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DYNAMICAL SYSTEMS OF QUADRATIC OPERATORS ON SET OF IDEMPOTENT MEASURES

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Abstract. We consider quadratic operators, which map the n–dimensional simplex of idempotent measures into itself. We introduce the concept of Volterra quadratic operator on the simplex of idempotent measures and provide some general properties of such operators.

We also consdier a special Volterra quadratic operator and provide a comprehensive analysis of the dynamical system generated by this operator. Moreover, the dynamical systems generated by general Volterra operators defined on 2 and 3–dimensional simplices of idempotent measures are studied. For each case, we find fixed points and limits of trajectories.

Keywords: quadratic operator, Volterra operator, simplex, idempotent measure, fixed point, trajectory, dynamical system, attracting, repelling.

Mathematics Subject Classification: 37C25, 47Axx

1. Introduction

Idempotent mathematics is developed by using a new set of basic associative operations, addition $\oplus$ and multiplication $\odot$, so that all the semifield or semiring axioms hold; moreover, the new addition is idempotent, that is, $x \oplus x = x$ for every element x of the corresponding semiring, see, for instance, [1], [2], [3], [4], [5], [6], [7].

One of extensively studied example is the semifield $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ known as the Max–Plus algebra. This semifield consists of all real numbers and an additional element $\mathbf{0} = -\infty$. The element $\mathbf{0}$ is the zero element in $\mathbb{R}_{\max}$, and the basic operations are defined by the formulas $x \oplus y = \max\{x, y\}$ and $x \odot y = x + y$; the identity (or unit) element $\mathbf{1}$ coincides with the usual zero 0. These operations give rise to a new algebraic structure known as the tropical semiring.

Tropical mathematics, often referred to as tropical geometry, related to tropical curves and varieties, which are polyhedral complexes that encode information about classical algebraic varieties. This field provides powerful tools for solving problems in various branches of mathematics, including algebraic geometry, combinatorics, optimization, and computational biology, see books [8], [9].

We consider idempotent measures in the sense of idempotent analysis [1], [4], [5], [6], [7]. Such measure theory is a new branch of mathematics analysis for studying deterministic control problems and first order nonlinear partial differential equations, such as Hamilton — Jacobi equations, with discontinuous initial data and low–lying eigenfunctions of Schrödinger operator.

We define the simplex $\mathcal{I}_n$ of idempotent measures on $\{1, 2, \dots, n\}$ as

$$\mathcal{I}_n = \big\{(x_1, \dots, x_n) \in \mathbb{R}^n_{\max} : \max_{1\leqslant i\leqslant n} x_i = 0\big\} = \big\{(x_1, \dots, x_n) \in \mathbb{R}^n_{\max} : x_1 \oplus \dots \oplus x_n = \mathbf{1}\big\}.$$

In [2] and [3] an idempotent analogue of the Markov chain was introduced; the linearity of the evolution operator was defined by the new operations $\oplus$ and $\odot$. In [10] all linear operators

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which map the n–dimensional simplex of idempotent measures into itself were described. These linear operators on the set of idempotent measures are "idempotent" analogues of Markov chains, where a state of the Markov chain is an idempotent measure, but linearity of the evolution operator is defined by the usual operations + and ·. Moreover, in [10], the dynamical systems generated by the linear maps of the set of idempotent measures were studied. In [11] and [12], the case, when the linear maps considered in [10] become quadratic operators, was investigated. In these works, quadratic operators that map the simplex of idempotent measures into itself were constructed, and their fixed points and corresponding dynamical systems were analyzed. However, since these studies focus on general quadratic operators, the investigation of dynamical systems becomes significantly more difficult for large values of n.

In this paper we consider special Volterra quadratic operators, which map $\mathcal{I}_n$ into itself and study dynamical systems generated by these operators.

The paper is organized as follows. In Section 2 we describe all quadratic operators, which map n–dimensional simplex of idempotent measures into itself, and define Volterra quadratic operators on the simplex of idempotent measures. Some general properties of such operators are provided. Section 3 is devoted to special Volterra quadratic operator. We make the complete analysis of dynamical system generated by this operator. The last section is devoted to the dynamical systems generated by general Volterra operators defined on 2 and 3–dimensional simplex of idempotent measures. In each case for arbitrary initial point we find limits of its trajectory.

2. Volterra quadratic operators

We consider a cubic matrix $P = (p_{ij,k})_{i,j,k=1}^n$ with $p_{ij,k} \in \mathbb{R}_{\max}$.

Definition 2.1. *The quadratic map*

$$Q : x = (x_1, \dots, x_n) \in \mathbb{R}^n_{\max} \to x' = Q(x) = (x'_1, \dots, x'_n) \in \mathbb{R}^n_{\max}$$

is defined as

$$x'_k = \sum_{i,j=1}^n p_{ij,k} x_i x_j, \quad k = 1, 2, \dots, n. \tag{2.1}$$

where $p_{ij,k} \in \mathbb{R}_{\max}$.

Denote

$$M_n = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n_{\max} : \quad x_i \leqslant 0, \quad \sum_{k=1}^n x_k = -1 \right\}.$$

Proposition 2.1. *Any quadratic map (2.1) with* $p_{ij,k} \leqslant 0$ *and*

$$\sum_{k=1}^n p_{ij,k} = -1 \tag{2.2}$$

maps M_n *into itself.*

Proof. Take arbitrary $x \in M_n$. Then by condition $p_{ij,k} \leqslant 0$ from (2.1) we get $x'_k \leqslant 0$ and

$$\sum_{k=1}^n x'_k = \sum_{k=1}^n \sum_{i,j=1}^n p_{ij,k} x_i x_j = \sum_{i,j=1}^n \sum_{k=1}^n p_{ij,k} x_i x_j = -\sum_{i=1}^n x_i \sum_{j=1}^n x_j = -(-1)^2 = -1.$$

The proof is complete. □

Remark 2.1. A quadratic stochastic operator (QSO), see for example [13], [14], [15], [16], [17], is defined by the identity (2.1) with the conditions

$$p_{ij,k} \geqslant 0, \quad \sum_{k=1}^{n} p_{ij,k} = 1.$$

Such a QSO maps the simplex

$$S^{n-1} = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : \quad x_i \geqslant 0, \quad \sum_{k=1}^{n} x_k = 1 \right\}$$

into itself.

Thus all known results related to QSOs can be reformulated for quadratic maps mentioned in Proposition 2.1 just by changing positive values (of parameters and variables) to negative ones. Below we study dynamical systems of quadratic maps on $\mathbb{R}^n_{\max} \setminus M_n$, too.

Definition 2.2. *The quadratic operator (2.1), corresponding to matrix $P = (p_{ij,k})^n_{i,j,k=1}$, with $p_{ij,k} \leqslant 0$ for all i, j, k, is called quadratic Volterra operator if it satisfies the condition*

$$p_{ij,k} = 0, \qquad k \notin \{i, j\}. \tag{2.3}$$

By this definition, it is easy to see that a Volterra operator (denoted by V) reads as

$$V: \quad x'_k = x_k \sum_{i=1}^{n} (p_{ik,k} + p_{ki,k}) x_i, \quad k = 1, 2, \dots, n. \tag{2.4}$$

Theorem 2.1. [18] *The quadratic operator V given by the cubic matrix $P = (p_{ij,k})^n_{i,j,k=1}$ with $p_{ij,k} \leqslant 0$ maps $\mathcal{I}_n$ into itself if and only if it satisfies one of following conditions:*

i) *Each k-th matrix of the cubic matrix P contains exactly one non–zero row and exactly one non–zero column.*
ii) *Cubic matrix P has at least one zero matrix, that is, there exists $m \leqslant n$ such that all elements of the m-th matrix $P_m = (p_{ij,m})^n_{i,j=1}$ consists of only zeroes.*

The condition (2.3) implies that Volterra operator satisfies Condition i) of Theorem 2.1.

Theorem 2.2. *A quadratic Volterra operator maps $\mathcal{I}_n$ into itself.*

Proof. Assume $x \in \mathcal{I}_n$, that is, $\max_{1 \leqslant i \leqslant n} x_i = 0$, for Volterra operator we show that $x' = V(x) \in \mathcal{I}_n$:

$$\max_{1 \leqslant k \leqslant n} x'_k = \max_{1 \leqslant k \leqslant n} \sum_{i,j=1}^{n} p_{ij,k} x_i x_j = \max_{1 \leqslant k \leqslant n} \left\{ x_k \sum_{i=1}^{n} (p_{ik,k} + p_{ki,k}) x_i \right\}. \tag{2.5}$$

By the condition $p_{ij,k} \leqslant 0$ and $x_i \leqslant 0$ we have

$$\sum_{i=1}^{n} (p_{ik,k} + p_{ki,k}) x_i \geqslant 0 \quad \text{for all} \quad k = 1, 2, \dots, n.$$

Moreover, since $x \in \mathcal{I}_n$, there is $m = 1, 2, \dots, n$ such that $x_m = 0$. Therefore, by (2.5) we get

$$\max_{1 \leqslant k \leqslant n} x'_k = x'_m = 0.$$

The proof is complete. □

For each $I \subset E = \{1, \dots, n\}$ we define the face Γ_I of $\mathcal{I}_n$ by

$$\Gamma_I = \{ x \in \mathcal{I}_n : \ x_i = 0, \ i \in I \}.$$

Proposition 2.2. *Let V be a Volterra QSO. Then each face of $\mathcal{I}_n$ is invariant set with respect to V.*

Proof. In view of (2.4) it is clear that if $x_i = 0$ then $x'_i = 0$. Hence $V(\Gamma_I) \subset \Gamma_I$. The proof is complete. □

Definition 2.3. *The quadratic Volterra operator (2.4) with the condition (2.2) is called the quadratic absolute stochastic Volterra operator (QASVO).*

It follows from the condition (2.2) that for QASVO we have $p_{ii,i} = -1$ for each $i = 1, \dots, n$.

Definition 2.4. *A solution to $V(x) = x$ is called the fixed point. The set of all fixed points is denoted by* $\mathrm{Fix}(V)$.

Proposition 2.3. *If $x = (x_1, \dots, x_n)$ is a fixed point of the quadratic map (2.1) with $p_{ij,k} \leqslant 0$ and (2.2), then*

$$\sum_{i=1}^{n} x_i \in \{-1, 0\}.$$

Proof. We take an arbitrary $x \in \mathrm{Fix}(V)$. Then from equation $V(x) = x$ we get

$$\sum_{k=1}^{n} x_k = -\sum_{i=1}^{n} x_i \sum_{j=1}^{n} x_j \quad \Leftrightarrow \quad \sum_{i=1}^{n} x_i = -1 \quad \text{or} \quad 0.$$

The proof is complete. □

3. Case $p_{ik,k} + p_{ki,k} = -1$

In this section we assume that

$$p_{ik,k} + p_{ki,k} = -1, \quad \text{for all} \quad i, k = 1, 2, \dots, n \tag{3.1}$$

In this case by (2.4) we get

$$V_0 : \quad x'_k = -x_k \sum_{i=1}^{n} x_i, \quad k = 1, 2, \dots, n. \tag{3.2}$$

3.1. Fixed points of V_0.

Proposition 3.1. *If (2.2), (2.3) and $p_{ij,i} + p_{ji,i} = -1$, then the set of all fixed points for the QASVO $V_0 : \mathcal{I}_n \to \mathcal{I}_n$ is*

$$\mathrm{Fix}(V_0) = \{(0, 0, \dots, 0)\} \cup \mathcal{J}_n,$$

where

$$\mathcal{J}_n = \{(x_1, x_2, \dots, x_n) \in \mathcal{I}_n : \sum_{i=1}^{n} x_i = -1\}.$$

Proof. The equation $V_0(x) = x$ can be rewritten as

$$x_k = -x_k \sum_{i=1}^{n} x_i.$$

Its solutions are $x_k = 0$ and $\sum_{i=1}^{n} x_i = -1$. The proof is complete. □

3.2. Dynamics under V_0. Let $x^{(0)} \in \mathcal{I}_n$ be an initial point, we define its trajectory (dynamical system) as $x^{(m+1)} = V_0(x^{(m)})$, $m = 0, 1, 2, \ldots$ Our aim is to study the limit points of the trajectory for any initial point $x^{(0)}$.

Denote

$$D_{n-1}^{\{n\}} = \{x \in \mathcal{I}_n : -1 < x_1 + x_2 + \ldots + x_{n-1} \leqslant 0\}.$$

Theorem 3.1. *For any initial point $x^{(0)} \in \mathcal{I}_n$ the identity holds*[1]

$$\lim_{m\to\infty} V_0^m(x^{(0)}) = \begin{cases} x^{(0)} & \text{if} \quad x^{(0)} \in \mathrm{Fix}(V_0), \\ (0, 0, \ldots, 0) & \text{if} \quad x^{(0)} \in D_{n-1}^{\{n\}}, \\ (-\infty, \ldots, -\infty, 0) & \text{if} \quad x^{(0)} = \left(x_1^{(0)}, \ldots, x_{n-1}^{(0)}, 0\right) \in \mathcal{I}_n \setminus \left(\mathrm{Fix}(V_0) \cup D_{n-1}^{\{n\}}\right). \end{cases}$$

Proof. According to Proposition 3.1, each fixed point except for $(0, 0, 0, \ldots, 0)$ satisfies the condition $x_1 + x_2 + \ldots + x_n = -1$. For such a point the limit is trivially the fixed point. Below we consider initial points, which are not fixed points. We need to find the limits for non–fixed points

$$\lim_{m\to\infty} V^m\left(x^{(0)}\right) = \lim_{m\to\infty} V\left(V\left(V\left(\cdots\left(V\left(x^{(0)}\right)\right)\cdots\right)\right)\right) = \lim_{n\to\infty} x^{(m)}.$$

We take a non–fixed point in the case $x_n^{(0)} = 0$, that is, $x^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, \ldots, x_{n-1}^{(0)}, 0\right) \in \mathcal{I}_n$. In this case by the formula (3.2) we get

$$1) \quad \sum_{i=1}^{n-1} x_i^{(1)} = -\sum_{i=1}^{n-1} x_i^{(0)} \sum_{j=1}^{n-1} x_j^{(0)} = -\left(\sum_{i=1}^{n-1} x_i^{(0)}\right)^2,$$

$$2) \quad \sum_{i=1}^{n-1} x_i^{(2)} = -\sum_{i=1}^{n-1} x_i^{(1)} \sum_{j=1}^{n-1} x_j^{(1)} = -\left(\sum_{i=1}^{n-1} x_i^{(1)}\right)^2 = -\left(\sum_{i=1}^{n-1} x_i^{(0)}\right)^4,$$

$$\cdots$$

$$m) \quad \sum_{i=1}^{n-1} x_i^{(m)} = -\sum_{i=1}^{n-1} x_i^{(m-1)} \sum_{j=1}^{n-1} x_j^{(m-1)} = -\left(\sum_{i=1}^{n-1} x_i^{(m-1)}\right)^2 = \cdots = -\left(\sum_{i=1}^{n-1} x_i^{(0)}\right)^{2^m}.$$

By the formula (3.2) and the above obtained results for each coordinates $x_k^{(0)}$ of the initial point $x^{(0)}$ we get

$$1) \quad x_k^{(1)} = -x_k^{(0)} \sum_{i=1}^{n-1} x_i^{(0)} = -x_k^{(0)}\left(x_1^{(0)} + x_2^{(0)} + \cdots + x_{n-1}^{(0)}\right),$$

$$2) \quad x_k^{(2)} = -x_k^{(1)} \sum_{i=1}^{n-1} x_i^{(1)} = -x_k^{(0)}\left(x_1^{(0)} + x_2^{(0)} + \cdots + x_{n-1}^{(0)}\right)^3,$$

$$3) \quad x_k^{(3)} = -x_k^{(2)} \sum_{i=1}^{n-1} x_i^{(2)} = -x_k^{(0)}\left(x_1^{(0)} + x_2^{(0)} + \cdots + x_{n-1}^{(0)}\right)^7,$$

$$\cdots$$

$$m) \quad x_k^{(m)} = -x_k^{(m-1)} \sum_{i=1}^{n-1} x_i^{(m-1)} = -x_k^{(0)}\left(x_1^{(0)} + x_2^{(0)} + \cdots + x_{n-1}^{(0)}\right)^{2^m - 1}.$$

[1] The third line is true in the case $x_k^{(0)} < 0$, if $x_k^{(0)} = 0$, then $x_k^{(m)} = 0$ for $k = \overline{1, n-1}$.

Therefore, we get the trajectories for each $x^{(0)} = \big(x_1^{(0)}, x_2^{(0)}, \dots, x_{n-1}^{(0)}, 0\big) \in \mathcal{I}_n$. Namely, let $x^{(0)} = \big(x_1^{(0)}, x_2^{(0)}, \dots, x_{n-1}^{(0)}, 0\big) \in \mathcal{D}_{n-1}^{\{n\}}$. The coordinates of this point satisfy

$$-1 < x_1^{(0)} + x_2^{(0)} + \ldots + x_{n-1}^{(0)} \leqslant 0.$$

Then we find

$$\lim_{m\to\infty} V^m\big(x_k^{(0)}\big) = \lim_{m\to\infty} x_k^{(m)} = -\lim_{m\to\infty} x_k^{(0)} \left(x_1^{(0)} + x_2^{(0)} + \ldots + x_{n-1}^{(0)}\right)^{2^m-1} = 0.$$

It implies that

$$\lim_{m\to\infty} V^m\big(x^{(0)}\big) = \lim_{m\to\infty} x^{(m)} = (0, 0, \dots, 0).$$

Now let $x^{(0)} = \big(x_1^{(0)}, x_2^{(0)}, \dots, x_{n-1}^{(0)}, 0\big) \in \mathcal{I}_n \setminus \big(\mathrm{Fix}(V_0) \cup D_{n-1}^{\{n\}}\big)$. The coordinates of this initial point satisfies the inequality

$$x_1^{(0)} + x_2^{(0)} + \ldots + x_{n-1}^{(0)} < -1.$$

We observe that if $x_k^{(0)} = 0$, then according to (2.1) we get $x_k^{(m)} = 0$ for each $m \in N$ and $k = \overline{1, n-1}$. This is why we study the case $x_k^{(0)} < 0$ for $k = \overline{1, n-1}$. We have

$$\lim_{m\to\infty} V^m\big(x_k^{(0)}\big) = \lim_{m\to\infty} x_k^{(m)} = -\lim_{m\to\infty} x_k^{(0)} \left(x_1^{(0)} + x_2^{(0)} + \ldots + x_{n-1}^{(0)}\right)^{2^m-1} = -\infty.$$

It follows that

$$\lim_{m\to\infty} V^m\big(x^{(0)}\big) = \lim_{m\to\infty} x^{(m)} = (-\infty, -\infty, \dots, -\infty, 0).$$

The proof is complete. □

4. The case $p_{ik,k} + p_{ki,k} \neq -1$

In this section we consider the case $p_{ik,k} + p_{ki,k} \neq -1$ for some i, k.

4.1. Fixed points of V in case $p_{ik,k} + p_{ki,k} \neq -1$. Finding all fixed points is a difficult problem for general n. We consider the cases $n = 2$ and $n = 3$.

Case $n = 2$. We are going to find fixed points of Volterra operator $V : \mathcal{I}_2 \to \mathcal{I}_2$ assuming that $p_{ii,i} \neq 0$, $i = 1, 2$.

Denote

$$a_{ij} = p_{ij,i} + p_{ji,i} \quad \text{if} \quad i \neq j.$$

For $n = 2$ the equation $V(x) = x$ by (2.4) becomes

$$\begin{cases} x_1 = p_{11,1}x_1^2 + a_{12}x_1x_2, \\ x_2 = p_{22,2}x_2^2 + a_{21}x_1x_2. \end{cases} \tag{4.1}$$

There arise three cases:

1a) If $x_1 \leqslant 0$ and $x_2 = 0$, then we get $x_1 = p_{11,1}x_1^2$. The solutions are $x_1 = \frac{1}{p_{11,1}}$ and $x_1 = 0$. Hence, the fixed points are $\left(\frac{1}{p_{11,1}}, 0\right)$ and $(0, 0)$.

1b) Assume, $x_1 = 0$ and $x_2 \leqslant 0$, then the fixed points are $\left(0, \frac{1}{p_{22,2}}\right)$ and $(0, 0)$.

1c) Let $x_1 = x_2 = 0$. It implies that the fixed point is $(0, 0)$.

Thus, for $n = 2$ we have

$$\mathrm{Fix}(V) = \left\{\left(\frac{1}{p_{11,1}}, 0\right), \left(0, \frac{1}{p_{22,2}}\right), (0, 0)\right\}.$$

Case $n=3$. Let us find all fixed points of the Volterra operator $V:\mathcal{I}_3\to\mathcal{I}_3$. Under the conditions (2.3), $p_{ii,i}=-1$ and $a_{ij}\neq -1$ we see that the equation $V(x)=x$ becomes

$$\begin{cases} x_1=-x_1^2+a_{12}x_1x_2+a_{13}x_1x_3,\\ x_2=-x_2^2+a_{21}x_1x_2+a_{23}x_2x_3,\\ x_3=-x_3^2+a_{31}x_1x_3+a_{32}x_2x_3.\end{cases}$$

Let $x_1\leqslant 0$, $x_2\leqslant 0$ and $x_3=0$, then we get the system of non-linear equations

$$\begin{cases} x_1=-x_1(x_1-a_{12}x_2),\\ x_2=-x_2(x_2-a_{21}x_1).\end{cases}$$

The following three cases are possible:

2a) If $x_1\leqslant 0$ and $x_2=0$, then solutions are $x_1=-1$ and $x_1=0$. Hence, the fixed points are $(-1,0,0)$ and $(0,0,0)$.

2b) Let $x_1=0$ and $x_2\leqslant 0$, then we obtain the fixed points $(0,-1,0)$ and $(0,0,0)$.

2c) Assume, $x_1<0$ and $x_2<0$. In this case we get the system of linear equations

$$\begin{cases} x_1-a_{12}x_2=-1,\\ a_{21}x_1-x_2=1.\end{cases} \tag{4.2}$$

From (4.2) we get fixed point

$$x=\Big(\frac{a_{12}+1}{a_{12}a_{21}-1},\frac{a_{21}+1}{a_{12}a_{21}-1},0\Big)\in\mathcal{I}_3$$

if

$$\big(a_{12}<-1,\ a_{21}<-1\big)\quad\text{or}\quad\big(a_{12}>-1,\ a_{21}>-1\big),$$

otherwise there is no fixed point.

In the same way for the cases $(x_1,0,x_3)$ and $(0,x_2,x_3)$ we find the fixed points $(-1,0,0)$, $(0,-1,0)$, $(0,0,-1)$ and also

$$\left(\frac{a_{13}+1}{a_{13}a_{31}-1},0,\frac{a_{31}+1}{a_{13}a_{31}-1}\right)\quad\text{if}\quad(a_{13}<-1,\quad a_{31}<-1)\quad\text{or}\quad(a_{13}>-1,\quad a_{31}>-1)\,,$$

$$\left(0,\frac{a_{23}+1}{a_{23}a_{32}-1},\frac{a_{32}+1}{a_{23}a_{32}-1}\right)\quad\text{if}\quad(a_{23}<-1,\ a_{32}<-1)\quad\text{or}\quad(a_{23}>-1,\ a_{32}>-1)\,.$$

Proposition 4.1. *Assume $n=3$, $p_{ii,i}=-1$ and $a_{ij}\neq-1$ for $i,j\in\{1,2,3\}$. Then the set of all fixed points for the Volterra operator $V:\mathcal{I}_3\to\mathcal{I}_3$ is*

$$\begin{aligned}\mathrm{Fix}(V)=&\{(0,0,0),(-1,0,0),(0,-1,0),(0,0,-1)\}\\ &\bigcup\Big\{(x_1,x_2,x_3)\in\mathcal{I}_3,\ \textit{where}\ x_i=\frac{a_{ij}+1}{a_{ij}a_{ji}-1},\ x_j=\frac{a_{ji}+1}{a_{ij}a_{ji}-1},\ x_k=0,\\ &\quad \textit{if}\ (a_{ij}<-1,\quad a_{ji}<-1)\ \textit{or}\ (a_{ij}>-1,\quad a_{ji}>-1)\\ &\quad \textit{for}\quad i\neq j,\quad i\neq k,\quad j\neq k,\quad i,j,k\in\{1,2,3\}\Big\}.\end{aligned}$$

4.2. Dynamics under QASVO V in case $p_{ij,i}+p_{ji,i}\neq-1$.

We recall some definitions. Consider a non–linear mapping of n–dimensional variables:

$$x'=V(x):=\begin{cases} x_1'=f_1\left(x_1;x_2;\ \ldots\ ;x_n\right),\\ x_2'=f_2\left(x_1;x_2;\ \ldots\ ;x_n\right),\\ \quad\cdots\\ x_n'=f_n\left(x_1;x_2;\ \ldots\ ;x_n\right).\end{cases}$$

The Jacobian matrix of operator V at a point x_0 is given by

$$J_V(x_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \frac{\partial f_1}{\partial x_2}(x_0) & \cdots & \frac{\partial f_1}{\partial x_n}(x_0) \\ \frac{\partial f_2}{\partial x_1}(x_0) & \frac{\partial f_2}{\partial x_2}(x_0) & \cdots & \frac{\partial f_2}{\partial x_n}(x_0) \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1}(x_0) & \frac{\partial f_n}{\partial x_2}(x_0) & \cdots & \frac{\partial f_n}{\partial x_n}(x_0) \end{pmatrix}.$$

Definition 4.1 (see [19]). *A fixed point $x_0 = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})$ of the operator V is called hyperbolic if the Jacobian $J_V(x_0)$ has no eigenvalues on the unit circle.*

Definition 4.2 (see [19]). *A hyperbolic fixed point $x_0 = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})$ is called*
1) *attracting if all eigenvalues of the Jacobi matrix $J_V(x_0)$ are less than 1 in absolute value;*
2) *repelling if all eigenvalues of the Jacobi matrix $J_V(x_0)$ are greater than 1 in absolute value;*
3) *saddle otherwise.*

Case $n = 2$.

Proposition 4.2. *For the Volterra operator $V : \mathcal{I}_2 \to \mathcal{I}_2$:*
a) *the fixed point $(0, 0)$ is an attracting fixed point;*
b) *fixed points $\left(\frac{1}{p_{11,1}}, 0\right)$ and $\left(0, \frac{1}{p_{22,2}}\right)$ are repelling fixed points.*

The proposition can be proved by the simple analysis of the eigenvalues.

We proceed to the limit points of trajectories. We calculate $x^{(m)} = V^m(x^{(0)})$ for the Volterra operator $V : \mathcal{I}_2 \to \mathcal{I}_2$, and for $m = 1$, $x_1^{(0)} = \frac{\alpha}{p_{11,1}} < 0$ and $x_2^{(0)} = 0$ we have

$$\left(x_1^{(1)}, x_2^{(1)}\right) = V\left(x^{(0)}\right) = \left(\frac{\alpha^2}{p_{11,1}}, 0\right).$$

By induction, for all natural values of m we get

$$V^m\left(x^{(0)}\right) = \left(\frac{\alpha^{2^m}}{p_{11,1}}, 0\right) \tag{4.3}$$

Hence, we obtain the following result.

Proposition 4.3. *For an arbitrary initial point $x^{(0)} = \left(\frac{\alpha}{p_{11,1}}, 0\right)$ we have*

$$\lim_{m\to\infty} V^m\left(x^{(0)}\right) = \lim_{m\to\infty}\left(\frac{\alpha^{2^m}}{p_{11,1}}, 0\right) = \begin{cases} (0,0) & \text{if} \quad \alpha \in [0,1), \\ \left(\frac{1}{p_{11,1}}, 0\right) & \text{if} \quad \alpha = 1, \\ (-\infty, 0) & \text{if} \quad \alpha > 1. \end{cases} \tag{4.4}$$

Similar results apply to te initial point as $x^{(0)} = (0, x_2^{(0)})$. The trajectories of points $x^{(0)} \in I_2$ are shown on Figure 1.

Case $n = 3$.

Now we define the characteristics of fixed points for QASVO in the case $n = 3$ under the condition $a_{ij} \neq -1$ for $i, j \in \{1, 2, 3\}$. In this case according to the condition (2.2) and Proposition 4.1, the fixed points for the operator QASVO $V : \mathcal{I}_3 \to \mathcal{I}_3$ are $(0, 0, 0)$, $(-1, 0, 0)$, $(0, -1, 0)$ and $(0, 0, -1)$.

Proposition 4.4. *Suppose that $a_{ij} \neq -1$ for $i, j \in \{1, 2, 3\}$. For the QASVO $V : \mathcal{I}_3 \to \mathcal{I}_3$:*
a) *the fixed point $(0, 0, 0)$ is an attracting fixed point;*

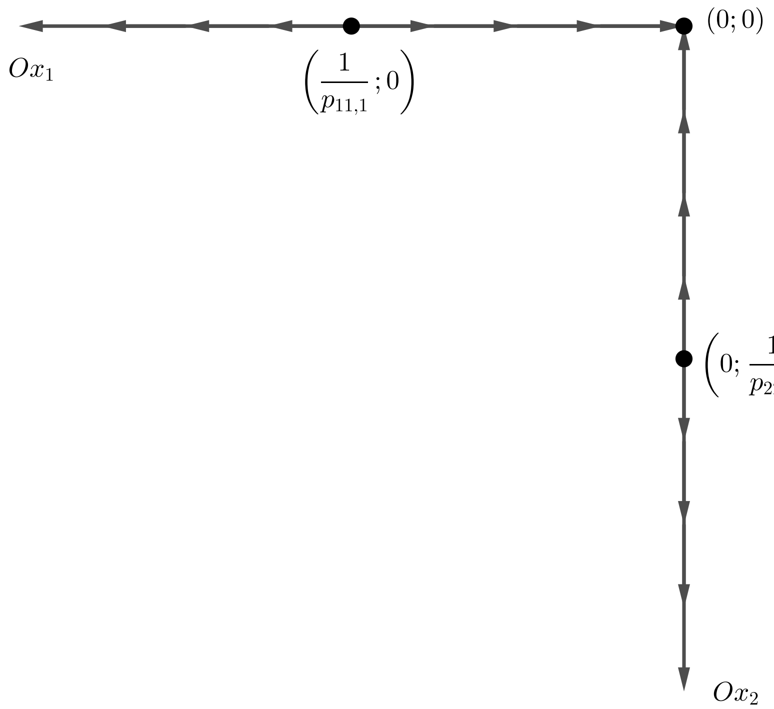

FIGURE 1. Trajectories in the case $n = 2$.

b) *the fixed point* $A_i = (x_1, x_2, x_3)$, *where* $x_i = -1$, $x_j = 0$, $x_k = 0$, $j \neq i$, $k \neq i$ *is repelling if* $a_{ij} < -1$ *and* $a_{ik} < -1$ *for* $i, j, k \in \{1, 2, 3\}$;

c) *the fixed point* $A_i = (x_1, x_2, x_3)$, *where* $x_i = -1$, $x_j = 0$, $x_k = 0$, $j \neq i$, $k \neq i$ *for* $i, j, k \in \{1, 2, 3\}$ *is a saddle point otherwise.*

Proof. By Proposition 4.1 we have four fixed points $(0,0,0)$, $(-1,0,0)$, $(0,-1,0)$ and $(0,0,-1)$ for QASVO under the conditions (2.2) and $a_{ij} \neq -1$.

Due to the formula (2.1) in the case $x_1 \leqslant 0$, $x_2 \leqslant 0$ and $x_3 = 0$ we get

$$\begin{cases} x_1^{(1)} = -\left(x_1^{(0)}\right)^2 + a_{12} x_1^{(0)} x_2^{(0)}, \\ x_2^{(1)} = -\left(x_2^{(0)}\right)^2 + a_{21} x_1^{(0)} x_2^{(0)}. \end{cases} \tag{4.5}$$

We check the characteristics of fixed points $(0,0,0)$, $(-1,0,0)$ and $(0,-1,0)$ according to the formula for the two–dimensional cases. We find eigenvalues of Jacobian for (4.5) by solving the equation

$$|J_V(x_0) - E\lambda| = \begin{vmatrix} -2x_1 + a_{12}x_2 - \lambda & a_{12}x_1 \\ a_{21}x_2 & -2x_2 + a_{21}x_1 - \lambda \end{vmatrix} = 0$$

with $a_{12} = p_{12,1} + p_{21,1}$, $a_{21} = p_{12,2} + p_{21,2}$. The eigenvalues of this Jacobian are

$$\begin{aligned} &\lambda_1 = \lambda_2 = 0, \quad \text{if} \quad x_0 = (0,0,0); \\ &\lambda_1 = 2, \ \lambda_2 = -a_{21}, \quad \text{if} \quad x_0 = (-1,0,0); \\ &\lambda_1 = 2, \ \lambda_2 = -a_{12}, \quad \text{if} \quad x_0 = (0,-1,0). \end{aligned}$$

Similar considerations holds in the cases $(x_1, 0, x_3)$ and $(0, x_2, x_3)$. The obtained results shows that the fixed point $(0,0,0)$ is always attracting. According to the conditoin (2.2) we have $a_{ij} + a_{ji} = -2$. Thus, the type of the fixed points $(-1,0,0)$, $(0,-1,0)$ and $(0,0,-1)$ depend each on the other. If one of them is repeller, then the others are saddle, else all of them are saddle points. The proof is complete. □

We assume that $x_3=0$ and in the plane Ox_1x_2 we consider a triangle with the vertices are at the fixed points $(0,0,0)$, $(-1,0,0)$ and $(0,-1,0)$. We denote this triangle by D_{12}. Then the set D_{12} is as follows:

$$D_{12}=\left\{x^{(0)}=\left(x_1^{(0)},x_2^{(0)},0\right):\ -1<x_1^{(0)}+x_2^{(0)}<0\right\}.$$

Theorem 4.1. *Let $n=3$ and $a_{12}\neq -1$. For an arbitrary $x^{(0)}=\left(x_1^{(0)},x_2^{(0)},0\right)$ the trajectory has the following limit: if*

$$x^{(0)}=\left(x_2^{(0)},x_2^{(0)},0\right)\in \bar{D}_{12},$$

then

$$\lim_{m\to\infty}V^m(x^{(0)})=\begin{cases}(0,0,0) & \text{if}\quad x_1^{(0)}+x_2^{(0)}>-1,\\ (-1,0,0) & \text{if}\quad x_1^{(0)}+x_2^{(0)}=-1,\quad a_{12}<-1,\\ (0,-1,0) & \text{if}\quad x_1^{(0)}+x_2^{(0)}=-1,\quad a_{12}>-1;\end{cases}$$

if

$$x^{(0)}=(x_2^{(0)},x_2^{(0)},0)\in\mathcal{I}_3\setminus\bar{D}_{12},\quad a_{12}=0,$$

then

$$\lim_{m\to\infty}V^m(x^{(0)})=\begin{cases}(0,-\infty,0) & \text{if}\quad -1<x_1^{(0)}<0,\\ (-1,-\infty,0) & \text{if}\quad x_1^{(0)}=-1;\end{cases}$$

if

$$x^{(0)}=(x_2^{(0)},x_2^{(0)},0)\in\mathcal{I}_3\setminus\bar{D}_{12},\qquad a_{21}=0,$$

then

$$\lim_{m\to\infty}V^m(x^{(0)})=\begin{cases}(-\infty,0,0) & \text{if}\quad -1<x_2^{(0)}<0,\\ (-\infty,-1,0) & \text{if}\quad x_2^{(0)}=-1;\end{cases}$$

and in all other cases

$$\lim_{m\to\infty}V^m(x^{(0)})=(-\infty,-\infty,0).$$

Proof. According to Proposition 4.1 and condition (2.2) there are four fixed points $(0,0,0)$, $(-1,0,0)$, $(0,-1,0)$ and $(0,0,-1)$ for QASVO $V:\mathcal{I}_3\to\mathcal{I}_3$. Below we consider initial points which are not fixed points. We need to find the limits for non–fixed points

$$\lim_{m\to\infty}V^m\left(x^{(0)}\right)=\lim_{m\to\infty}V\left(V\left(V\left(\cdots\left(V\left(x^{(0)}\right)\right)\cdots\right)\right)\right)=\lim_{m\to\infty}x^{(m)}.$$

We take non–fixed point in case $x_3^{(0)}=0$, that is, by the formula (2.1) we get

$$\begin{cases}x_1^{(1)}=-\left(x_1^{(0)}\right)^2+a_{12}x_1^{(0)}x_2^{(0)},\\ x_2^{(1)}=-\left(x_2^{(0)}\right)^2-\left(a_{12}+2\right)x_1^{(0)}x_2^{(0)}.\end{cases}\tag{4.6}$$

By the system (4.6) we obtain the recurrent identities

$$x_1^{(1)}+x_2^{(1)}=-\left(\left(x_1^{(0)}\right)^2+\left(x_2^{(0)}\right)^2+2x_1^{(0)}x_2^{(0)}\right)=-\left(x_1^{(0)}+x_2^{(0)}\right)^2,$$
$$x_1^{(2)}+x_2^{(2)}=-\left(\left(x_1^{(1)}\right)^2+\left(x_2^{(1)}\right)^2+2x_1^{(1)}x_2^{(1)}\right)=-\left(x_1^{(1)}+x_2^{(1)}\right)^2=-\left(x_1^{(0)}+x_2^{(0)}\right)^4,$$
$$\cdots$$
$$x_1^{(m)}+x_2^{(m)}=-\left(x_1^{(m-1)}+x_2^{(m-1)}\right)^2=\cdots=-\left(x_1^{(0)}+x_2^{(0)}\right)^{2^m}$$

We consider a non–fixed point $x^{(0)}=\left(x_1^{(0)},x_2^{(0)},0\right)\in D_{12}$. The coordinates of this point satisfy double inequality $-1<x_1^{(0)}+x_2^{(0)}<0$. It implies

$$\lim_{m\to\infty}\left(x_1^{(m)}+x_2^{(m)}\right)=-\lim_{m\to\infty}\left(x_1^{(0)}+x_2^{(0)}\right)^{2^m}=0.$$

Hence,

$$\lim_{m\to\infty}\left(x_1^{(m)}\right)=0 \quad \text{and} \quad \lim_{m\to\infty}\left(x_2^{(m)}\right)=0.$$

This gives

$$\lim_{m\to\infty} V^m\left(x^{(0)}\right)=(0,0,0).$$

It shows that the trajectory of every initial point $x^{(0)}\in D_{12}$ under QASVO approaches to $(0,0,0)$.

We consider the non–fixed point $x^{(0)}=\left(x_1^{(0)},x_2^{(0)},0\right)\in\mathcal{I}_3\setminus D_{12}$. The coordinates of this point satisfy the inequality $x_1^{(0)}+x_2^{(0)}<-1$. In this case, according to the above obtained result, we get

$$\lim_{m\to\infty}\left(x_1^{(m)}+x_2^{(m)}\right)=-\infty. \tag{4.7}$$

Now we are going to determine the limits for each coordinates $x_i^{(m)}$, $i=1,2$. Denoting $a_{ij}=p_{ij,i}+p_{ji,i}$, we get $a_{ij}+a_{ji}=-2$. By (4.5) we obtain

$$\begin{cases} x_1^{(1)}=x_1^{(0)}\big(-x_1^{(0)}+a_{12}x_2^{(0)}\big),\\ x_2^{(1)}=x_2^{(0)}\big(-x_2^{(0)}+a_{21}x_1^{(0)}\big).\end{cases} \tag{4.8}$$

We note that if $x_k^{(0)}=0$, then according to (2.1) we get $x_k^{(m)}=0$ for every $m\in N$ and $k\in\{1,2\}$. This is why we study case $x_1^{(0)}<0$, $x_2^{(0)}<0$ and $x_1^{(0)}+x_2^{(0)}<-1$. We analyse the system (4.8) in the cases $a_{ij}=0$ and $a_{ij}\neq 0$ separately.

b1) Let $a_{12}a_{21}=0$.

b1.1) Assume that $a_{12}=0$. Then we get $a_{21}=-2$ and (4.8) becomes

$$\begin{cases} x_1^{(1)}=-\big(x_1^{(0)}\big)^2,\\ x_2^{(1)}=x_2^{(0)}\big(-x_2^{(0)}-2x_1^{(0)}\big).\end{cases} \tag{4.9}$$

This allows us to get the mth iteration of the operator

$$\begin{cases} x_1^{(m)}=-\big(x_1^{(m-1)}\big)^2,\\ x_2^{(m)}=-x_2^{(m-1)}\big(x_2^{(m-1)}+2x_1^{(m-1)}\big).\end{cases} \tag{4.10}$$

By the first equation in the above system we get

$$x_1^{(m)}=-\big(x_1^{(m-1)}\big)^2=-\big(\big(x_1^{(m-2)}\big)^2\big)^2=\ldots=-\big(x_1^{(0)}\big)^{2^m}.$$

The inequality $x_1^{(0)}<0$ implies

$$\lim_{m\to\infty}x_1^{(m)}=-\lim_{m\to\infty}\big(x_1^{(0)}\big)^{2^m}=\begin{cases}0 & \text{if } \ -1<x_1^{(0)}\leqslant 0,\\ -1 & \text{if } \ x_1^{(0)}=-1,\\ -\infty & \text{if } \ x_1^{(0)}<-1.\end{cases} \tag{4.11}$$

By (4.7), (4.10) and (4.11) we obtain that $x_2^{(n)}\to-\infty$ as $m\to\infty$.

We have $x_1^{(0)}+x_2^{(0)}<-1$. Under this condition and by the second equation in the system (4.9) we get

$$\begin{aligned} x_2^{(1)}&=-x_2^{(0)}\big(x_2^{(0)}+2x_1^{(0)}\big)<-x_2^{(0)}\big(x_2^{(0)}+x_1^{(0)}\big),\\ x_2^{(2)}&=-x_2^{(1)}\big(x_2^{(1)}+2x_1^{(1)}\big)<-x_2^{(1)}\big(x_2^{(1)}+x_1^{(1)}\big)<-\big(-x_2^{(0)}\big(x_2^{(0)}+x_1^{(0)}\big)\big)\big(x_2^{(1)}+x_1^{(1)}\big).\end{aligned}$$

Repeating this estimating, we get

$$x_2^{(m)} < -x_2^{(m-1)}\big(x_2^{(m-1)}+x_1^{(m-1)}\big) < -\big(-x_2^{(m-2)}\big(x_2^{(m-2)}+x_1^{(m-2)}\big)\big)\big(x_2^{(m-1)}+x_1^{(m-1)}\big)$$
$$< \ldots < (-1)^m x_2^{(0)}\big(x_2^{(0)}+x_1^{(0)}\big)\big(x_2^{(1)}+x_1^{(1)}\big)\cdots\big(x_2^{(m-1)}+x_1^{(m-1)}\big) < (-1)^m x_2^{(0)}\big(x_2^{(0)}+x_1^{(0)}\big)^m.$$

It follows that

$$\lim_{m\to\infty} x_2^{(m)} < \lim_{m\to\infty}(-1)^m x_2^{(0)}\left(x_2^{(0)}+x_1^{(0)}\right)^m = -\infty.$$

As a result, if $a_{12}=0$, then we obtain

$$\lim_{m\to\infty} V^m(x^{(0)}) = \begin{cases} (0,-\infty,0) & \text{if } \ -1 < x_1^{(0)} \leqslant 0,\\ (-1,-\infty,0) & \text{if } \ x_1^{(0)}=-1,\\ (-\infty,-\infty,0) & \text{if } \ x_1^{(0)}<-1. \end{cases}$$

b1.2) We consider the case $a_{21}=0$. Then similarly to the above considered case we show that

$$\lim_{m\to\infty} V^m(x^{(0)}) = \begin{cases} (-\infty,0,0), & \text{if } \ -1 < x_2^{(0)} \leqslant 0,\\ (-\infty,-1,0), & \text{if } \ x_2^{(0)}=-1,\\ (-\infty,-\infty,0), & \text{if } \ x_2^{(0)}<-1. \end{cases}$$

b2) Let $a_{ij}\neq 0$. We have

$$x_1^{(0)}<0, \qquad x_2^{(0)}<0, \qquad x_1^{(0)}+x_2^{(0)}<-1.$$

b2.1) If $a_{12}<-1$, then the identity $a_{ij}+a_{ji}=-2$ implies that $a_{21}>-1$. We denote $a_{12}=-(1+\alpha)$, where $\alpha\in(0;1)$, then we get $a_{21}=\alpha-1$ and we rewrite (4.8) for each m as

$$\begin{cases} x_1^{(m)} = x_1^{(m-1)}\big(-x_1^{(m-1)}-(\alpha+1)x_2^{(m-1)}\big),\\ x_2^{(m)} = x_2^{(m-1)}\big(-x_1^{(m-1)}+(\alpha-1)x_1^{(m-1)}\big). \end{cases} \tag{4.12}$$

By the system (4.12) and condition $x_1^{(0)}+x_2^{(0)}<-1$ we find

$$x_1^{(m)}+x_2^{(m)} = -\big(x_1^{(m-1)}+x_2^{(m-1)}\big)^2 = \cdots = -\big(x_1^{(0)}+x_2^{(0)}\big)^{2^m} \to -\infty. \tag{4.13}$$

Using (4.13) for the first equation in the system (4.12), we obtain

$$-\big(x_1^{(m-1)}+(\alpha+1)x_1^{(m-1)}\big) = -\big(x_1^{(m-1)}+x_1^{(m-1)}+\alpha x_1^{(m-1)}\big) > -\big(x_1^{(m-1)}+x_2^{(m-1)}\big) > 1.$$

Under this inequality, the first equation of the system (4.12) implies

$$x_1^{(m)} < x_1^{(m-1)}\big(-x_1^{(m-1)}-x_2^{(m-1)}\big) < x_1^{(m-1)}.$$

for every m. And as a result we obtain the inequalities

$$x_1^{(1)} < x_1^{(0)}\big(-x_1^{(0)}-x_2^{(0)}\big),$$
$$x_1^{(2)} < x_1^{(1)}\big(-x_1^{(1)}-x_2^{(1)}\big) < x_1^{(0)}\big(-x_1^{(0)}-x_2^{(0)}\big)\big(-x_1^{(1)}-x_2^{(1)}\big).$$

Continuing this process and using (4.13), we find

$$x_1^{(m)} < x_1^{(0)}\big(-x_1^{(0)}-x_2^{(0)}\big)\big(-x_1^{(1)}-x_2^{(1)}\big)\cdots\big(-x_1^{(m-1)}-x_2^{(m-1)}\big) < x_1^{(0)}\big(-x_1^{(0)}-x_2^{(0)}\big)^{2^m-1}.$$

Since $x_1^{(0)}+x_2^{(0)}<-1$, it follows that

$$\lim_{m\to\infty} x_1^{(m)} < \lim_{m\to\infty}\big(x_1^{(0)}\big(-x_1^{(0)}-x_2^{(0)}\big)^{2^m-1}\big) = -\infty.$$

As a result we obtain

$$x_1^{(0)} > x_1^{(1)} > x_1^{(2)} > \ldots > x_1^{(m)}.$$

We consider the second equation of the system (4.12)

$$x_2^{(m)} = x_2^{(m-1)}\big(-x_2^{(m-1)} + (\alpha-1)x_1^{(m-1)}\big).$$

We proved above that it follows from $x_1^m \to -\infty$ that there exits a number $m_0 \in \mathbb{N}$, which satisfies the condition

$$-x_2^{(m_0-1)} + (\alpha-1)x_1^{(m_0-1)} \geqslant 1.$$

Then we get the inequality

$$x_2^{(m_0-1)} > x_2^{(m_0)}.$$

Therefore, we get the inequalities

$$x_2^{(m_0-1)} > x_2^{(m_0)} > x_2^{(m_0+1)} > \ldots > x_2^{(m)} > \ldots,$$

that is

$$\lim_{m\to\infty} x_2^{(m)} = -\infty.$$

Now assume that there exists m_0 such that $-x_2^{(m_0)} + (\alpha-1)x_1^{(m_0)} < 1$. Let

$$b = -x_1^{(m_0)} - x_2^{(m_0)} > 1,$$

then by (4.12) for $m = m_0+1$ we have

$$\begin{cases} x_1^{(m_0+1)} = x_1^{(m_0)}\big(b - \alpha x_2^{(m_0)}\big), \\ x_2^{(m_0+1)} = x_2^{(m_0)}\big(b + \alpha x_1^{(m_0)}\big). \end{cases} \tag{4.14}$$

Under the condition $b+\alpha x_1^{(m_0)} < 1$ the second equality of the above system yields the inequality $x_2^{(m_0+1)} > x_2^{(m_0)}$. But, we have $b - \alpha x_2^{(m_0)} > 1$, and we repeat this process for $m = m_0+2$ to obtain

$$\begin{aligned} x_2^{(m_0+2)} =& x_2^{(m_0+1)}\big(b+\alpha x_1^{(m_0+1)}\big) = x_2^{(m_0)}\big(b+\alpha x_1^{(m_0)}\big)\big(b+\alpha x_1^{(m_0)}\big(b-\alpha x_2^{(m_0)}\big)\big) \\ <& x_2^{(m_0)}\big(b+\alpha x_1^{(m_0)}\big)\big(b+\alpha x_1^{(m_0)}\big) = x_2^{(m_0+1)}\big(b+\alpha x_1^{(m_0)}\big)^2. \end{aligned}$$

The last inequality shows that the sequence $\{x_2^{(i)}\}_{i=0}^{\infty}$ is not strictly monotone and

$$\lim_{m\to\infty} x_1^{(m)} = -\infty.$$

Thus, there exits a number N such that

$$x_1^{(N)} \geqslant \frac{1}{\alpha-1}, \qquad x_2^{(N+1)} = x_2^{(N)}\left(-x_2^{(N)} + (\alpha-1)x_1^{(N)}\right) < x_2^{(N)}.$$

It follows that for every $m_N > N$ we have the inequalities

$$x_1^{(m_N)} > x_1^{(m_N+1)} > x_1^{(m_N+2)} > \ldots$$

In other words, if $a_{12} \neq 0$ and $x_1^{(m)}$ approaches to the negative infinity, then $x_2^{(m)}$ also approaches to the negative infinity.

b2.2) If $a_{21} < -1$, then the equation $a_{ij} + a_{ji} = -2$ implies that $a_{12} > -1$. Then similarly to the above case we show that

$$\lim_{m\to\infty} x_1^{(m)} = -\infty, \qquad \lim_{m\to\infty} x_2^{(m)} = -\infty.$$

We also note that in the case $a_{12} = -1$, $a_{21} = -1$ the dynamics is given in Theorem 3.1. By the above obtained results we conclude that for $a_{ij} \neq 0$ we have

$$\lim_{m\to\infty} V^m(x^{(0)}) = (-\infty, -\infty, 0).$$

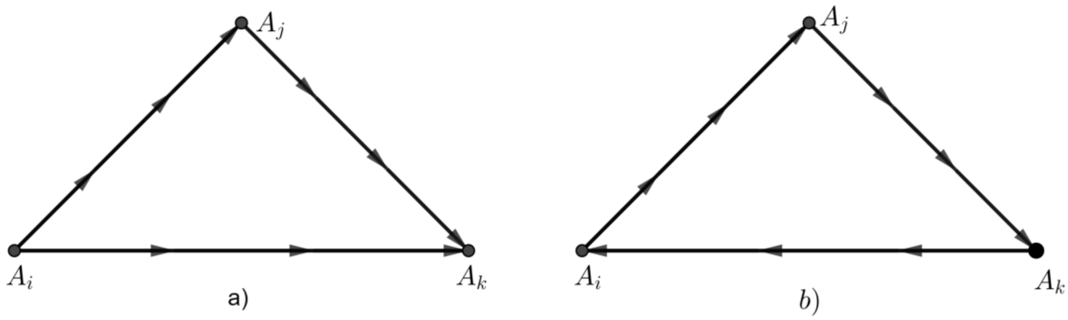

FIGURE 2. Trajectories of points on the lines $x_i + x_j = -1$.

c) Finally, let a non–fixed point $x^{(0)} = \big(x_1^{(0)}, x_2^{(0)}, 0\big) \in \partial D_{12}$ lie on the line $x_1 + x_2 = -1$. Assume that $a_{12} > -1$ and $\alpha \in \big(0;1\big)$. Let $a_{12} = -1 + \alpha$. Then the equation $x^{(1)} = V\big(x^{(0)}\big)$ is rewritten as

$$\begin{cases} x_1^{(1)} = x_1^{(0)}\big(-x_1^{(0)} + (\alpha - 1)x_2^{(0)}\big), \\ x_2^{(1)} = x_2^{(0)}\big(-x_2^{(0)} - (\alpha + 1)x_1^{(0)}\big). \end{cases}$$

By (4.13) we have $x_1^{(m)} + x_2^{(m)} = -1$ and we can rewrite this system as the one–dimensional system

$$x_1^{(m+1)} = f\big(x_1^{(m)}\big),$$

where

$$f(x) = x((1 - \alpha) - \alpha x), \quad x \in [-1, 0].$$

The fixed points of $f(x)$ are -1 and 0. We have

$$f'\big(0\big) = 1 - \alpha < 1 \quad \text{and} \quad f'(-1) = 1 + \alpha > 1.$$

Thus, 0 is the attractor and -1 is the repeller. Moreover, $f(x)$ is monotone increasing on $[-1, 0]$. Therefore, it is easy to see that

$$\lim_{m\to\infty} f^m(x) = 0 \quad \text{for all} \quad x \in (-1, 0].$$

In conclusion, if we iterate the non–fixed point $x^{(0)} = \big(x_1^{(0)}, x_2^{(0)}, 0\big)$, which lies on the line $x_1 + x_2 = -1$, under the condition $a_{12} > -1$ we obtain

$$\lim_{m\to\infty} x_1^{(m)} = 0, \qquad \lim_{m\to\infty} x_2^{(m)} = -1.$$

In the same way we show that in the case $a_{12} < -1$

$$\lim_{m\to\infty} x_1^{(m)} = -1, \qquad \lim_{m\to\infty} x_2^{(m)} = 0.$$

Similar arguments hold for the cases $x_0 = (x_1, 0, x_3)$ and $x_0 = (0, x_2, x_3)$.

Summarizing the above results, we conclude that, if $a_{12} > -1$ and $a_{13} > -1$, then the non–fixed points, which lie on the lines $x_1 + x_2 = -1$ and $x_1 + x_3 = -1$ move far from the fixed $A_i = (-1, 0, 0)$ and this fixed point is repelling, see Figure 2.a). And if $a_{12} > -1$ and $a_{13} < -1$, then the non–fixed points which lie on the line $x_1 + x_2 = -1$ move far from the fixed $A_i = (-1, 0, 0)$, but the non–fixed points, which lie on the line $x_1 + x_3 = -1$, tend to the point $A_i = (-1, 0, 0)$; in this case the fixed point $A_i = (-1, 0, 0)$ is saddle, see Figure 2.b). Possible trajectories of the non–fixed point on line $x_i + x_j = -1$ are shown on Figures 2.a) and 2.b). □

Dynamics of initial point by iteration under QASVO on planes Ox_1x_2, Ox_1x_3 and Ox_2x_3 are shown on Figures 3 and 4. Figure 3 shows that the case $(0, 0, 0)$ is attracting, the case $(0, -1, 0)$

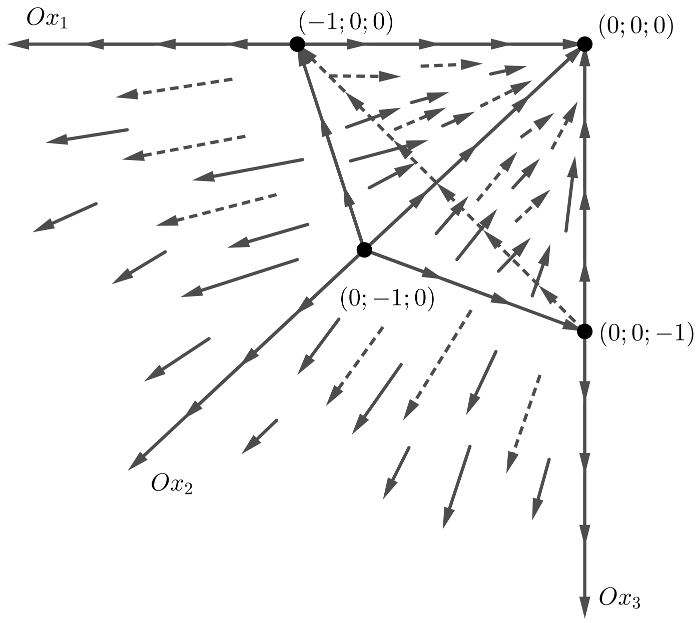

FIGURE 3. Trajectories of points on planes Ox_1x_2, Ox_1x_3 and Ox_2x_3.

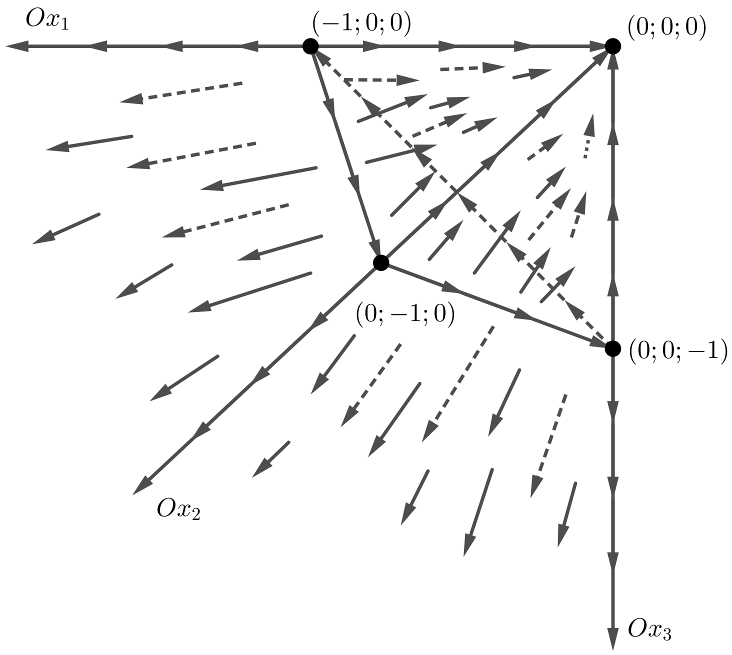

FIGURE 4. Trajectories of points on planes Ox_1x_2, Ox_1x_3 and Ox_2x_3.

is repelling, the cases $(-1, 0, 0)$ and $(0, 0, -1)$ are saddle points. Figure 4 shows that the case $(0, 0, 0)$ is attracting, the cases $(0, -1, 0)$, $(-1, 0, 0)$ and $(0, 0, -1)$ are saddle points.

BIBLIOGRAPHY

1. M. Akian. *Densities of idempotent measures and large deviations* // Trans. Amer. Math. Soc. **351**:11, 4515–454 (1999). https://doi.org/10.1090/S0002-9947-99-02153-4
2. P. Del Moral, M. Doisy. *On applications of Maslov optimization theory* // Math. Notes. **69**:2, 232–244 (2001). https://doi.org/10.1023/A:1002828503858

3. P. Del Moral, M. Doisy. *Maslov idempotent probability calculus* // Theory Probab. Appl. **43**:4, 562–576 (1998). https://doi.org/10.1137/S0040585X97977161
4. G. L. Litvinov, V.P. Maslov (eds.) *Idempotent Mathematics and Mathematical Physics*. Amer. Math. Soc., Providence, RI (2005). https://doi.org/10.1090/conm/377
5. G. L. Litvinov. *Maslov dequantization, idempotent and tropical mathematics: a brief introduction* // J. Math. Sci., New York **140**:3, 426–444 (2007). https://doi.org/10.1007/s10958-007-0450-5
6. V. P. Maslov, S. N. Samborskii (eds.). *Idempotent analysis* Amer. Math. Soc., Providence, RI (1992).
7. M.M. Zarichnyi. *Spaces and maps of idempotent measures* // Izv. Math. **74**:3, 481–499 (2010). https://doi.org/10.1070/IM2010v074n03ABEH002495
8. I. Itenberg, G. Mikhalkin, E. Shustin. *Tropical Algebraic Geometry*. Birkhäuser, Basel (2009).
9. M. E. Kazaryan. *Tropical geometry*. MCCME, Moscow (2012). (in Russian).
10. U.A. Rozikov, M.M. Karimov. *Dinamics of linear maps of idempotent measures* // Lobachevskii J. Math. **34**:1, 20–28 (2013). https://doi.org/10.1134/S1995080212040130
11. I.T. Juraev. *Fixed points of quadratic operators defined on a three-dimensional simplex of idempotent measures* // Lobachevskii J. Math. **43**:3, 762–769 (2022). https://doi.org/10.1134/S1995080222060130
12. I.T. Juraev. *Dynamics of quadratic operators of idempotent measures* // Lobachevskii J. Math. **43**:8, 2145–2154 (2022). https://doi.org/10.1134/S1995080222110130
13. H. Kesten. *Quadratic transformations: a model for population growth. I.* // Adv. Appl.Probab. **2**, 1–82 (1970). https://doi.org/10.2307/3518344
14. H. Kesten. *Quadratic transformations: a model for population growth. II.* // Adv. Appl.Probab. **2**, 179–228 (1970). https://doi.org/10.2307/1426318
15. Yu.I.Lyubich. *Mathematical Structures in Population Genetics*. Springer–Verlag, Berlin (1992).
16. U.A. Rozikov. *Population Dynamics: Algebraic and Probabilistic Approach*. World Sci. Publ., Singapore (2020).
17. R.N. Ganikhodzhaev, F.M. Mukhamedov, U.A. Rozikov. *Quadratic stochastic operators: Results and open problems* // Infin. Dimens. Anal. Quantum Probab. Relat. Top. **14**:2, 279–335 (2011). https://doi.org/10.1142/S0219025711004365
18. I.T. Juraev, M.M. Karimov. *Quadratic Operators Defined on a Finite-dimensional Simplex of Idempotent Measures* // Discontin. Nonlinearity Complex. **8**:3, 279–286 (2019). https://doi.org/10.5890/DNC.2019.09.004
19. R.L. Devaney. *A First Course in Chaotic Dynamical Systems. Theory and experiment*. CRC Press, Boca Raton, FL (2020). https://doi.org/10.1201/9780429280665

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