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MULTIPLICITY OF SOLUTIONS FOR RESONANT DISCRETE $2n$-TH ORDER PERIODIC BOUNDARY VALUE PROBLEM

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Abstract. We examine a class periodic boundary value problems for a discrete equation of order $2n$. We demonstrate the existence of multiple solutions by using the critical point theory and variational methods. Additionally, we consider two examples, in which we discuss the fundamental characteristics of the multiplicity of solutions.

Keywords: Discrete boundary value problems, equation of $2n$-th order, variational methods, critical point theory.

Mathematics Subject Classification: 39A10, 34B08, 34B15

1. Introduction

In the work we study the following boundary value problem for a nonlinear difference equation of $2n$-th order

$$\begin{cases} \sum_{k=0}^{n}(-1)^k\Delta^{2k}y(t-k) = \gamma_j y(t) + h(t, y(t)), & t \in [1, N]_{\mathbb{Z}}, \\ \Delta^i y(-(n-1)) = \Delta^i y(N-(n-1)), & i \in [0, 2n-1]_{\mathbb{Z}}. \end{cases} \tag{1.1}$$

where $n \in \mathbb{N}$, $N \geqslant n$ is an integer, $[1, N]_{\mathbb{Z}}$ is the discrete interval $\{1, 2, \ldots, N\}$, Δ is the forward difference operator defined by

$$\Delta y(t) = y(t+1) - y(t), \qquad \Delta^0 y(t) = y(t), \qquad \Delta^i y(t) = \Delta^{i-1}(\Delta y(t))$$

for $i \in \{1, 2, 3, \ldots, 2n\}$; $h : [1, N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ is a function such that for each fixed $t \in [1, N]_{\mathbb{Z}}$ the function $h(t, \cdot)$ is continuous, and γ_j, $j \in [1, q-1]_{\mathbb{Z}}$, is the $(j+1)$-th eigenvalue of the linear boundary value problem

$$\begin{cases} \sum_{k=0}^{n}(-1)^k\Delta^{2k}y(t-k) = \gamma y(t), & t \in [1, N]_{\mathbb{Z}}, \\ \Delta^i y(-(n-1)) = \Delta^i y(N-(n-1)), & i \in [0, 2n-1]_{\mathbb{Z}}. \end{cases} \tag{1.2}$$

where $q = \frac{N-1}{2}$ when N is odd and $q = \frac{N}{2}$ when N is even. A function $y : [-(n-1), N+n]_{\mathbb{Z}} \to \mathbb{R}$, which satisfies both equations of in (1.1) is a solution to (1.1).

The problem (1.1) is said to be resonant on γ_j at infinity if

$$\lim_{|x|\to\infty} \frac{h(t,x)}{x} = 0,$$

for each $t \in [1, N]_{\mathbb{Z}}$.

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We stress that the problem (1.1) can be regarded as a discrete analog of the following boundary value problem for a differential equation of $2n$-th order

$$\begin{cases} \sum_{k=0}^{n}(-1)^k \frac{d^{2k}y(t)}{dt^{2k}} = \gamma_j y(t) + h(t, y(t)), & t \in (0,1), \\ y^{(i)}(0) = y^{(i)}(1), & i \in [0, 2n-1]_{\mathbb{Z}}. \end{cases} \tag{1.3}$$

Difference equations emerge inevitably in the mathematical modeling of important problems in mechanical engineering, control systems, artificial or biological neural networks, economics, and other fields [1], [14]. The existence and variety of solutions to boundary value for difference equations nowadays attracts more and more attention. One of important tools in studying the difference equations is the fixed point theorems in cones, which are frequently employed. Another tool for the study of nonlinear difference equations is the approach based on upper and lower solution. It is widely accepted that critical point theory, variational approaches, and monotonicity methods are useful tools for determining whether and how different solutions exist for a variety of problems; for more detail, see [2]–[8], [10]–[13], [15]–[27].

In this paper we study the existence and multiplicity of solutions to discrete nonlinear problem (1.1).

We make the following assumptions.

(h_1) The identity $h(t, -x) = -h(t, x)$ holds for all $(t, x) \in [1, N]_{\mathbb{Z}} \times \mathbb{R}$.

(h_2) There exists γ_l, $1 \leqslant l \leqslant j$ such that

$$\limsup_{x \to 0} \frac{2H(t,x)}{x^2} < \gamma_l - \gamma_j,$$

where

$$H(t,x) = \int_0^x h(t,s)\, ds$$

for $(t, x) \in [1, N]_{\mathbb{Z}} \times \mathbb{R}$.

(h_3) The identity

$$\lim_{|x| \to \infty} \frac{h(t,x)}{x} = 0$$

holds for each $t \in [1, N]_{\mathbb{Z}}$.

(h_4) If $\|y_m\| \to \infty$ such that $\frac{\|y_m^{(j)}\|}{\|y_m\|} \to 1$ as $m \to \infty$, then there exists $\tau > 0$ and $m_0 \in \mathbb{N}$ such that

$$\sum_{t=1}^{N} h(t, y_m(t)) y_m^{(j)}(t) \geqslant \tau \ \text{ for } \ m \geqslant m_0,$$

where

$$y_m = y_m^{(j)} + y_m^{\star}, \qquad y_m^{(j)} \in E^{(j)}, \qquad y_m^{\star} \in E^- \bigoplus E^+,$$

$$E^{(j)} = E(\gamma_j), \qquad E^- = \bigoplus_{i=0}^{j-1} E(\gamma_i), \qquad E^+ = \bigoplus_{i=j+1}^{N-1} E(\gamma_i).$$

(h_5) $H(t, x) \to +\infty$ as $|x| \to +\infty$ uniformly for $t \in [1, N]_{\mathbb{Z}}$.

(h_6) The inequalities

$$0 < \liminf_{|x| \to \infty} \frac{h(t,x)}{x} \leqslant \limsup_{|x| \to \infty} \frac{h(t,x)}{x} < \gamma_j$$

hold uniformly in $t \in [1, N]_{\mathbb{Z}}$.

(h_7) The identity

$$\lim_{|x|\to\infty} h(t,x)x - 2H(t,x) = +\infty$$

holds uniformly in $t \in [1, N]_{\mathbb{Z}}$.

Our main results are formulated in the next two theorems.

Theorem 1. *Assume that $h(t, x)$ satisfies (h_1)–(h_5), then the problem (1.1) possesses at least $2(j - l + 1)$ nontrivial solutions.*

Theorem 2. *Assume that $h(t, x)$ satisfies (h_1), (h_2), (h_5)–(h_7), then the problem (1.1) possesses at least $2(j - l + 1)$ nontrivial solutions.*

The paper is organized as follows. In Section 2 we prove an auxiliary lemma. In Section 3 we prove the main results. In the end of paper we provide the examples to illustrate our results.

2. Preliminaries

We define the vector space E^N

$$E^N = \{y : [-(n-1), N+n]_{\mathbb{Z}} \to \mathbb{R} \mid \Delta^i y(-(n-1)) = \Delta^i y(N-(n-1)),\ i \in [0, 2n-1]_{\mathbb{Z}}\},$$

The inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ in E^N are

$$\langle y, z\rangle = \sum_{t=1}^{N} y(t)z(t), \qquad \|y\| = \left(\sum_{t=1}^{N} |y(t)|^2\right)^{\frac{1}{2}}, \quad y, z \in E^N.$$

Remark 1. *It is clear that for each $y \in E^N$ we have*

$$\begin{aligned} y(-(n-1)) &= y(N-(n-1))\\ y(-(n-2)) &= y(N-(n-2))\\ y(-(n-3)) &= y(N-(n-3))\\ &\ \vdots \quad \vdots\\ y(0) &= y(N)\\ y(1) &= y(N+1)\\ &\ \vdots \quad \vdots\\ y(n) &= y(N+n). \end{aligned} \tag{2.1}$$

The space E^N is isomorphic to the finite dimensional space $\mathbb{R}^N$ and this makes E^N the N–dimensional Hilbert space. By writing $y = (y(1), \ldots, y(N)) \in \mathbb{R}^N$ we mean that y can be extended to a vector in E^N so that (2.1) holds, namely, y can be extended to the vector

$$(y(N-(n-1)), y(N-(n-2)), \ldots, y(N), y(1), y(2), \ldots, y(N), y(1), \ldots, y(n)) \in E^N.$$

Let us discuss the eigenvalues and eigenvectors of (1.2). By applying the results of [2] we see that the problem (1.2) has precisely N real eigenvalues γ_j, $j \in [0, N-1]_{\mathbb{Z}}$,

$$\begin{cases} \gamma_j = \mu_0 + 2\displaystyle\sum_{l=1}^{n} \mu_l \cos\left(\frac{2\pi l j}{N}\right), & j \in [0, N-1]_{\mathbb{Z}},\\ \gamma_j = \gamma_{N-j}, & j \in [1, N-1]_{\mathbb{Z}}, \end{cases} \tag{2.2}$$

where $\mu_l = (-1)^l \sum_{j=l}^{n} C_{2j}^{j+l}$, $l \in [0,n]_{\mathbb{Z}}$, and C_k^p are the binomial coefficients. The eigenspaces $E(\gamma_j)$ associated with γ_j, $j \in [0, N-1]_{\mathbb{Z}}$ read

$$\begin{cases} E(\gamma_0) = \operatorname{span}(\varphi_0), \\ E(\gamma_j) = \operatorname{span}(\varphi_j, \chi_j), \quad j \in [1, N-1]_{\mathbb{Z}}. \end{cases} \tag{2.3}$$

where

$$\varphi_j = (\varphi_j(0), \varphi_j(1), \varphi_j(2), \dots, \varphi_j(N-1))^t, \qquad \chi_j = (\chi_j(0), \chi_j(1), \chi_j(2), \dots, \chi_j(N-1))^t$$

for $j \in [0, N-1]_{\mathbb{Z}}$, and with

$$\varphi_j(r) = \cos \frac{2\pi r j}{N}, \qquad \chi_j(r) = \sin \frac{2\pi r j}{N}, \qquad r \in [0, N-1]_{\mathbb{Z}}.$$

Since $\gamma_j = \gamma_{N-j}$ for $j \in [1, N-1]_{\mathbb{Z}}$, the problem (1.2) has $q+1$ different eigenvalues. These eigenvalues are ordered as follows:

$$0 < \gamma_0 < \gamma_1 < \gamma_2 < \dots < \gamma_q.$$

We have the decomposition

$$E^N = E^- \bigoplus E^{(j)} \bigoplus E^+,$$

where

$$E^- = \bigoplus_{i=0}^{j-1} E(\gamma_i), \qquad E^{(j)} = E(\gamma_j), \qquad E^+ = \bigoplus_{i=j+1}^{N-1} E(\gamma_i).$$

Hence, we have

$$\gamma_{j+1} \|y^+\|^2 \leqslant \sum_{t=1}^{N} \sum_{k=0}^{n} |\Delta^k y^+(t-k)|^2 \leqslant \gamma_{N-1} \|y^+\|^2, \ \ \forall y^+ \in E^+. \tag{2.4}$$

On E^N we define the functional Ψ by the formula

$$\Psi(y) = \frac{1}{2} \sum_{t=1}^{N} \sum_{k=0}^{n} |\Delta^k y(t-k)|^2 - \frac{1}{2} \gamma_j \sum_{t=1}^{N} |y(t)|^2 - \sum_{t=1}^{N} H(t, y(t)). \tag{2.5}$$

It is easy to see that $\Psi \in C^1(E^N, \mathbb{R})$ and its derivative $\Psi'(y)$ at $y \in E^N$ is given by

$$\langle \Psi'(y), z \rangle = \sum_{t=1}^{N} \left[\sum_{k=0}^{n} \Delta^k y(t-k) \Delta^k z(t-k) - \gamma_j y(t) z(t) - h(t, y(t)) z(t) \right] \tag{2.6}$$

for $z \in E^N$.

By [2, Lm. 2.3], the derivative Ψ' can be written as

$$\langle \Psi'(y), z \rangle = \sum_{t=1}^{N} \left[\sum_{k=0}^{n} (-1)^k \Delta^{2k} y(t-k) - \gamma_j y(t) - h(t, y(t)) \right] z(t), \qquad z \in E^N.$$

The solutions to Equation (1.1) are critical points of the functional Ψ.

The space of functionals from E into $\mathbb{R}$, which are continuously Fréchet differentiable, is denoted by $C^1(E, \mathbb{R})$. By S_l we denote the sphere in E of the radius l centered at the origin.

Definition 1. *Let E be a real Banach space, and $\Psi \in C^1(E, \mathbb{R})$. The functional Ψ is said to satisfy the Cerami condition (C) if each sequence $(y_t) \subset E$ for which $(\Psi(y_t))$ is bounded and $(1 + \|u_t\|)\|\Psi'(y_t)\| \to 0$ as $t \to \infty$, possesses a convergent subsequence.*

Remark 2. *We note that the Cerami condition is weaker than the Smale palate condition.*

Lemma 1 ([9]). *Let E be a reflexive Banach space, $\Psi \in C^1(E, \mathbb{R})$ with $\Psi(0) = 0$. Assume that Ψ is an even functional satisfying Condition (C) and the following conditions:*
1) *There exist constants l, $c > 0$ and a closed linear subspace E_1 of E such that* $\operatorname{codim} E_1 = l$ *and $\Psi|_{E_1 \cap S_l} \geqslant c$.*
2) *There exists a subspace E_2 with* $\dim E_2 = k$, *$k > l$, such that $\Psi(y) \to -\infty$ as $\|y\| \to \infty$, $y \in E_2$.*

Then Ψ possesses at least $2(k - l)$ nontrivial critical points.

3. Proof of main results

Here we prove our main results.

3.1. Proof of Theorem 1.

We begin with the following lemma.

Lemma 2. *Under Conditions (h_3) and (h_4) the functional Ψ satisfies Condition (C).*

Proof. Let $(y_n) \subset E^N$ be such that $\{\Psi(y_m)\}$ is bounded and $(1 + \|y_m\|) \|\Psi'(y_m)\| \to 0$ as $m \to +\infty$. It is sufficient to show that $\{y_m\}$ remains bounded in E^N. We argue by contradiction.

Suppose that $\|y_m\| \to \infty$ as $m \to \infty$. For each $z \in E^N$ we have

$$\langle \Psi'(y_m), z \rangle = \sum_{t=1}^{N} \sum_{k=0}^{n} \Delta^k y_m(t-k) \Delta^k z(t-k) - \gamma_j \sum_{t=1}^{N} y_m(t) z(t) - \sum_{t=1}^{N} h(t, y_m(t)) z(t) = o(\|z\|). \tag{3.1}$$

Let $y_m = y_m^- + y_m^{(j)} + y_m^+$, where $y_m^- \in E^-, y_m^{(j)} \in E^{(j)}$ and $y_m^+ \in E^+$. Taking $z = y_m^+$ in (3.1), by (2.4) we get

$$(\gamma_{j+1} - \gamma_j) \|y_m^+\|^2 \leqslant \sum_{k=1}^{N} \sum_{k=0}^{n} \left|\Delta^k y_m^+(t-k)\right|^2 - \gamma_j \|y_m^+\|^2 \leqslant \|y_m^+\| + \sum_{t=1}^{N} h(t, y_m(t)) y_m^+(t). \tag{3.2}$$

By (h_3), for each $\varepsilon > 0$, there exists $C_1 \in \mathbb{R}$ such that

$$h(t, x) \leqslant \varepsilon |x| + C_1, \qquad (t, x) \in [1, N]_{\mathbb{Z}} \times \mathbb{R}.$$

By combining the above inequality with (3.2) we deduce

$$\begin{aligned}(\gamma_{j+1} - \gamma_j) \|y_m^+\|^2 &\leqslant \|y_m^+\| + \sum_{t=1}^{N} (\varepsilon |y_m(t)| + C_1) |y_m^+(t)| \\ &\leqslant \|y_m^+\| + \varepsilon \|y_m\| \|y_m^+\| + C_2 \|y_m^+\| \\ &\leqslant (1 + C_2) \|y_m^+\| + \varepsilon \|y_m\| \|y_m^+\|,\end{aligned}$$

where C_2 is a positive constant independent of m. Since ε is arbitrary, we obtain

$$\frac{\|y_m^+\|}{\|y_m\|} \to 0 \quad \text{as} \quad m \to \infty. \tag{3.3}$$

Letting $z = y_m^-$ in (3.1), by similar reasons we demonstrate that

$$\frac{\|y_m^-\|}{\|y_m\|} \to 0 \quad \text{as} \quad m \to \infty. \tag{3.4}$$

According to (3.3) and (3.4),

$$\frac{\|y_m^+ + y_m^-\|}{\|y_m\|} \to 0, \qquad \frac{\|y_m^{(j)}\|}{\|y_m\|} \to 1, \qquad m \to \infty.$$

By (h_4) there exists $\tau > 0$ and $m_0 \in \mathbb{N}$ such that

$$\langle \Psi'(y_m), y_m^{(j)} \rangle = -\sum_{t=1}^{N} h(t, y_m(t)) y_m^{(j)}(t) \leqslant -\tau \quad \text{for} \quad m \geqslant m_0.$$

Then, for each $m \geqslant m_0$, we get

$$\|\Psi'(y_m)\| \|y_m\| \geqslant \|\Psi'(y_m)\| \|y_m^{(j)}\| \geqslant |\langle \Psi'(y_m), y_m^{(j)} \rangle| \geqslant |\langle -\Psi'(y_m), y_m^{(j)} \rangle| \geqslant \tau.$$

This contradicts the assumption $(1 + \|y_m\|) \|\Psi'(y_m)\| \to 0$ as $m \to +\infty$. The proof is complete. □

Lemma 3. *Under Conditions (h_2) and (h_3) there exists constants ρ, $a > 0$ such that $\Psi(y) \geqslant a$ for $y \in E_l^+$ with $\|y\| = \rho$, where*

$$E_l^+ = \bigoplus_{i=l}^{N-1} E(\gamma_i).$$

Proof. It follows from Condition (h_2) that for each $\varepsilon > 0$ there exists $R > 0$ such that

$$H(t, x) < \frac{1}{2}(\gamma_l - \gamma_j - \varepsilon)|x|^2, \qquad (t, |x|) \in [1, N] \times [0, R].$$

By (h_3), for $\varepsilon = 1$ there exist $b \in \mathbb{R}$ such that

$$|h(t, x)| \leqslant |x| + b.$$

Therefore, there exists $C_3 > 0$ such that

$$|H(t, x)| \leqslant C_3 |x|^p$$

for $|x| > R$, where $p > 2$.

Let $y \in E_l^+$ and

$$S_1 = \{t \in [1, N]_{\mathbb{Z}} : |y(t)| \leqslant R\}, \qquad S_2 = \{t \in [1, N]_{\mathbb{Z}} : |y(t)| > R\}.$$

We get

$$\begin{aligned}
\Psi(y) &= \frac{1}{2} \sum_{t=1}^{N} \sum_{k=0}^{n} \left|\Delta^k y(t-k)\right|^2 - \frac{1}{2}\gamma_j \sum_{t=1}^{N} |y(t)|^2 - \sum_{t=1}^{N} H(t, y(t)) \\
&\geqslant \frac{1}{2}\gamma_l \|y\|^2 - \frac{1}{2}\gamma_j \|y\|^2 - \sum_{t \in S_1} H(t, y(t)) - \sum_{t \in S_2} H(t, y(t)) \\
&\geqslant \frac{1}{2}(\gamma_l - \gamma_j) \|y\|^2 - \frac{1}{2}(\gamma_l - \gamma_j - \varepsilon) \|y\|^2 + \frac{1}{2}(\gamma_l - \gamma_j - \varepsilon) \sum_{t \in S_2} |y(t)|^2 - C_3 \sum_{t \in S_2} |y(t)|^p \\
&\geqslant \frac{1}{2}\varepsilon \|y\|^2 + \frac{1}{2}(\gamma_l - \gamma_j - \varepsilon) \sum_{t \in S_2} |y(t)|^p |y(t)|^{2-p} - C_3 \sum_{t \in S_2} |y(t)|^p \\
&\geqslant \frac{1}{2}\varepsilon \|y\|^2 + \frac{1}{2}(\gamma_l - \gamma_j - \varepsilon) R^{2-p} \sum_{t \in S_2} |y(t)|^p - C_3 \sum_{t \in S_2} |y(t)|^p \\
&\geqslant \frac{1}{2}\varepsilon \|y\|^2 - \frac{1}{2}(\gamma_j - \gamma_l + \varepsilon + C_3) \sum_{t \in S_2} |y(t)|^p \\
&\geqslant \frac{1}{2}\varepsilon \|y\|^2 - C_4 \|y\|^p \\
&= \|y\|^2 \left[\frac{1}{2}\varepsilon - C_4 \|y\|^{p-2}\right],
\end{aligned}$$

where C_4 is a constant positive.

Since $p > 2$, the function $x \to \frac{1}{2}\varepsilon - C_4 x^{p-2}$ is strictly positive in a neighborhood of zero. Hence, there exist $\rho > 0$ and $a > 0$ such that $\Psi(y) \geqslant a$ for all $y \in E_l^+$ with $\|y\| = \rho$. The proof is complete. □

Lemma 4. *Under Conditions*(h_5) *the convergence* $\Psi(y) \to -\infty$ *holds as* $\|y\| \to \infty$, $y \in \widehat{E} = E^- \oplus E^{(j)}$.

Proof. By Condition (h_5) there exists $A > 0$ such that

$$-H(t,x) \leqslant 0, \qquad (t,|x|) \in [1,N]_{\mathbb{Z}} \times [A,+\infty)$$

However, by the continuation of H we have

$$-H(t,x) \leqslant \overline{H} = \max_{|x| \leqslant A} |-H(t,x)|, \qquad \forall (t,|x|) \in [1,N]_{\mathbb{Z}} \times [0,A].$$

Then,

$$-H(t,x) \leqslant \overline{H}, \qquad l(t,x) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}. \tag{3.5}$$

Let $y \in \hat{E} = E^- \oplus E^{(j)}$. Assume that $\|y\| \to \infty$, then there exists a nonempty subset $\Lambda \subset [1,N]_{\mathbb{Z}}$ such that for $|y(t)| \to \infty$ as $t \in \Lambda$.

Hence, by Condition (h_5),

$$-\sum_{t \in \Lambda} H(t, y(t)) \to -\infty \tag{3.6}$$

By (3.5) and (3.6) we obtain

$$\begin{aligned}\Psi(y) &\leqslant -\sum_{t=1}^{N} H(t,y(t)) \leqslant -\sum_{t \in \Lambda} H(t,y(t)) - \sum_{t \notin \Lambda} H(t,y(t)) \\ &\leqslant -\sum_{t \in \Lambda} H(t,y(t)) + \text{mes}\{[1,N] \setminus \Lambda\}\overline{H} \to -\infty.\end{aligned}$$

where $\text{mes}\{[1,N] \setminus \Lambda\}$ is the number of the element of the set $[1,N] \setminus \Lambda$. The proof is complete. □

Proof of Theorem 1. It follows from the definition (2.5) of Ψ that $\Psi(0) = 0$ and Ψ is even by Condition (h_1). Lemma 2 implies that Ψ satisfies Condition (C). We take $E = E^N$, $E_1 = E_l^+$ and $E_2 = \widehat{E}$. Lemmas 3 and 4 show that Ψ satisfies assumptions 1) and 2) of Lemma 1, respectively. Then the functional Ψ possesses at least $2(j - l + 1)$ nontrivial critical points, which are nontrivial solutions of the problem (1.1). The proof is complete. □

3.2. Proof of Theorem 2.

We begin with two auxiliary statements.

Lemma 5. *Under Condition* (h_7) *holds the functional* Φ *satisfies Condition* (C).

Proof. Let $(y_m) \subset E^N$ be such that for some $C_5 > 0$ the inequality and convergence hold

$$|\Psi(y_m)| \leqslant C_5, \qquad (1 + \|y_m\|)\,\|\Psi'(y_m)\| \to 0, \qquad m \to +\infty.$$

It is easy to see that

$$2\Psi(y_m) - \langle \Psi'(y_m), y_m \rangle = \sum_{t=1}^{N} h(t, y_m(t))\, y_m(t) - 2H(t, y_m(t)).$$

By the Cauchy inequality we have

$$\begin{aligned}\left|\sum_{t=1}^{n} h\left(t, y_m(t)\right) y_m(t)-2 H\left(t, y_m(t)\right)\right| &\leqslant 2\left|\Psi\left(y_m\right)\right|+\left|\left\langle\Psi^{\prime}\left(y_m\right), y_m\right\rangle\right| \\ &\leqslant 2 C_5+\left\|\Psi^{\prime}\left(y_m\right)\right\| \cdot\left\|y_m\right\| \\ &\leqslant 2 C_5+\left(1+\left\|y_m\right\|\right)\left\|\Psi^{\prime}\left(y_m\right)\right\| \\ &\leqslant 2 C_5+C_6,\end{aligned} \tag{3.7}$$

where C_6 is a positive constant.

It remains to demonstrate that $\{y_m\}$ is bounded. Suppose that $\|y_m\| \to \infty$ as $m \to \infty$. Then there exists $t_0 \in [1, N]_{\mathbb{Z}}$ such that

$$|y_m(t_0)| \to +\infty, \qquad m \to +\infty.$$

By Condition (h_7) we obtain

$$h(t_0, y_m(t_0))\, y_m(t_0) - 2H(t_0, y_m(t_0)) \to +\infty \quad \text{as} \quad m \to \infty.$$

On the other hand, by the continuity of h and Copndition (h_7) there exists a constant $C_7 \in \mathbb{R}$ such that

$$h(t,x)x - 2H(t,x) \geqslant C_7, \qquad (t,x) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}.$$

This yields

$$\begin{aligned}\sum_{t=1}^{N} h\left(t, y_m(t)\right) y_m(t)-2 H\left(t, y_m(t)\right)= & h\left(t_0, y_m\left(t_0\right)\right) y_m\left(t_0\right)-2 H\left(t_0, y_m\left(t_0\right)\right) \\ & +\sum_{t \neq t_0}^{N} h\left(t_0, y_m(t)\right) y_m(t)-2 H\left(t_0, y_m(t)\right) \\ \geqslant & h\left(t_0, y_m\left(t_0\right)\right) y_m\left(t_0\right) \\ & -2 H\left(t_0, y_m\left(t_0\right)\right)+(N-1) C_7 \rightarrow+\infty, \quad m \rightarrow+\infty,\end{aligned}$$

which contradicts (3.7). This completes the proof. □

Lemma 6. *Under Conditions (h_2) and (h_6) there exist constants ρ, $a \geqslant 0$ such that $\Phi(y) \geqslant a$ for $u \in E_l^+$ with $\|u\| = \rho$.*

Proof. By Condition (h_2) for each $\epsilon > 0$ there exists $R > 0$ such that

$$H(t,t) \leqslant \frac{1}{2}(\lambda_l - \lambda_j - \varepsilon)|x|^2, \qquad (t,|x|) \in [1,z]_{\mathbb{Z}} \times [0,R].$$

By Condition (h_6), for each $\epsilon > 0$ there exists $r > 0$ such that

$$-\varepsilon|x|^2 \leqslant xh(t,x) \leqslant (\gamma_j + \varepsilon)|x|^2, \qquad (t,|x|) \in [1,N]_{\mathbb{I}} \times [r,+\infty).$$

On the other hand, by the continuity of $x \to h(t,x)$, there exists $C_8 > 0$ such that

$$|h(t,x)| \leqslant C_8, \qquad (t,|x|) \in [1,N]_{\mathbb{Z}} \times [0,r].$$

Hence,

$$-\varepsilon|x|^2 - C_8|x| \leqslant xh(t,x) \leqslant (\gamma_j + \varepsilon)|x|^2 + C_8|x|.$$

This implies

$$|h(t,x)| \leqslant (\gamma_j + \varepsilon)|x| + C_8, \qquad l(t,x) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}. \tag{3.8}$$

Therefore, there exists $C_9 > 0$ such that

$$|H(t,x)| \leqslant C_9|x|^q, \quad \forall (t,|x|) \in [1,N]_{\mathbb{Z}} \times [R,+\infty),$$

where $q > 2$. The rest of the proof is similar to that of Lemma 3. The proof is complete. □

Proof of Theorem 2. The functional Ψ defined by (2.5) satisfies the identity $\Psi(0) = 0$ and is even by Condition (h_1). Lemma 5 implies that Ψ satisfies Condition (C). We take $E = E^N$, $E_1 = E_l^+$ and $E_2 = \widehat{E}$. Lemmas 4 and 6 imply that Ψ satisfies assumptions 1) and 2) of Lemma 1, respectively. Then the functional Ψ possesses at least $2(j-l+1)$ nontrivial critical points, which are nontrivial solutions of the problem (1.1). The proof is complete. □

4. Examples

4.1. Example for Theorem 1. We define $h : [1,N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ as

$$\begin{cases} h(t,x) = \dfrac{-\gamma_j x^3}{2t(1+x^2)} + \dfrac{1}{2t}\gamma_j x, & |x| \geqslant 1, \quad t \in [1,N]_{\mathbb{Z}}, \\ h(t,x) = \dfrac{-\gamma_j x}{2t(1+x^2)} + \dfrac{1}{2t}\gamma_j x^3, & |x| < 1, \quad t \in [1,N]_{\mathbb{Z}}. \end{cases} \tag{4.1}$$

Then we have

$$\begin{cases} H(t,x) = \dfrac{\gamma_j}{4t}\ln\left(1+x^2\right), & |x| \geqslant 1, \quad t \in [1,N]_{\mathbb{Z}}, \\ H(t,x) = -\dfrac{\gamma_j}{4t}\ln\left(1+x^2\right) + \dfrac{1}{4t}\gamma_j x^4, & |x| < 1, \quad t \in [1,N]_{\mathbb{Z}}. \end{cases} \tag{4.2}$$

It is easy to verify that that the function $h(t,x)$ satisfies Conditions (h_1), (h_3) and the function $H(t,x)$ satisfies Condition (h_5).

By simple calculations we find

$$\limsup_{x\to 0} \frac{2H(t,x)}{x^2} = \frac{-\gamma_j}{4t} < \gamma_j - \gamma_j = 0.$$

Therefore, Condition (h_2) is satisfied with $\gamma_l = \gamma_j$.

We are going to prove that $h(t,x)$ satisfies Condition (h_4). By Condition (h_4) for each sufficiently small ε we have the estimate

$$\|y_m^\star\| \leqslant \varepsilon \|y_m\| \tag{4.3}$$

for sufficiently large m. We denote

$$\Omega_1 = \{t \in [1,N]_{\mathbb{Z}} : |y_m(t)| \geqslant 1\}, \qquad \Omega_2 = \{t \in [1,N]_{\mathbb{Z}} : |y_m(t)| < 1\}.$$

We have

$$\sum_{t=1}^{N} h(t, y_m(t))\, y_m^{(j)}(t) = \sum_{t\in\Omega_1} h(t, y_m(t))\, y_m^{(j)}(t) + \sum_{t\in\Omega_2} h(t, y_m(t))\, y_m(t) - \sum_{t\in\Omega_2} h(t, y_m(t))\, y_m^\star(t).$$

On the other hand,

$$\sum_{t\in\Omega_2} h(t, y_m(t))\, y_m(t) = \sum_{t\in\Omega_2} \frac{-\gamma_j}{2t}\frac{y_m^2(t)}{1+y_m^2(t)} + \frac{1}{2t}\gamma_j y_m^4(t) \geqslant C, \tag{4.4}$$

where C is constant.

By the Cauchy — Bunyakovsky — Schwarz inequality we obtain

$$\begin{aligned} -\sum_{t\in\Omega_2} h(t, y_m(t))\, y_m^\star(t) &= \frac{\gamma_j}{2}\sum_{t\in\Omega_2} \frac{y_m(t) y_m^\star(t)}{t\left(1+y_m^2(t)\right)} + \frac{\gamma_j}{2}\sum_{t\in\Omega_2}\frac{1}{t} y_m^3(t) y_m^\star(t) \\ &\geqslant \frac{\gamma_j}{2N}\sum_{t\in\Omega_2}\frac{y_m(t)y_m^\star(t)}{1+y_m^2(t)} + \frac{\gamma_j}{2N}\sum_{t\in\Omega_2} y_m(t) y_m^\star(t) \\ &\geqslant \frac{3\gamma_j}{4N}\sum_{t\in\Omega_2} y_m(t) y_m^\star(t) \end{aligned}$$

$$\geqslant -\frac{3\gamma_j}{4N}\|y_m\|\,\|y_m^\star\|.$$

By (4.3) we get

$$-\sum_{t\in\Omega_2} h\left(t, y_m(t)\right) y_m^\star(t) \geqslant \frac{-3\gamma_j}{4N}\varepsilon\|y_m\|^2. \tag{4.5}$$

In the other hand, using Cauchy-Schwarz inequality, it is clear to see that

$$\begin{aligned}\sum_{t\in\Omega_1} h\left(t, y_m(t)\right) y_m^{(j)}(t) &= \sum_{t\in\Omega_1}\left[-\frac{\gamma_j}{2t}\frac{y_m^3(t)}{1+y_m^2(t)}y_m^{(j)}(t) + \frac{1}{2t}\gamma_j y_m(t) y_m^{(j)}(t)\right]\\ &= \sum_{t\in\Omega_1}\left[\frac{-\gamma_j y_m^4(t)}{2t\left(1+y_m^2(t)\right)} + \frac{\gamma_j y_n^3(t) y_m^\star(t)}{2t\left(1+y_m^2(t)\right)} + \frac{1}{2t}\gamma_j y_m^2(t) - \frac{1}{2t}\gamma_j y_m(t) y_m^\star(t)\right]\\ &\geqslant \quad \gamma_j\left[\frac{1}{2N}-\varepsilon\right]\|y_m\|^2.\end{aligned}$$

By the above inequality and (4.4), (4.5) we find

$$\sum_{t=1}^{N} h\left(t, y_m(t)\right) y_m^{(j)}(t) \geqslant \frac{\gamma_j}{2N}\left[1-\left(2N+\frac{3}{2}\right)\varepsilon\right]\|y_m\|^2 + C.$$

Since ε is small enough, Condition (h_4) holds. Hence, by Theorem 1, the problem (1.1) has at least two nontrivial solutions.

4.2. Example for Theorem 2. We consider a continuous function $h : [1, N]_{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}$ given by the formula:

$$h(t, x) = \left[\frac{-x}{1+x^2} + \frac{1}{2}x\right]\gamma_j e^{-t}, \quad (t, x) \in [1, N]_{\mathbb{Z}} \times \mathbb{R}.$$

We obviously have

$$H(t, x) = \left[\frac{-\ln(1+x^2)}{2} + \frac{1}{4}x^2\right]\gamma_j e^{-t}, \quad (t, x) \in [1, N]_{\mathbb{Z}} \times \mathbb{R}.$$

It is easy to see that $h(t, x)$ satisfies Condition (h_1).

We obtain

$$\limsup_{x\to 0}\frac{2H(t,x)}{x^2} = \limsup_{x\to 0}\left[\left(\frac{-\ln(1+x^2)}{x^2}+\frac{1}{2}\right)\gamma_j e^{-t}\right] = \frac{-\gamma_j}{2}e^{-t} < \gamma_j - \gamma_j = 0,$$

and hence, Condition (h_1) is satisfied with $\gamma_l = \gamma_j$. On the other hand,

$$\lim_{|x|\to+\infty} H(t, x) = \lim_{|x|\to+\infty} x^2\left[\frac{1}{4} - \frac{\ln(1+x^2)}{2x^2}\right]\gamma_j e^{-t} = +\infty,$$

$$0 < \lim_{|x|\to\infty}\inf\frac{h(t,x)}{x} = \lim_{|x|\to\infty}\sup\frac{h(t,x)}{x} = \frac{1}{2}\gamma_j e^{-t} < \gamma_j,$$

and

$$\lim_{|x|\to+\infty} h(t, x)x - 2H(t, x) = \lim_{|x|\to+\infty}\left[\frac{-x^2}{1+x^2} + \ln(1+x^2)\right]\gamma_j e^{-t} = +\infty.$$

Thus all assumptions of Theorem 2 are satisfied and the problem (1.1) has at least two nontrivial solutions.

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