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HOMOGENIZATION OF ATTRACTORS TO REACTION–DIFFUSION EQUATIONS IN DOMAINS WITH RAPIDLY OSCILLATING BOUNDARY: SUPERCRITICAL CASE

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Abstract. This paper is devoted to studying the reaction–diffusion systems with rapidly oscillating coefficients in the equations and in boundary conditions in domains with locally periodic oscillating boundary; on this boundary a Robin boundary condition is imposed. We consider the supercritical case, when the homogenization changes the Robin boundary condition on the oscillating boundary is to the homogeneous Dirichlet boundary condition in the limit as the small parameter, which characterizes oscillations of the boundary, tends to zero. In this case, we prove that the trajectory attractors of these systems converge in a weak sense to the trajectory attractors of the limit (homogenized) reaction–diffusion systems in the domain independent of the small parameter. For this aim we use the homogenization theory, asymptotic analysis and the approach of V.V. Chepyzhov and M.I. Vishik concerning trajectory attractors of dissipative evolution equations. The homogenization method and asymptotic analysis are used to derive the homogenized reaction–diffusion system and to prove the convergence of solutions. First we define the appropriate auxiliary functional spaces with weak topology, then, we prove the existence of trajectory attractors for these systems and formulate the main Theorem. Finally, we prove the main convergence result with the help of auxiliary lemmas.

Keywords: attractors, homogenization, reaction–diffusion systems, nonlinear equations, weak convergence, rapidly oscillating boundary.

Mathematics Subject Classification: 34B45, 81Q15

1. Introduction

This paper is the next step in our investigations of homogenization problem for reaction–diffusion systems in domains with very rapidly oscillating boundary, for detailed geometric settings see [18]. In [5] we studied the critical case, in which the Robin condition was imposed on the oscillating part of the boundary and under the homogenization the type of boundary condition was preserve and only the coefficients changed. The subcritical case, when the Robin condition becomes the Neumann condition under the homogenization, will be considered separately.

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In this paper we deal with the supercritical case, when the Robin boundary condition becomes the Dirichlet boundary condition under the homogenization. We prove the existence of trajectory attractors for both perturbed and homogenized problems, construct the attractor for the latter problem, and prove the convergence of the attractors as the small parameter characterizing the oscillations, tends to zero, namely, we prove the Hausdorff convergence of attractors. In many pure mathematical papers, one can find the asymptotic analysis of problems in domains with oscillating (rough) boundaries, see, for example, [12], [13], [28], [29], [37] for rapidly oscillating boundary, [5], [18] for very rapidly oscillating boundary, [33] for spectral problems, [14], [15], [16] for operator convergence, [35] for a general geometry, [22], [23] for multilevel oscillating boundary, [36] for fractal structure, [1], [2], [3], [4] for eigenvalue problems). We also mention the basic frameworks [19], [32], [38], [39], where one can find the detail bibliography.

Concerning attractors see, for example, [6], [26], [40], and the bibliography in these monographs. Homogenization of attractors were studied in [27], [30], [31], see also [7], [8], [10], [11], [24].

The first section is devoted to basic settings, in the second section we describe the limiting (homogenized) reaction–diffusion system and its trajectory attractor. The third section contains auxiliary results including integral estimates (for analogous estimates see [20], [21]) and in the fourth section the proof of main theorem is given.

2. Statement of the problem

Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geqslant 2$, with a smooth boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where Ω lies in a half–space $x_d > 0$ and $\Gamma_1 \subset \{x : x_d = 0\}$. Given a smooth non–positive 1–periodic in the variable $\hat{\xi}$ function $F(\hat{x}, \hat{\xi})$, $\hat{x} = (x_1, \dots, x_{d-1}), \hat{\xi} = (\xi_1, \dots, \xi_{d-1})$, we define the domain Ω_ε via its boundary $\partial\Omega_\varepsilon = \Gamma_1^\varepsilon \cup \Gamma_2$, where

$$\Gamma_1^\varepsilon = \big\{x = (\hat{x}, x_d) \,:\, (\hat{x}, 0) \in \Gamma_1, x_d = \varepsilon^\alpha F(\hat{x}, \hat{x}\varepsilon^{-1})\big\}, \quad 0 \leqslant \alpha < 1,$$

that is, we add a thin oscillating layer

$$\Pi_\varepsilon = \big\{x = (\hat{x}, x_d) \,:\, (\hat{x}, 0) \in \Gamma_1, x_d \in [0, \varepsilon^\alpha F(\hat{x}, \hat{x}\varepsilon^{-1}))\big\}$$

to the domain Ω. We assume that $F(\hat{x}, \hat{\xi})$ is compactly supported on Γ_1 uniformly in $\hat{\xi}$. We consider the boundary-value problem

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} = \lambda\Delta u_\varepsilon - a\left(x, \dfrac{x}{\varepsilon}\right) f(u_\varepsilon) + h\left(x, \dfrac{x}{\varepsilon}\right), & x \in \Omega_\varepsilon, \quad t > 0,\\ \dfrac{\partial u_\varepsilon}{\partial \nu} + \varepsilon^\beta p\left(\hat{x}, \dfrac{\hat{x}}{\varepsilon}\right) u_\varepsilon = \varepsilon^{1-\alpha} g(\hat{x}, \dfrac{\hat{x}}{\varepsilon}), & x \in \Gamma_1^\varepsilon, \quad t > 0,\\ u_\varepsilon = 0, & x \in \Gamma_2, \quad t > 0,\\ u_\varepsilon = U(x), & x \in \Omega_\varepsilon, \quad t = 0, \end{cases} \tag{2.1}$$

where $x = (\hat{x}, x_d)$, $u_\varepsilon = u_\varepsilon(x,t) = (u^1, \dots, u^n)^\top$ is an unknown vector function, the nonlinear function $f = (f^1, \dots, f^n)^\top$ is given, $h = (h^1, \dots, h^n)^\top$ is the known function, $\beta > 0$, and λ is an $n \times n$–matrix with constant coefficients and having a positive symmetrical part:

$$\frac{1}{2}(\lambda + \lambda^\top) \geqslant \varpi I, \quad \varpi > 0,$$

and I is the unit matrix with of size $n \times n$. We assume that $p = \operatorname{diag}\{p^1, \dots, p^n\}$, $g = (g^1, \dots, g^n)^\top$, and $p^i = p^i(\hat{x}, \hat{\xi})$, $g^i = g^i(\hat{x}, \hat{\xi})$, $i = 1, \dots n$, are continuous 1–periodic in $\hat{\xi}$

functions and $p^i(\hat{x},\hat{\xi})$, $i=1,\dots n$, are positive. By

$$\frac{\partial u_\varepsilon}{\partial \nu}=\left(\frac{\partial u_\varepsilon^1}{\partial \nu},\dots,\frac{\partial u_\varepsilon^n}{\partial \nu}\right)^\top$$

we denote the normal derivative of the vector function u_ε multiplied by the matrix λ, that is,

$$\frac{\partial u_\varepsilon^i}{\partial \nu}:=\sum_{j=1}^{n}\sum_{k=1}^{d}\lambda_{ij}\frac{\partial u_\varepsilon^j}{\partial x_k}N_k, \qquad i=1,\dots,n,$$

and $N=(N_1,\dots,N_d)$ is the unit outer normal to the boundary of the domain. By $p_{\max}=const$ we denote the maximum of p^i on Γ_1 with respect to x and i. By U we denote a vector function in $(L_2(\Omega))^n$.

The function $a=a(x,\xi)$ is supposed to belong to $C(\overline{\Omega}_\varepsilon\times\mathbb{R}^d)$ and obey the ellipticity condition $0<a_0\leqslant a(x,\xi)\leqslant A_0$ with some constants a_0, A_0. We assume that the function $a_\varepsilon(x)=a\left(x,\frac{x}{\varepsilon}\right)$ has the average $\overline{a}(x)$ when $\varepsilon\to 0+$ in the space $L_{\infty,*w}(\Omega)$, that is

$$\int\limits_\Omega a\left(x,\frac{x}{\varepsilon}\right)\varphi(x)dx\to\int\limits_\Omega \overline{a}(x)\varphi(x)\,dx, \quad \varepsilon\to 0+, \tag{2.2}$$

for each function $\varphi\in L_1(\Omega)$.

We denote by V (respectively V_ε) the Sobolev space $H^1(\Omega,\Gamma_2)$ (respectively $H^1(\Omega_\varepsilon,\Gamma_2)$), that is, the space of functions from the Sobolev space $H^1(\Omega)$ (respectively $H^1(\Omega_\varepsilon)$) with the zero trace on Γ_2. We also denote by V' (respectively V_ε') the dual space for V (respectively V_ε), that is, the space of linear bounded functionals on V (respectively V_ε).

Let Ω^+ be a domain such that $\Omega_\varepsilon\subset\Omega^+$ for each ε. For the vector function $h(x,\xi)$ we suppose that for each $\varepsilon>0$ and i the function $h_\varepsilon^i(x)=h^i\left(x,\frac{x}{\varepsilon}\right)$ belongs to $\in L_2(\Omega^+)$ and has the average $\overline{h^i}(x)$ in the space $L_2(\Omega^+)$ for $\varepsilon\to 0+$, that is,

$$h^i\left(x,\frac{x}{\varepsilon}\right)\rightharpoonup\overline{h^i}(x)\quad\text{weakly in}\quad L_2(\Omega^+)\quad\text{as}\quad\varepsilon\to 0+,$$

or

$$\int\limits_{\Omega^+} h^i\left(x,\frac{x}{\varepsilon}\right)\varphi(x)dx\to\int\limits_{\Omega^+}\overline{h^i}(x)\varphi(x)dx\quad\text{as}\quad\varepsilon\to 0+, \tag{2.3}$$

for each function $\varphi\in L_2(\Omega^+)$ and for all $i=1,\dots,n$.

From the condition (2.3) it follows that the norms of the functions $h_\varepsilon^i(x)$ are bounded uniformly in ε, in the space $L_2(\Omega_\varepsilon)$, that is,

$$\|h_\varepsilon^i(x)\|_{L_2(\Omega_\varepsilon)}\leqslant M_0\quad\text{for all}\quad\varepsilon\in(0,1]. \tag{2.4}$$

We assume that the components of vector function $f\in C(\mathbb{R}^n;\mathbb{R}^n)$ satisfy the inequalities

$$\sum_{i=1}^{n}|f^i(v)|^{\frac{p_i}{(p_i-1)}}\leqslant C_0\left(\sum_{i=1}^{n}|v^i|^{p_i}+1\right), \quad 2\leqslant p_1\leqslant\ldots\leqslant p_{n-1}\leqslant p_n, \tag{2.5}$$

$$\sum_{i=1}^{n}\gamma_i|v^i|^{p_i}-C\leqslant\sum_{i=1}^{n}f^i(v)v^i, \quad v\in\mathbb{R}^n, \tag{2.6}$$

with $\gamma_i>0$ for each $i=1,\dots,n$. The inequality (2.5) is motivated by the fact that in real reaction–diffusion systems, the functions f^i are polynomials with possibly different degrees. The inequality (2.6) is called the *dissipativity condition* for the reaction–diffusion system (2.1). In a simple model case $p_i\equiv p$ for $i=1,\dots,n$, the conditions (2.5) and (2.6) reduce to the equalities

$$|f(v)|\leqslant C_0\left(|v|^{p-1}+1\right), \qquad \gamma|v|^p-C\leqslant f(v)v, \quad v\in\mathbb{R}^n. \tag{2.7}$$

We stress that the Lipschitz condition for the function $f(v)$ is *not assumed*.

Remark 2.1. *Our technique can also applied for studying the systems, in which nonlinear terms are of the form* $\sum_{j=1}^{m} a_j\left(x, \frac{x}{\varepsilon}\right) f_j(u)$, *where* a_j *are matrices with the entries allowing averaging and* $f_j(u)$ *are polynomial vectors of* u *satisfying conditions of form* (2.5), (2.6). *For brevity, we study the case* $m = 1$ *and* $a_1\left(x, \frac{x}{\varepsilon}\right) = a\left(x, \frac{x}{\varepsilon}\right) I$, *where* I *is the identity matrix.*

We introduce the notation

$$\mathbf{H} := [L_2(\Omega)]^n, \qquad \mathbf{H}_\varepsilon := [L_2(\Omega_\varepsilon)]^n, \qquad \mathbf{V} := [H^1(\Omega, \Gamma_2)]^n, \qquad \mathbf{V}_\varepsilon := [H^1(\Omega_\varepsilon; \Gamma_2)]^n.$$

The norms in these spaces are introduced as

$$\|v\|^2 := \int\limits_\Omega \sum_{i=1}^n |v^i(x)|^2 dx, \qquad \|v\|_\varepsilon^2 := \int\limits_{\Omega_\varepsilon} \sum_{i=1}^n |v^i(x)|^2 dx,$$

$$\|v\|_1^2 := \int\limits_\Omega \sum_{i=1}^n |\nabla v^i(x)|^2 dx, \qquad \|v\|_{1,\varepsilon}^2 := \int\limits_{\Omega_\varepsilon} \sum_{i=1}^n |\nabla v^i(x)|^2 dx.$$

Let $q_i = \frac{p_i}{(p_i-1)}$, $i = 1, \dots, n$. We shall employ the notation $\mathbf{p} = (p_1, \dots, p_n)$ and $\mathbf{q} = (q_1, \dots, q_n)$, and define the spaces

$$\begin{aligned}
&\mathbf{L_p} := L_{p_1}(\Omega) \times \ldots \times L_{p_n}(\Omega),\\
&\mathbf{L}_{\mathbf{p},\varepsilon} := L_{p_1}(\Omega_\varepsilon) \times \ldots \times L_{p_n}(\Omega_\varepsilon),\\
&\mathbf{L_p}(\mathbb{R}_+; \mathbf{L_p}) := L_{p_1}(\mathbb{R}_+; L_{p_1}(\Omega)) \times \ldots \times L_{p_n}(\mathbb{R}_+; L_{p_n}(\Omega)),\\
&\mathbf{L_p}(\mathbb{R}_+; \mathbf{L}_{\mathbf{p},\varepsilon}) := L_{p_1}(\mathbb{R}_+; L_{p_1}(\Omega_\varepsilon)) \times \ldots \times L_{p_n}(\mathbb{R}_+; L_{p_n}(\Omega_\varepsilon)).
\end{aligned}$$

As in [25], [26], we study weak solutions of the initial boundary value problem (2.1), that is, functions

$$u_\varepsilon = u_\varepsilon(x,t), \qquad u_\varepsilon \in \mathbf{L}_\infty^{loc}(\mathbb{R}_+; \mathbf{H}_\varepsilon) \cap \mathbf{L}_2^{loc}(\mathbb{R}_+; \mathbf{V}_\varepsilon) \cap \mathbf{L}_\mathbf{p}^{loc}(\mathbb{R}_+; \mathbf{L}_{\mathbf{p},\varepsilon}),$$

which satisfy Equation (2.1) in the distributional sense (the sense of generalized functions), that is, for which the integral identity

$$\begin{aligned}
&-\int\limits_{\Omega_\varepsilon \times \mathbb{R}_+} u_\varepsilon \cdot \frac{\partial \psi}{\partial t}\, dxdt + \int\limits_{\Omega_\varepsilon \times \mathbb{R}_+} \lambda \nabla u_\varepsilon \cdot \nabla \psi\, dxdt + \int\limits_{\Omega_\varepsilon \times \mathbb{R}_+} a_\varepsilon(x) f(u_\varepsilon) \cdot \psi\, dxdt\\
&+\varepsilon^\beta \int\limits_{\Gamma_1^\varepsilon \times \mathbb{R}_+} p\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right) u_\varepsilon \cdot \psi\, dsdt = \int\limits_{\Omega_\varepsilon \times \mathbb{R}_+} h_\varepsilon(x) \cdot \psi\ dxdt + \varepsilon^{1-\alpha} \int\limits_{\Gamma_1^\varepsilon \times \mathbb{R}_+} g\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right) \cdot \psi\, dsdt
\end{aligned} \tag{2.8}$$

holds for each function $\psi \in \mathbf{C}_0^\infty(\mathbb{R}_+; \mathbf{V}_\varepsilon \cap \mathbf{L}_{\mathbf{p},\varepsilon})$. Here $y_1 \cdot y_2$ stands for the scalar product of vectors y_1, $y_2 \in \mathbb{R}^n$.

If $u_\varepsilon \in \mathbf{L_p}(0, M; \mathbf{L}_{\mathbf{p},\varepsilon})$, then it follows from the condition (2.5) that $f(u) \in \mathbf{L_q}(0, M; \mathbf{L}_{\mathbf{q},\varepsilon})$. At the same time, if $u_\varepsilon \in \mathbf{L}_2(0, M; \mathbf{V}_\varepsilon)$, then $\lambda \Delta u_\varepsilon + h_\varepsilon \in \mathbf{L}_2(0, M; \mathbf{V}'_\varepsilon)$. Therefore, each weak solution $u_\varepsilon(x,s)$ to problem (2.1) satisfies

$$\frac{\partial u_\varepsilon}{\partial t} \in \mathbf{L_q}(0, M; \mathbf{L}_{\mathbf{q},\varepsilon}) + \mathbf{L}_2(0, M; \mathbf{V}'_\varepsilon).$$

The Sobolev embedding theorem implies

$$\mathbf{L_q}(0, M; \mathbf{L}_{\mathbf{q},\varepsilon}) + \mathbf{L}_2(0, M; \mathbf{V}'_\varepsilon) \subset \mathbf{L_q}\left(0, M; \mathbf{H}_\varepsilon^{-\mathbf{r}}\right),$$

where

$$\mathbf{H}_\varepsilon^{-\mathbf{r}} := H^{-r_1}(\Omega_\varepsilon) \times \ldots \times H^{-r_n}(\Omega_\varepsilon), \qquad \mathbf{r} = (r_1, \ldots, r_n),$$
$$r_i = \max\left\{1, d\left(\frac{1}{q_i} - \frac{1}{2}\right)\right\}, \quad i = 1, \ldots, n.$$

Here $H^{-r}(\Omega_\varepsilon)$ denotes the dual space for the Sobolev space $H_0^r(\Omega_\varepsilon)$, $r > 0$, of functions in the domain Ω_ε with zero trace on the boundary.

Hence, for each weak solution $u_\varepsilon(x,t)$ to problem (2.1) its time derivative $\frac{\partial u_\varepsilon(x,t)}{\partial t}$ belongs to $\mathbf{L_q}(0, M; \mathbf{H}_\varepsilon^{-\mathbf{r}})$.

Remark 2.2. *Existence of a weak solution $u(x,t)$ to problem (2.1) for an arbitrary initial data $U \in \mathbf{H}_\varepsilon$ and fixed ε can be proved in the standard way, see, for example, [6], [25]. This solution need not be unique, since the function $f(v)$ satisfies only the conditions (2.5), (2.6) and the Lipschitz condition with respect to v is not supposed.*

The next lemma can be proved similarly to the proof of from [26, Prop. XV.3.1].

Lemma 2.1. *Let $u_\varepsilon \in \mathbf{L}_2^{loc}(\mathbb{R}_+; \mathbf{V}_\varepsilon) \cap \mathbf{L}_\mathbf{p}^{loc}(\mathbb{R}_+; \mathbf{L}_{\mathbf{p},\varepsilon})$ be a weak solution of the problem (2.1). Then the following statements are true.*

(i) $u_\varepsilon \in \mathbf{C}(\mathbb{R}_+; \mathbf{H}_\varepsilon)$;

(ii) *the function $\|u_\varepsilon(\,\cdot\,, t)\|^2$ is absolutely continuous on $\mathbb{R}_+$ and*

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_\varepsilon(\,\cdot\,,t)\|^2 &+ \int\limits_{\Omega_\varepsilon} \lambda \nabla u_\varepsilon(x,t) \cdot \nabla u_\varepsilon(x,t) dx \\ &+ \int\limits_{\Omega_\varepsilon} a_\varepsilon(x) f(u_\varepsilon(x,t)) \cdot u_\varepsilon(x,t) dx + \varepsilon^\beta \int\limits_{\Gamma_1^\varepsilon} p\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right) u_\varepsilon(x,t) \cdot u_\varepsilon(x,t)\, ds \\ = \int\limits_{\Omega_\varepsilon} h_\varepsilon(x) \cdot u_\varepsilon(x,t) dx &+ \varepsilon^{1-\alpha} \int\limits_{\Gamma_1^\varepsilon} g\left(\hat{x}, \frac{\hat{x}}{\varepsilon}\right) \cdot u_\varepsilon(x,t)\, ds \end{aligned} \tag{2.9}$$

for almost all $t \in \mathbb{R}_+$.

To define the trajectory space $\mathcal{K}_\varepsilon^+$ for (2.1), we use the general approaches of [9, Sect. 2]. For each segment $[t_1, t_2] \in \mathbb{R}$ we consider the Banach spaces

$$\mathcal{F}_{t_1,t_2} := \mathbf{L_p}(t_1, t_2; \mathbf{L_p}) \cap \mathbf{L}_2(t_1, t_2; \mathbf{V}) \cap \mathbf{L}_\infty(t_1, t_2; \mathbf{H}) \cap \left\{ v \;\Big|\; \frac{\partial v}{\partial t} \in \mathbf{L_q}\left(t_1, t_2; \mathbf{H}^{-r}\right) \right\}$$

(sometimes we omit the parameter ε for brevity) with the norm

$$\|v\|_{\mathcal{F}_{t_1,t_2}} := \|v\|_{\mathbf{L_p}(t_1,t_2;\mathbf{L_p})} + \|v\|_{\mathbf{L}_2(t_1,t_2;\mathbf{V})} + \|v\|_{\mathbf{L}_\infty(0,M;\mathbf{H})} + \left\|\frac{\partial v}{\partial t}\right\|_{\mathbf{L_q}(t_1,t_2;\mathbf{H}^{-r})}.$$

Letting $\mathcal{D}_{t_1,t_2} = \mathbf{L_q}\left(t_1, t_2; \mathbf{H}^{-r}\right)$, we obtain $\mathcal{F}_{t_1,t_2} \subseteq \mathcal{D}_{t_1,t_2}$ and for $u(t) \in \mathcal{F}_{t_1,t_2}$ we have $A(u(t)) \in \mathcal{D}_{t_1,t_2}$. We consider weak solutions to (2.1) as solutions of an equation in the general scheme in [9, Sect. 2].

We consider the spaces

$$\mathcal{F}_+^{loc} = \mathbf{L}_\mathbf{p}^{loc}(\mathbb{R}_+; \mathbf{L_p}) \cap \mathbf{L}_2^{loc}(\mathbb{R}_+; \mathbf{V}) \cap \mathbf{L}_\infty^{loc}(\mathbb{R}_+; \mathbf{H}) \cap \left\{ v \;\Big|\; \frac{\partial v}{\partial t} \in \mathbf{L}_\mathbf{q}^{loc}(\mathbb{R}_+; \mathbf{H}^{-r}) \right\},$$

$$\mathcal{F}_{\varepsilon,+}^{loc} = \mathbf{L}_\mathbf{p}^{loc}(\mathbb{R}_+; \mathbf{L}_{\mathbf{p},\varepsilon}) \cap \mathbf{L}_2^{loc}(\mathbb{R}_+; \mathbf{V}_\varepsilon) \cap \mathbf{L}_\infty^{loc}(\mathbb{R}_+; \mathbf{H}_\varepsilon) \cap \left\{ v \;\Big|\; \frac{\partial v}{\partial t} \in \mathbf{L}_\mathbf{q}^{loc}(\mathbb{R}_+; \mathbf{H}_\varepsilon{}^{-r}) \right\}.$$

Let $\mathcal{K}_\varepsilon^+$ be the set of all weak solutions to (2.1). For each $U \in \mathbf{H}$ there exists at least one trajectory $u(\cdot) \in \mathcal{K}_\varepsilon^+$ such that $u(0) = U(x)$, Remark 2.2. Hence, the trajectory space $\mathcal{K}_\varepsilon^+$ of (2.1) is not empty. It is easy to see that $\mathcal{K}_\varepsilon^+ \subset \mathcal{F}_{\varepsilon,+}^{loc}$ and the space $\mathcal{K}_\varepsilon^+$ is translation invariant, i.e., if $u(t) \in \mathcal{K}_\varepsilon^+$, then $u(\tau + t) \in \mathcal{K}_\varepsilon^+$ for all $\tau \geqslant 0$. Hence, $S(\tau)\mathcal{K}_\varepsilon^+ \subseteq \mathcal{K}_\varepsilon^+$ for all $\tau \geqslant 0$.

We define metrics $\rho_{t_1,t_2}(\cdot,\cdot)$ in the spaces $\mathcal{F}_{t_1,t_2}$ by means of the norms from $\mathbf{L}_2(t_1,t_2;\mathbf{H})$

$$\rho_{t_1,t_2}(u,v) = \left(\int_{t_1}^{t_2} \|u(t) - v(t)\|_{\mathbf{H}}^2 dt\right)^{\frac{1}{2}}, \qquad u(\cdot),\, v(\cdot) \in \mathcal{F}_{t_1,t_2}.$$

The topology Θ_+^{loc} in $\mathcal{F}_+^{loc}$ is generated by these metrics. We recall that $\{v_k\} \subset \mathcal{F}_+^{loc}$ converges to $v \in \mathcal{F}_+^{loc}$ as $k \to \infty$ in Θ_+^{loc} if $\|v_k(\cdot) - v(\cdot)\|_{\mathbf{L}_2(t_1,t_2;\mathbf{H})} \to 0$ as $k \to \infty$ for all $[t_1,t_2] \subset \mathbb{R}_+$. The topology Θ_+^{loc} is metrizable and the corresponding metric space is complete. We consider this topology in the trajectory space $\mathcal{K}_\varepsilon^+$ of (2.1). In the same way we define the topology $\Theta_{\varepsilon,+}^{loc}$ in the space $\mathcal{F}_{\varepsilon,+}^{loc}$.

We consider the translation semigroup $\{S(\tau)\}$ on $\mathcal{K}_\varepsilon^+$, $S(\tau) : \mathcal{K}_\varepsilon^+ \to \mathcal{K}_\varepsilon^+$, $\tau \geqslant 0$. The translation semigroup $\{S(\tau)\}$ acting on $\mathcal{K}_\varepsilon^+$ is continuous in the topology $\Theta_{\varepsilon,+}^{loc}$. This fact is implied by the definition of this topology.

Following the lines of [9, Sect. 2], we define bounded sets in the space $\mathcal{K}_\varepsilon^+$ by means of the norm of Banach space $\mathcal{F}_{\varepsilon,+}^b$. We obtain

$$\mathcal{F}_{\varepsilon,+}^b = \mathbf{L}_{\mathbf{p}}^b(\mathbb{R}_+;\mathbf{L}_{\mathbf{p},\varepsilon}) \cap \mathbf{L}_2^b(\mathbb{R}_+;\mathbf{V}_\varepsilon) \cap \mathbf{L}_\infty(\mathbb{R}_+;\mathbf{H}_\varepsilon) \cap \left\{v \;\middle|\; \frac{\partial v}{\partial t} \in \mathbf{L}_{\mathbf{q}}^b(\mathbb{R}_+;\mathbf{H}_\varepsilon^{-r})\right\}$$

and the space $\mathcal{F}_{\varepsilon,+}^b$ is a subspace of $\mathcal{F}_{\varepsilon,+}^{loc}$.

We denote by $\mathcal{K}_\varepsilon$ the kernel to (2.1), which, by definition, consists of all complete weak solutions, i.e. $u(t)$, $t \in \mathbb{R}$, to our system, bounded in

$$\mathcal{F}_\varepsilon^b = \mathbf{L}_{\mathbf{p}}^b(\mathbb{R};\mathbf{L}_{\mathbf{p},\varepsilon}) \cap \mathbf{L}_2^b(\mathbb{R};\mathbf{V}_\varepsilon) \cap \mathbf{L}_\infty(\mathbb{R};\mathbf{H}_\varepsilon) \cap \left\{v \;\middle|\; \frac{\partial v}{\partial t} \in \mathbf{L}_{\mathbf{q}}^b(\mathbb{R};\mathbf{H}_\varepsilon^{-r})\right\}.$$

Lemma 2.2. *The problem (2.1) has the trajectory attractors $\mathfrak{A}_\varepsilon$ in the topological space $\Theta_{\varepsilon,+}^{loc}$. The set $\mathfrak{A}_\varepsilon$ is bounded in $\mathcal{F}_{\varepsilon,+}^b$ and compact in $\Theta_{\varepsilon,+}^{loc}$. Moreover, $\mathfrak{A}_\varepsilon = \Pi_+\mathcal{K}_\varepsilon$, the kernel $\mathcal{K}_\varepsilon$ is non–empty and bounded in $\mathcal{F}_\varepsilon^b$. We recall that the spaces $\mathcal{F}_{\varepsilon,+}^b$ and $\Theta_{\varepsilon,+}^{loc}$ depend on ε.*

To prove this lemma, we use the approach in the proof from [26, Ch. XV, Sect. 3, Thm. 3.2]. To prove the existence of an absorbing set (bounded in $\mathcal{F}_{\varepsilon,+}^b$ and compact in $\Theta_{\varepsilon,+}^{loc}$) one can use Lemma 2.1 similarly to [26, Ch. XV, Sect. 3, Prop. 3.1, Cor. 3.1].

It is easy to verify that $\mathfrak{A}_\varepsilon \subset \mathcal{B}_0(R)$ for all $\varepsilon \in (0,1)$. Here $\mathcal{B}_0(R)$ is an open ball in $\mathcal{F}_{\varepsilon,+}^b$ with a sufficiently large radius R. Due to the Aubin — Lions — Simon Lemma [5, Lm. 3.1] we have

$$\mathcal{B}_0(R) \Subset \mathbf{L}_2^{loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon^{1-\delta}), \tag{2.10}$$

$$\mathcal{B}_0(R) \Subset \mathbf{C}^{loc}(\mathbb{R}_+;\mathbf{H}_\varepsilon^{-\delta}), \quad 0 < \delta \leqslant 1. \tag{2.11}$$

Bearing in mind these embeddings, the attraction to the constructed trajectory attractor can be strengthen.

Corollary 2.1. *For each bounded set $\mathcal{B} \subset \mathcal{K}_\varepsilon^+$ in $\mathcal{F}_{\varepsilon,+}^b$ we have*

$$\mathrm{dist}_{\mathbf{L}_2(0,M;\mathbf{H}_\varepsilon^{1-\delta})}\left(\Pi_{0,M}S(\tau)\mathcal{B}, \Pi_{0,M}\mathcal{K}_\varepsilon\right) \to 0,$$
$$\mathrm{dist}_{\mathbf{C}([0,M];\mathbf{H}_\varepsilon^{-\delta})}\left(\Pi_{0,M}S(\tau)\mathcal{B}, \Pi_{0,M}\mathcal{K}_\varepsilon\right) \to 0, \qquad \tau \to \infty,$$

where M is a positive constant.

We recall that $\Omega \subset \Omega_\varepsilon$ and Ω lies in the positive half–space $\{x_d > 0\}$. Therefore, for each function $u(x,t)$ belonging to the space $\mathcal{F}^b_{\varepsilon,+}$ with $x \in \Omega_\varepsilon$, its restriction to the domain Ω belongs to the space $\mathcal{F}^b_+$ and

$$\|u\|_{\mathcal{F}^b_+} \leqslant \|u\|_{\mathcal{F}^b_{\varepsilon,+}}.$$

In view of this observation, we have the next statement.

Corollary 2.2. *The trajectory attractors $\mathfrak{A}_\varepsilon$ are bounded in $\mathcal{F}^b_+$ uniformly in $\varepsilon \in (0,1)$. The kernels $\mathcal{K}_\varepsilon$ are bounded in $\mathcal{F}^b$ uniformly in $\varepsilon \in (0,1)$.*

3. Homogenized reaction–diffusion system and its trajectory attractor: case $\beta < 1 - \alpha$

In the next sections, we study the behaviour of the problem (2.1) as $\varepsilon \to 0$ in the supercritical case $\beta < 1-\alpha$. We have the formal limiting problem with the Dirichlet boundary condition

$$\begin{cases} \dfrac{\partial u_0}{\partial t} = \lambda \Delta u_0 - \overline{a}\,(x)\, f(u_0) + \overline{h}\,(x)\,, & x \in \Omega, \quad t>0, \\ u_0 = 0, & x \in \partial\Omega, \quad t > 0, \\ u_0 = U(x), & x \in \Omega, \quad t = 0, \end{cases} \tag{3.1}$$

Here $\overline{a}(x)$ and $\overline{h}(x)$ are defined in (2.2) and (2.3), respectively. The limiting boundary condition arises due to the relation between small parameters, see similarly [18].

We note that in the supercritical case the influence of the boundary layer on the part of the boundary Γ_1 completely disappears (compare with the critical case [5] and subcritical case mentioned in the introductory part).

As before, we consider weak solutions of the problem (3.1), that is, functions

$$u_0(x,t) \in \mathbf{L}^{loc}_\infty(\mathbb{R}_+;\mathbf{H}) \cap \mathbf{L}^{loc}_2(\mathbb{R}_+;\mathbf{V}) \cap \mathbf{L}^{loc}_\mathbf{p}(\mathbb{R}_+;\mathbf{L_p})\,,$$

which satisfy the integral identity

$$-\int\limits_{\Omega\times\mathbb{R}_+} u_0 \cdot \frac{\partial \psi}{\partial t}\, dxdt + \int\limits_{\Omega\times\mathbb{R}_+} \lambda \nabla u_0 \cdot \nabla\psi\, dxdt + \int\limits_{\Omega\times\mathbb{R}_+} \bar{a}(x) f(u_0)\cdot\psi\, dxdt = \int\limits_{\Omega\times\mathbb{R}_+} \bar{h}(x)\cdot \psi\, dxdt \tag{3.2}$$

for each function $\psi \in \mathbf{C}^\infty_0(\mathbb{R}_+;\mathbf{V}\cap\mathbf{L_p})$. For each weak solution $u(x,t)$ to problem (3.1) we have $\frac{\partial u_0(x,t)}{\partial t} \in \mathbf{L_q}(0,M;\mathbf{H^{-r}})$ (see Section 2). Recall, that the limiting domain Ω in (3.1) and (3.2) is independent of ε and its boundary contains the plain part Γ_1.

Similarly (2.1), for each initial data $U \in \mathbf{H}$ the problem (3.1) has at least one weak solution, see Remark 2.2. Lemma 2.1 also holds for the problem (3.1), in which ε–depending coefficients a, h, p and g are to be replaced by the corresponding averaged coefficients $\overline{a}(x)$, $\overline{h}(x)$.

Let $\overline{\mathcal{K}}^+$ be the trajectory space for (3.1) (the set of all weak solutions), which belongs to the corresponding spaces $\mathcal{F}^{loc}_+$ and $\mathcal{F}^b_+$, see [9, Sect. 2]. Recall that $\overline{\mathcal{K}}^+ \subset \mathcal{F}^{loc}_+$ and the space $\overline{\mathcal{K}}^+$ is translation invariant with respect to translation semigroup $\{S(\tau)\}$, that is, $S(\tau)\overline{\mathcal{K}}^+ \subseteq \overline{\mathcal{K}}^+$ for all $\tau \geqslant 0$. We now construct the trajectory attractor in the topology Θ^{loc}_+ for the problem (3.1), Section 2 and [9, Sect. 2].

Similarly to Lemma 2.2 we have the next statement.

Lemma 3.1. *The problem (3.1) has the trajectory attractor $\overline{\mathfrak{A}}$ in the topological space Θ^{loc}_+. The set $\overline{\mathfrak{A}}$ is bounded in $\mathcal{F}^b_+$ and compact in Θ^{loc}_+. Moreover,*

$$\overline{\mathfrak{A}} = \Pi_+ \overline{\mathcal{K}},$$

the kernel $\overline{\mathcal{K}}$ of the problem (3.1) is non–empty and bounded in $\mathcal{F}^b$.

We also have $\overline{\mathfrak{A}} \subset \mathcal{B}_0(R)$, where $\mathcal{B}_0(R)$ is a ball in $\mathcal{F}_+^b$ with a sufficiently large radius R. Finally, the analogue of Corollary 2.1 holds for the trajectory attractor $\overline{\mathfrak{A}}$.

Corollary 3.1. *For each bounded set $\mathcal{B} \subset \overline{\mathcal{K}}^+$ in $\mathcal{F}_+^b$ we have*

$$\begin{aligned}
&\operatorname{dist}_{\mathbf{L}_2(0,M;\mathbf{H}^{1-\delta})}\left(\Pi_{0,M}S(\tau)\mathcal{B}, \Pi_{0,M}\overline{\mathcal{K}}\right) \to 0,\\
&\operatorname{dist}_{\mathbf{C}([0,M];\mathbf{H}_\varepsilon^{-\delta})}\left(\Pi_{0,M}S(\tau)\mathcal{B}, \Pi_{0,M}\overline{\mathcal{K}}\right) \to 0\ (\tau \to \infty), \qquad M > 0.
\end{aligned}$$

4. Preliminary lemmas: case $\beta < 1 - \alpha$

We consider auxiliary elliptic problems

$$\begin{cases}
\lambda \Delta v_\varepsilon + h\left(x, \dfrac{x}{\varepsilon}\right) = 0, & x \in \Omega_\varepsilon,\\
\dfrac{\partial v_\varepsilon}{\partial \nu} + \varepsilon^\beta p(\hat{x}, \dfrac{\hat{x}}{\varepsilon}) v_\varepsilon = \varepsilon^{1-\alpha} g(\hat{x}, \dfrac{\hat{x}}{\varepsilon}), & x \in \Gamma_1^\varepsilon,\\
v_\varepsilon = 0, & x \in \Gamma_2,
\end{cases} \tag{4.1}$$

where $x = (\hat{x}, x_d)$ and

$$\begin{cases}
\lambda \Delta v_0 + \overline{h}(x) = 0, & x \in \Omega,\\
v_0 = 0, & x \in \partial\Omega,
\end{cases} \tag{4.2}$$

and $\overline{h}(x)$ is defined in (2.3).

The next lemma was proved in [18].

Lemma 4.1. *Let $\beta < 1 - \alpha$, $F(\hat{x}, \hat{\xi}), g(\hat{x}, \hat{\xi}), p(\hat{x}, \hat{\xi})$ be periodic in ξ smooth functions, λ be a given matrix, the function $h(x, \frac{x}{\varepsilon})$ satisfy the condition (2.3). Then for all sufficiently small $\varepsilon > 0$ the problem (4.1) has no unique solution. The family of solutions is uniformly bounded in the $\mathbf{V}_\varepsilon$–norm and the strong convergence*

$$v_\varepsilon \to v_0 \tag{4.3}$$

holds in $\mathbf{V}_\varepsilon$ as $\varepsilon \to 0$.

Lemma 4.2. *The following statements are true.*

(1) *All solutions $u_\varepsilon(t)$ to (2.1) satisfy*

$$\|u_\varepsilon(t)\|_\varepsilon^2 \leqslant \|u_\varepsilon(0)\|_\varepsilon^2 e^{-\varkappa_1 t} + R_1^2, \tag{4.4}$$

$$\begin{aligned}
\varpi \int_t^{t+1} \|u_\varepsilon(s)\|_{\varepsilon,1}^2\, ds + 2a_0 \sum_{i=1}^N \gamma_i \int_t^{t+1} \|u_\varepsilon^i(s)\|_{L_{p_i}(\Omega_\varepsilon)}^{p_i}\, ds&\\
+ 2p_{\max}\varepsilon^{1-\alpha} \int_t^{t+1} \|u_\varepsilon(s)\|_{\mathbf{L}_2(\Gamma_1^\varepsilon)}^2\, ds \leqslant \|u_\varepsilon(t)\|_\varepsilon^2 + R_2^2,&
\end{aligned} \tag{4.5}$$

where $\varkappa_1 > 0$ is a constant independent of ε. Positive constants R_1 and R_2 depend on M_0, see (2.4), and are independent of $u_\varepsilon(0)$ and ε.

(2) *All solutions $u(t)$ to (3.1) satisfy the same inequalities (4.4) and (4.5) with the norms in the functional spaces on the domain Ω instead of Ω_ε.*

Proof. We give a brief outline of the proof, for detail see [26, Ch. XV, Sect. 3, Prop. 3.1, Cor. 3.1]. The integral over the part of the boundary Γ_1^ε in the left hand side of (2.9) is nonnegative since the matrix p is positive definite. We integrate (2.9) with respect to t. Then, to estimate the terms

$$\varepsilon^{1-\alpha}\int\limits_{\Gamma_1^\varepsilon} g\cdot w\,ds \qquad \text{and} \qquad \varepsilon^{\beta}\int\limits_{\Gamma_1^\varepsilon} pu_\varepsilon\cdot w\,ds$$

we use the Cauchy inequality and the compactness of the embedding $\mathbf{L}_2(\Gamma_1^\varepsilon)\Subset\mathbf{V}_\varepsilon$. For other terms we use a standard procedure, see [26, Ch. XV, Sect. 3, Prop. 3.1, Cor. 3.1]. The proof is complete. $\square$

5. Main result

In this section we formulate the main result on the limiting behaviour of the trajectory attractors $\mathfrak{A}_\varepsilon$ of reaction–diffusion systems (2.1) as $\varepsilon\to 0$ in the supercritical case $\beta<1-\alpha$.

Theorem 5.1. *The convergence*

$$\mathfrak{A}_\varepsilon\to\overline{\mathfrak{A}} \quad as \quad \varepsilon\to 0+ \tag{5.1}$$

holds in the topological space Θ_+^{loc}, and

$$\mathcal{K}_\varepsilon\to\overline{\mathcal{K}} \quad as \quad \varepsilon\to 0+ \quad in \quad \Theta^{loc}. \tag{5.2}$$

Proof. It is easy to see that (5.2) implies (5.1). Hence, it is sufficient to prove (5.2), i.e., for every neighbourhood $\mathcal{O}(\overline{\mathcal{K}})$ in Θ^{loc} there exists $\varepsilon_1=\varepsilon_1(\mathcal{O})>0$ such that

$$\mathcal{K}_\varepsilon\subset\mathcal{O}(\overline{\mathcal{K}}) \quad \text{for} \quad \varepsilon<\varepsilon_1. \tag{5.3}$$

Assume that (5.3) is false. Then there exists a neighbourhood $\mathcal{O}'(\overline{\mathcal{K}})$ in Θ^{loc}, a sequence $\varepsilon_k\to 0+$ $(k\to\infty)$, and a sequence $u_{\varepsilon_k}(\cdot)=u_{\varepsilon_k}(t)\in\mathcal{K}_{\varepsilon_k}$ such that

$$u_{\varepsilon_k}\notin\mathcal{O}'(\overline{\mathcal{K}}) \quad \text{for all} \quad k\in\mathbb{N}.$$

The function $u_{\varepsilon_k}(x,t)$, $t\in\mathbb{R}$ is a solution to

$$\begin{cases} \dfrac{\partial u_{\varepsilon_k}}{\partial t}=\lambda\Delta u_{\varepsilon_k}-a\left(x,\dfrac{x}{\varepsilon_k}\right)f(u_{\varepsilon_k})+h\left(x,\dfrac{x}{\varepsilon_k}\right), & x\in\Omega_{\varepsilon_k},\\ \dfrac{\partial u_{\varepsilon_k}}{\partial\nu}+\varepsilon_k^{\beta}p(\hat{x},\dfrac{\hat{x}}{\varepsilon_k})u_{\varepsilon_k}=\varepsilon_k^{1-\alpha}g(\hat{x},\dfrac{\hat{x}}{\varepsilon_k}), & x\in\Gamma_1^{\varepsilon_k},\\ u_{\varepsilon_k}=0, & x\in\Gamma_2, \end{cases} \tag{5.4}$$

where $\beta<1-\alpha$. To get the uniform in ε estimate of the solution, we use Lemma 4.2. By means of (4.4) and (4.5) we find that the sequence $\{u_{\varepsilon_k}(x,t)\}$ is bounded in $\mathcal{F}^b$

$$\begin{aligned} \|u_{\varepsilon_k}\|_{\mathcal{F}^b}=&\sup_{t\in\mathbb{R}}\|u_{\varepsilon_k}(t)\|+\sup_{t\in\mathbb{R}}\left(\int\limits_t^{t+1}\|u_{\varepsilon_k}(\vartheta)\|_1^2\,d\vartheta\right)^{\frac12}\\ &+\sup_{t\in\mathbb{R}}\left(\int\limits_t^{t+1}\|u_{\varepsilon_k}(\vartheta)\|_{\mathbf{L_p}}^p\,d\vartheta\right)^{\frac1p}+\varepsilon^{\beta}\sup_{t\in\mathbb{R}}\int\limits_t^{t+1}\int\limits_{\Gamma_1^\varepsilon}p\Big(\hat{x},\frac{\hat{x}}{\varepsilon}\Big)u_\varepsilon(x,\vartheta)\cdot u_\varepsilon(x,\vartheta)\,ds\,d\vartheta \\ &+\sup_{t\in\mathbb{R}}\left(\int\limits_t^{t+1}\left\|\frac{\partial u_{\varepsilon_k}}{\partial t}(\vartheta)\right\|_{\mathbf{H}^{-r}}^q d\vartheta\right)^{\frac1q}\leqslant C, \qquad k\in\mathbb{N}. \end{aligned} \tag{5.5}$$

We recall that $\beta < 1-\alpha$. The constant C is independent of ε. Hence, there exists a subsequence $\{u_{\varepsilon'_k}(x,t)\} \subset \{u_{\varepsilon_k}(x,t)\}$ such that $u_{\varepsilon'_k}(x,t) \to \overline{u}(x,t)$ as $k \to \infty$ in Θ^{loc}. Here $\overline{u}(x,t) \in \mathcal{F}^b$ and $\overline{u}(t)$ satisfies (5.5) with the same constant C. Because of (5.5) we get

$$u_{\varepsilon'_k}(x,t) \rightharpoonup \overline{u}(x,t) \quad \text{as} \quad k \to \infty$$

weakly in $\mathbf{L}_2^{loc}(\mathbb{R};\mathbf{V})$, weakly in $\mathbf{L}_{\mathbf{p}}^{loc}(\mathbb{R};\mathbf{L}_{\mathbf{p}})$ amd $*$-weakly in $\mathbf{L}_\infty^{loc}(\mathbb{R}_+;\mathbf{H})$, and

$$\frac{\partial u_{\varepsilon'_k}(x,t)}{\partial t} \rightharpoonup \frac{\partial \overline{u}(x,t)}{\partial t} \quad \text{as} \quad \varkappa \to \infty$$

weakly in $\mathbf{L}_{\mathbf{q},w}^{loc}(\mathbb{R};\mathbf{H}^{-r})$. We claim that $\overline{u}(x,t) \in \overline{\mathcal{K}}$. We have $\|\overline{u}\|_{\mathcal{F}^b} \leqslant C$. Hence, we have to verify that $\overline{u}(x,t) = u_0(x,t)$, i.e., that it is a weak solution to (3.1).

Using (5.5) and (2.3), we find

$$\frac{\partial u_{\varepsilon_k}}{\partial t} - \lambda\Delta u_{\varepsilon_k} - h_{\varepsilon_k}(x) \to \frac{\partial \bar{u}}{\partial t} - \lambda\Delta\bar{u} - \overline{h}(x) \quad \text{as} \quad k \to \infty \tag{5.6}$$

in the space $D'(\mathbb{R};\mathbf{H}_\varepsilon^{-\mathbf{r}})$ since the differentiation operators are continuous in the space of distributions.

We are going to prove that

$$a\left(x, \frac{x}{\varepsilon_k}\right) f(u_{\varepsilon_k}) \rightharpoonup \bar{a}(x) f(\bar{u}) \quad \text{as} \quad k \to \infty \tag{5.7}$$

weakly in $\mathbf{L}_{\mathbf{q},w}^{loc}(\mathbb{R};\mathbf{L}_{\mathbf{q}})$. We fix an arbitrary number $M > 0$. The sequence $\{u_{\varepsilon_k}(x,t)\}$ is bounded in $\mathbf{L}_{\mathbf{p}}(-M,M;\mathbf{L}_{\mathbf{p}})$, see (5.5). Then, due to (2.5), the sequence $\{f(u_{\varepsilon_k}(t))\}$ is bounded in $\mathbf{L}_{\mathbf{q}}(-M,M;\mathbf{L}_{\mathbf{q}})$. Since $\{u_{\varepsilon_k}(x,t)\}$ is bounded in $\mathbf{L}_2(-M,M;\mathbf{V})$ and $\left\{\frac{\partial u_{\varepsilon_k}}{\partial t}(t)\right\}$ is bounded in $\mathbf{L}_{\mathbf{q}}(-M,M;\mathbf{H}^{-\mathbf{r}})$, we can suppose that $u_{\varepsilon_k}(x,t) \to \bar{u}(x,t)$ as $k \to \infty$ strongly in $\mathbf{L}_2(-M,M;\mathbf{L}_2) = \mathbf{L}_2(\Omega\times]-M,M[)$ and, therefore,

$$u_{\varepsilon_k}(x,t) \to \bar{u}(x,t) \ \text{ as } k \to \infty \quad \text{for almost all} \quad (x,t) \in \Omega\times]-M,M[.$$

Since the function $f(v)$ is continuous in $v \in \mathbb{R}$, we conclude that

$$f(u_{\varepsilon_k}(x,t)) \to f(\bar{u}(x,t)) \quad \text{as} \quad k \to \infty \quad \text{for almost all} \quad (x,t) \in \Omega\times]-M,M[. \tag{5.8}$$

We have

$$a\left(x, \frac{x}{\varepsilon_k}\right) f(u_{\varepsilon_k}) - \bar{a}(x) f(\bar{u}) = a\left(x, \frac{x}{\varepsilon_k}\right)\left(f(u_{\varepsilon_k}) - f(\bar{u})\right) + \left(a\left(x, \frac{x}{\varepsilon_k}\right) - \bar{a}(x)\right) f(\bar{u}). \tag{5.9}$$

Let us show that both terms in the right–hand side of (5.9) tend to zero as $k \to \infty$ weakly in $\mathbf{L}_{\mathbf{q}}(-M,M;\mathbf{L}_{\mathbf{q}}) = \mathbf{L}_{\mathbf{q}}(\Omega\times]-M,M[)$. First, the sequence $a\left(x, \frac{x}{\varepsilon_k}\right)(f(u_{\varepsilon_k}) - f(\bar{u}))$ tends to zero as $k \to \infty$ for almost all $(x,t) \in \Omega\times]-M,M[$, see (5.8). Applying [34, Ch. 1, Sec. 1, Lm. 1.3], we conclude that

$$a\left(x, \frac{x}{\varepsilon_k}\right)\Big(f(u_{\varepsilon_k}) - f(\bar{u})\Big) \rightharpoonup 0 \quad \text{as} \quad k \to \infty$$

weakly in $\mathbf{L}_{\mathbf{q}}(\Omega\times]-M,M[)$. Second, the sequence $\left(a\left(x, \frac{x}{\varepsilon_k}\right) - \bar{a}(x)\right) f(\bar{u})$ also tends to zero a $k \to \infty$ weakly in $\mathbf{L}_{\mathbf{q}}(\Omega\times]-M,M[)$ since $a\left(x, \frac{x}{\varepsilon_k}\right) \rightharpoonup \bar{a}(x)$ as $k \to \infty$ $*$-weakly in $\mathbf{L}_{\infty,*w}(-M,M;\mathbf{L}_2)$ and $f(\bar{u}) \in \mathbf{L}_{\mathbf{q}}(\Omega\times]-M,M[)$. This proves (5.7).

Hence, due to Lemma 4.1, see also [18], for $\overline{u}(x,t) = u_0(x,t)$ we have

$$
-\int_{-M}^{M}\int_{\Omega_{\varepsilon_k}} u_{\varepsilon_k}\cdot\frac{\partial\psi}{\partial t}\,dxdt + \int_{-M}^{M}\int_{\Omega_{\varepsilon_k}} \lambda\nabla u_{\varepsilon_k}\cdot\nabla\psi\;dxdt + \int_{-M}^{M}\int_{\Omega_{\varepsilon_k}} a_{\varepsilon_k}(x)f(u_{\varepsilon_k})\cdot\psi\,dxdt
$$

$$
+\,{\varepsilon_k}^{\beta}\int_{-M}^{M}\int_{\Gamma_1^{\varepsilon_k}} p\left(\hat{x},\frac{\hat{x}}{\varepsilon_k}\right)u_{\varepsilon_k}\cdot\psi\,dsdt - \varepsilon_k^{1-\alpha}\int_{-M}^{M}\int_{\Gamma_1^{\varepsilon_k}} g\left(\hat{x},\frac{\hat{x}}{\varepsilon_k}\right)\cdot\psi\,dsdt \to
$$

$$
-\int_{-M}^{M}\int_{\Omega} u_0\cdot\frac{\partial\psi}{\partial t}\,dxdt + \int_{-M}^{M}\int_{\Omega} \lambda\nabla u_0\cdot\nabla\psi\,dxdt + \int_{-M}^{M}\int_{\Omega} \overline{a}(x)f(u_0)\cdot\psi\,dxdt
$$

as $k\to\infty$.

Using (5.8), we pass to the limit in Equation (5.4) as $k\to\infty$ in the space $D'(\mathbb{R};\mathbf{H}^{-r})$ and obtain that the function $u_0(x,t)$ satisfies the integral identity (3.2) and, therefore, it is a complete trajectory of Equation (3.1). Hence, $u_0\in\overline{\mathcal{K}}$.

We have proved above that $u_{\varepsilon_k}\to u_0$ as $k\to\infty$ in Θ^{loc}. The assumption $u_{\varepsilon_k}\notin\mathcal{O}'(\overline{\mathcal{K}})$ (see [17]) implies $u_0\notin\mathcal{O}'(\overline{\mathcal{K}})$, and, hence, $u_0\notin\overline{\mathcal{K}}$. We arrive at the contradiction that completes the proof. □

Using the compact embeddings (2.10) and (2.11), we can improve the convergence (5.1).

Corollary 5.1. *For each* $0<\delta\leqslant 1$ *and for all* $M>0$

$$
\mathrm{dist}_{\mathbf{L}_2([0,M];\mathbf{H}^{1-\delta})}\left(\Pi_{0,M}\mathfrak{A}_\varepsilon,\Pi_{0,M}\overline{\mathfrak{A}}\right)\to 0, \tag{5.10}
$$

$$
\mathrm{dist}_{\mathbf{C}([0,M];\mathbf{H}^{-\delta})}\left(\Pi_{0,M}\mathfrak{A}_\varepsilon,\Pi_{0,M}\overline{\mathfrak{A}}\right)\to 0,\quad \varepsilon\to 0+\,. \tag{5.11}
$$

To prove (5.10) and (5.11), we reproduce the proof of Theorem 5.1 replacing the topology of Θ^{loc} to that of $\mathbf{L}_2^{loc}(\mathbb{R}_+;\mathbf{H}^{1-\delta})$ or $\mathbf{C}^{loc}(\mathbb{R}_+;\mathbf{H}^{-\delta})$.

Finally, we consider the Cauchy problem for reaction–diffusion systems, for which the uniqueness theorem is true. It sufficient to assume that the nonlinear term $f(u)$ in (2.1) satisfies the condition

$$
(f(v_1)-f(v_2),v_1-v_2)\geqslant -C|v_1-v_2|^2\quad\text{for each}\quad v_1,v_2\in\mathbb{R}^n, \tag{5.12}
$$

see [25], [26]. It was proved in [25] that if (5.12) is true, then (2.1) and (3.1) generate dynamical semigroups acting in the spaces of initial data $\mathbf{H}_\varepsilon$ and $\mathbf{H}$, possessing global attractors $\mathcal{A}_\varepsilon$ and $\overline{\mathcal{A}}$ being bounded in the spaces $\mathbf{V}_\varepsilon$ and $\mathbf{V}$, respectively (see also [40], [6]). Moreover,

$$
\mathcal{A}_\varepsilon=\{u(0)\mid u\in\mathfrak{A}_\varepsilon\},\qquad \overline{\mathcal{A}}=\{u(0)\mid u\in\overline{\mathfrak{A}}\}.
$$

The convergence (5.11) implies the following statement.

Corollary 5.2. *Under the assumptions of Theorem 5.1 the convergence*

$$
\mathrm{dist}_{\mathbf{H}^{-\delta}}\left(\mathcal{A}_\varepsilon,\overline{\mathcal{A}}\right)\to 0,\qquad \varepsilon\to 0+,
$$

holds.

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