ISSN 2304-0122 Ufa Mathematical Journal. Vol. 17. No 2 (2025). P. 69-90.

doi:10.13108/2025-17-2-69

# ON ONE METHOD OF RATIONAL APPROXIMATIONS OF RIEMANN — LIOUVILLE TYPE INTEGRAL ON SEGMENT

**P.G. POTSEIKO, Y.A. ROVBA**

**Abstract.** We study rational approximations of functions defined by a Riemann — Liouville integral on the interval $[-1,1]$ with a density belonging to some classes of continuous functions. As the approximation apparatus, the Riemann — Liouville type integral with a density being a rational Fourier — Chebyshev integral operator serves. We find upper bounds for approximations of the Riemann — Liouville type integral with a bounded density, which depends on the poles and the position of a point in the segment.

As a separate problem we study of approximations of Riemann — Liouville type integrals with a density being a function with a power singularity. We obtain uniform upper bounds for approximations with a certain majorant that depends on the position of a point in a segment. We find an asymptotic expression for this majorant, which depends on the poles of approximating rational function. We study the case, when the poles are some modifications of the Newman parameters. We find optimal values of the parameters, for which the approximations have the greatest decay rate. The rate of best rational approximations by the considered method is higher in comparison with the corresponding polynomial analogues.

**Keywords:** Riemann — Liouville integral, rational Fourier — Chebyshev integral operator, uniform rational approximation, asymptotic estaimtes, Laplace method.

**Mathematics Subject Classification:** 53A04, 52A40, 52A10

## 1. Introduction

The operator of Riemann — Liouville fractional differentiation [14]

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)\tau}{(t-\tau)^{1-\alpha}}, \quad \alpha > 0,$$

where $\Gamma(\cdot)$ is the Euler Gamma function, has wide applications in various fields of science and technics [2], [5]. A number of problems in fluid mechanics, chemistry, physics and other scientific fields are described by models using mathematical tools from the theory of fractional calculus. As a rule, the analytical solution of these problems is difficult, and the development of approximate methods for solving them is relevant [25], [23], [26], [28].

The functions represented by the Riemann— Liouville integral are widely used in the theory of both polynomial [6], [20] and rational approximation [16], [10], [18], [17], [13]. With their help, new classes of continuous functions were found, on which the rate of uniform rational approximation is higher than for the corresponding polynomial analogues. At the same

P.G. Potseiko, Y.A. Rovba, On one method of rational approximations of Riemann — Liouville type integral on segment.



The work is supported by the State Programm of Scientific Researches «Convergence 2020», no. 20162269 (Republich of Belarus).

*Submitted May 5, 2024.*

time, in the approximation of Riemann — Liouville integrals, the Fourier series are employed occasionally.

In the rational approximation, Fejer, Jackson, and Vallée Poussin integral operators [12], [11], [19] are applied, which are analogues of well–known polynomial periodic operators based on Fourier series and methods of their summation. In 1979, Rovba [9] introduced an integral operator on the interval $[-1,1]$ associated with the Chebyshev — Markov system of rational functions, which is a natural generalization of partial sums of the Chebyshev — polynomial Fourier series.

Suppose we are given an arbitrary set of numbers $\{a_k\}_{k=1}^n$, where either $a_k$ are real and $|a_k| < 1$ or they are complex conjugate. On the set of functions $f(x)$ summable with the weight $1/\sqrt{1-x^2}$ on the segment $[-1,1]$ we consider the rational Fourier — Chebyshev integral operator of order at most $n$, see [9],

$$s_n(f,x) = \frac{1}{2\pi}\int\limits_{-\pi}^{+\pi} f(\cos v)\frac{\sin\lambda_n(u,v)}{\sin\dfrac{v-u}{2}}\,dv, \quad x=\cos u, \tag{1.1}$$

where

$$\lambda_n(u,v) = \int\limits_u^v \lambda_n(y)\,dy, \qquad \lambda_n(y) = \frac{1}{2} + \sum_{k=1}^n \frac{1-|\alpha_k|^2}{1+2|\alpha_k|\cos(y-\arg\alpha_k)+|\alpha_k|^2},$$

$$\alpha_k = \frac{a_k}{1+\sqrt{1-a_k^2}}.$$

The branch of root is fixed by the requirement $|\alpha_k| < 1$. The operator $s_n$ sends $f$ into the set $\mathbb{R}_n(A)$, which consists of rational functions

$$\frac{p_n(x)}{\prod\limits_{k=1}^n (1+a_k x)}, \qquad p_n(x)\in\mathbb{P}_n,$$

$A = (a_1,\ldots,a_n)$, and $s_n(1,x)\equiv 1$. If $a_k = 0$, $k=1,2,\ldots,n$, then the operator $s_n(\cdot,\cdot)$, is the Dirichlet integral of polyanomial Fourier — Chebyshev series.

Gorskaya and Galimyanov developed methods for approximate calculation of the Riemann— Liouville integral on the real axis by using orthogonal Fourier series [3], [24]. A distinctive feature of these studies was the approach based on the representation of density of Riemann— Liouville integral by a Fourier series. In our opinion, this method of approximating the Riemann— Liouville integral has been little studied and is of scientific interest. In [8], approximations of the Riemann— Liouville type integral on the interval $[-1,1]$ were introduced and investigated by a method based on representing its density by partial sums of the Fourier — Chebyshev polynomial series. An integral representation of approximations was established and estimates for pointwise and uniform approximations were obtained in the case, when the density belongs to some classes of continuous functions on a segment.

The aim of this paper is to study rational approximations of the Riemann — Liouville type integral on the interval $[-1,1]$ by a method based on the representation of its density by the rational Fourier— Chebyshev integral operator (1.1). We obtain an integral representation for the approximations and their pointwise and uniform estimates. The dependence of estimates on the choice of poles of the approximating function is established. We find estimates for uniform rational approximations in the case when the poles represent some modifications of the Newman parameters.

## 2. Riemann — Liouville integral on segment

We consider the class of functions

$$f(x)=\frac{1}{\Gamma(r)}\int\limits_{-1}^{x}(x-t)^{r-1}\varphi(t)\frac{dt}{\sqrt{1-t^2}},\qquad x\in[-1,1],\qquad r\in[1,+\infty).\tag{2.1}$$

It is obvious that the integral in the right hand side is a Riemann— Liouville type integral on the interval $[-1,1]$ with density $\varphi(t)\in C[-1,1]$. It easily follows from (2.1) that

$$\varphi(x)=\sqrt{1-x^2}f^{(r)}(x),\qquad r=1,2,\ldots.$$

Suppose that the density of integral (2.1) is represented by a rational Fourier — Chebyshev integral operator (1.1) Then the operator

$$\tilde{s}_n(\varphi,x)=\frac{1}{\Gamma(r)}\int\limits_{-1}^{x}(x-t)^{r-1}s_n(\varphi,t)\frac{dt}{\sqrt{1-t^2}},\qquad x\in[-1,1],\tag{2.2}$$

defines some function, rational for $r=1,2,\ldots$, with the same poles as $s_n(\varphi,t)$. We introduce the notation

$$\begin{aligned}&\tilde{\varepsilon}_n(\varphi,x,A)=f(x)-\tilde{s}_n(\varphi,x),\qquad x\in[-1,1],\\&\tilde{\varepsilon}_n(\varphi,A)=\|f(x)-\tilde{s}_n(\varphi,x)\|_{C[-1,1]},\qquad n\in\mathbb{N}.\end{aligned}\tag{2.3}$$

We suppose that $\alpha_1=\alpha_2=\ldots=\alpha_p=0$, $p=[r-1]$, where $[\cdot]$ denotes the integer part of a number. Let us study the quantity $\tilde{\varepsilon}_n(\varphi,x,A)$.

**Theorem 2.1.** *For each $r\in[1,+\infty)$ the approximations of the Riemann— Liouville type integral (2.1) on the interval $[-1,1]$ by the operator (2.2) satisfy the integral representation*

$$\begin{aligned}\tilde{\varepsilon}_n(\varphi,x,A)=&\frac{2^{1-r}}{\pi\Gamma(r)}\int\limits_{-\pi}^{\pi}\varphi(\cos v)\int\limits_{0}^{1}(1-t)^{r-1}t^{1-r}\sqrt{\prod_{k=1}^{n}\frac{t^2+2t|\alpha_k|\cos(u-\theta_k)+|\alpha_k|^2}{1+2t|\alpha_k|\cos(u-\theta_k)+|\alpha_k|^2t^2}}\\&\cdot\frac{(1-2t\cos 2u+t^2)^{\frac{r-1}{2}}}{\sqrt{1-2t\cos(v-u)+t^2}}\sin\psi_n(x,t,v)\,dt\,dv,\quad x=\cos u,\ \theta_k=\arg\alpha_k,\end{aligned}\tag{2.4}$$

*where*

$$\psi_n(x,t,v)=\arg\frac{(\xi t-\bar{\xi})^{r-1}\omega(\xi t)}{(t-\zeta\bar{\xi})\omega(\zeta)},\qquad \omega(\zeta)=\prod_{k=1}^{n}\frac{\zeta+\alpha_k}{1+\alpha_k\zeta},\qquad \xi=\mathrm{e}^{iu},$$

*$\Gamma(\cdot)$ is the Euler Gamma function.*

*Proof.* It is known [9] that the rational Fourier — Chebyshev integral operator satisfies the representation

$$s_n(\varphi,t)=\frac{1}{2\pi}\int\limits_{0}^{\pi}\varphi(\cos v)D_n(v,\tau)\,dv,\quad t=\cos\tau,\quad n=0,1,\ldots,\tag{2.5}$$

where

$$D_n(v,\tau)=\frac{\zeta\dfrac{\omega(\zeta)}{\omega(z)}-z\dfrac{\omega(z)}{\omega(\zeta)}}{\zeta-z}+\frac{\zeta z\omega(\zeta)\omega(z)-\dfrac{1}{\omega(\zeta)\omega(z)}}{\zeta z-1},\quad z=\mathrm{e}^{i\tau},\quad \zeta=\mathrm{e}^{iv},$$

and the function $\omega(\cdot)$ is defined in the formulation of the theorem. We substitute this representation into (2.2) and use the Fubini theorem to interchange the integration order

$$\tilde{s}_n(\varphi, x) = \frac{1}{2\pi\Gamma(r)} \int\limits_0^\pi \varphi(\cos v) I_n(v, x)\, dv, \quad x \in [-1, 1], \tag{2.6}$$

where

$$I_n(v, x) = \int\limits_{-1}^x (x-t)^{r-1} D_n(v, \tau) \frac{dt}{\sqrt{1-t^2}}, \quad t = \cos\tau.$$

We transform the inner integral $I_n(v, x)$ by making the change of variable $t = \cos\tau$

$$I_n(v, x) = \int\limits_u^\pi (\cos u - \cos\tau)^{r-1} D_n(v, \tau)\, d\tau, \quad x = \cos u.$$

The integrand is even and hence,

$$I_n(v, x) = \frac{1}{2} \int\limits_{[-\pi,-u]\sqcup[u,\pi]} (\cos u - \cos\tau)^{r-1} D_n(v, \tau)\, d\tau, \quad x = \cos u.$$

Passing to integration over the variable $z$, $z = \mathrm{e}^{i\tau}$, in the integral in the right hand side, we have

$$I_n(v, x) = \frac{(-1)^{r-1}}{2^r i} \int\limits_\Gamma (z-\xi)^{r-1} (z-\overline{\xi})^{r-1} z^{1-r} D_n(v, \tau) \frac{dz}{z}, \quad \xi = \mathrm{e}^{iu},$$

where $\Gamma$ the arc of unit circumference from the point $\xi$ to the point $1/\xi$ passed counterclockwise, see Figure 1.

It is obvious that for a fixed value of the parameter $v$ the integral $I_n(v, x)$ is a function of the parameter $x$ with first order poles at the points (see (1.1))

$$a_k = -\left(\frac{2z_k}{(1+z_k^2)}\right)^{-1}, \qquad k = 1, 2, \ldots, n.$$

This is why it is sufficient to study the integral $I_n(x, v, \rho)$, which differs from $I_n(x, v)$ by $\zeta = \rho\mathrm{e}^{iv}$, $\rho \in (0, 1)$, and then to use the identity

$$I_n(v, x) = \lim_{\rho\to 1} I_n(v, x, \rho). \tag{2.7}$$

We represent the integral in $I_n(v, x, \rho)$ as a sum of four integrals

$$I_n(v, x, \rho) = \frac{(-1)^{r-1}}{2^r i} \left[\overline{\omega(\zeta)} J_1 - \zeta\omega(\zeta) J_2 + \omega(\zeta) J_3 - \overline{\zeta\omega(\zeta)} J_4\right], \tag{2.8}$$

where

$$J_1 = \int\limits_\Gamma \frac{(z-\xi)^{r-1}(z-\overline{\xi})^{r-1} z^{1-r}}{z-\zeta} \omega(z)\, dz, \qquad J_2 = \int\limits_\Gamma \frac{(z-\xi)^{r-1}(z-\overline{\xi})^{r-1}}{(z-\zeta) z^r \omega(z)}\, dz,$$

$$J_3 = \int\limits_\Gamma \frac{(z-\xi)^{r-1}(z-\overline{\xi})^{r-1} z^{1-r}}{z-\frac{1}{\zeta}} \omega(z)\, dz, \qquad J_4 = \int\limits_\Gamma \frac{(z-\xi)^{r-1}(z-\overline{\xi})^{r-1}}{(z-\frac{1}{\zeta}) z^r \omega(z)}\, dz.$$

We note that in the case when the parameter $r$, $r \in (1, +\infty)$, is not natural, the integrands of each of the integrals have branching points $z = 0$, $z = \xi$, $z = \overline{\xi}$ and $z = \infty$. If $r = 1, 2, \ldots$, then the integrands are rational functions of the integration variable and the reasoning in this case is simpler. Since, obviously, the approximations (2.3) have a similar integral representation, we single out single–valued branches of multivalued functions only if $r \in (1, +\infty)\setminus\mathbb{N}$.

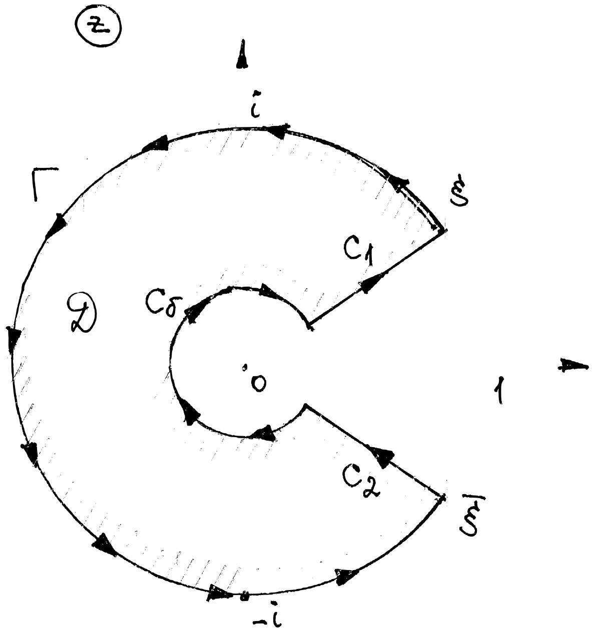


FIGURE 1. The contour $C$ for the integral $J_1$.

We transform each of the four integrals in (2.8) separately. Let us study the integral $J_1$. We fix the parameter $\xi$ and consider the domain bounded by the contour on Figure 1

$$C = C_1 \cup \Gamma \cup C_2^- \cup C_\delta^-,$$

where

$$C_1 = \{z : z = \xi t,\ t \in [\delta, 1]\}, \qquad C_2 = \left\{z : z = \overline{\xi} t,\ t \in [\delta, 1]\right\},$$
$$C_\delta = \left\{z : z = \delta \mathrm{e}^{i\tau},\ \tau \in [\theta, 2\pi - \theta]\right\}.$$

In this domain the function $g_r(z, \xi) = (z - \xi)^{r-1}(z - \overline{\xi})^{r-1} z^{1-r}$, splits into regular branches fixed by the conditions $g_a(1, \mathrm{e}^{i\frac{\pi}{3}}) = \mathrm{e}^{i\pi(2k-1)(r-1)}$, $k \in \mathbb{Z}$. We choose the branch, which obeys the condition $g_a^*(1, \mathrm{e}^{i\frac{\pi}{3}}) = (-1)^{r-1}$, and apply the Cauchy residue theorem to the integral $J_1$

$$\left(\int\limits_{C_1} + \int\limits_{\Gamma} + \int\limits_{C_2^-} + \int\limits_{C_\delta^-}\right) \frac{(z - \xi)^{r-1}(z - \overline{\xi})^{r-1} z^{1-r}}{z - \zeta} \omega(z)\, dz = 2\pi i \omega(\zeta) r(\zeta, \xi),$$

where

$$r(\zeta, \xi) = \begin{cases} (\zeta - \xi)^{r-1}(\zeta - \overline{\xi})^{r-1}\zeta^{1-r}, & \zeta \in \mathfrak{D}, \\ 0, & \zeta \notin \mathfrak{D}. \end{cases}$$

We consider the integral over the arc $C_\delta$. Making the change of variable $z = \delta \mathrm{e}^{i\tau}$, we easily get

$$\int\limits_{C_\delta} \frac{(z - \xi)^{r-1}(z - \overline{\xi})^{r-1} z^{1-r}}{z - \zeta} \omega(z)\, dz$$
$$= i\delta^{p+2-r} \int\limits_{-u}^{u} \frac{(\delta \mathrm{e}^{i\tau} - \xi)^{r-1}(\delta \mathrm{e}^{i\tau} - \overline{\xi})^{r-1} \mathrm{e}^{(p+2-r)i\tau}}{\delta \mathrm{e}^{i\tau} - \zeta} \prod_{k=p+1}^{n} \frac{\delta \mathrm{e}^{i\tau} + \alpha_k}{1 + \alpha_k \delta \mathrm{e}^{i\tau}}\, d\tau.$$

Since by the assumption $p + 2 - r > 0$, as $\delta \to 0$ we arrive at the asymptotic identity

$$\int\limits_{C_\delta} \frac{(z - \xi)^{r-1}(z - \overline{\xi})^{r-1} z^{1-r}}{z - \zeta} \omega(z)\, dz \sim -\frac{2i \sin(p + 2 - r)u}{\zeta} \delta^{p+2-r} \prod_{k=p+1}^{n} \alpha_k \underset{\delta \to 0}{\longrightarrow} 0.$$

At the same time we find

$$\int_0^{\xi} \frac{(z-\xi)^{r-1}(z-\overline{\xi})^{r-1}z^{1-r}}{z-\zeta}\omega(z)\,dz + J_1 + \int_{\overline{\xi}}^{0} \frac{(z-\xi)^{r-1}(z-\overline{\xi})^{r-1}z^{1-r}}{z-\zeta}\omega(z)\,dz = 2\pi i\omega(\zeta)r(\zeta,\xi),$$

where the first and third integrals are taken over the corresponding rays of the complex plane.

In the first integral we make the change of variable $z=\xi t$, and the change $z=\overline{\xi}t$ in the third integral and we arrive at the expression

$$J_1 = -\int_0^1 (t-1)^{r-1}t^{1-r}\left[\frac{\xi(\xi t-\overline{\xi})^{r-1}}{\xi t-\zeta}\omega(\xi t) - \frac{\overline{\xi}(\overline{\xi}t-\xi)^{r-1}}{\overline{\xi}t-\zeta}\omega(\overline{\xi}t)\right]dt + 2\pi i\omega(\zeta)r(\zeta,\xi). \quad (2.9)$$

We proceed to studying the integral $J_2$. As above, we fix $\xi$ and consider the domain $D$ enveloped by the contour shown in Figure 2,

$$C = C_1 \cup C_R \cup C_2^- \cup \Gamma^-,$$

where

$$C_1 = \{z : z = \xi t,\ t\in[1,R]\},\quad C_2 = \left\{z : z = \overline{\xi}t,\ t\in[1,R]\right\},$$
$$C_R = \left\{z : z = Re^{i\tau},\ \tau\in[\theta, 2\pi-\theta]\right\}.$$

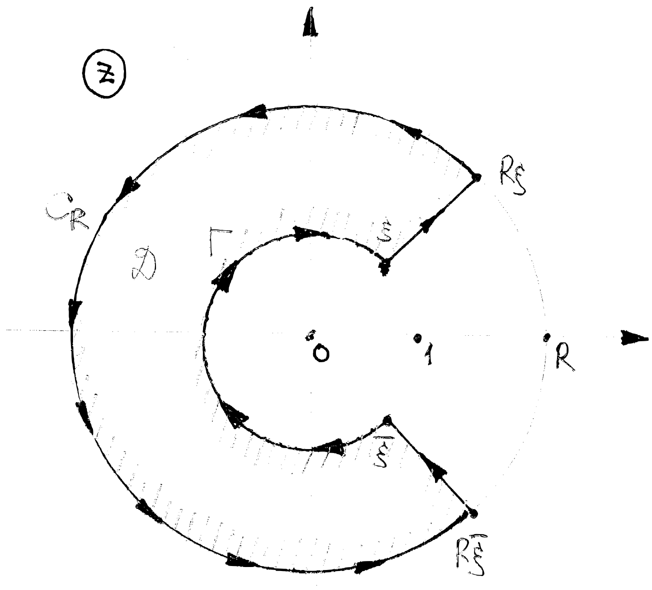


FIGURE 2. The contour $C$ for the integral $J_2$.

In this domain the function

$$g_r(z,\xi) = (z-\xi)^{r-1}(z-\overline{\xi})^{r-1}z^{1-r},$$

splits into regular branches. Arguing as in the case of the integral $J_1$, we single out its single–valued branch. Applying the Cauchy integral theorem to the integral $J_2$, we obtain

$$\left(\int_{C_1} + \int_{C_R} + \int_{C_2^-} + \int_{\Gamma^-}\right) \frac{(z-\xi)^{r-1}(z-\overline{\xi})^{r-1}}{(z-\zeta)z^r\omega(z)}\,dz = 0. \quad (2.10)$$

We consider the integral over the arc $C_R$. Making the change of variable $z = Re^{i\tau}$, we get

$$\int\limits_{C_R} \frac{(z-\xi)^{r-1}(z-\overline{\xi})^{r-1}}{(z-\zeta)z^r\omega(z)}\,dz = \int\limits_{u}^{-u} \frac{(Re^{i\tau}-\xi)^{r-1}(Re^{i\tau}-\overline{\xi})^{r-1}}{(Re^{i\tau}-\zeta)(Re^{i\tau})^r\omega(Re^{i\tau})}\,Rie^{i\tau}\,d\tau$$

$$= \frac{i}{R^{p+2-r}} \int\limits_{u}^{-u} \frac{\left(e^{i\tau}-\frac{\xi}{R}\right)^{r-1}\left(e^{i\tau}-\frac{\overline{\xi}}{R}\right)^{r-1}}{\left(e^{i\tau}-\frac{\zeta}{R}\right)e^{i\tau(r-1)}} \prod_{k=p+1}^{n} \frac{\frac{1}{R}+\alpha_k e^{i\tau}}{e^{i\tau}+\frac{\alpha_k}{R}}\,d\tau.$$

Passing to the limit as $R\to\infty$, in view of the inequality $p+2-r>0$ we obtain

$$\int\limits_{C_R} \frac{(z-\xi)^{r-1}(z-\overline{\xi})^{r-1}}{(z-\zeta)z^r\omega(z)}\,dz \sim -\frac{2i}{(2-r)R^{p+2-r}}\sin((p+2-r)\theta)\prod_{k=p+1}^{n}\alpha_k \xrightarrow[R\to\infty]{} 0\,.$$

At the same time by (2.10) we find

$$\int\limits_{\xi}^{+\xi\infty} \frac{(z-\xi)^{r-1}(z-\overline{\xi})^{r-1}}{(z-\zeta)z^r\omega(z)}\,dz + \int\limits_{+\overline{\xi}\infty}^{\overline{\xi}} \frac{(z-\xi)^{r-1}(z-\overline{\xi})^{r-1}}{(z-\zeta)z^r\omega(z)}\,dz - J_2 = 0,$$

where the first and second integrals are taken over the corresponding rays of the complex plane. By making the changes $z=\xi t$ and $z=\overline{\xi}t$ in the first and second integrals, respectively, we obtain

$$J_2 = \int\limits_{1}^{+\infty} (t-1)^{r-1}t^{-r}\left[\frac{(\xi t-\overline{\xi})^{r-1}}{(\xi t-\zeta)\omega(\xi t)} - \frac{(\overline{\xi}t-\xi)^{r-1}}{(\overline{\xi}t-\zeta)\omega(\overline{\xi}t)}\right]dt.$$

One more change of variable $t\mapsto 1/t$ gives

$$J_2 = (-1)^{1-r}\int\limits_{0}^{1}(1-t)^{r-1}t^{1-r}\left[\frac{(\overline{\xi}t-\xi)^{r-1}\omega(\overline{\xi}t)}{\xi-\zeta t} - \frac{(\xi t-\overline{\xi})^{r-1}\omega(\xi t)}{\overline{\xi}-\zeta t}\right]dt. \tag{2.11}$$

Proceeding similarly for the integrals $J_3$ and $J_4$, we conclude

$$J_3 = -\int\limits_{0}^{1}(t-1)^{r-1}t^{1-r}\left[\frac{\xi(\xi t-\overline{\xi})^{r-1}\omega(\xi t)}{\xi t-\frac{1}{\zeta}} - \frac{\overline{\xi}(\overline{\xi}t-\xi)^{r-1}\omega(\overline{\xi}t)}{\overline{\xi}t-\frac{1}{\zeta}}\right]dt, \tag{2.12}$$

$$\begin{aligned} J_4 =&(-1)^{r-1}\int\limits_{0}^{1}(1-t)^{r-1}t^{1-r}\left[\frac{(\overline{\xi}t-\xi)^{r-1}\omega(\overline{\xi}t)}{\xi-\frac{t}{\zeta}} - \frac{(\xi t-\overline{\xi})^{r-1}\omega(\xi t)}{\overline{\xi}-\frac{t}{\zeta}}\right]dt \\ &-2\pi i\zeta\omega(\zeta)r(\zeta,\xi). \end{aligned} \tag{2.13}$$

The representations (2.8), (2.9), (2.11), (2.12) and (2.13) imply

$$\begin{aligned} I_n(v,x,\rho) = &-\frac{1}{2^r i}\int\limits_{0}^{1}(1-t)^{r-1}t^{1-r}\left[\frac{\xi(\xi t-\overline{\xi})^{r-1}\omega(\xi t)}{(\xi t-\zeta)\omega(\zeta)} - \frac{\overline{\xi}(\overline{\xi}t-\xi)^{r-1}\omega(\overline{\xi}t)}{(\overline{\xi}t-\zeta)\omega(\zeta)}\right. \\ &+\frac{(\overline{\xi}t-\xi)^{r-1}\omega(\overline{\xi}t)\zeta\omega(\zeta)}{\xi-\zeta t} - \frac{(\xi t-\overline{\xi})^{r-1}\omega(\xi t)\zeta\omega(\zeta)}{\overline{\xi}-\zeta t} \\ &+\frac{\xi(\xi t-\overline{\xi})^{r-1}\omega(\xi t)\omega(\zeta)}{\xi t-\frac{1}{\zeta}} - \frac{\overline{\xi}(\overline{\xi}t-\xi)^{r-1}\omega(\overline{\xi}t)\omega(\zeta)}{\overline{\xi}t-\frac{1}{\zeta}} \end{aligned}$$

$$+\frac{(\overline{\xi}t-\xi)^{r-1}\omega(\overline{\xi}t)}{(\xi-\frac{t}{\zeta})\zeta\omega(\zeta)}-\frac{(\xi t-\overline{\xi})^{r-1}\omega(\xi t)}{\left(\overline{\xi}-\frac{t}{\zeta}\right)\zeta\omega(\zeta)}\Bigg]\,dt+2\pi r_1(u,v),$$

where

$$r_1(u,v)=\begin{cases}(\cos u-\cos v)^{r-1}, & |u|<v,\\ 0, & |u|>v.\end{cases}$$

By appropriate transformations the above integral is reduced to

$$I_n(v,x,\rho)=-\frac{2^{1-r}}{i}\int\limits_0^1(1-t)^{r-1}t^{1-r}\left[\frac{(\xi t-\overline{\xi})^{r-1}\omega(\xi t)}{(t-\zeta\overline{\xi})\omega(\zeta)}-\frac{(\overline{\xi}t-\xi)^{r-1}\omega(\overline{\xi}t)}{(t-\zeta\xi)\omega(\zeta)}\right.$$
$$\left.-\frac{(\overline{\xi}t-\xi)^{r-1}\omega(\zeta)\omega(\overline{\xi}t)}{t-\overline{\zeta}\xi}+\frac{(\xi t-\overline{\xi})^{r-1}\omega(\zeta)\omega(\xi t)}{t-\overline{\zeta\xi}}\right]dt+2\pi r_1(u,v),\ \ \zeta=\rho e^{iv},\xi=e^{iu}.$$

For each fixed $t\in(0,1)$ the expression in the square brackets of the integrand is continuous in the variable $\zeta$ and hence, the passage to the limit (2.7) is valid. In this case, the representation (2.6) yields

$$\tilde{s}_n(\varphi,x)=-\frac{1}{2^r\pi i\Gamma(r)}\int\limits_0^\pi\varphi(\cos v)\int\limits_0^1(1-t)^{r-1}t^{1-r}\left[\frac{(\xi t-\overline{\xi})^{r-1}\omega(\xi t)}{(t-\zeta\overline{\xi})\omega(\zeta)}-\frac{(\overline{\xi}t-\xi)^{r-1}\omega(\overline{\xi}t)}{(t-\zeta\xi)\omega(\zeta)}\right.$$
$$\left.-\frac{(\overline{\xi}t-\xi)^{r-1}\omega(\zeta)\omega(\overline{\xi}t)}{t-\overline{\zeta}\xi}+\frac{(\xi t-\overline{\xi})^{r-1}\omega(\zeta)\omega(\xi t)}{t-\overline{\zeta\xi}}\right]dt\,dv$$
$$+\frac{1}{\Gamma(r)}\int\limits_u^\pi\varphi(\cos v)(\cos u-\cos v)^{r-1}\,dv,\quad x\in[-1,1].$$

The second integral in the right hand side is the Riemann — Liouville type integral (2.1). By the relation (2.3) this yields

$$\tilde{\varepsilon}_n(\varphi,x,A)=\frac{1}{2^r\pi i\Gamma(r)}\int\limits_0^\pi\varphi(\cos v)\int\limits_0^1(1-t)^{r-1}t^{1-r}\left[\frac{(\xi t-\overline{\xi})^{r-1}\omega(\xi t)}{(t-\zeta\overline{\xi})\omega(\zeta)}-\frac{(\overline{\xi}t-\xi)^{r-1}\omega(\overline{\xi}t)}{(t-\zeta\xi)\omega(\zeta)}\right.$$
$$\left.-\frac{(\overline{\xi}t-\xi)^{r-1}\omega(\zeta)\omega(\overline{\xi}t)}{t-\overline{\zeta}\xi}+\frac{(\xi t-\overline{\xi})^{r-1}\omega(\zeta)\omega(\xi t)}{t-\overline{\zeta\xi}}\right]dt\,dv,\quad \xi=e^{iu},\ x=\cos u.$$

We represent the outer integral as the sum of two integrals, respectively, by the term in the square bracket of the integrand and then after a simple change of variable $v\mapsto -v$ we arrive at the expression

$$\begin{aligned}\tilde{\varepsilon}_n(\varphi,x,A)=&\frac{1}{2^r\pi i\Gamma(r)}\int\limits_{-\pi}^\pi\varphi(\cos v)\\ &\cdot\int\limits_0^1(1-t)^{r-1}t^{1-r}\left[\frac{(\xi t-\overline{\xi})^{r-1}\omega(\xi t)}{(t-\zeta\overline{\xi})\omega(\zeta)}-\frac{(\overline{\xi}t-\xi)^{r-1}\omega(\zeta)\omega(\overline{\xi}t)}{t-\overline{\zeta}\xi}\right]dt\,dv.\end{aligned}\tag{2.14}$$

The expressions in square brackets are complex conjugate, and to arrive at (2.4) it is sufficient to perform simple transformations in the found integral representation. The proof is complete. □

In Theorem 2.1 we let $\alpha_k = 0$, $k = 1, 2, \ldots, n$. In this case $A = (0, 0, \ldots, 0) = O$, and the quantity $\tilde{\varepsilon}_n(\varphi, x, O) = \tilde{\varepsilon}_n^{(0)}(\varphi, x)$ is the approximation for the Riemann — Liouville integral (2.1) by the operator, which is the image of polynomial Fourier — Chebyshev series under the transformation (2.2). In this case the following corollary holds.

**Corollary 2.1.** *The integral representation*

$$\tilde{\varepsilon}_n^{(0)}(\varphi, x) = \frac{2^{1-r}}{\pi\Gamma(r)} \int\limits_{-\pi}^{\pi} \varphi(\cos v) \int\limits_{0}^{1} (1-t)^{r-1} t^{n+1-r} \frac{(1 - 2t\cos 2u + t^2)^{\frac{r-1}{2}}}{\sqrt{1 - 2t\cos(v-u) + t^2}}$$
$$\cdot \sin\left(n(u-v) + \arg\frac{(\xi t - \overline{\xi})^{r-1}}{t - \zeta\overline{\xi}}\right) dt\, dv, \qquad \xi = e^{iu}, \qquad x = \cos u,$$

*holds.*

The latter integral representation was given in [8].

**Theorem 2.2.** *If* $\max\limits_{|x|\leqslant 1} |\varphi(x)| = K$, *and the poles of approximating rational functions satisfies the condition*

$$\sigma_n(A) = \sum_{k=1}^{n} (1 - |\alpha_k|) \underset{n\to\infty}{\longrightarrow} \infty, \tag{2.15}$$

*then for each* $r \in (1, +\infty)$ *as* $n \to \infty$ *the upper bounds*

$$|\tilde{\varepsilon}_n(\varphi, x, A)| \leqslant \frac{2K(\sqrt{1-x^2})^{r-1} \ln \sigma_n(A)}{\pi[\sigma_n(A)]^r} + O\left(\frac{(\sqrt{1-x^2})^{r-1}}{[\sigma_n(A)]^r}\right) \tag{2.16}$$

*hold if* $x \in (-1, 1)$ *and*

$$|\tilde{\varepsilon}_n(\varphi, 1, A)| \leqslant \frac{2^{2-r} K\Gamma(2r-1)}{\pi\Gamma(r)} \frac{\ln \sigma_n(A)}{[\sigma_n(A)]^{2r-1}} + O\left(\frac{1}{[\sigma_n(A)]^{2r-1}}\right), \tag{2.17}$$

*for* $x = 1$.

*Proof.* We note that the condition (2.15) is necessary and sufficient for the completeness of the system of rational functions $\{1/(z - \alpha_k)\}_{k=1}^{+\infty}$ [1]. We use the integral representation of approximations (2.4). The $2\pi$–periodicity of the integrand in the outer integral over the variable $v$ implies

$$|\tilde{\varepsilon}_n(\varphi, x, A)| \leqslant \frac{2^{1-r} K}{\pi\Gamma(r)} \int\limits_{0}^{2\pi} \int\limits_{0}^{1} (1-t)^{r-1} t^{1-r} \sqrt{\prod_{k=1}^{n} \frac{t^2 + 2t|\alpha_k|\cos(u - \theta_k) + |\alpha_k|^2}{1 + 2t|\alpha_k|\cos(u-\theta_k) + |\alpha_k|^2 t^2}}$$
$$\cdot \frac{(1 - 2t\cos 2u + t^2)^{\frac{r-1}{2}}}{\sqrt{1 - 2t\cos v + t^2}} |\sin \psi_n(x, t, v+u)|\, dt\, dv, \quad x = \cos u, \;\; \theta_k = \arg \alpha_k,$$

where $\psi_n(x, t, v+u)$ and $\omega(\zeta)$ were defined in Theorem 2.1.

Let us estimate the root of the integrand. We apply the method proposed in [22] and obtain

$$\begin{aligned}\sqrt{\prod_{k=1}^{n} \frac{t^2 + 2t|\alpha_k|\cos(u-\theta_k) + |\alpha_k|^2}{1 + 2t|\alpha_k|\cos(u-\theta_k) + |\alpha_k|^2 t^2}} &\leqslant \sqrt{\prod_{k=1}^{n} \left(1 - (1-t^2)(1-|\alpha_k|)\frac{1+|\alpha_k|}{(1+|\alpha_k|t)^2}\right)} \\ &\leqslant \sqrt{\prod_{k=1}^{n} (1 - (1-t^2)(1-|\alpha_k|))} \\ &\leqslant e^{\frac{1}{2}(t^2-1)\sigma_n(A)}, \quad n = 1, 2, \ldots,\end{aligned} \tag{2.18}$$

where the latter inequality is implied by the estimate $1-t \leqslant \mathrm{e}^{-t}$ for all $t$. We represent the outer integral as the sum of three integrals over the segments

$$\left[0, \frac{1}{\sigma_n(A)}\right], \quad \left[\frac{1}{\sigma_n(A)}, 2\pi - \frac{1}{\sigma_n(A)}\right] \quad \text{and} \quad \left[2\pi - \frac{1}{\sigma_n(A)}, 2\pi\right],$$

and we find

$$|\tilde{\varepsilon}_n(\varphi, x, A)| \leqslant \frac{2^{1-r}K}{\pi\Gamma(r)}\left(I_n^{(1)} + I_n^{(2)} + I_n^{(3)}\right), \tag{2.19}$$

where

$$I_n^{(1)} = \int\limits_0^{\frac{1}{\sigma_n(A)}} \int\limits_0^1 \frac{(1-t)^{r-1}t^{1-r}(1-2t\cos 2u+t^2)^{\frac{r-1}{2}}}{\sqrt{1-2t\cos v+t^2}} \mathrm{e}^{\frac{1}{2}(t^2-1)\sigma_n(A)} |\sin\psi_n(x,t,v+u)|\,dt\,dv,$$

$$I_n^{(2)} = \int\limits_{\frac{1}{\sigma_n(A)}}^{2\pi-\frac{1}{\sigma_n(A)}} \int\limits_0^1 \frac{(1-t)^{r-1}t^{1-r}(1-2t\cos 2u+t^2)^{\frac{r-1}{2}}}{\sqrt{1-2t\cos v+t^2}} \mathrm{e}^{\frac{1}{2}(t^2-1)\sigma_n(A)} |\sin\psi_n(x,t,v+u)|\,dt\,dv,$$

$$I_n^{(3)} = \int\limits_{2\pi-\frac{1}{\sigma_n(A)}}^{2\pi} \int\limits_0^1 \frac{(1-t)^{r-1}t^{1-r}(1-2t\cos 2u+t^2)^{\frac{r-1}{2}}}{\sqrt{1-2t\cos v+t^2}} \mathrm{e}^{\frac{1}{2}(t^2-1)\sigma_n(A)} |\sin\psi_n(x,t,v+u)|\,dt\,dv.$$

Let us study separately each of these integrals.

The integral $I_n^{(1)}$ obeys the estimate

$$I_n^{(1)} \leqslant \frac{1}{\sigma_n(A)} \int\limits_0^1 (1-t)^{r-2}t^{1-r}(1-2t\cos 2u+t^2)^{\frac{r-1}{2}} \mathrm{e}^{\frac{1}{2}(t^2-1)\sigma_n(A)}\,dt, \qquad r \in (1, +\infty).$$

To study the integral in right hand side, we employ the Laplace method [4], [21]. The function $S(t) = \frac{1}{2}(t^2-1)$ increases as $t \in (0,1)$ and hence, it attains its maximal value as $t=1$. In view of asymptotic identities $S(t) \sim t-1$,

$$(1-t)^{r-2}t^{1-r}(1-2t\cos 2u+t^2)^{\frac{r-1}{2}} \sim (2\sin u)^{r-1}(1-t)^{r-2},$$

valid $t \to 1$, for a sufficiently small $\varepsilon > 0$ we find

$$I_n^{(1)} \leqslant \frac{(2\sin u)^{r-1}}{\sigma_n(A)}(1+o(1)) \int\limits_{1-\varepsilon}^1 (1-t)^{r-2}\mathrm{e}^{(t-1)\sigma_n(A)}\,dt, \qquad n \to \infty.$$

After appropriate transformations of the integral in the right hand side, in view of the condition (2.15) it is easy to obtain

$$I_n^{(1)} \leqslant \frac{(2\sin u)^{r-1}\Gamma(r-1)}{[\sigma_n(A)]^r}(1+o(1)), \qquad n \to \infty. \tag{2.20}$$

We proceed to the integral $I_n^{(3)}$. The change of the variable $v \mapsto 2\pi - v$ gives

$$I_n^{(3)} = \int\limits_0^{\frac{1}{\sigma_n(A)}} \int\limits_0^1 \frac{(1-t)^{r-1}t^{1-r}(1-2t\cos 2u+t^2)^{\frac{r-1}{2}}}{\sqrt{1-2t\cos v+t^2}} \mathrm{e}^{\frac{1}{2}(t^2-1)\sigma_n(A)} |\sin\psi_n(x,t,u-v)|\,dt\,dv.$$

This implies that the integral $I_n^{(3)}$ obeys an estimate similar to that for $I_n^{(1)}$, namely,

$$I_n^{(3)} \leqslant \frac{(2\sin u)^{r-1}\Gamma(r-1)}{[\sigma_n(A)]^r}(1+o(1)), \qquad n \to \infty. \tag{2.21}$$

Finally, we proceed to the integral $I_n^{(2)}$ and represent it as

$$I_n^{(2)} = \left( \int\limits_{\frac{1}{\sigma_n(A)}}^{\pi} + \int\limits_{\pi}^{2\pi - \frac{1}{\sigma_n(A)}} \right) \int\limits_0^1 \frac{(1-t)^{r-1} t^{1-r} (1 - 2t\cos 2u + t^2)^{\frac{r-1}{2}}}{\sqrt{1 - 2t\cos v + t^2}}$$
$$\cdot\, \mathrm{e}^{\frac{1}{2}(t^2-1)\sigma_n(A)} |\sin \psi_n(x, t, v+u)|\, dt\, dv.$$

Obviously, it is sufficient to consider the first of them, since after replacing $v \mapsto 2\pi - v$, the same arguing shows that the second integral satisfies the same estimate. Applying the Laplace method to study the asymptotic behavior as $n \to \infty$ of the inner integral, we find

$$J_n^{(2)} = \frac{(2\sin u)^{r-1}\Gamma(r)}{2\sin\frac{v}{2}[\sigma_n(A)]^r} \left| \sin\left( \arg(\xi - \overline{\xi})^{r-1} - \arg(1-\zeta) - \arg\frac{\omega(\xi\zeta)}{\omega(\xi)} \right) \right| (1 + o(1)),$$

where $\xi = \mathrm{e}^{iu}$, $\zeta = \mathrm{e}^{iv}$.

Taking into consideration the identities

$$\arg(\xi - \overline{\xi})^{r-1} = \frac{\pi}{2}(r-1), \qquad \arg(1-\zeta) = -\frac{\pi}{2} + \frac{v}{2},$$
$$\arg\frac{\omega(\xi\zeta)}{\omega(\xi)} = \int\limits_u^{v+u} \sum_{k=1}^n \frac{1 - |\alpha_k|^2}{1 + 2|\alpha_k|\cos(y - \arg\alpha_k) + |\alpha_k|^2}\, dy,$$

we arrive at the estimate

$$I_n^{(2)} \leqslant \frac{(2\sin u)^{r-1}\Gamma(r)}{[\sigma_n(A)]^r}(1 + o(1)) \int\limits_{\frac{1}{\sigma_n(A)}}^{\pi} \frac{\left|\sin\left(\lambda_n(u, v+u) - \frac{\pi r}{2}\right)\right|}{\sin\frac{v}{2}}\, dv,$$

where $\lambda_n(u, v+u)$ is from (1.1).

Since

$$\frac{1}{\sin t} - \frac{1}{t} = O(t), \qquad t \in [0, \pi/2],$$

we get

$$\int\limits_{\frac{1}{\sigma_n(A)}}^{\pi} \frac{\left|\sin\left(\lambda_n(u, v+u) - \frac{\pi r}{2}\right)\right|}{\sin\frac{v}{2}}\, dv = \int\limits_{\frac{1}{\sigma_n(A)}}^{\pi} \frac{\left|\sin\left(\lambda_n(u, v+u) - \frac{\pi r}{2}\right)\right|}{v}\, dv + O(1),$$

and therefore,

$$I_n^{(2)} \leqslant \frac{2(2\sin u)^{r-1}\Gamma(r)}{[\sigma_n(A)]^r} \int\limits_{\frac{1}{\sigma_n(A)}}^{\pi} \frac{\left|\sin\left(\lambda_n(u, v+u) - \frac{\pi r}{2}\right)\right|}{v}\, dv + O\left(\frac{(\sin u)^{r-1}}{[\sigma_n(A)]^r}\right).$$

This yields

$$\begin{aligned} I_n^{(2)} &\leqslant \frac{2(2\sin u)^{r-1}\Gamma(r)}{[\sigma_n(A)]^r} \int\limits_{\frac{1}{\sigma_n(A)}}^{\pi} \frac{dv}{v} + O\left(\frac{(\sin u)^{r-1}}{[\sigma_n(A)]^r}\right) \\ &= \frac{2(2\sin u)^{r-1}\Gamma(r)\ln\sigma_n(A)}{[\sigma_n(A)]^r} + O\left(\frac{(\sin u)^{r-1}}{[\sigma_n(A)]^r}\right), \quad n \to \infty. \end{aligned} \tag{2.22}$$

By the representation (2.19) and the estimates (2.20), (2.21) and (2.22) we obtain

$$|\tilde{\varepsilon}_n(\varphi, x, A)| \leqslant \frac{2K(\sin u)^{r-1}}{\pi} \frac{\ln \sigma_n(A)}{[\sigma_n(A)]^r} + O\left(\frac{(\sin u)^{r-1}}{[\sigma_n(A)]^r}\right), \quad n \to \infty.$$

The latter relation yields the estimate (2.16).

To prove the inequality (2.17) in the representation (2.4) we let $x = 1$, which corresponds to $u = 0$. Then

$$\tilde{\varepsilon}_n(\varphi, 1, A) = \frac{2^{1-r}}{\pi\Gamma(r)} \int_0^{2\pi} \varphi(\cos v) \int_0^1 \frac{(1-t)^{2r-2}t^{1-r}}{\sqrt{1-2t\cos v + t^2}}$$
$$\cdot \sqrt{\prod_{k=1}^{n} \frac{t^2 + 2t|\alpha_k|\cos\theta_k + |\alpha_k|^2}{1 + 2t|\alpha_k|\cos\theta_k + |\alpha_k|^2 t^2}} \sin\psi_n(1, t, v)\, dt\, dv, \quad \theta_k = \arg\alpha_k,$$

where $\psi_n(1, t, v)$ was defined in Theorem 2.1.

Using the estimate (2.18) and again representing the outer integral as a sum of three integrals, we find

$$|\tilde{\varepsilon}_n(\varphi, 1, A)| \leqslant \frac{2^{1-r}K}{\pi\Gamma(r)} \left( \int_0^{\frac{1}{\sigma_n(A)}} + \int_{\frac{1}{\sigma_n(A)}}^{2\pi - \frac{1}{\sigma_n(A)}} + \int_{2\pi - \frac{1}{\sigma_n(A)}}^{2\pi} \right) I_n^{(4)}(v)\, dv, \tag{2.23}$$

where

$$I_n^{(4)}(v) = \int_0^1 \frac{(1-t)^{2r-2}t^{1-r}}{\sqrt{1-2t\cos v + t^2}} \mathrm{e}^{\frac{1}{2}(t^2-1)\sigma_n(A)} |\sin\psi_n(1, t, v)|\, dt.$$

Now the problem is reduced to studying the asymptotic behavior of each of the three terms in estimate (2.23). Using similar methods that were used to prove the inequality (2.16), we obtain (2.17). The proof is complete. $\square$

In Theorem 2.2 we let $\alpha_k = 0$, $k = 1, 2, \ldots, n$. In this case $A = (0, 0, \ldots, 0) = O$ and the quantity $\tilde{\varepsilon}_n(\varphi, x, O) = \tilde{\varepsilon}_n^{(0)}(\varphi, x)$ is the approximation for the Riemann — Liouville type integral (2.1) by a polynomial analogue of the operator (2.2).

**Corollary 2.2.** *If* $\max\limits_{|x|\leqslant 1} |\varphi(x)| = K$, *then for* $r \in (1, +\infty)$ *as* $n \to \infty$ *the upper estimates*

$$|\tilde{\varepsilon}_n^{(0)}(\varphi, x)| \leqslant \frac{2K(\sqrt{1-x^2})^{r-1}\ln n}{\pi n^r} + O\left(\frac{(\sqrt{1-x^2})^{r-1}}{n^r}\right)$$

*hold if* $x \in (-1, 1)$ *and*

$$|\tilde{\varepsilon}_n^{(0)}(\varphi, 1)| \leqslant \frac{2^{2-r}K\Gamma(2r-1)}{\pi\Gamma(r)} \frac{\ln n}{n^{2r-1}} + O\left(\frac{1}{n^{2r-1}}\right),$$

*as* $x = 1$, *where* $\Gamma(\cdot)$ *is the Euler Gamma function.*

A similar in order estimate is contained in [8]. We note that the estimate in Corollary 2.2 is less accurate in the constant. To improve the results, it is seems that we need to seek other ways for estimating (2.22).

## 3. Approximations of a Riemann — Liouville type integral with a density having a power singularity

We are going to study the approximations (2.3) in the case $\varphi_\gamma(x) = (1-x)^\gamma$, $\gamma \in (0, +\infty)$, that the approximations of the functions

$$f_\gamma(x) = \frac{1}{\Gamma(r)} \int\limits_{-1}^{x} (x-t)^{r-1}(1-t)^\gamma \frac{dt}{\sqrt{1-t^2}}, \quad x \in [-1,1], \quad r \in (1, +\infty). \tag{3.1}$$

We introduce the notation

$$\begin{aligned} \tilde{\varepsilon}_n(\varphi_\gamma, x, A) &= f_\gamma(x) - \tilde{s}_n(\varphi_\gamma, x), && x \in [-1,1], \\ \tilde{\varepsilon}_n(\varphi_\gamma, A) &= ||f_\gamma(x) - \tilde{s}_n(\varphi_\gamma, x)||_{C[-1,1]}, && n \in \mathbb{N}. \end{aligned}$$

We suppose that

$$\alpha_k \in [0,1), \qquad k = 1, 2, \ldots, n,$$

and

$$\alpha_1 = \alpha_2 = \ldots = \alpha_p = 0, \qquad p = \max\{[\gamma], [r-1]\},$$

where $[\cdot]$ denotes the integer part of a number.

**Theorem 3.1.** *For each $r \in (1, +\infty)$ and $\gamma \in (0, +\infty)$, for the approximations of integral of the Riemann — Liouville type integral (3.1) on the interval $[-1,1]$ by the image of the rational Fourier — Chebyshev integral operator (2.2) we have*

1) *the integral representation*

$$\begin{aligned} \tilde{\varepsilon}_n(\varphi_\gamma, x, A) = &\frac{2^{2-r-\gamma}\sin\pi\gamma}{\pi\Gamma(r)} \int\limits_0^1 (1-y)^{2\gamma} y^{-\gamma}\omega(y) \\ &\cdot \int\limits_0^1 (1-t)^{r-1} t^{1-r} \frac{(1-2t\cos 2u + t^2)^{\frac{r-1}{2}} \pi_n(t,x,A) \sin\Omega_n(x,t,y)}{\sqrt{1-2yt\cos u + t^2y^2}}\, dt\, dy, \end{aligned} \tag{3.2}$$

*where*

$$\omega(y) = \prod_{k=1}^{n} \frac{y-\alpha_k}{1-\alpha_k y}, \qquad \pi_n(t,x,A) = \prod_{k=1}^{n} \sqrt{\frac{t^2 - 2\alpha_k t\cos u + \alpha_k^2}{1 - 2t\alpha_k\cos u + t^2\alpha_k^2}},$$

$$\Omega_n(x,t,y) = \arg\frac{(\xi t - \bar{\xi})^{r-1}\xi\omega(\xi t)}{1 - ty\xi}, \qquad \xi = e^{iu}, \qquad x = \cos u;$$

2) *estimate for pointwise approximations*

$$\begin{aligned} |\tilde{\varepsilon}_n(\varphi_\gamma, x, A)| \leqslant &\frac{2^{2-r-\gamma}|\sin\pi\gamma|}{\pi\Gamma(r)} \int\limits_0^1 (1-y)^{2\gamma} y^{-\gamma}|\omega(y)| \\ &\cdot \int\limits_0^1 (1-t)^{r-1} t^{1-r} \frac{(1-2t\cos 2u + t^2)^{\frac{r-1}{2}} \pi_n(t,x,A)}{\sqrt{1-2yt\cos u + t^2y^2}}\, dt\, dy; \end{aligned} \tag{3.3}$$

3) *uniform in $x \in (-1,1)$ estimate of approximations*

$$|\tilde{\varepsilon}_n(\varphi_\gamma, x, A)| \leqslant (\sqrt{1-x^2})^{r-1} \tilde{\varepsilon}_n^*(\varphi_\gamma, A), \quad n \in \mathbb{N}, \tag{3.4}$$

*where*

$$\tilde{\varepsilon}_n^*(\varphi_\gamma, A) = \frac{2^{1-\gamma}|\sin\pi\gamma|}{\pi[\nu_n(A)]^r}(1+o(1))\int\limits_0^1 (1-y)^{2\gamma-1}y^{-\gamma}|\omega(y)|\,dy, \tag{3.5}$$

$$\nu_n(A) = \sum_{k=1}^n \frac{1-\alpha_k}{1+\alpha_k};$$

4) *estimate of approximations for* $x=1$

$$|\tilde{\varepsilon}_n(\varphi_\gamma, 1, A)| \leqslant \frac{2^{2-r-\gamma}|\sin\pi\gamma||\sin\pi r|}{\pi\Gamma(r)}\int\limits_0^1 (1-y)^{2\gamma-1}y^{-\gamma}|\omega(y)|\,dy\int\limits_0^1 (1-t)^{2r-2}t^{1-r}|\omega(t)|\,dt. \tag{3.6}$$

*Proof.* We employ the integral representation of approximation established in (2.14). For the density $\varphi_\gamma(x)$ it becomes

$$\tilde{\varepsilon}_n(\varphi_\gamma, x, A) = \frac{1}{2^r\pi i\Gamma(r)}\int\limits_{-\pi}^{\pi}(1-\cos v)^\gamma\int\limits_0^1(1-t)^{r-1}t^{1-r}$$

$$\cdot\left[\frac{(\xi t-\bar{\xi})^{r-1}\omega(\xi t)}{(t-\zeta\bar{\xi})\omega(\zeta)} - \frac{(\bar{\xi}t-\xi)^{r-1}\omega(\zeta)\omega(\bar{\xi}t)}{t-\bar{\zeta}\xi}\right]dt\,dv, \quad \zeta = \mathrm{e}^{iv},\ \xi = \mathrm{e}^{iu},\ x=\cos u.$$

Using the Fubini theorem, we interchange the integration order

$$\tilde{\varepsilon}_n(\varphi_\gamma, x, A) = \frac{1}{2^r\pi i\Gamma(r)}\int\limits_0^1(1-t)^{r-1}t^{1-r}I_n(t,x)\,dt, \tag{3.7}$$

where

$$I_n(t,x) = \int\limits_{-\pi}^{\pi}(1-\cos v)^\gamma\left[\frac{(\xi t-\bar{\xi})^{r-1}\omega(\xi t)}{(t-\zeta\bar{\xi})\omega(\zeta)} - \frac{(\bar{\xi}t-\xi)^{r-1}\omega(\zeta)\omega(\bar{\xi}t)}{t-\bar{\zeta}\xi}\right]dv, \quad \zeta=\mathrm{e}^{iv}. \tag{3.8}$$

Let us transform the integral $I_n(t,x)$. In order to do this, we pass to the integration in the variable $\zeta$

$$I_n(t,x) = \frac{(-1)^\gamma}{2^\gamma i}\oint\limits_\Gamma(1-\zeta)^{2\gamma}\zeta^{-\gamma}\left[\frac{(\xi t-\bar{\xi})^{r-1}\omega(\xi t)}{(t-\zeta\bar{\xi})\omega(\zeta)} - \frac{(\bar{\xi}t-\xi)^{r-1}\omega(\zeta)\omega(\bar{\xi}t)}{t-\bar{\zeta}\xi}\right]\frac{d\zeta}{\zeta},$$

where $\Gamma = \{\zeta : |\zeta|=1\}$, $\xi = \mathrm{e}^{iu}$, $t\in(0,1)$.

We note that in the case $\gamma\in(0,+\infty)\backslash\mathbb{N}$, the integrand has branching points at $\zeta=0$, $\zeta=1$ and $\zeta=\infty$. We represent the latter integral as the sum of two integrals so that

$$I_n(t,x) = -\frac{(-1)^\gamma}{2^\gamma i}\left[(\xi t-\bar{\xi})^{r-1}\omega(\xi t)\xi J_n^{(1)} + \frac{(\bar{\xi}t-\xi)^{r-1}\omega(\bar{\xi}t)}{t}J_n^{(2)}\right], \tag{3.9}$$

where

$$J_n^{(1)} = \oint\limits_\Gamma\frac{(1-\zeta)^{2\gamma}\zeta^{-\gamma}}{(\zeta-\xi t)\zeta\omega(\zeta)}\,d\zeta, \qquad J_n^{(2)} = \oint\limits_\Gamma\frac{(1-\zeta)^{2\gamma}\zeta^{-\gamma}}{\zeta-\xi/t}\omega(\zeta)\,d\zeta.$$

Applying to each of the integrals the methods similar to ones employed in the proof of Theorem 2.1, we find

$$J_n^{(1)} = (\mathrm{e}^{-2\pi i\gamma}-1)\int\limits_0^1\frac{(1-y)^{2\gamma}y^{-\gamma}}{1-ty\xi}\omega(y)\,dy, \qquad J_n^{(2)} = \bar{\xi}t(1-\mathrm{e}^{-2\pi i\gamma})\int\limits_0^1\frac{(1-y)^{2\gamma}y^{-\gamma}}{1-ty\bar{\xi}}\omega(y)\,dy.$$

Relation (3.9) and the latter integral representations yield

$$I_n(t,x) = -2^{1-\gamma}\sin\pi\gamma \int_0^1 (1-y)^{2\gamma}y^{-\gamma}\omega(y)\left[\frac{(\xi t-\overline{\xi})^{r-1}\xi\omega(\xi t)}{1-ty\xi} - \frac{(\overline{\xi}t-\xi)^{r-1}\overline{\xi}\omega(\overline{\xi}t)}{1-ty\overline{\xi}}\right] dy. \quad (3.10)$$

The expressions in the square brackets are mutually complex conjugate. After appropriate calculations we find

$$\begin{aligned} I_n(t,x) = &- 2^{2-\gamma} i\sin\pi\gamma \int_0^1 (1-y)^{2\gamma}y^{-\gamma}\omega(y) \\ &\cdot \frac{(1-2t\cos 2u+t^2)^{\frac{r-1}{2}}\pi_n(t,x,A)\sin\Omega_n(x,t,y)\,dy}{\sqrt{1-2yt\cos u+t^2y^2}}, \end{aligned} \quad (3.11)$$

where $\pi_n(t,x,A)$ and $\Omega_n(x,t,y)$ were defined in Theorem 2.2.

The integral representations (3.7) and (3.11) yield (3.2). The estimate (3.3) is easily implied by (3.2).

Let us prove the estimate (3.4). In order to do this, we employ the arguing from (2.18)

$$\pi_n(t,x,A) = \prod_{k=1}^{n}\sqrt{\frac{t^2-2\alpha_k t\cos u+\alpha_k^2}{1-2t\alpha_k\cos u+t^2\alpha_k^2}} \leqslant \mathrm{e}^{\frac{t^2-1}{2}\nu_n(A)},$$

where $\nu_n(A)$ is from (3.5). In view of the latter inequality and the estimate (3.3) we obtain

$$\begin{aligned} |\tilde{\varepsilon}_n(\varphi_\gamma,x,A)| \leqslant &\frac{2^{2-r-\gamma}|\sin\pi\gamma|}{\pi\Gamma(r)}\int_0^1 (1-y)^{2\gamma}y^{-\gamma}|\omega(y)| \\ &\cdot \int_0^1 \frac{(1-t)^{r-1}t^{1-r}}{1-yt}(1-2t\cos 2u+t^2)^{\frac{r-1}{2}}\mathrm{e}^{\frac{t^2-1}{2}\nu_n(A)}\,dt\,dy, \;\; x=\cos u, \;\; x\in(-1,1). \end{aligned}$$

To study the asymptotic behavior of the inner integral as $n\to\infty$ we use the Laplace method [4], [21]. We do not provide them because they literally reproduce the proof of the estimates (2.20) and (2.21). After needed calculations we obtain (3.4).

By the integral representation (3.2) with $x=1$, that corresponds to $u=0$, we have

$$\tilde{\varepsilon}_n(\varphi_\gamma,1,A) = \frac{2^{2-r-\gamma}\sin\pi\gamma(\pm\sin\pi r)}{\pi\Gamma(r)}\int_0^1 (1-y)^{2\gamma}y^{-\gamma}\omega(y)\int_0^1 \frac{(1-t)^{2r-2}t^{1-r}}{1-yt}\omega(t)\,dt\,dy.$$

This integral representation easily implies the estimate (3.6). The proof is complete. □

In Theorem 3.1 we let $\alpha_k=0$, $k=1,2,\ldots,n$. In this case $A=(0,0,\ldots,0)=O$ and the quantity $\tilde{\varepsilon}_n(\varphi_\gamma,x,O)=\tilde{\varepsilon}_n^{(0)}(\varphi_\gamma,x)$ is the approximation for the function $f_\gamma(x)$ on the segment $[-1,1]$ by the image of partial sums of polynomial Fourier — Chebyshev series under the transformation (2.2). Then Theorem 3.1 implies the following corollary.

**Corollary 3.1.** *For the approximation for the function $f_\gamma(x)$ on the segment $[-1,1]$ by the image of partial sums of polynomial Fourier — Chebyshev series under the transformation (2.2) we have*

1) *the integral representation*

$$\tilde{\varepsilon}_n^{(0)}(\varphi_\gamma, x) = \frac{2^{2-r-\gamma}\sin\pi\gamma}{\pi\Gamma(r)}\int\limits_0^1 (1-y)^{2\gamma}y^{n-\gamma} \cdot \int\limits_0^1 (1-t)^{r-1}t^{n+1-r}\frac{(1-2t\cos 2u+t^2)^{\frac{r-1}{2}}\sin\Omega_n^{(0)}(x,t,y)}{\sqrt{1-2yt\cos u+t^2y^2}}\,dt\,dy, \tag{3.12}$$

*where*

$$\Omega_n^{(0)}(x,t,y) = (n+1)u + \arg\frac{(\xi t-\overline{\xi})^{r-1}}{1-yt\xi}, \qquad \xi = \mathrm{e}^{iu}, \qquad x = \cos u;$$

2) *the estimate for pointwise approximations*

$$|\tilde{\varepsilon}_n^{(0)}(\varphi_\gamma, x)| \leqslant \frac{2^{2-r-\gamma}|\sin\pi\gamma|}{\pi\Gamma(r)}\int\limits_0^1 (1-y)^{2\gamma}y^{n-\gamma} \cdot \int\limits_0^1 (1-t)^{r-1}t^{n+1-r}\frac{(1-2t\cos 2u+t^2)^{\frac{r-1}{2}}}{\sqrt{1-2yt\cos u+t^2y^2}}\,dt\,dy;$$

3) *uniform in $x \in (-1,1)$ estimate for approximations*

$$|\tilde{\varepsilon}_n^{(0)}(\varphi_\gamma, x)| \leqslant \frac{2^{1-\gamma}|\sin\pi\gamma|(\sqrt{1-x^2})^{r-1}\Gamma(2\gamma)}{\pi n^{r+2\gamma}}(1+o(1)), \qquad n\in\mathbb{N},$$

4) *the estimate of approximations for $x=1$*

$$|\tilde{\varepsilon}_n(\varphi_\gamma, 1, A)| \leqslant \frac{2^{2-r-\gamma}|\sin\pi\gamma||\sin\pi r|\Gamma(2\gamma)\Gamma(2r-1)}{\pi\Gamma(r)n^{2r+2\gamma-1}}(1+o(1)), \qquad n\in\mathbb{N}.$$

The integral representation and the estimate of pointwise approximations follow directly from the corresponding results of Theorem 3.1 for $\alpha_k = 0$, $k = 1, 2, \ldots, n$. The upper bounds are implied by the corresponding results of rational approximations in Theorem 3.1 if we apply the well–known Laplace method.

## 4. Asymptotics of majorant for uniform approximations

It is interesting to study the asymptotic behavior of quantity (3.5) and of the right hand side in the estimate (3.6) as $n\to\infty$. In order to do this, in the corresponding integrals we make the change of variable

$$y = \frac{1-u}{1+u}, \qquad dy = -\frac{2du}{(1+u)^2}$$

and we obtain

$$\tilde{\varepsilon}_n^*(\varphi_\gamma, A) = \frac{2^{1+\gamma}|\sin\pi\gamma|}{\pi[p+\nu_m(A)]^r}(1+o(1))\int\limits_0^1 g_\gamma(u)\left|\prod_{k=1}^m\frac{\beta_k-u}{\beta_k+u}\right|du, \qquad n = m+p, \tag{4.1}$$

$$g_\gamma(u) = \frac{u^{2\gamma-1}}{(1+u)(1-u^2)^\gamma}\left(\frac{1-u}{1+u}\right)^p, \qquad \nu_m(A) = \sum_{k=1}^m\beta_k, \qquad \beta_k = \frac{1-\alpha_k}{1+\alpha_k}.$$

In the same the estimate (3.6) becomes

$$
\begin{aligned}
|\tilde{\varepsilon}_n(\varphi_\gamma, 1, A)| \leqslant & \frac{2^{1+r+\gamma}|\sin \pi\gamma||\sin \pi r|}{\pi\Gamma(r)} \int\limits_0^1 g_\gamma(u) \left| \prod_{k=1}^m \frac{1-\beta_k}{1+\beta_k} \right| du \\
& \cdot \int\limits_0^1 \frac{u^{2r-2}}{(1+u)^2(1-u^2)^{r-1}} \left( \frac{1-u}{1+u} \right)^p \left| \prod_{k=1}^m \frac{1-\beta_k}{1+\beta_k} \right| du, \quad n \in \mathbb{N},
\end{aligned}
\tag{4.2}
$$

where $r \in (1, +\infty)$, $\gamma \in (0, +\infty)$.

We suppose that the parameters $\beta_k$, $k = 1, 2, \dots, m$, are ordered as follows

$$
0 < \beta_m < \beta_{m-1} < \ldots < \beta_1 \leqslant 1.
$$

Moreover, for each value of $n$, a corresponding set of parameters $\beta_k$ can be chosen, $k = 1, 2 \dots, m$, that is, generally speaking, $\beta_k = \beta_k(n)$. Because of this, we suppose the conditions

$$
\mu_m(A) = \sum_{k=1}^m \frac{1}{\beta_k} \underset{n\to\infty}{\longrightarrow} \infty, \tag{4.3}
$$

$$
\nu_m(A) = \sum_{k=1}^m \beta_k \underset{n\to\infty}{\longrightarrow} \infty. \tag{4.4}
$$

The conditions (4.3) and (4.4) are not contradictory. Below we shall consider a sequence of parameters $\{\beta_k\}_{k=1}^m$ satisfying both relations.

**Theorem 4.1.** *Under the conditions* (4.3) *and* (4.4) *the upper bounds*

$$
\tilde{\varepsilon}_n^*(\varphi_\gamma, A) \leqslant \frac{2^{1-\gamma}|\sin \pi\gamma|\Gamma(2\gamma)}{\pi[p+\nu_m(A)]^r} \left[ \frac{1}{(\mu_m(A))^{2\gamma}} + \frac{\Gamma(1+p-\gamma)}{\Gamma(1+p+\gamma)} \prod_{k=1}^m \frac{1-\beta_k}{1+\beta_k} \right] (1+o(1)); \tag{4.5}
$$

$$
\begin{aligned}
|\tilde{\varepsilon}_n(\varphi_\gamma, 1, A)| \leqslant & \frac{2^{2-r-\gamma}|\sin \pi\gamma||\sin \pi r|\Gamma(2\gamma)\Gamma(2r-1)}{\pi\Gamma(r)} \\
& \cdot \left[ \frac{1}{(\mu_m(A))^{2\gamma}} + \frac{\Gamma(1+p-\gamma)}{\Gamma(1+p+\gamma)} \prod_{k=1}^n \frac{1-\beta_k}{1+\beta_k} \right] \\
& \cdot \left[ \frac{1}{(\mu_m(A))^{2r-1}} + \frac{\Gamma(2+p-r)}{\Gamma(1+p+r)} \prod_{k=1}^n \frac{1-\beta_k}{1+\beta_k} \right] (1+o(1)), \quad n \to \infty.
\end{aligned}
\tag{4.6}
$$

*hold.*

*Proof.* We begin with the estimate (4.5). We represent (4.1) as

$$
\tilde{\varepsilon}_n^*(\varphi_\gamma, A) = \frac{2^{1+\gamma}|\sin \pi\gamma|}{\pi[p+\nu_m(A)]^r} [I_n^{(1)}(A) + I_n^{(2)}(A)](1+o(1)), \quad n \in \mathbb{N}, \tag{4.7}
$$

where

$$
I_n^{(1)}(A) = \int\limits_0^{\beta_m} g_\gamma(u) \prod_{k=1}^m \frac{\beta_k - u}{\beta_k + u} du, \qquad I_n^{(2)}(A) = \int\limits_{\beta_m}^1 g_\gamma(u) \left| \prod_{k=1}^m \frac{\beta_k - u}{\beta_k + u} \right| du.
$$

Let us study the asymptotic behavior as $n \to \infty$ of each of the two integrals separately. We represent the integral $I_n^{(1)}(A)$ as

$$
I_n^{(1)}(A) = \int\limits_0^{\beta_m} g_\gamma(u) e^{S(u)} du, \qquad S(u) = \sum_{k=1}^m \ln \frac{\beta_k - u}{\beta_k + u}.
$$

The function $S(u)$ decreases on the interval $[0, \beta_m]$ since $S'(u) < 0$, and therefore, it attains its maximum value at $u = 0$. In view of the expansion

$$S(u) = -2u \sum_{k=1}^{m} \frac{1}{\beta_k} - \frac{2}{3} u^3 \sum_{k=1}^{m} \frac{1}{\beta_k^3} + O(u^5), \qquad u \to 0,$$

and the obvious asymptotic identity $g_\gamma(u) \sim u^{2\gamma-1}$, $u \to 0$, for some sufficiently small $\varepsilon > 0$ and $n \to \infty$ we find

$$I_n^{(1)}(A) \sim \int_0^{\varepsilon} u^{2\gamma-1} \mathrm{e}^{-2u\mu_m(A)}\, du.$$

In the integrals in the right hand side we make the change $2u\mu_m(A) \mapsto t$

$$I_n^{(1)}(A) = \frac{1}{(2\mu_m(A))^{2\gamma}} \int_0^{\varphi(m,\varepsilon)} t^{s-1} \mathrm{e}^{-t}\, dt, \quad n \to \infty,$$

where $\varphi(m, \varepsilon) = 2\varepsilon\mu_m(A) \to \infty$ as $n \to \infty$ by the condition (4.3). Taking into consideration the identity

$$\int_0^{+\infty} u^{s-1} \mathrm{e}^{-u}\, du = \Gamma(s), \quad s > 0,$$

by the latter asymptotic identity we obtain

$$I_n^{(1)}(A) = \frac{\Gamma(2\gamma)}{(2\mu_m(A))^{2\gamma}} (1 + o(1)), \quad n \to \infty. \tag{4.8}$$

We proceed to studying the integral $I_n^{(2)}(A)$. It is obvious that

$$I_n^{(2)}(A) \leqslant \max_{u \in [0,1]} \left| \prod_{k=1}^{m} \frac{\beta_k - u}{\beta_k + u} \right| \int_{\beta_m}^{1} g_\gamma(u)\, du.$$

The integral in the right hand side is well–defined. Its integrand is non–negative. Hence, its value is majorized by the integral with the same integrand over the segment $[0, 1]$. It is known [27] that

$$\max_{u \in [0,1]} \left| \prod_{k=1}^{m} \frac{\beta_k - u}{\beta_k + u} \right| \leqslant \prod_{k=1}^{m} \frac{1 - \beta_k}{1 + \beta_k}.$$

In view of this relation we arrive at the estimate

$$I_n^{(2)}(A) \leqslant c(\gamma) \prod_{k=1}^{m} \frac{1 - \beta_k}{1 + \beta_k}, \quad n \to \infty, \tag{4.9}$$

where

$$c(\gamma) = \int_0^1 \frac{u^{2\gamma-1}}{(1+u)(1-u^2)^\gamma} \left( \frac{1-u}{1+u} \right)^p du = \frac{\Gamma(2\gamma)\Gamma(1+p-\gamma)}{2^{2\gamma}\Gamma(1+p+\gamma)}.$$

Substituting the estimates (4.8) and (4.9) into the representation (4.7), we arrive at the inequality (4.5).

We proceed to the estimate (4.6). The estimate of the first integral in the right hand side in (4.2) is essentially contained in the asymptotic identity (4.8). Applying similar reasoning to the second integral, we arrive at (4.6). The proof is complete. □

Letting $\alpha_k = 0$, $k = 1, 2, \ldots, n$, in Theorem 4.1 in view of the inequality (3.4) we arrive at the third and fourth estimates in Corollary 3.1.

## 5. Case of Newman parameters of approximating function

We study the asymptotic expression of majorant of uniform approximations (4.5) and the upper bound for the approximations at the point $x = 1$ (4.6) in the case when the values taken by the parameters $\beta_k$, $k = 1, 2, \ldots, m$,, are some modification of the parameters introduced by Newman in the work [27]. We study this problem below. Let $A_N$ be a set of parameters $\alpha_k$, $k = 1, 2, \ldots, n$, which for each fixed $n \in \mathbb{N}$, satisfies the conditions

$$\begin{aligned} &\alpha_1 = \alpha_2 = \ldots = \alpha_p = 0, \qquad p = \max\{[\gamma], [r-1]\}; \\ &\alpha_{p+k} = \frac{1-\beta_k}{1+\beta_k}, \qquad \beta_k = \mathrm{e}^{-\frac{ck}{\sqrt{m}}}, \qquad k = 1, 2, \ldots, m, \end{aligned} \tag{5.1}$$

$c$ is some positive constant independent of $n$, $n = m + p$ as $n > p$.

**Theorem 5.1.** *For all $\gamma \in (0, +\infty)$ and $r \in (1, +\infty)$ the approximations for the function $f_\gamma(x)$ on the segment $[-1, 1]$ by the rational integral operator (2.2) with the parameters (5.1) satisfy the upper bound*

$$|\tilde{\varepsilon}_n(\varphi_\gamma, x, A_N)| \leqslant (\sqrt{1-x^2})^{r-1} \tilde{\varepsilon}_n^*(\varphi_\gamma, A_N), \qquad n \in \mathbb{N}, \qquad x \in (-1, 1), \tag{5.2}$$

*where*

$$\begin{aligned} \tilde{\varepsilon}_n^*(\varphi_\gamma, A_N) &\leqslant \frac{2^{1-\gamma}|\sin\pi\gamma|\Gamma(2\gamma)c^r}{\pi n^{\frac{r}{2}}} \left[\frac{c^{2\gamma}}{n^\gamma}\mathrm{e}^{-2\gamma c\sqrt{n}} + \frac{\Gamma(1+p-\gamma)}{\Gamma(1+p+\gamma)}\sqrt{n}\mathrm{e}^{-\frac{\pi^2}{4c}\sqrt{n}}\right](1+o(1)); \\ |\tilde{\varepsilon}_n(\varphi_\gamma, 1, A_N)| &\leqslant \frac{2^{2-r-\gamma}|\sin\pi\gamma||\sin\pi r|\Gamma(2\gamma)\Gamma(2r-1)}{\pi\Gamma(r)} \\ &\cdot \left[\frac{c^{2\gamma}}{n^\gamma}\mathrm{e}^{-2\gamma c\sqrt{n}} + \frac{\Gamma(1+p-\gamma)}{\Gamma(1+p+\gamma)}\sqrt{n}\mathrm{e}^{-\frac{\pi^2}{4c}\sqrt{n}}\right] \\ &\cdot \left[\frac{c^{2r-1}}{n^{\frac{2r-1}{2}}}\mathrm{e}^{-(2r-1)c\sqrt{n}} + \frac{\Gamma(2+p-r)}{\Gamma(1+p+r)}\sqrt{n}\mathrm{e}^{-\frac{\pi^2}{4c}\sqrt{n}}\right](1+o(1)), \quad n \to \infty. \end{aligned} \tag{5.3}$$

*Proof.* Let us study the asymptotic behavior of the right hand side of identity (4.5), when the parameters $\beta_k$, $k = 1, 2, \ldots, m$, are given by the formulas (5.1). It is known [7] that in this case the asymptotic identities

$$\prod_{k=1}^{m} \frac{1-\beta_k}{1+\beta_k} \sim \sqrt{m}\mathrm{e}^{-\frac{\pi^2}{4c}\sqrt{m}}, \qquad n \to \infty,$$

$$\mu_m(A_N) = \sum_{k=1}^{m} \frac{1}{\beta_k} \sim \frac{\sqrt{m}}{c}\mathrm{e}^{c\sqrt{m}}, \qquad n \to \infty,$$

hold. Let us establish the asymptotic identity for the quantity $\nu_m(A_N)$, see (4.4). We use the methods of studying the asymptotic behavior of similar sums described in [15]:

$$\nu_m(A_N) = \sum_{k=1}^{m} \beta_k = \int_1^m \mathrm{e}^{-\frac{ct}{\sqrt{m}}}\, dt + O\left(\mathrm{e}^{-c\sqrt{m}}\right) + O(1), \quad n \to \infty.$$

After appropriate calculations we find

$$\nu_m(A_N) \sim \frac{\sqrt{m}}{c}, \quad n \to \infty.$$

In view of the above asymptotic identities and the inequality (4.5) we obtain

$$\tilde{\varepsilon}_n^*(\varphi_\gamma, A_N) \leqslant \frac{2^{1-\gamma}|\sin\pi\gamma|\Gamma(2\gamma)c^r}{\pi m^{\frac{r}{2}}} \left[\frac{c^{2\gamma}}{m^\gamma}\mathrm{e}^{-2\gamma c\sqrt{m}} + \frac{\Gamma(1+p-\gamma)}{\Gamma(1+p+\gamma)}\sqrt{m}\mathrm{e}^{-\frac{\pi^2}{4c}\sqrt{m}}\right](1+o(1)).$$

Taking into consideration that $n = m + p$ by the latter identity we arrive at the estimate (5.2).

In order to obtain the estimate (5.3), it is necessary to apply the above arguing to the estimate (4.6). The proof is complete. $\square$

It is of interest to minimize the majorant in the estimate (5.2) and the right hand side of the estimate (5.3) by choosing the optimal parameter $c$ for each of these problems; in other words, to find the best estimates of the approximations with parameters (5.1). We let

$$\hat{\varepsilon}_n^*(\varphi_\gamma) = \inf_c \hat{\varepsilon}_n^*(\varphi_\gamma, A_N).$$

**Theorem 5.2.** *For each $\gamma \in (0, +\infty)$ and $r \in (1, +\infty)$ for the approximations of the function $f_\gamma(x)$ on the interval $(-1, 1)$ by the rational integral operator (2.2) with the parameters (5.1) as $n \to \infty$ the upper bounds*

$$|\tilde{\varepsilon}_n(\varphi_\gamma, x, A_N)| \leqslant \begin{cases} (\sqrt{1-x^2})^{r-1} c_1(\gamma, r) \dfrac{\mathrm{e}^{-\frac{\pi}{2}\sqrt{2\gamma n}}}{n^{\frac{r-1}{2}}}(1+o(1)), & x \in (-1,1), \\ c_2(\gamma, r) n \mathrm{e}^{-\pi\sqrt{(r+\gamma-\frac{1}{2})n}}(1+o(1)), & x = 1, \end{cases} \tag{5.4}$$

*hold, where*

$$c_1(\gamma, r) = \frac{2^{1-r-\gamma}\pi^{r-1}|\sin\pi\gamma|\Gamma(2\gamma)\Gamma(1+p-\gamma)}{(\sqrt{2\gamma})^r\Gamma(1+p+\gamma)},$$
$$c_2(\gamma, r) = \frac{2^{2-r-\gamma}|\sin\pi\gamma||\sin\pi r|\Gamma(2\gamma)\Gamma(2r-1)\Gamma(1+p-\gamma)\Gamma(1+p+\gamma)}{\pi\Gamma(r)\Gamma(2+p-r)\Gamma(1+p+r)}.$$

*Proof.* We choose the constant $c$ in (5.2) by the identity $2\gamma c = \frac{\pi^2}{4c}$, that is,

$$c = \frac{\pi}{2\sqrt{2\gamma}}. \tag{5.5}$$

Then, after some transformations, it follows from (5.2) that

$$|\tilde{\varepsilon}_n(\varphi_\gamma, x, A_N)| \leqslant \frac{2^{1-r-\gamma}\pi^{r-1}|\sin\pi\gamma|(\sqrt{1-x^2})^{r-1}\Gamma(2\gamma)}{n^{\frac{r-1}{2}}(\sqrt{2\gamma})^r} \frac{\Gamma(1+p-\gamma)}{\Gamma(1+p+\gamma)} \mathrm{e}^{-\frac{\pi}{2}\sqrt{2\gamma n}}$$
$$+ O\left(\frac{(\sqrt{1-x^2})^{r-1}\mathrm{e}^{-\frac{\pi}{2}\sqrt{2\gamma n}}}{n^{\frac{r}{2}+\gamma}}\right), \qquad n \in \mathbb{N}, \qquad x \in (-1,1).$$

In order to prove that the constant $c$ defined in (5.5) does provide the asymptotically minimal value for right hand side of (5.2), it is sufficient to use the method described in [7]. Therefore,

$$\hat{\varepsilon}_n^*(\varphi_\gamma, A_N^*) = \inf_c \hat{\varepsilon}_n^*(\varphi_\gamma, A_N) = \hat{\varepsilon}_n^*(\varphi_\gamma)$$

and we arrive at (5.4).

We proceed to proving the second estimate in (5.4). We consider the relation (5.3). The expression in its right hand side consists of the four terms, the decay rate of which is expressed by an exponential and some power of $n$. We seek the optimal parameter $c$ we seek by the condition of coincidence of powers of the exponentials in the terms with the smallest and largest decay rate. That is,

$$(2\gamma + 2r - 1)c = \frac{\pi^2}{2c}, \qquad c = \frac{\pi}{\sqrt{2(2\gamma+2r-1)}}.$$

In order to prove that the found constant $c$ provides the asymptotically minimal value for the right hand side in the estimate (5.3), we again need to use the method described in [7]. This leads us to the second estimate in (5.4). The proof is complete. $\square$

## 6. Conclusion

In the work we study the approximations of the Riemann — Liouville integral on the interval $[-1, 1]$ by the Riemann — Liouville integral with a density being a rational Fourier — Chebyshev integral operator.

We find the estimate is found for the approximations of Riemann — Liouville integral with a bounded density depending on the poles of the approximating rational function. This estimate depends significantly on the position of point on the interval $[-1, 1]$. We establish that the approximation rate at the ends of the interval is higher than on the entire interval.

We study the rational approximations of the Riemann — Liouville integral with a density being a function with a power singularity. The integral representation and approximation estimates depending on the poles of the approximating function are established. The consider the case, when the poles are some modifications of the Newman parameters. We find their optimal values, at which the uniform rational approximations have the greatest decay rate.

As a corollary, we find estimates for approximations of the integral of Riemann— Liouville type by the polynomial analogue of considered rational integral operator.

The made research allows us to conclude that the class of functions defined by the Riemann — Liouville integrals with a density having a power singularity on the interval $[-1, 1]$ reflects the features of rational approximation by the introduced method in the sense that, for a certain choice of parameters of the approximating function, the rate of uniform rational approximations (Theorem 5.2) is higher in comparison with their polynomial analogues (Corollary 3.1).

## BIBLIOGRAPHY


1. N.I. Akhiezer. *Lectures on the theory of approximation.* Nauka, Moscow (1965). *English translation:* N.I. Achieser. *Theory of approximation.* Frederick Ungar Publishing Co., New York (1956).
2. Yu.I. Babenko. *Fractional differentiation method in applied problems of heat and mass transfer theory.* NPO "Professional", St.–Petersburg (2009). (in Russian).
3. T.Yu. Gorskaya, A.F. Galimyanov. *Approximation of fractional integrals by partial sums of Fourier series* // Iz. Kazan. Arkh.–Stroit. Univ. **3**:41, 261–265 (2017). (in Russian).
4. M.A. Evgrafov. *Asymptotic estimates and entire functions.* Nauka, Moscow (1979). *English translation:* Dover Publications, Mineola, NY (2020).
5. D.A. Zenyuk, Yu.N. Orlov. *On application of fractional Riemann — Liouville fractional calculus for description of probability distribution* // Preprints Keldysh Inst. Appl. Math. **18**, (2014). (in Russian).
6. S.M. Nikolskii. *On best approximation of function, the sth derivative of which has discontinuities of first kind* // Dokl. Akad. Nauk SSSR. **55**, 99–102 (1947). (in Russian).
7. P.G. Potseiko, Y.A. Rovba. *On estimates of uniform approximations by rational Fourier — Chebyshev integral operators for a certain choice of poles* // Math. Notes **113**:6, 815–830 (2023). https://doi.org/10.1134/S0001434623050231
8. P.G. Potseiko, E.A. Rovba. Approximation of Riemann — Liouville type integrals on an interval by methods based on Fourier — Chebyshev sums //Math. Notes **116**:1, 104–118 (2024). https://doi.org/10.1134/S0001434624070095
9. E.A. Rovba. *One direct method in the rational approximation* // Dokl. Akad. Nauk BSSR **23**:11, 968–971 (1979). (in Russian).
10. E.A. Rovba. *Approximation of functions that are differentiable in the Riemann-Liouville sense by rational operators* // Dokl. Akad. Nauk Belarusi **40**:6, 18–22 (1996). (in Russian).
11. E.A. Rovba. *Rational integral operators on a segment* // Vestn. Beloruss. Gos. Univ., Ser. 1, Fiz. Mat. Inform. **1**, 34–39 (1996). (in Russian).
12. V.N. Rusak. *On a method of approximation by rational functions* // Izv. Akad. Nauk BSSR, Ser. Fiz.–Mat. Nauk **3**, 36–43 (1978). (in Russian).

13. I.V. Rybachenko. *Rational interpolation of functions with the Riemann – Liouville derivative in* $L_p$ // Vestn. Beloruss. Gos. Univ., Ser. 1, Fiz. Mat. Inform. 2, 69–74, 127 (2006). (in Russian).
14. St.G. Samko, A.A. Kilbas, O.I. Marichev. *Fractional integrals and derivatives: theory and applications.* Nauka i Tekhnika, Minsk (1987). *English translation:* Gordon and Breach, New York, NY (1993).
15. Yu.V. Sidorov, M.V. Fedoryuk, M.I. Shabunin. *Lectures on the theory of functions of a complex variable.* Nauka, Moscow (1989). (in Russian).
16. A.P. Starovoĭtov. *Comparison of the rates of rational and polynomial approximations of differentiable functions* // Math. Notes **44**:3–4, 770–774 (1988). https://doi.org/10.1007/BF01158923
17. A.P. Starovoitov. *Rational approximations of Riemann – Liouville and Weyl fractional integrals* // Math. Notes **78**:3, 391–402 (2005). https://doi.org/10.1007/s11006-005-0138-4
18. K.A. Smotritskiĭ. *On the approximation of functions differentiable in the Riemann – Liouville sense* // Vestsi Nats. Akad. Navuk Belarusi, Ser. Fiz.–Mat. Navuk 4, 42–47 (2002). (in Russian).
19. K.A. Smotritskiĭ. *On the approximation of convex functions by rational integral operators on an interval* // Vestn. Beloruss. Gos. Univ., Ser. 1, Fiz. Mat. Inform. 3, 64–70, 125 (2005). (in Russian).
20. A.A. Tyuleneva. *Approximation of Riemann – Liouville integrals by algebraic polynomials on segment* // Izv. Saratov Univ., Ser. Mat. Mekh. Inform. **14**:3, 305–311 (2014). (in Russian). https://doi.org/10.18500/1816-9791-2014-14-3-305-311
21. M.V. Fedoryuk. *Asymptotics: integrals and series.* Nauka, Moscow (1987). (in Russian). (in Russian).
22. H. Akçay, B. Ninness. *Rational basis functions for robust identification from frequency and time–domain measurements* // Automatica **34**:9, 1101–1117 (1998). https://doi.org/10.1016/S0005-1098(98)00052-1
23. A. Atangana, J.F. Gómez–Aguilar. *Numerical approximation of Riemann – Liouville definition of fractional derivative: From Riemann – Liouville to Atangana – Baleanu* // Numer. Methods Partial Differ. Equations **34**:5, 1502–1523 (2018). https://doi.org/10.1002/num.22195
24. A. Galimyanov, T. Gorskaya. *Calculation of fractional integrals using partial sums of Fourier series for structural mechanics problems* // E3S Web of Conferences. **274**, 03011 (2021). https://doi.org/10.1051/e3sconf/202127403011
25. L. Khitri–Kazi–Tani, H. Dib. *On the approximation of Riemann – Liouville integral by fractional nabla h-sum and applications* // Mediterr. J. Math. **14**:2, 86 (2017). https://doi.org/10.1007/s00009-017-0887-9
26. T. Marinov, N. Ramirez, F. Santamaria. *Fractional integration toolbox* // Fract. Calc. Appl. Anal. **16**:3, 670–681 (2013). https://doi.org/10.2478/s13540-013-0042-7
27. D.J. Newman. *Rational approximation to* $|x|$ // Mich. Math. J. **11**:1, 11–14 (1964). https://doi.org/10.1307/mmj/1028999029
28. K.M. Owolabi, A. Atangana. *Numerical approximation of Riemann – Liouville differentiation* // In "Numerical Methods for Fractional Differentiation", Springer, Singapore **54**, 139–160 (2019). https://doi.org/10.1007/978-981-15-0098-5_3



Pavel Gennadievich Potseiko,
Yanka Kupala State University of Grodno,
Ozheshko str. 22,
230023, Grodno, Belarus
E-mail: `pahamatby@gmail.com`

Evgeny Alexeevich Rovba,
Yanka Kupala State University of Grodno,
Ozheshko str. 22,
230023, Grodno, Belarus
E-mail: `rovba.ea@gmail.com`