ON BASIC SUMMABILITY IN $\mathbb{R}$

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Abstract. The paper deals with the concept of basic summability of residue function of interval function, which is a synonym for its differential form. As one comprehensive concept, it includes not only all known concepts of integrability, such as *Newton*'s, generalized *Riemann* and generalized *Riemann — Stieltjes* integrability, but also arithmetic series.

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1. Introduction

In the middle of the last century, the concept of *Riemann* integral was slightly redefined, which led us to the concept of generalized *Riemann* integrals [1], [2], [4]. At the same time, the notion of absolutely continuous function (AC) was redefined to the concept of functions of negligible variation (NV) [3], [6], [10]. The connection between the indefinite integral and the generalized *Riemann* integral, in the form of the *Newton — Leibniz* formula, can be established only in the case that the primitive function F of the integrand f is an NV function on every null subset E of the compact interval of integration I. Otherwise, if the function F is a function of bounded variation (BV) on I and is not an NV function on every null subset E of I, the cumulative change in the value of the primitive F on I (the value of the associated interval function ΔF on the interval I [12]) differs from the value of the generalized *Riemann* integral of the function f over the interval of integration I. The difference of those values is equal to the sum of the residue function $\Re_{\Delta F}$ of the interval function ΔF on the singularity set S, and the sum of the values of the generalized *Riemann* integral and the sum of the function $\Re_{\Delta F}$ on the set S is the total value of the generalized *Riemann* integral of the function f over I, which is obviously equal to $\Delta F(I)$, unconditionally [7], [8], [9]. In other words, the *Newton — Leibnitz* formula, in which the total value of the *Riemann* integral figures, is valid unconditionally. The value of the residue function $\Re_{\Delta F}$ of interval function ΔF at the point $x \in I$ is the limit $\lim\limits_{I\to x} \Delta F(I)$, so that the limit $\lim\limits_{I\to x} f\mu_I$ is the value of the residue function $\Re_{f\mu_I}$ at the point $x \in I$ of the point-interval function $f\mu_I$, where μ_I is the *Lebesgue* measure on the interval I [9]. Clearly, only in special cases, for the class of ACG functions F [4], the sums of values of functions $\Re_{\Delta F}$ and $\Re_{f\mu_I}$, on the compact interval I, are equal and the total integral of derivative f of primitive F over I is equal to the generalized *Riemann* integral of function f over I. Accordingly, the paper presents the concept of basically summable functions, which is comparable to the concept of functions of bounded variation, see [3], with its peculiarities, which lead us, in case the sum of the residue function exists on a compact set, to the concept of total integrability.

The paper is organized as follows. Section 2 is preliminary. Section 3, as the main one, begins with the definition of the differential form, which is renamed to the residue function, in the second part of this section. Finally, after defining the concept of basic summability, at the end

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of this section the necessary and sufficient conditions are given, in the form of a theorem, for the residue function to be basically summable.

2. Preliminaries

Given a compact interval $[a,b]$ in $\mathbb{R}$, let $\mathcal{I}([a,b])$ be a family of all compact subintervals I of $[a,b]$ and let $\mathcal{P}([a,b])$ be the power set of $[a,b]$. We set $\mathbb{R}_+ := [0,+\infty)$ and $\mathbb{N} := \{1,2,3,\ldots\}$. The interior, closure and boundary of a set $E \in \mathcal{P}([a,b])$ are denoted by $\dot{E}$, $\bar{E}$ and ∂E, respectively, and the characteristic function (the indicator) of E is denoted by $\chi_E : \mathbb{R} \to \{0,1\}$; it is equal to 1 if x is in E, and to 0 if x is not in E. A set function $\mathfrak{G}$ on $\mathcal{P}([a,b])$ is said to be a countably additive if it satisfies the condition

$$\mathfrak{G}\left(\cup_{1\leqslant n} E_n\right) = \sum_{1\leqslant n} \mathfrak{G}\left(E_n\right),$$

for each countable collection $\{E_n\}_{1\leqslant n}$ of pairwise disjoint sets E_n such that $\cup_{1\leqslant n} E_n \in \mathcal{P}([a,b])$. Any countably additive set function $\mathfrak{G}$ on $\mathcal{P}([a,b])$ is in fact a finite signed measure. In the theory of measures a signed measure is sometimes called a charge. Accordingly, by a charge on $\mathcal{P}([a,b])$ we mean a countably additive set function on $\mathcal{P}([a,b])$. Non-negative measures can be thought of as making precise a notion of "size"for sets. The *Lebesgue* measure on $\mathbb{R}$ is denoted by μ, however, for any $I \subset \mathcal{I}([a,b])$ we sometimes write $\Delta x(I)$ or $|I|$ instead of μ_I. If null set is a set of *Lebesgue*measure zero, then the point function f on $[a,b]$ is said to be null function on $[a,b]$, if the set $\{x \in [a,b] \mid f(x) \neq 0\}$ is a null set, see Definition 2.4 in [1]. If $\{x \in [a,b] \mid f(x) \neq 0\}$ is an empty set, then f is a zero function on $[a,b]$. A function, which is a finite combination of set and point functions, connected by elementary operations, all the way to a composition of functions, is called the set–point function. Given $\delta > 0$ on $[a,b]$, named a gauge, an interval–point pair (I,x) is called δ–fine if $I \subset (x-\delta(x), x+\delta(x))$. A countable partition $P[a,b]$ of some compact interval $[a,b] \in \mathbb{R}$ is a countable set of interval–point pairs $([a_n,b_n], x_n)_{1\leqslant n}$ such that the subintervals $[a_n,b_n]$ are non-overlapping, $\cup_{1\leqslant n}[a_n,b_n] = [a,b]$ and $x_n \in [a_n,b_n]$. The points $\{x_n\}_{1\leqslant n}$ are the tags of $P[a,b]$ [1]. It is clear that there are many different ways to arrange the position of the tags x_n with respect to $[a_n,b_n]$. Each of these positions leads to one of a *Riemann* type definition of the generalized *Riemann* integral. If $E \in \mathcal{P}([a,b])$, then the restriction of $P[a,b]$ to E is a countable collection of $([a_n,b_n], x_n) \in P[a,b]$ such that all x_n are tagged in E:

$$P[a,b]\,|_E = \{([a_n,b_n], x_n) \in P[a,b] \mid x_n \in E\}.$$

Let h be an interval–point function on $\mathcal{I}([a,b]) \times [a,b]$ and let $E \in \mathcal{P}([a,b])$. In what follows, a set of values and the sum of the values of h on $P[a,b]|_E \subset P[a,b]$ are denoted by $h(P[a,b]\,|_E)$ and $s(h, P[a,b]\,|_E)$ instead of $\langle h\chi_E\rangle(P[a,b])$ and $s(\langle h\chi_E\rangle, P[a,b])$, respectively:

$$h(P[a,b]\,|_E) = \{h(([a_n,b_n], x_n)) \mid ([a_n,b_n], x_n) \in P[a,b]\,|_E\},$$

as well as

$$s(h, P[a,b]\,|_E) = \sum_{([a_n,b_n],x_n)\in P[a,b]|_E} h(([a_n,b_n], x_n)).$$

By $\rho(x,y)$ we denote the distance between two points x and y in $\mathbb{R}$, then

$$\rho(x,E) := \inf\{\rho(x,y) \mid y \in E\}$$

is the distance between x and a set $E \in \mathcal{P}([a,b])$. For a point function f on $[a,b]$ and the *Lebesgue* measure μ, throughout the rest of the paper, the interval–point function $\langle \mu f\rangle$ is denoted by $\mathfrak{F}$.

Definition 2.1. *Given a compact interval* $[a,b]$ *in* $\mathbb{R}$, *let* $E \in \mathcal{P}([a,b])$. *A countable* δ*–fine partition* $P([a,b])$ *is said to be fully tagged in* $\bar{E}$ *if a gauge* $\delta > 0$ *on* $[a,b]$ *satisfies the condition* $\delta(x) \leqslant \rho(x, \partial\bar{E})$ *on* $[a,b] \setminus \partial\bar{E}$.

The following lemma is a form of the well–known Vitali Covering Lemma (VCL) related to a countable δ–fine partition of a compact interval $[a,b]$ in $\mathbb{R}$ fully tagged in a measurable set $E \in \mathcal{P}([a,b])$ with *Lebesgue* measure $\mu_E < (b-a)$.

Lemma 2.1. *For a compact interval $[a,b]$ in $\mathbb{R}$ let $\mathcal{V}_{[a,b]}$ be a Vitali cover of $[a,b]$ and let $E \in \mathcal{P}([a,b])$ be a measurable set having Lebesgue measure $\mu_E < (b-a)$. Then for every $\varepsilon \in (0, b-a-\mu_E)$ there exist a gauge $\delta_\varepsilon > 0$ on $[a,b]$ and a countable sequence $\left\{I_n \mid I_n \in \mathcal{V}_{[a,b]}\right\}_{1\leqslant n}$ of non–overlapping intervals I_n such that*

$$s\left(\mu, P[a,b]\,|_{[a,b]\setminus \bar{E}}\right) > (1-\varepsilon')(b-a), \tag{2.1}$$

where $\varepsilon'(b-a) = \varepsilon + \mu_E$, $P[a,b]\,|_{[a,b]\setminus \bar{E}} \subset P[a,b]$ and $P[a,b]$ is a countable δ_ε–fine partition of $[a,b]$ fully tagged in $\bar{E}$.

Proof. Let $\mathcal{V}_{[a,b]}$ be a *Vitali* cover of $[a,b]$ and let $E \subset [a,b]$ be a set of *Lebesgue* measure $\mu_E < (b-a)$. For every $\varepsilon \in (0, b-a-\mu_E)$ there is an open set $\mathcal{O}_\varepsilon$, such that $\bar{E} \subset \mathcal{O}_\varepsilon$ and $\mu(\mathcal{O}_\varepsilon) < \varepsilon + \mu_E$. Then for every $\varepsilon \in (0, b-a-\mu_E)$ we choose a gauge $\delta_\varepsilon > 0$ on $[a,b]$ such that $\delta_\varepsilon(x) \leqslant \rho(x, \partial\bar{E})$ on $[a,b] \setminus \partial\bar{E}$ and $\delta_\varepsilon(x) \leqslant \rho(x, \partial\mathcal{C}\mathcal{O}_\varepsilon)$ on $\partial\bar{E}$, where $\mathcal{C}\mathcal{O}_\varepsilon$ is the closure of $\mathcal{O}_\varepsilon$. For the relative complement $[a,b]/\bar{E}$ of $\bar{E}$, with respect to $[a,b]$, we take a collection of intervals $I \in \mathcal{V}_{[a,b]}$, which paired with $x \in [a,b]/\bar{E}$ form δ_ε–fine interval-point pairs for every $\varepsilon \in (0, b-a-\mu_E)$, to be a *Vitali* cover $\mathcal{V}_{[a,b]\setminus\bar{E}}$ of $[a,b]\setminus\bar{E}$. If $\mathcal{V}_{\bar{E}} \in \mathcal{V}_{[a,b]}$ is a *Vitali* cover of $\bar{E}$, which means that for every $\varepsilon \in (0, b-a-\mu_E)$ there is a collection of intervals $I \in \mathcal{V}_{\bar{E}}$, which, paired with $x \in \bar{E}$, form δ_ε–fine interval–point pairs, then each countable set

$$\left\{(I_n, x_n) \mid I_n \in \mathcal{V}_{[a,b]\setminus\bar{E}} \cup \mathcal{V}_{\bar{E}} \wedge \cup_{1\leqslant n} I_n = [a,b]\right\}$$

is a countable δ_ε–fine partition $P[a,b]$ of $[a,b]$ fully tagged in $\bar{E}$.

Since

$$s(\mu, P[a,b]) - s\left(\mu, P[a,b]\,|_{[a,b]\setminus\bar{E}}\right) = s\left(\mu, P[a,b]\,|_{\bar{E}}\right) < \mu(\mathcal{O}_\varepsilon) < \varepsilon + \mu_E,$$

where $s(\mu, P[a,b]) = b-a$, it follows that for every $\varepsilon \in (0, b-a-\mu_E)$ there exists a countable δ_ε–fine subpartition $P[a,b]\,|_{[a,b]\setminus\bar{E}}$ consisting of the non–overlapping intervals $I_n \in \mathcal{V}_{[a,b]\setminus\bar{E}} \subset \mathcal{V}_{[a,b]}$ such that

$$s\left(\mu, P[a,b]\,|_{[a,b]\setminus\bar{E}}\right) > (1-\varepsilon')(b-a),$$

where $\varepsilon'(b-a) = \varepsilon + \mu_E$. The proof is complete. □

If $E \in \mathcal{P}([a,b])$ is a measurable set having *Lebesgue* measure $\mu_E < (b-a)$, then for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon > 0$ on $[a,b]$, such that any δ_ε–fine partition $P[a,b]$ of $[a,b]$ consists of two countable subpartitions, one of which is $P[a,b]\,|_{[a,b]\setminus\bar{E}}$ consisting of, generally speaking, countably many countable δ_ε–fine partitions, and the other is $P[a,b]\,|_{\bar{E}}$ fully tagged in $\bar{E}$. Thus, in what follows, unless otherwise stated, without loss of generality, a set E is supposed to be closed.

3. The concept of basic summability

For some compact interval $[a,b]$ in $\mathbb{R}$, let $E \in \mathcal{P}([a,b])$ and let h and k be two interval–point functions on $\mathcal{I}([a,b]) \times [a,b]$. We begin with the definition of the differential form on E of $\langle h-k\rangle$ denoted by $\delta\langle h-k\rangle$.

Definition 3.1. *Given a compact interval $[a,b]$ in $\mathbb{R}$, let $E \in \mathcal{P}([a,b])$ and let h and k be two interval–point functions on $\mathcal{I}([a,b]) \times [a,b]$. The point function $\delta\langle h-k\rangle$ on E is a differential form on E of $\langle h-k\rangle$ if for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon > 0$ on $[a,b]$ such that*

$$\left|\left(\langle h-k\rangle - \delta\langle h-k\rangle\right)(P[a,b]|_E)\right| < \varepsilon, \tag{3.1}$$

whenever $P[a,b]|_E \subset P[a,b]$ *and* $P[a,b]$ *is a countable* δ_ε*–fine partition of* $[a,b]$ *fully tagged in* E*. Here* $\delta\langle h-k\rangle = \lim\limits_{\delta_\varepsilon\to 0^+}\langle h-k\rangle$.

In the case $|\delta\langle h-k\rangle| < +\infty$ on E we say that h and k are differentially comparable on E. If, in addition, $\delta\langle h-k\rangle$ is identically zero on E, then h and k are differential equivalents on E. For $E \in \mathcal{P}\,[a,b]$, let ς be a strictly positive charge on $\mathcal{I}\,[a,b]$ such that its differential form $\delta\varsigma$ is identically zero on $[a,b]$. If the set of values on E of the point function $\delta_\varsigma\langle h-k\rangle$, as the limit $\lim\limits_{\delta_\varepsilon\to 0^+}\langle\langle h-k\rangle/\varsigma\rangle$, is a set of defined values, which means that $|\delta_\varsigma\langle h-k\rangle| < +\infty$ on E, then h and k are derivatively comparable on E with respect to ς. For a measurable set $E \subset [a,b]$ having *Lebesgue* measure $\mu_E < (b-a)$ the next definition is that of derivative equivalence on E for h and k, with respect to ς.

Definition 3.2. *For a compact interval* $[a,b]$ *in* $\mathbb{R}$*, let* $E \subset [a,b]$ *be a measurable set having Lebesgue measure* $\mu_E < (b-a)$*, let* h *and* k *be two interval-point functions on* $\mathcal{I}\,([a,b])\times[a,b])$ *and let* ς *be a strictly positive charge on* $\mathcal{I}\,[a,b]$ *such that its differential form* $\delta\varsigma$ *is identically zero on* $[a,b]$*. Then the interval–point functions* h *and* k *are said to be derivative equivalents on* $[a,b]\setminus E$ *with respect to* ς *if for every* $\varepsilon > 0$ *there exists a gauge* $\delta_\varepsilon > 0$ *on* $[a,b]$ *such that*

$$\left|\langle h-k\rangle\left(P[a,b]|_{[a,b]\setminus E}\right)\right| < \varepsilon\varsigma(P[a,b]|_{[a,b]\setminus E}), \tag{3.2}$$

whenever $P[a,b]|_{[a,b]\setminus E} \subset P[a,b]$ *and* $P[a,b]$ *is a countable* δ_ε*–fine partition of* $[a,b]$ *fully tagged in* E*. Here*

$$\delta_\varsigma\langle h-k\rangle = \lim_{\delta_\varepsilon\to 0^+}\langle\langle h-k\rangle/\varsigma\rangle \equiv 0.$$

It is clear that for each interval–point function h the point function $\delta_\mu h$ on $[a,b]$, as the limit $\lim\limits_{\delta_\varepsilon\to 0^+}\langle h/\mu\rangle$, is the derivative function on $[a,b]$ of h. Of course, the point function f is a derivative on $[a,b]$ of h if f is the derivative of h at every point $x \in [a,b]$. The interval–point function h is said to be differentiable on $[a,b]$ to the point function f if h has f as a derivative on $[a,b]$.

On the other hand, it is obviously possible that on some subset of $[a,b]$, where h and k are or are not, it is irrelevant, differential equivalents, but are not derivative equivalents, they are differentially comparable, which means that the limit $\delta\langle h-k\rangle$ is defined on this subset. Such a set is a set of singularities defined below.

Definition 3.3. *For a compact interval* $[a,b]$ *in* $\mathbb{R}$*, let* $S \subset [a,b]$ *be a measurable set having Lebesgue measure* $\mu_S < (b-a)$*, let* h *and* k *be two interval–point functions on* $\mathcal{I}\,([a,b])\times[a,b])$ *and let* ς *be a strictly positive charge on* $\mathcal{I}\,[a,b]$ *such that its differential form* $\delta\varsigma$ *is identically zero on* $[a,b]$*. The set* S *is said to be a set of singularities of* $\langle h-k\rangle$*, with respect to* ς *if for every* $\varepsilon > 0$ *there exists a gauge* $\delta_\varepsilon > 0$ *on* $[a,b]$ *such that*

$$\varepsilon\varsigma(P[a,b]|_S) \leqslant |\langle h-k\rangle\,(P[a,b]|_S)|\,, \tag{3.3}$$

whenever $P[a,b]|_S \subset P[a,b]$ *and* $P[a,b]$ *is a countable* δ_ε*–fine partition of* $[a,b]$ *fully tagged in* S*.*

As previously stated, if the point function $\delta\,\langle h-k\rangle$ is identically zero on the set S, then h and k are differential equivalents on S. However, if h and k are not derivative equivalents on S, the previous inequality (3.3) becomes the double inequality

$$\varepsilon\varsigma(P[a,b]|_S) \leqslant |\langle h-k\rangle\,(P[a,b]|_S)| < \varepsilon. \tag{3.4}$$

In case the point function $\delta\,\langle h-k\rangle$ is a nonzero function on the set S, and h and k are differentially comparable on S, then the following double inequality is satisfied

$$\varepsilon\varsigma(P[a,b]|_S) \leqslant |\langle\langle h-k\rangle - \delta\langle h-k\rangle\rangle\,(P[a,b]|_S)| < \varepsilon, \tag{3.5}$$

and the singularity set S is a set of removable singularities of $\langle h-k\rangle$ with respect to ς.

For an interval $[a,b]$ in $\mathbb{R}$ let h be an interval–point function on $\mathcal{I}([a,b])\times[a,b])$. The absolute sum $\sigma_{\delta|h|}([a,b])$ over $[a,b]$ of the point function $\delta|h|$, which is a limit on $[a,b]$ of $|h|$, is defined to be $\lim\limits_{\delta\to 0^+} s(|h|,P[a,b])$. If the set E is a set in $[a,b]$ with indicator χ_E, then $\sigma_{\langle\delta|h|\chi_E\rangle}([a,b])$ is the absolute sum of δh over E. In what follows we write $\sigma_{\delta|h|}(E)$ instead of $\sigma_{\langle\delta|h|\chi_E\rangle}([a,b])$. If $\sigma_{\delta|h|}(E)$ has a finite value, then we say that δh is absolutely summable (AS) over E. Hence, $\delta h\in AS([a,b])$ if $\sigma_{\delta|h|}([a,b])<+\infty$, and if $\sigma_{\delta|h|}([a,b])\leqslant+\infty$, then δh is said to be absolutely extendedly summable ($A_{ex}S$) over $[a,b]$. Define

$$\|\delta h\|_{AS}=\delta|h|+\sigma_{\delta|h|}([a,b]).$$

One can show that with the usual operations of addition and scalar multiplication of functions that $(AS([a,b]),+,\cdot)$ is a vector space and $\|\cdot\|_{AS}$ is a norm on it, so that $(AS([a,b]),+,\cdot,\|\cdot\|_{AS})$ is a normed vector space. Therefore, the space $AS([a,b])$ is the space of absolutely summable functions. If $\sigma_{\delta|h|}(E)=0$, then δh is negligibly absolutely summable over E. Now, let $\delta h\in AS([a,b])$. Then δh is a null function on $[a,b]$. If, in addition, δh is negligibly absolutely summable either over each set $E\in\mathcal{P}([a,b])$ or over $[a,b]$, then δh is identically zero on $[a,b]$.

Definition 3.4. *For a compact interval $[a,b]$ in $\mathbb{R}$, let h be an interval–point function on $\mathcal{I}([a,b])\times[a,b])$ and let $E\in\mathcal{P}([a,b])$. The real number $\sigma_{\delta h}(E)$ is a basic sum over E of the limit δh of h, if for every $\varepsilon>0$ there exists a gauge $\delta_\varepsilon>0$ on $[a,b]$ such that*

$$|s(h,P[a,b]|_E)-\sigma_{\delta h}(E)|<\varepsilon, \tag{3.6}$$

whenever $P[a,b]|_E\subset P[a,b]$ and $P[a,b]$ is a countable δ_ε–fine partition of $[a,b]$ fully tagged in E. Here,

$$\sigma_{\delta h}(E)=\lim_{\delta_\varepsilon\to 0^+} s(h,P[a,b]|_E)=(B)\sum_{x\in[a,b]}\langle\delta h\chi_E\rangle(x).$$

Obviously, if $s(|h|,P[a,b]|_E)$ replaces $s(h,P[a,b]|_E)$ in (3.6), then Definition 3.4 becomes the definition of the absolute sum $\sigma_{\delta|h|}$ over E of $\delta|h|$. In the case $\sigma_{\delta h}(E)$ has a finite value, which means that $|\sigma_{\delta h}(E)|<+\infty$, we say that δh is basically summable (BS) over E. We define

$$\|\delta h\|_{BS}:=|\delta h|+|\sigma_{\delta h}([a,b])|.$$

Then, the space $(BS([a,b]),+,\cdot)$ of basically summable functions is a vector space and $\|\cdot\|_{BS}$ is a norm on it, so that $(BS([a,b]),,+,\cdot\|\cdot\|_{BS})$ is a normed vector space. If $|\sigma_{\delta h}(E)|\leqslant+\infty$, then δh is basically extendedly summable ($B_{ex}S$) over E. In the case $\sigma_{\delta h}(E)=0$ we say that δh is negligibly basically summable over E. The differential form δh is said to be negligibly basically summable on $[a,b]$ if and only if $\sigma_{\delta h}(E)=0$ for every $E\in\mathcal{P}([a,b])$.

Some point functions may be negligibly basically summable over a set, but not negligibly absolutely summable over that set. A point function, which is negligibly absolutely summable over a set, is also negligibly absolutely summable over all its subsets; this does not apply to a function, which is negligibly basically summable over a set, but not on a set.

Two differential forms δh and δk are absolutely (basically) summable equivalents over a set if $\delta\langle h-k\rangle$ is negligibly absolutely (basically) summable over that set. If δh and δk are absolutely summable equivalents over $[a,b]$, then $\delta\langle h-k\rangle$ is identically zero on $[a,b]$ and this means that h and k are differential equivalents on $[a,b]$. In the general case the opposite is not true.

A statement is absolutely (basically) true δh–almost everywhere if it is true everywhere except in a set E, with

$$\sigma_{\delta|h|}(E)=0,\qquad \sigma_{\delta h}(E)=0.$$

Such a set is said to be absolutely (basically) a δh–negligible set. If h is replaced by μ, then E is a negligible set. Clearly, each negligible set is a null set. Theabove defined notion of the basic sum $\sigma_{\delta h}$ of the limit δh of h, leads us to the identity

$$\begin{aligned}\lim_{\delta\to 0^+}\sum_{(I,x)\in P[a,b]|_E} h\left(P[a,b]|_E\right) &= \lim_{\delta\to 0^+} s\left(h, P[a,b]|_E\right)\\ &=\sigma_{\delta h}\left(E\right) = (B)\sum_{x\in[a,b]}\left\langle \delta h\chi_E\right\rangle(x) \\ &= (B)\sum_{x\in E}\lim_{\delta\to 0^+} h\left(P[a,b]|_E\right),\end{aligned} \tag{3.7}$$

which explicitly shows us that the limit of the sum of the values of h is equal to the sum of the values of its limit δh.

The next thing we are going to look at is the *Cauchy* criterion for the basic sum. This theorem is important because it allows us to prove that a certain interval–point function is basically summable without knowing the value of the basic sum. It will be used in some proofs later on.

Theorem 3.1. *For a compact interval $[a,b]$ in $\mathbb{R}$, let h be an interval–point function on $\mathcal{I}\left([a,b]\right)\times[a,b])$. The differential form δh, as a limit on $[a,b]$ of h, is basically summable over $[a,b]$ if and only if for every $\varepsilon>0$ there exists a gauge $\delta_\varepsilon>0$ on $[a,b]$ such that*

$$\left|s\left(h, P_1[a,b]\right) - s\left(h, P_2[a,b]\right)\right| < \varepsilon, \tag{3.8}$$

whenever $P_1\left[a,b\right]$ and $P_2\left[a,b\right]$ are countable δ_ε–fine partitions of $[a,b]$.

Proof. Let δh be basically summable over $[a,b]$. Accordingly, for every $\varepsilon>0$ there exists a gauge $\delta_\varepsilon>0$ on $[a,b]$, such that

$$\left|s\left(h, P_1[a,b]\right) - \sigma_{\delta h}\left([a,b]\right)\right| < \frac{\varepsilon}{2}, \qquad \left|s\left(h, P_2[a,b]\right) - \sigma_{\delta h}\left([a,b]\right)\right| < \frac{\varepsilon}{2},$$

whenever $P_1\left[a,b\right]$ and $P_2\left[a,b\right]$ are countable δ_ε–fine partitions of $[a,b]$. Hence,

$$\begin{aligned}\left|s\left(h, P_1[a,b]\right) - s\left(h, P_2[a,b]\right)\right| \leqslant&\left|s\left(h, P_1[a,b]\right) - \sigma_{\delta h}\left([a,b]\right)\right| + \left|s\left(h, P_2[a,b]\right) - \sigma_{\delta h}\left([a,b]\right)\right|\\ <&\frac{\varepsilon}{2}+\frac{\varepsilon}{2}<\varepsilon.\end{aligned}$$

And vice versa, assume that for every $\varepsilon>0$ there exists a gauge $\delta_\varepsilon>0$ on $[a,b]$ such that

$$\left|s\left(h, P_1[a,b]\right) - s\left(h, P_2[a,b]\right)\right| < \varepsilon,$$

whenever $P_1\left[a,b\right]$ and $P_2\left[a,b\right]$ are countable δ_ε-fine partitions of $[a,b]$. For each $n\in\mathbb{N}$ we choose a gauge $\delta_n>0$ on $[a,b]$ such that

$$\left|s\left(h, P_1[a,b]\right) - s\left(h, P_2[a,b]\right)\right| < \frac{1}{n},$$

whenever $P_1\left[a,b\right]$ and $P_2\left[a,b\right]$ are δ_n–fine. Let the sequence $\{\delta_n\}$ be non–increasing and let $P_n[a,b]$ be a countable δ_n–fine partition of $[a,b]$ for each $n\in\mathbb{N}$.

If $m>n\geqslant N$, then $\delta_N\geqslant\delta_n\geqslant\delta_m$. Now, $P_n[a,b]$ is δ_n–fine and thus also δ_N–fine. The same holds for $P_m[a,b]$: $P_m[a,b]$ is δ_m–fine and thus also δ_N–fine. It implies that

$$\left|s\left(h, P_n[a,b]\right) - s\left(h, P_m[a,b]\right)\right| < \frac{1}{N},$$

for $m>n\geqslant N$. So, the sequence $\{s\left(h,P_n[a,b]\right)\}$ is a *Cauchy* sequence. Further, let $\sigma_{\delta h}\left([a,b]\right)$ be the limit of this sequence and let $\varepsilon>0$. Choose an integer N such that $\frac{1}{N}<\frac{\varepsilon}{2}$ and

$$\left|s\left(h, P_n[a,b]\right) - \sigma_{\delta h}\left([a,b]\right)\right| < \frac{\varepsilon}{2}$$

for all $n \geqslant N$. Therefore, for each countable δ_ε–fine partitions $P[a,b]$ of $[a,b]$

$$|s(h,P[a,b]) - \sigma_{\delta h}([a,b])| \leqslant |s(h,P[a,b]) - s(h,P_n[a,b])| + + |s(h,P_n[a,b]) - \sigma_{\delta h}([a,b])| < \frac{1}{N} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon.$$

The proof is complete. □

If we have a differential form δh basically summable over an interval $[a,b]$, then intuitively we suppose that it is also basically summable over every subinterval $I \in \mathcal{I}([a,b])$. It turns out that this statement is indeed true. This is the content of the following lemma. Of course, we are going to use the previous theorem, the *Cauchy* criterion, to prove this.

Lemma 3.1. *For a compact interval $[a,b]$ in $\mathbb{R}$, let h be an interval–point function on $\mathcal{I}([a,b]) \times [a,b])$. If the differential form δh, as a limit on $[a,b]$ of h, is basically summable over $[a,b]$, then δh is basically summable over every compact interval $I \in \mathcal{I}([a,b])$.*

Proof. Let $\varepsilon > 0$. Choose a gauge $\delta_\varepsilon > 0$ on $[a,b]$ such that

$$|s(h,P_1[a,b]) - s(h,P_2[a,b])| < \frac{\varepsilon}{2},$$

whenever $P_1[a,b]$ and $P_2[a,b]$ are countable δ_ε–fine partitions of $[a,b]$. We can do this owing to the *Cauchy* criterion. For arbitrary points c and d in (a,b), choose countable δ_ε–fine partitions $P_c[a,c]$ and $P_d[d,b]$. Let $\tilde{P}_1[c,d]$ and $\tilde{P}_2[c,d]$ be two countable δ_ε–fine partitions of $[c,d]$. We define

$$P_1[a,b] := P_c[a,c] \cup \tilde{P}_1[c,d] \cup P_d[d,b]$$

and

$$P_2[a,b] := P_c[a,c] \cup \tilde{P}_2[c,d] \cup P_d[d,b].$$

Obviously, $P_1[a,b]$ and $P_2[a,b]$ are countable δ_ε–fine partitions of $[a,b]$ and

$$\left| s(h,\tilde{P}_1[c,d]) - s(h,\tilde{P}_2[c,d]) \right| = |s(h,P_1[a,b]) - s(h,P_2[a,b])| < \varepsilon.$$

By using the *Cauchy* criterion again, we conclude that δh is basically summable over $I = [c,d]$. The proof is complete. □

If h is a charge $\mathfrak{S}$ on $\mathcal{P}([a,b])$, we need to define a total differential form (briefly, a total differential) on a set $E \in \mathcal{P}([a,b])$ of $\mathfrak{S}$.

Definition 3.5. *For a compact interval $[a,b]$ in $\mathbb{R}$, let $\mathfrak{S}$ be a charge on $\mathcal{P}([a,b])$. The differential form $\delta\mathfrak{S}$, as a limit on $[a,b]$ of $\mathfrak{S}$, is said to be a total differential on $[a,b]$ if it is basically summable over $[a,b]$ and $\sigma_{\delta\mathfrak{S}} = \mathfrak{S}$ on $\mathcal{P}([a,b])$.*

The next lemma, as a logical link in the chain, points to the fact that summation and differentiation are, in general, two inverse operations, for the reason that $\delta\sigma_{\delta h}$ and δh are basically summable equivalents on $[a,b]$. It is an alternative form of the well–known lemma in the theory of the generalized *Riemann* integrals called the Straddle Lemma [1].

Lemma 3.2. *For a compact interval $[a,b]$ in $\mathbb{R}$, let h be an interval–point function on $\mathcal{I}([a,b]) \times [a,b])$. If the differential form δh, as a limit on $[a,b]$ of h, is basically summable over $[a,b]$, then $\delta\sigma_{\delta h}$ and δh are basically summable equivalents on $[a,b]$.*

Proof. Let h be an interval–point function on $\mathcal{I}([a,b]) \times [a,b])$, the limit δh of which on $[a,b]$ is basically summable over $[a,b]$ and let $I \in \mathcal{I}([a,b])$. Since

$$s(\sigma_{\delta h}, P_I) = \sigma_{\delta h}(I)$$

for any countable partition P_I of I, it follows, from Lemma 3.1 that for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon > 0$ on $[a, b]$ such that

$$|s(\langle h - \sigma_{\delta h}\rangle, P_I)| = |s(h, P_I) - \sigma_{\delta h}(I)| < \varepsilon,$$

whenever P_I is a countable δ_ε–fine partition of I. Therefore, $\sigma_{\delta\langle h-\sigma_{\delta h}\rangle}(I) = 0$ for every $I \in \mathcal{I}([a, b])$, and this means that $\delta\sigma_{\delta h}$ and δh are basically summable equivalents on $[a, b]$. □

As was noted above, the limit of an interval–point function is a point function. A *Baire* class consists of functions, which are limits of convergent interval functions [4], [5]. Thus, if h is an interval function ΔF, associated to a point function F, being continuous and has bounded variation, then $\sigma_{\delta|\mathcal{F}|}$ is the total variational measure induced by $\mathcal{F} = \Delta F$, more precisely, the *Lebesgue-Stieltjes* measure [10], [11]. Accordingly, we can take the absolute value of the basic sum $|\sigma_{\delta h}|$ and the absolute sum $\sigma_{\delta|h|}$ to be the basic summary measure induced by h and the absolute summary measure induced by h, respectively. We recall that by a charge on $\mathcal{P}([a, b])$ we mean a countably additive set function on $\mathcal{P}([a, b])$. Hence, both $|\sigma_{\delta h}|$ and $\sigma_{\delta|h|}$ are charges on $\mathcal{P}([a, b])$. Additionally, the signed summary measure $\sigma_{\delta h}$ induced by h is also a charge on $\mathcal{P}([a, b])$. Let $E \in \mathcal{P}([a, b])$. If a charge $\mathcal{F}$ on $\mathcal{I}([a, b])$ has the property: *each negligible subset of E is at the same time absolutely a $\delta\mathcal{F}$-negligible set too.* Then $\delta\mathcal{F}$ is a zero function on E, and $\mathcal{F}$ is an absolutely continuous (AC_δ) function, that is, $\mathcal{F}$ satisfies the*Strong Lusin SL condition* [5] on the set E. If, in addition, the set E is a countable union of sets, on each of which $\mathcal{F}$ is AC_δ, then $\mathcal{F}$ is generalized absolutely continuous (ACG_δ) on the set E, [4], [11]. Due to the Straddle Lemma, if $\delta\mathcal{F}$, as a limit on $[a, b]$ of $\mathcal{F}$, is basically summable over $[a, b]$, then the limit $\delta\langle\mathcal{F} - \sigma_{\delta\mathcal{F}}\rangle$ is identically zero on $[a, b]$.

The following lemma is a slightly modified form of the well–known lemma in the theory of the generalized *Riemann* integrals called the Saks — Henstock Lemma [4].

Lemma 3.3. *For a compact interval $[a, b]$ in $\mathbb{R}$, let $E \subset [a, b]$. If two differential forms δh and δk are basically summable equivalents on $[a, b]$, then the set E is both basically and absolutely a $\delta\langle h - k\rangle$–negligible set.*

Proof. Let δh and δk be basically summable equivalents on $[a, b]$ and let $E \subset [a, b]$. Then, by Definition 3.4, for every $\varepsilon > 0$ there exists a gauge $\delta_\varepsilon > 0$ on $[a, b]$ such that

$$|s(\langle h - k\rangle, P[a, b])| < \varepsilon,$$

whenever $P[a, b]$ is a countable δ_ε–fine partition of $[a, b]$ fully tagged in E.

Let

$$P_0 = \{([a_{\widehat{n}}, b_{\widehat{n}}], x_{\widehat{n}}) \in P[a, b] \mid x_{\widehat{n}} \in E\} = P[a, b]|_E.$$

The set $[a, b] \setminus \cup_{\widehat{n}} (a_{\widehat{n}}, b_{\widehat{n}})$ consists of a countably many pairwise disjoint closed intervals $S_{\widetilde{n}}$. Since δh and δk are basically summable equivalents on $[a, b]$, they are basically summable equivalents on each $S_{\widetilde{n}}$, too. Hence, there exists a countable δ_ε–fine partition $P_{\widetilde{n}}$ of each $S_{\widetilde{n}}$ such that

$$|s(\langle h - k\rangle, \delta_\varepsilon, P_{\widetilde{n}})| < \frac{\varepsilon}{2^{\widetilde{n}}}.$$

Then, the partition $P = P_0 \cup P_1 \; \ldots \; \cup P_{\widetilde{n}} \cup \; \ldots$ is a countable δ_ε–fine partition of $[a, b]$ fully tagged in E. Accordingly, we have

$$\begin{aligned}|s(\langle h - k\rangle, P_0)| &= \left| s(\langle h - k\rangle, P_0) + \sum_{1 \leqslant \widetilde{n}} s(\langle h - k\rangle, P_{\widetilde{n}}) - \sum_{1 \leqslant \widetilde{n}} s(\langle h - k\rangle, P_{\widetilde{n}}) \right| \\ &\leqslant |s(\langle h - k\rangle, P)| + \sum_{1 \leqslant \widetilde{n}} |s(\langle h - k\rangle, P_{\widetilde{n}})| < \varepsilon + \sum_{1 \leqslant \widetilde{n}} \frac{\varepsilon}{2^{\widetilde{n}}} < 2\varepsilon.\end{aligned}$$

By Definition 3.4, $\sigma_{\delta\langle h-k\rangle}(E) = 0$.

To address the second part of the lemma, let P_0^+ be the collection of the interval point pairs $([a_{\hat{n}}, b_{\hat{n}}], x_{\hat{n}})$, fully tagged in E, such that

$$\langle h-k\rangle\left(\left(\left[a_{\hat{n}}, b_{\hat{n}}\right], x_{\hat{n}}\right)\right) \geqslant 0$$

since $P_0^- \leqslant 0$. Then

$$0 \leqslant s\left(\langle h-k\rangle, P_0^+\right)=\sum_{([a_{\hat{n}}, b_{\hat{n}}], x_{\hat{n}}) \in P_0^+}|\langle h-k\rangle|\left(\left(\left[a_{\hat{n}}, b_{\hat{n}}\right], x_{\hat{n}}\right)\right)<2 \varepsilon,$$

$$-\left[s\left(\langle h-k\rangle, P_0^-\right)\right]=\sum_{([a_{\hat{n}}, b_{\hat{n}}], x_{\hat{n}}) \in P_0^-}|\langle h-k\rangle|\left(\left(\left[a_{\hat{n}}, b_{\hat{n}}\right], x_{\hat{n}}\right)\right)<2 \varepsilon.$$

Therefore,

$$s\left(|\langle h-k\rangle|, P_0\right)=\sum_{([a_{\hat{n}}, b_{\hat{n}}], x_{\hat{n}}) \in P_0}|\langle h-k\rangle|\left(\left(\left[a_{\hat{n}}, b_{\hat{n}}\right], x_{\hat{n}}\right)\right)<4 \varepsilon,$$

that is, $\sigma_{\delta|\langle h-k\rangle|}(E)=0$. The proof is complete. □

We recall that if some point function is negligibly absolutely summable over a set, it is negligibly absolutely summable over all its subsets, too. Accordingly, it follows from the Saks — Henstock Lemma that if δh and δk are basically summable equivalents on $[a, b]$, then δh and δk are absolutely summable equivalents over $[a, b]$. Hence, a corollary of Lemma 3.3 is as follows.

Corollary 3.1. *Let* $[a, b]$ *be a compact interval in* $\mathbb{R}$. *Differential forms* δh *and* δk *are basically summable equivalents on* $[a, b]$, *if and only if they are absolutely summable equivalents over* $[a, b]$.

The next two lemmas are also very important in the theory of the generalized *Riemann* integrals.

Lemma 3.4. *For a point function* f *on* $[a, b]$, *let* $E \subset[a, b]$ *be a set on which* $f \neq 0$, *and let* δh *be a differential form on* $[a, b]$ *of* h. *The set* E *is absolutely a* δh*–negligible set if and only if* E *is absolutely a* $\delta\langle f h\rangle$*–negligible set.*

Proof. Let f be a finite–valued point function, which does not vanish on $E \subset[a, b]$. Then, for each $\hat{n} \in \mathbb{N}$, let $E_{\hat{n}}=\{x \in E \mid |f| \geqslant 1 / \hat{n}\}$. Fix $\hat{n}$ and let $\varepsilon=\frac{\varepsilon^{\prime}}{\left(2^{\hat{n}} \hat{n}\right)}>0$. By the *Archimedean* property of $\mathbb{R}$ we have

$$E=\bigcup_{\hat{n}=1}^{+\infty} E_{\hat{n}}.$$

If E is absolutely a δh–negligible set, then each $E_{\hat{n}}$ is absolutely a δh–negligible set and there exists a gauge $\delta_{\varepsilon}>0$ on $[a, b]$ such that

$$s\left(|h|, P[a, b]|_{E_{\hat{n}}}\right)<\frac{\varepsilon^{\prime}}{2^{\hat{n}} \hat{n}},$$

whenever $P[a, b]|_{E_{\hat{n}}} \subset P([a, b])$ and $P([a, b])$ is a countable δ_{ε}–fine partition of $[a, b]$, fully tagged in E, that is, there exists a gauge $\delta_{\varepsilon}>0$ on $[a, b]$ such that

$$s\left(|\langle f h\rangle|, P[a, b]|_E\right) \leqslant \sum_{1 \leqslant \hat{n}} \hat{n} s\left(|h|, P[a, b]|_{E_{\hat{n}}}\right)<\sum_{1 \leqslant \hat{n}} \frac{\varepsilon^{\prime}}{2^{\hat{n}}}<\varepsilon^{\prime},$$

whenever $P[a, b]|_E \subset P([a, b])$ and $P([a, b])$ is a countable δ_{ε}–fine partition of $[a, b]$ fully tagged in E. Hence, $\sigma_{\delta|\langle f h\rangle|}(E)=0$. Secondly, let E be absolutely a $\delta\langle f h\rangle$-negligible set. Then there exists a gauge $\delta_{\varepsilon}>0$ on $[a, b]$ such that

$$s\left(|\langle f h\rangle|, P[a, b]|_{E_{\hat{n}}}\right)<\frac{\varepsilon^{\prime}}{2^{\hat{n}} \hat{n}},$$

whenever $P[a,b]|_{E_{\hat{n}}} \subset P([a,b])$ and $P([a,b])$ is a countable δ_ε–fine partition of $[a,b]$ fully tagged in E, that is, there exists a gauge $\delta_\varepsilon > 0$ on $[a,b]$ such that

$$s\left(|h|, P[a,b]|_E\right) \leqslant \sum_{1\leqslant \hat{n}} \hat{n}s\left(|\langle fh\rangle|, P[a,b]|_{E_{\hat{n}}}\right) < \sum_{1\leqslant \hat{n}} \frac{\varepsilon'}{2^{\hat{n}}} < \varepsilon',$$

whenever $P[a,b]|_E \subset P([a,b])$ and $P([a,b])$ is a countable δ_ε–fine partition of $[a,b]$ fully tagged in E. This implies $\sigma_{\delta|h|}(E) = 0$. The proof is complete. □

The proof of the next lemma is based on the trivial inequalities

$$\sigma_{\delta|\langle h-k\rangle|}([a,b]) \leqslant \sigma_{\delta|h|}([a,b]) + \sigma_{\delta|k|}([a,b])$$

and

$$\sigma_{\delta|h|}([a,b]) \leqslant \sigma_{\delta|\langle h-k\rangle|}([a,b]) + \sigma_{\delta|k|}([a,b])$$

and is trivial. This is why we formulate the lemma without the proof.

Lemma 3.5. *Let $[a,b]$ be a compact interval in $\mathbb{R}$ and let $E \in \mathcal{P}[a,b]$ be absolutely (basically) a δh–negligible set. The set E is absolutely (basically) a δk–negligible set if and only if the set E is absolutely (basically) a $\delta\langle h-k\rangle$–negligible set.*

We are now able to redefine the previously defined concept of basic summability through a basic summable primitive (a BS primitive). This is how we come to one comprehensive concept that includes within itself not only all the most well–known integrabilities such as *Newton*'s and generalized *Riemann* and *Riemann* — *Stieltjes* integrability, but also arithmetic series.

Definition 3.6. *For a compact interval $[a,b]$ in $\mathbb{R}$, let f be a point function on $[a,b]$ and let ς be a strictly positive charge on $\mathcal{I}([a,b])$ such that its differential form $\delta\varsigma$ is identically zero on $[a,b]$. Then, a charge $\mathfrak{G}$ on $\mathcal{P}([a,b])$, whose limit $\delta\mathfrak{G}$ is the total differential on $[a,b]$, is said to be the BS primitive on $\mathcal{P}([a,b])$ for f, with respect to ς, if $\mathfrak{G}$ and $\langle f\varsigma\rangle$ are basically summable equivalents on $[a,b]$. Here*

$$\mathfrak{G}(E) = (B)\sum_{x\in[a,b]} \langle \delta\langle f\varsigma\rangle \chi_E\rangle(x)$$

for each $E \in \mathcal{P}([a,b])$.

Despite the fact that the result our next theorem is trivial, as can be seen from its proof, this theorem is important because it offers us opportunities to analyze special cases important in the theory of the generalized *Riemann* integrals, which will be presented in the second paper. First, the differential form δh will be renamed to the residue function, denoted by $\mathfrak{R}_h$. Then the limit of h will be denoted, as needed, by either δh or $\mathfrak{R}_h$.

Theorem 3.2. *For a compact interval $[a,b]$ in $\mathbb{R}$ let f be a point function on $[a,b]$ and let $\mathfrak{G}$ be a charge on $\mathcal{P}([a,b])$, whose limit $\delta\mathfrak{G}$ is the total differential on $[a,b]$. Then, the residue function $\mathfrak{R}_{\langle\mathfrak{G}-\mathfrak{F}\rangle}$ is basically summable over $[a,b]$ if and only if there is a charge $\mathfrak{C}$ on $\mathcal{P}([a,b])$, whose limit $\delta\mathfrak{C}$ is the total differential on $[a,b]$, and $\langle\mathfrak{G}-\mathfrak{C}\rangle$ is the BS primitive on $\mathcal{P}[a,b]$ for f, which means that*

$$(B)\sum_{x\in[a,b]} \langle \delta\mathfrak{F}\chi_E\rangle(x) = \langle\mathfrak{G}-\mathfrak{C}\rangle(E), \tag{3.9}$$

whenever $E \in \mathcal{P}([a,b])$.

Proof. Let f be a point function on $[a,b]$ and let $\mathfrak{G}$ be a charge on $\mathcal{P}([a,b])$, whose limit $\delta\mathfrak{G}$ is the total differential on $[a,b]$. If $\mathfrak{R}_{\langle\mathfrak{G}-\mathfrak{F}\rangle}$ is basically summable over $[a,b]$, then by the Straddle Lemma $\sigma_{\delta\sigma_{\mathfrak{R}_{\langle\mathfrak{G}-\mathfrak{F}\rangle}}} = \sigma_{\mathfrak{R}_{\langle\mathfrak{G}-\mathfrak{F}\rangle}}$ on $\mathcal{P}([a,b])$. Hence, $\sigma_{\mathfrak{R}_{\langle\mathfrak{G}-\mathfrak{F}\rangle}}$, whose limit $\delta\sigma_{\mathfrak{R}_{\langle\mathfrak{G}-\mathfrak{F}\rangle}}$ is the total differential on $[a,b]$, is a charge $\mathfrak{C}$ on $\mathcal{P}([a,b])$ such that $\langle\mathfrak{G}-\mathfrak{C}\rangle$ is the BS primitive on

$\mathcal{P}[a,b]$ for f. Now, let $\mathfrak{E}$ be a charge on $\mathcal{P}([a,b])$ such that its limit $\delta\mathfrak{E}$ is total differential on $[a,b]$ and $\langle\mathfrak{S}-\mathfrak{E}\rangle$ is the BS primitive on $\mathcal{P}[a,b]$ for f. Then,

$$\sigma_{\mathfrak{R}_{\langle\mathfrak{S}-\mathfrak{F}\rangle}} = \sigma_{\delta\mathfrak{E}} = \mathfrak{E}$$

on $\mathcal{P}([a,b])$. The proof is complete. □

Hence, if δh is basically summable over $[a,b]$, with the sum $\sigma_{\delta h}([a,b])$, then δh is said to be basically integrable over any compact interval $I \in \mathcal{I}([a,b])$ and $\sigma_{\delta h}(I)$ is a signed integral measure of I, that is,

$$(B)\int_I \delta h(x) = \sigma_{\delta h}(I).$$

Let us now return to the beginning of this section, where we have stated indirectly that it is possible that at a point of the interval $[a,b]$, at which the derivative $f = \delta_\mu h$ of an arbitrary interval–point function h is not defined, the residue function $\mathfrak{R}_h$ is defined. Accordingly, suppose that $E \in \mathcal{P}([a,b])$ is a measurable set of Lebesgue measure $\mu_E < (b-a)$, on which the interval–point function h is not differentiable. In this case we can extend the function f from the set $[a,b]\setminus E$ to the set $[a,b]$ so that the extended function f_{ex} is defined on $[a,b]$. If the differential forms δh and $\delta\mathfrak{F}_{ex}$, as limits on $[a,b]$ of h and $\mathfrak{F}_{ex} = \langle f_{ex}\mu\rangle$, respectively, are basically summable over $[a,b]$, then

$$(B)\int_a^b \delta h(x) - (B)\int_a^b \delta\mathfrak{F}_{ex}(x) = (B)\sum_{x\in[a,b]} \delta\langle h-\mathfrak{F}_{ex}\rangle(x), \tag{3.10}$$

that is,

$$(B)\int_a^b \delta\langle h-\mathfrak{F}_{ex}\rangle(x) = (B)\sum_{x\in[a,b]} \mathfrak{R}_{\langle h-\mathfrak{F}_{ex}\rangle}(x) = \sigma_{\mathfrak{R}_{\langle h-\mathfrak{F}_{ex}\rangle}}([a,b]). \tag{3.11}$$

Accordingly, we conclude that if δh and $\delta\mathfrak{F}_{ex}$ are not basically summable equivalents on $[a,b]$, then the charge $\sigma_{\mathfrak{R}_{\langle h-\mathfrak{F}_{ex}\rangle}}$ gives us the difference between the integral value of $\delta_\mu h$ and the BS primitive for f_{ex}. Namely, according to the Straddle Lemma, if there is a charge $\mathfrak{S}$ on $\mathcal{P}([a,b])$, whose limit $\delta\mathfrak{S}$ is the total differential on $[a,b]$, such that $\mathfrak{S}$ and $\mathfrak{F}_{ex}$ are not differential equivalents on $[a,b]$, but they are derivative equivalents on $[a,b]\setminus E$, with respect to μ, then $\mathfrak{R}_{\langle\mathfrak{S}-\mathfrak{F}_{ex}\rangle}$ does not vanish on $[a,b]$, more precisely on the set E, and $\mathfrak{S} \neq \sigma_{\delta\mathfrak{F}_{ex}}$ on $\mathcal{P}([a,b])$ and $\delta_\mu\mathfrak{S} = \delta_\mu\mathfrak{F}_{ex}$ on $[a,b]\setminus E$. Thus, if the charge $\mathfrak{S}$ is a primitive for f, then by integrating the extended function $\mathfrak{F}_{ex}$ and taking that integral value, which is the BS primitive for f_{ex}, to be the primitive for f, we make an integral error. According to Theorem 3.2, the sum of the residue function $\mathfrak{R}_{\langle\mathfrak{S}-\mathfrak{F}_{ex}\rangle}$ on the set E, gives us an estimate of the made integral error.

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