ISSN 2304-0122 Ufa Mathematical Journal. Vol. 17. No 3 (2025). P. 1-8.

doi:10.13108/2025-17-3-1

DIMENSIONS OF RIESZ PRODUCTS AND PLURIHARMONIC MEASURES

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Abstract. On the unit sphere in $\mathbb{C}^n$, $n \geqslant 2$, we consider the Riesz products generated by the Ryll — Wojtaszczyk polynomials. We obtain the lower bound for the energy dimension of such Riesz products. The obtained inequality implies immediately an estimate for the Hausdorff dimension of the considered products. This results is also obtained in another way, by means of known one–dimensional estimates and decomposition into slice–products. These decompositions are employed for sharp estimating of the Hausdorff dimension of pluriharmonic measures on an n–dimensional torus, $n \geqslant 2$.

Keywords: Riesz product on sphere, energy dimension, Hausdorff dimenison, pluriharmonic measure.

Mathematics Subject Classification: 42A55, 28A78, 31C10, 43A85

1. Introduction

The present work is motivated by the following general question: how does the spectrum of a measure μ influence the size of support of measure μ? We first of all consider the Riesz products and related objects on the unit sphere

$$S = S_n = \{\zeta \in \mathbb{C}^n : |\zeta| = 1\}, \qquad n \geqslant 2,$$

and we begin with the definition of the spaces $H(p,q)$, $(p,q) \in \mathbb{Z}_+^2$.

1.1. Complex spherical harmonics.

Let $\mathcal{U}(n)$ be the group of unitary operators on the Hilbert space $\mathbb{C}^n$, $n \geqslant 2$. We note that $S_n = \mathcal{U}(n)/\mathcal{U}(n-1)$, and therefore, S_n is a homogeneous space. The general constructions of an abstract harmonic analysis are explicitly realized on the sphere S_n in terms of the spaces $H(p,q)$, $(p,q) \in \mathbb{Z}_+^2$.

Definition 1.1. *Let us fix the dimension n, $n \geqslant 2$, and let $H(p,q) = H(p,q;n)$ be the space of all homogeneous harmonic polynomials of bi–degree $(p,q) \in \mathbb{Z}_+^2$. By the definition this means that the considered polynomials have the degree p in the variables z_1, z_2, $\dots$, z_n, the degree q in the variable $\overline{z}_1$, $\overline{z}_2$, $\dots$, $\overline{z}_n$, and also have the total degree $p+q$.*

For the restriction of space $H(p,q)$ to the sphere S we use the same symbol. The elements of the space $H(p,q)$ are often called complex spherical harmonics.

1.2. Riesz products on spher.

The Zygmund dichotomy [15] hints that the homogeneous holomorphic polynomials introduced by Ryll and Wojtaszczyk [14] can be used for constructing singular products of Riesz type on the sphere.

We say that $\{R_j\}_{j=1}^{\infty}$ is the Ryll — Wojtaszczyk sequence with a constant $\delta \in (0,1)$ if

1. $R_j \in H(j,0)$, that is, R_j is a homogeneous holomorphic polynomial of degree j;

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Submitted August 10, 2024.

2. $\|R_j\|_{L^\infty(S)} = 1$;
3. $\|R_j\|_{L^2(S)} \geqslant \delta$ for all $j = 1, 2, \dots$.

Definition 1.2. *Let $R = \{R_j\}_{j=1}^\infty$ be a Ryll — Wojtaszczyk sequence,*

$$J = \{j_k\}_{k=1}^\infty \subset \mathbb{N}, \qquad \frac{j_{k+1}}{j_k} \geqslant 3 \qquad \text{and} \qquad a = \{a_k\}_{k=1}^\infty \subset \mathbb{D}.$$

Then (R, J, a) is called the Riesz triple.

Each Riesz triple (R, J, a) generates a (standard) Riesz product $\Pi(R, J, a)$ by means of the formal identity

$$\Pi(R, J, a) = \prod_{k=1}^\infty \left(\frac{\overline{a}_k \overline{R}_{j_k}}{2} + 1 + \frac{a_k R_{j_k}}{2} \right),$$

where the product converges in $\star$–weak sense, see [7] for more detail.

There exists a rather rich set of singular Riesz products on the sphere. Indeed, if (R, J, a) is a Riesz triple and $a \notin \ell^2$, then by Corollary 1 in [6] there exists a sequence $U = \{U_j\}_{j=1}^\infty$, $U_j \in \mathcal{U}(n)$ such that the product $\Pi(R \circ U, J, a)$ is singular with respect to the Lebesgue measure on the sphere S_n. Here by the definition

$$R \circ U = \{R_j \circ U_j\}_{j=1}^\infty.$$

For a singular Riesz product, it is natural to ask the question about its dimension. In the present work we provide the estimates for the energy and Hausdorff dimensions of Riesz products on the sphere. To the best of the author's knowledge, the problems of such kind were not studied earlier.

Let $\dim_{\mathcal{H}}\Pi$ denote the Hausdorff dimension of a measure Π. In particular, we obtain the following result.

Theorem 1.1. *Let (R, J, a) be a Riesz triple. Then*

$$\dim_{\mathcal{H}}\Pi(R, J, a) \geqslant 2n - 1 - \limsup_{k\to\infty} \left(\frac{1}{2\log j_k} \sum_{\ell=1}^{k-1} |a_\ell|^2 \right).$$

1.3. Structure of work. Auxiliary results including the definitions and basic facts in the harmonic analysis on the sphere S_n are collected in Section 2. The estimates for energy and Hausdorff dimensions of Riesz products on the unit sphere are obtained in Section 3. The related problem on Hausdorff dimensions of pluriharmonic measures on the torus is considered in Section 4. We note that Lemma 2.1 on the disintegration of Riesz products serves as a link between the results of Section 3, in which this lemma is used directly, and Section 4, where analogues of Lemma 2.1 are applied.

1.4. Notation. For $F, G > 0$ the writing $F \lesssim G$ means that $F \leqslant CG$ with some constant $C > 0$. If $F \lesssim G$ and $G \lesssim F$, we use the notation $F \approx G$.

2. Auxiliary results

2.1. Basic facts in harmonic analysis on sphere S_n. Let $\sigma = \sigma_n$ denote the normalized Lebesgue measure on the unit sphere S_n. We note that

$$L^2(\sigma) = \bigoplus_{(p,q)\in\mathbb{Z}_+^2} H(p, q). \tag{2.1}$$

The features of harmonic analysis on the sphere S are well illustrated by the following multiplication rule for the spaces $H(p,q)$: if $f \in H(p,q)$ and $g \in H(r,s)$, then

$$fg \in \sum_{\ell=0}^{L} H(p+r-\ell, q+s-\ell),$$

where $L = \min(p,s) + \min(q,r)$. The proofs of the formulated facts and of further results are provided in [13, Ch. 12].

Let $M(S_n)$ be the space of all complex Borel measures defined on the sphere S_n. Let $K_{p,q}(z,\zeta)$ be the reproducing kernel of the Hilbert space $H(p,q) \subset L^2(S)$. The polynomial

$$\mu_{p,q}(z) = \int_S K_{p,q}(z,\zeta)\, d\mu(\zeta), \quad z \in S,$$

is called the $H(p,q)$–projection of the measure $\mu \in M(S)$. For $\mu \in M(S)$, in terms of the spaces $H(p,q)$, the spectrum $\operatorname{spec}(\mu)$ is defined by the identity

$$\operatorname{spec}(\mu) = \left\{(p,q) \in \mathbb{Z}_+^2 : \mu_{p,q} \neq \mathbf{0}\right\}.$$

2.2. Desintegration in terms of slice–products. Let $\mathbb{T} = S_1$ be the unit circle. For $\xi \in S_n$ the slice–product

$$\Pi_\xi(R,J,a)(\lambda) := \Pi(R(\lambda\xi),J,a), \quad \lambda \in \mathbb{T},$$

is the classical Riesz product defined by the formula

$$\prod_{k=1}^{\infty} \left(\frac{\overline{a}_k \overline{R}_{j_k}(\xi) \overline{\lambda}^{j_k}}{2} + 1 + \frac{a_k R_{j_k}(\xi) \lambda^{j_k}}{2} \right), \quad \lambda \in \mathbb{T}. \tag{2.2}$$

The convergence of the product (2.2) is to be treated in $\star$–weak sense. In particular, $\Pi_\xi(R,J,a)$ is a well–defined probability measure.

Let $\mathbb{CP}^{n-1}$ be the complex projective space of the dimension $n-1$, that is, the set of all one–dimensional linear subspaces of the space $\mathbb{C}^n$. Let $\pi = \pi_n$ be the canonical projection from S_n to $\mathbb{CP}^{n-1}$. We observe that $\Pi_{\lambda\xi}(w) = \Pi_\xi(\lambda w)$ for $\xi \in S_n$, $\lambda, w \in \mathbb{T}$. Therefore, the probabilistic slice–measure Π_ζ is well–defined for $\zeta \in \mathbb{CP}^{n-1}$ as an element of $M_+(S_n)$, the support of which is the unit circle $\pi^{-1}(\zeta) \subset S_n$.

Let $\widehat{\sigma}_n$ be the unique probabilistic measure on $\mathbb{CP}^{n-1}$, which is invariant with respect to all unitary transformations of the space $\mathbb{C}^n$. The next lemma shows that the product $\Pi(R,J,a)$ is the integral of its sections.

Lemma 2.1 ([6, Lm. 1])**.** *Let (R,J,a) be a Riesz triple on the sphere S_n, $n \geqslant 2$, and let $\Pi(R,J,a)$ denote the corresponding Riesz product. Then*

$$\Pi(R,J,a) = \int_{\mathbb{CP}^{n-1}} \Pi_\zeta(R,J,a)\, d\widehat{\sigma}_n(\zeta)$$

in the following weak sense

$$\int_{S_n} f\, d\Pi(R,J,a) = \int_{\mathbb{CP}^{n-1}} \int_{S_n} f\, d\Pi_\zeta(R,J,a)\, d\widehat{\sigma}_n(\zeta)$$

for all $f \in C(S_n)$.

3. Dimensions of Riesz products

3.1. Energy dimension. Let $M_+(S_n)$ be the set of all positive Borel measures on the sphere S_n. For a measure $\mu \in M_+(S)$, its t–energy $I_t(\mu)$, $t>0$, is defined by the identity

$$I_t(\mu) := \int\limits_S \int\limits_S \frac{d\mu(x)d\mu(y)}{|x-y|^t}.$$

We say that d is the energy dimension of the measure μ and use the notation $\dim_{\mathcal{E}}\mu = d$ if

$$d = \sup\{t : I_t(\mu) < \infty\}.$$

To estimate the energy dimension of the product $\Pi(R, J, a)$, we need the next theorem.

Theorem 3.1 ([12, Thm. 3.1])**.** *For each t, $0 < t < 2n-1$, there are constants $C_1, C_2 > 0$ such that*

$$C_1 I_t(\mu) \leqslant \|\mu_{0,0}\|_2^2 + \sum_{j=1}^{\infty} j^{t-2n+1} \sum_{p+q=j} \|\mu_{p,q}\|_2^2 \leqslant C_2 I_t(\mu)$$

for all $\mu \in M_+(S_n)$, where $n \geqslant 2$.

In the work [11], an analogue of Theorem 3.2 for the unit sphere was used to calculate the energy dimension of the classical Riesz product on the circle $\mathbb{T}$. In the next result we apply Theorem 3.1 to estimate from below the energy dimension of the Riesz product on the sphere. The used arguing is similar to the corresponding arguing in the proof of Theorem 3.1 in [11], however, the technical details differ.

Theorem 3.2. *Let (R, J, a) be a Riesz triple. Then*

$$\dim_{\mathcal{E}}\Pi(R, J, a) \geqslant 2n - 1 - \alpha_0,$$

where

$$\alpha_0 = \max\left[0,\ \limsup_{k\to\infty}\left(\frac{\log\frac{|a_k|^2}{2} + \sum\limits_{\ell=1}^{k-1}\log\left(1+\frac{|a_\ell|^2}{2}\right)}{\log j_k}\right)\right]. \tag{3.1}$$

In particular,

$$\dim_{\mathcal{E}}\Pi(R, J, a) \geqslant 2n - 1 - \limsup_{k\to\infty}\left(\frac{1}{2\log j_k}\sum_{\ell=1}^{k-1}|a_\ell|^2\right).$$

Proof. Let $\Pi := \Pi(R, J, a)$. For $k = 1, 2, \dots$ we denote

$$\Gamma_k := \left\{\pm j_k + \sum_{\ell=1}^{k-1}\varepsilon_\ell j_\ell : \varepsilon_\ell = 0, \pm 1\right\}.$$

Let $\gamma \in \Gamma_k$. Without loss of generality we suppose that $\gamma > 0$. Then there exists exactly one representation

$$\gamma = j_k + \sum_{\ell=1}^{k-1}\varepsilon_\ell j_\ell.$$

By the multiplication rule for the spherical harmonics, see Section 2.1, we have

$$\sum_{p-q=\gamma}\Pi_{p,q} = R_{j_k}\prod_{\ell=1}^{k-1}R_{j_\ell}(\varepsilon_\ell),$$

where

$$\begin{aligned} R_{j_\ell}(\varepsilon_\ell) &= R_{j_\ell}, && \varepsilon_\ell = 1,\\ R_{j_\ell}(\varepsilon_\ell) &= 1, && \varepsilon_\ell = 0,\\ R_{j_\ell}(\varepsilon_\ell) &= \overline{R}_{j_\ell}, && \varepsilon_\ell = -1. \end{aligned}$$

Applying the property (2.1), we obtain

$$\sum_{p-q=\gamma} \|\Pi_{p,q}\|_2^2 = \left\| R_{j_k} \prod_{\ell=1}^{k-1} R_{j_\ell}(\varepsilon_\ell) \right\|_2^2 \leqslant \left(\frac{|a_k|}{2} \prod_{\ell:\,\varepsilon_\ell\neq 0} \frac{|a_\ell|}{2} \right)^2,$$

where the empty product is equal to one.

Now we fix a number $\alpha > \alpha_0$. Since $p+q \geqslant p-q$, we have

$$\sum_{\gamma\in\Gamma_k} |p+q|^{-\alpha} \sum_{p-q=\gamma} \|\Pi_{p,q}\|_2^2 \leqslant \sum_{\gamma\in\Gamma_k} |\gamma|^{-\alpha} \sum_{p-q=\gamma} \|\Pi_{p,q}\|_2^2. \tag{3.2}$$

If $\gamma \in \Gamma_k$, then $|\gamma| \approx j_k$. Hence,

$$\sum_{\gamma\in\Gamma_k} |\gamma|^{-\alpha} \sum_{p-q=\gamma} \|\Pi_{p,q}\|_2^2 \lesssim j_k^{-\alpha} \frac{|a_k|^2}{2} \prod_{\ell=1}^{k-1} \left(1 + \frac{|a_\ell|^2}{2}\right). \tag{3.3}$$

Combining the properties (3.2) and (3.3), we obtain

$$I_{2n-1-\alpha}(\Pi) \lesssim 1 + j_k^{-\alpha} \frac{|a_k|^2}{2} \prod_{\ell=1}^{k-1} \left(1 + \frac{|a_\ell|^2}{2}\right) \tag{3.4}$$

by Theorem 3.1.

We have $\log j_k \geqslant (k-1)\log 3$, and thus, choosing the number A, $0 < A < 1$, sufficiently close to one, by means of the inequality $\alpha > \alpha_0$ we obtain that

$$\alpha \geqslant \frac{\log \frac{|a_k|^2}{2} + \sum_{\ell=1}^{k-1} \log\left(1 + \frac{|a_\ell|^2}{2}\right) + k|\log A|}{\log j_k},$$

in other words,

$$j_k^{-\alpha} \frac{|a_k|^2}{2} \prod_{\ell=1}^{k-1} \left(1 + \frac{|a_\ell|^2}{2}\right) \leqslant A^k.$$

Combining the latter property and (3.4), we conclude that

$$I_{2n-1-\alpha}(\Pi) < \infty.$$

Since the quantity $\alpha > \alpha_0$ is chosen arbitrarily, we finally have

$$\dim_{\mathcal{E}} \Pi \geqslant 2n - 1 - \alpha_0,$$

and this completes the proof. □

3.2. Hausdorff dimension. For a Borel set E by the symbol $\dim_{\mathcal{H}} E$ we denote its Hausdorff dimension. The Hausdorff dimension of a measure $\mu \in M_+(S_n)$ is defined by the identity

$$\dim_{\mathcal{H}}\mu = \inf\{\dim_{\mathcal{H}} E : E \text{ is a Borel set suuch that } \mu(E) > 0\}.$$

The properties of Hausdorff dimension of a measure are presented in [9, Ch. 10].

For the Riesz product $\Pi = \Pi(R, J, a)$ direct calculations show that there exists a sufficiently small $\varepsilon > 0$ such that $\Pi_{p,q} = \mathbf{0}$ under the condition

$$\left|\frac{p}{q} - 1\right| < \varepsilon, \qquad (p, q) \neq (0, 0).$$

This is why the application of Theorem 1.1 from [3] to the product Π ensures that $\dim_{\mathcal{H}}\Pi \geqslant 2n-2$. In fact, a sharper estimate can be obtained by means of Theorem 3.2.

3.2.1. Application of Theorem 3.2. If $I_t(\mu) < \infty$, then $\dim_{\mathcal{H}}\mu > t$, cf. [8, Sect. 4.3]. This is why the Hausdorff dimension of a measure is always not less than its energy dimension.

Corollary 3.1. *Let (R, J, a) be the Riesz triple. Then*

$$\dim_{\mathcal{H}}\Pi(R, J, a) \geqslant 2n - 1 - \alpha_0,$$

where the quantity α_0 is given by the identity (3.1).

Proof. Since $\dim_{\mathcal{H}}\Pi \geqslant \dim_{\mathcal{E}}\Pi$, it is sufficient to apply Theorem 3.2. □

Remark 3.1. *It is clear that Theorem 1.1 is a particular case of the above corollary.*

3.2.2. Reduction of problem to slice–products. An alternative approach to Corollary 3.1 is to consider slice–products $\Pi(R, J, a)$. We need the following theorem, which follows from Sections [10, Sects. 2.10.2, 2.10.17].

Theorem 3.3. *Let $K \subset \mathbb{CP}^{n-1} \times \mathbb{T}$ be a compact set and $K_\zeta = \{w \in \mathbb{T} : (\zeta, w) \in K\}$. Suppose that $\dim_{\mathcal{H}} K_\zeta > \beta$ for $\zeta \in X \subset \mathbb{CP}^{n-1}$. If $\widehat{\sigma}_n(X) > 0$, then*

$$\dim_{\mathcal{H}} K \geqslant 2n - 2 + \beta.$$

We fix some number $\alpha > \alpha_0$. It is sufficient to prove that

$$\dim_{\mathcal{H}}\Pi(R, J, a) \geqslant 2n - 1 - \alpha. \tag{3.5}$$

Suppose that the needed estimate fails. Then there exists a compact set $E \subset S_n$ such that $\Pi(E) > 0$ and

$$\dim_{\mathcal{H}} E < 2n - 1 - \alpha.$$

We identify the sphere S_n and the product $\mathbb{CP}^{n-1} \times \mathbb{T}$. We consider the sets

$$E_\zeta = \{w \in \mathbb{T} : (\zeta, w) \in E\}.$$

We state that

$$\dim_{\mathcal{H}} E_\zeta \leqslant 1 - \alpha \quad \text{for } \widehat{\sigma}_n\text{–almost all} \quad \zeta \in \mathbb{CP}^{n-1}. \tag{3.6}$$

Indeed, if the formulated property fails, then

$$\dim_{\mathcal{H}} E_\zeta > 1 - \alpha \quad \text{for} \quad \zeta \in X, \quad \text{where} \quad \widehat{\sigma}_n(X) > 0.$$

Applying Theorem 3.3, we obtain

$$\dim_{\mathcal{H}} E \geqslant 2n - 1 - \alpha,$$

that leads to a contradiction. Thus, the property (3.6) holds.

Since $\Pi(E) > 0$, Lemma 2.1 ensures that

$$\Pi_\zeta(E_\zeta) > 0 \quad \text{for} \quad \zeta \in Y_0, \quad \text{where} \quad \widehat{\sigma}_n(Y_0) > 0.$$

Combining this fact and (3.6), we obtain

$$\dim_{\mathcal{H}}\Pi_\zeta \leqslant 1-\alpha \quad \text{for} \quad \zeta \in Y, \quad \text{where} \quad \widehat{\sigma}_n(Y) > 0. \tag{3.7}$$

Now we recall that $\Pi_\zeta(R, J, a)$ is the classical Riesz product defined by the formula (2.2). Therefore,

$$\dim_{\mathcal{H}}\Pi_\zeta(R, J, a) \geqslant 1-\alpha_0 > 1-\alpha$$

by Theorem 3.1 in [11]. This contradicts the property (3.7). Thus, the property (3.5) holds for each number $\alpha > \alpha_0$.

In other words, the consideration of slice–products allows us to obtain Corollary 3.1 by means of Theorem 3.1 in [11], which is a corresponding result on classical Riesz products.

4. Hausdorff dimensions of pluriharmonic measures

This section is motivated by analogues of Lemma 2.1 and their applications in studying the pluriharmonic measures.

4.1. Pluriharmonic measures on unit sphere. A measure $\mu \in M(S_n)$ is called pluriharmonic if

$$\operatorname{spec}(\mu) \subset \{(p,q) \in \mathbb{Z}^2_+ : pq = 0\}.$$

An analogue of Lemma 2.1 is known for pluriharmonic measures on the sphere S_n. It follows from this fact that $\dim_{\mathcal{H}}\mu \geqslant 2n-2$ for each pluriharmonic measure $\mu \in M(S_n)$; further details can be found in [1], [3]. However, to the best of the author's knowledge, the sharpness of this estimate remains an open question.

4.2. Pluriharmonic measures on torus. A measure $\mu \in M(\mathbb{T}^n)$ is called pluriharmonic if $\hat{\mu}(k_1, \dots, k_n) = 0$ for $(k_1, \dots, k_n) \in \mathbb{Z}^n \setminus (\mathbb{Z}^n_- \cup \mathbb{Z}^n_+)$. It is well–known that $\mu \in M(\mathbb{T}^n)$ is a pluriharmonic measure if and only if the Poisson integral of measure μ is a pluriharmonic function in the polydisk $\mathbb{D}^n$. Therefore, an analogue of Lemma 2.1 holds for each pluriharmonic measure μ on $\mathbb{T}^n$; see the proof of Proposition 2.1 in [2] for a measure μ on the unit sphere S_n, as well as [5], where the pluriharmonic measures are considered on the Shilov boundary of a circular bounded symmetric domain. Arguing as in the case of sphere S_n, we conclude that

$$\dim_{\mathcal{H}}\mu \geqslant n-1, \tag{4.1}$$

see also [5, Cor. 3.3]. For $n = 2$ the estimate (4.1) and its sharpness were established in work [4]. Simple examples show that the estimate (4.1) is sharp for all $n \geqslant 2$. Indeed, we let

$$\mu = \delta_1(\xi_1) \otimes m(\xi_2) \otimes \cdots \otimes m(\xi_n),$$

where δ_1 is the Dirac measure at the point $1 \in \mathbb{T}$, and $m = \sigma_1$ is the normalized Lebesgue measure on the circle $\mathbb{T}$. It is clear that μ is a pluriharmonic measure and

$$\dim_{\mathcal{H}}\mu = n-1.$$

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