5.84\cdot10^{-8}$
0.9999	$6.59\cdot10^{-4}$ $2.89\cdot10^{-4}$	$3.26\cdot10^{-3}$ $3.26\cdot10^{-5}$	$1.64\cdot10^{-2}$ $3.87\cdot10^{-6}$	$7.34\cdot10^{-2}$ $4.73\cdot10^{-7}$	$3.79\cdot10^{-1}$ $5.84\cdot10^{-8}$
0.99999	$5.87\cdot10^{-1}$ $2.89\cdot10^{-4}$	$1.12\cdot10^{0}$ $3.26\cdot10^{-5}$	$2.04\cdot10^{1}$ $3.88\cdot10^{-6}$	$2.13\cdot10^{1}$ $4.73\cdot10^{-7}$	$7.28\cdot10^{2}$ $5.84\cdot10^{-8}$
-0.01	$2.90\cdot10^{-4}$ $5.86\cdot10^{0}$	$3.68\cdot10^{-5}$ $1.15\cdot10^{1}$	$4.99\cdot10^{-6}$ $1.93\cdot10^{1}$	$6.40\cdot10^{-7}$ $1.98\cdot10^{1}$	$6.65\cdot10^{-8}$ $7.33\cdot10^{0}$
-0.1	$2.86\cdot10^{-4}$ $1.73\cdot10^{0}$	$2.83\cdot10^{-5}$ $4.80\cdot10^{-1}$	$3.20\cdot10^{-6}$ $3.47\cdot10^{-2}$	$3.91\cdot10^{-7}$ $1.54\cdot10^{-4}$	$4.83\cdot10^{-8}$ $4.83\cdot10^{-8}$
-1	$7.33\cdot10^{-5}$ $1.06\cdot10^{-4}$	$8.17\cdot10^{-6}$ $8.15\cdot10^{-6}$	$9.70\cdot10^{-7}$ $9.69\cdot10^{-7}$	$1.18\cdot10^{-7}$ $1.18\cdot10^{-7}$	$1.46\cdot10^{-8}$ $1.46\cdot10^{-8}$
-10	$2.36\cdot10^{-6}$ $2.39\cdot10^{-6}$	$2.68\cdot10^{-7}$ $2.69\cdot10^{-7}$	$3.20\cdot10^{-8}$ $3.20\cdot10^{-8}$	$3.91\cdot10^{-9}$ $3.91\cdot10^{-9}$	$4.83\cdot10^{-10}$ $4.83\cdot10^{-10}$
-100	$2.80\cdot10^{-8}$ $2.84\cdot10^{-8}$	$3.18\cdot10^{-9}$ $3.19\cdot10^{-9}$	$3.80\cdot10^{-10}$ $3.80\cdot10^{-10}$	$4.64\cdot10^{-11}$ $4.64\cdot10^{-11}$	$5.74\cdot10^{-12}$ $5.73\cdot10^{-12}$

to the appearance of a false boundary, which is extremely undesirable in calculations near and at the boundary [27], [26].

BIBLIOGRAPHY

1. M.N. Bakhshaliyeva, E.H. Khalilov. *Justification of the collocation method for an integral equation of the exterior Dirichlet problem for the Laplace equation* // Comput. Math. Math. Phys. **61**:6, 923–937 (2021). https://doi.org/10.1134/S0965542521030039
2. I.S. Berezin, N.P. Zhidkov. *Computing methods. Vol. I.* GIFML, Moscow (1982); *English translation:* Pergamon Pres, Oxford (1965).
3. C.A. Brebbia, J.C.F. Telles, L.A. Wrobel. *Boundary element techniques. Theory and applications in engineering* Springer–Verlag, Berlin (1984). https://doi.org/10.1007/978-3-642-48860-3
4. D.Yu. Ivanov. *A refinement of the boundary element collocation method near the boundary of domain in the case of two-dimensional problems of non–stationary heat conduction with boundary conditions of the second and third kind* // Vestn. Tomsk. Gos. Univ., Mat. Mekh. 57, 5–25 (2019). (in Russian). https://doi.org/10.17223/19988621/57/1
5. D.Yu. Ivanov. *A refinement of the boundary element collocation method near the boundary of a two–dimensional domain using semianalytic approximation of the double layer heat potential* // Vestn. Tomsk. Gos. Univ., Mat. Mekh. 65, 30-52 (2020). (in Russian). https://doi.org/10.17223/19988621/65/3

6. D.Yu. Ivanov. *On one semi–analytical approximation of the normal derivative of the simple layer potential near the boundary of a two–dimensional domain* // Vestn. Udmurt. Univ., Mat. Mekh. Komp'yut. Nauki **33**:3, 434–451 (2023). (in Russian). https://doi.org/10.35634/vm220103
7. D.Yu. Ivanov. *On approximation of the normal derivative of the single layer heat potential near the boundary of a two–dimensional domain* // Vestn. Tomsk. Gos. Univ., Mat. Mekh. 83, 31–51 (2023). https://doi.org/10.17223/19988621/83/4
8. D.Yu. Ivanov. *On uniform convergence of semi–analytic solution of Dirichlet problem for dissipative Helmholtz equation in vicinity of boundary of two–dimensional domain* // Ufa Math. J. **15**:4, 76–99 (2023). https://doi.org/10.13108/2023-15-4-76
9. D.Yu. Ivanov. *On one semi–analytical approximation of the normal derivative of the simple layer potential near the boundary of a two–dimensional domain* // Vestn. Udmurt. Univ., Mat. Mekh. Komp'yut. Nauki **33**:3, 434–451 (2023). https://doi.org/10.35634/vm230304
10. D. Colton, R. Kress. *Integral equation methods in scattering theory.* John Wiley & Sons, New York (1983).
11. P.A. Krutitskii, A.D. Fedotova, V.V. Kolybasova. *Quadrature formula for the simple layer potential* // Differ. Equ. **55**:9, 1226–1241 (2019). https://doi.org/10.1134/S0012266119090106
12. P.A. Krutitskii, I.O. Reznichenko. *Quadrature formula for the harmonic double layer potential* // Differ. Equ. **57**:7, 901–920 (2021). https://doi.org/10.1134/S0012266121070077
13. P.A. Krutitskii, I.O. Reznichenko. *Quadrature formula for a double layer potential in the case of the Helmholtz equation* // Comput. Math. Math. Phys. **62**:3, 411–426 (2022). https:// doi.org /10.1134/S0965542522030095
14. P.A. Krutitskii, I.O. Reznichenko. *Improved quadrature formula for a single–layer potential* // Comput. Math. Math. Phys. **63**:2, 187–201 (2023). https://doi.org/10.31857/S0044466923020114
15. I.K. Lifanov. *Method of singular integral equations and numerical experiment.* "Yanus", Moscow (1995).
16. V.G. Maz'ya. *Boundary integral equations* // Enzycl. Math. Sci. **27**, 127–222 (1991).
17. I.O. Reznichenko, P.A. Krutitskii. *Quadrature formula for the direct value of the double–layer potential* // Program. Comput. Softw. **48**:3, 227–233 (2022). https://doi.org/10.1134/S0361768822030094
18. W.I. Smirnow. *Course in higher mathematics. Part IV/2.* Nauka, Moscow (1981); *English translation:* H. Deutsch, Frankfurt am Main (1995).
19. G.M. Fikhtengolts. *Course of differential and integral calculus. Vol. 2.* Fizmatlit, Moscow (2003). (in Russian).
20. G.M. Fikhtengolts. *Course of differential and integral calculus. Vol. 1.* Fizmatlit, Moscow (2003). (in Russian).
21. È.H. Khalilov, M.N. Bakhshalieva. *On the derivative of the double–layer logarithmic potential* // Vestn. Tomsk. Gos. Univ., Mat. Mekh. 62, 38–54 (2019). https://doi.org/10.17223/19988621/62/4
22. È.G. Khalilov. *Quadrature formula for normal derivative of double layer potentia* // Ufa Math. J. **15**:4, 100–111 (2023). https://doi.org/10.13108/2023-15-4-100
23. C. Cheng, D. Pan, Z. Han, M. Wu, Z. Niu. *A state space boundary element method with analytical formulas for nearly singular integrals* // Acta Mech. Solida Sin. **31**:4, 433–444 (2018). https://doi.org/10.1007/s10338-018-0040-8
24. X.W. Gao, K. Yang, J. Wang. *An adaptive element subdivision technique for evaluation of various 2D singular boundary integrals* // Eng. Anal. Bound. Elem. **32**:8, 692–696 (2008). https://doi.org/10.1016/j.enganabound.2007.12.004
25. X.–W. Gao, J.–B. Zhang, B.–J. Zheng, C. Zhang. *Element-subdivision method for evaluation of singular integrals over narrow strip boundary elements of super thin and slender structures* // Eng. Anal. Bound. Elem. **66**, 145–154 (2016). https://doi.org/10.1016/j.enganabound.2016.02.002
26. Y.P. Gong, C.Y. Dong, Y. Bai. *Evaluation of nearly singular integrals in isogeometric boundary element method* // Eng. Anal. Bound. Elem. **75**, 21–35 (2017). https://doi.org/10.1016/j.enganabound.2016.11.005
27. Y. Gu, W. Chen, B. Zhang, W. Qu. *Two general algorithms for nearly singular integrals in two dimensional anisotropic boundary element method* // Comput. Mech. **53**:6, 1223–1234 (2014). https://doi.org/10.1007/s00466-013-0965-1

28. J. Lv, Y. Miao, H. Zhu. *The distance sinh transformation for the numerical evaluation of nearly singular integrals over curved surface elements* // Comput. Mech. **53**:2, 359–367 (2014). https://doi.org/10.1007/s00466-013-0913-0
29. Z. Niu, C. Cheng, H. Zhou, Z. Hu. *Analytic formulations for calculating nearly singular integrals in two-dimensional BEM* // Eng. Anal. Bound. Elem. **31**:12, 949–964 (2007). https://doi.org/10.1016/j.enganabound.2007.05.001
30. Z. Niu, Z. Hu, C. Cheng, H. Zhou. *A novel semi–analytical algorithm of nearly singular integrals on higher order elements in two dimensional BEM* // Eng. Anal. Bound. Elem. **61**, 42–51 (2015). https://doi.org/10.1016/j.enganabound.2015.06.007
31. Y.–M. Zhang, Y. Gu, J.–T. Chen. *Stress analysis for multilayered coating systems using semi-analytical BEM with geometric non–linearities* // Comput. Mech. **47**:5, 493–504 (2011). https://doi.org/10.1007/s00466-010-0559-0
32. Y. Zhang, Y. Gong, X. Gao. *Calculation of 2D nearly singular integrals over high–order geometry elements using the sinh transformation* // Eng. Anal. Bound. Elem. **60**, 144–153 (2015). https://doi.org/10.1016/j.enganabound.2014.12.006

Dmitry Yurievich Ivanov,
Russian University of Transport,
Obraztsova str. 9, bld. 9,
127994, Moscow, Russia
E-mail: ivanovdyu@yandex.ru