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ON CONVERGENCE RATE IN ERGODIC THEOREM FOR SOME STATISTICALLY AVERAGING SEQUENCES IN $\mathbb{R}$

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Abstract. In this work we consider two types of averaging of unitary representation of the group $\mathbb{R}$ constructed by some sequences of probability measures on $\mathbb{R}$. The first sequence of measures generalizes the uniform distribution. The densities of the measures in this sequence are convolutions of finitely many indicators of segments. The second sequence is defined by the exponential decay of Fourier transform. For such averagings we obtain the estimates for the convergence rate in the norm depending on the singularities of spectral measure of the unitary representation in a neighbourhood of zero and of the asymptotics of sequence of Fourier transforms of averaging probabilistic measures. At the same time, the maximal possible rates are powers with the exponents $m > 1$ and the exponential rate, respectively, and this is significantly better than the maximal convergence rate in the classical von Neumann ergodic theorem.

Keywords: convergence rates in ergodic theorems, statistically averaging sequence, Fourier transform, slowly varying functions, asymptotics of integrals.

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1. Introduction

Let $\mathcal{H}$ be a Hilbert space, on which the group $\mathbb{R}$ acts by unitary transformations U_t, $t \in \mathbb{R}$. The von Neumann ergodic theorem (statistical ergodic theorem) states that for each vector $h \in \mathcal{H}$

$$\left\| \frac{1}{2t} \int\limits_{[-t,t]} U_s h\, ds - Ph \right\|_{\mathcal{H}} \to 0$$

as $t \to \infty$, where P is the orthogonal projection onto the subspace of fixed vectors of group $\{U_t\}_{t\in\mathbb{R}}$. The convergence rate in this theorem is well–studied: it is determined by the singularity of spectral measure $\sigma_h, h \in \mathcal{H}$ in a neighbourhood of zero, see the reviews [3], [4], as well as recent works [19], [12].

The statistical ergodic theorem is generalized as follows. According to the terminology of Tempelman [22], for a locally–compact commutative group $\mathcal{G}$, the statistically averaging sequence is the sequence (or net) of probability measures $\{\nu_n\}_{n\geqslant 1}$ on the group $\mathcal{G}$ such that for each unitary representation $\{U_g\}_{g\in\mathcal{G}}$ in the Hilbert space $\mathcal{H}$ the convergence

$$\left\| \int\limits_{\mathcal{G}} U_s h\, d\nu_n(s) - Ph \right\|_{\mathcal{H}} \to 0$$

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holds as $n \to \infty$ for each $h \in \mathcal{H}$, where P is the orthogonal projection onto the subspace of fixed vectors of group $\{U_g\}_{g\in\mathcal{G}}$. The classical von Neumann theorem considers uniform distributions on the segment $[-t,t]$, that is, the net of absolutely continuous probability measures

$$d\nu_t(s) = \frac{1}{2t}\chi_{[-t,t]}(s)\,ds.$$

By the Blum — Eisenberg criterion [13], the net of probability measures ν_t on locally compact commutative group $\mathcal{G}$ is statistically averaging if and only if

$$\lim_{t\to\infty}\int\limits_{\mathcal{G}} \chi(g)d\nu_t(g) = 0$$

for each character $\chi \in \mathcal{G}^\wedge$, $\chi \neq 1$. For the group $\mathbb{R}^m$, $m \geqslant 1$, this condition is written as

$$\lim_{t\to\infty}\int\limits_{\mathbb{R}^m} e^{i(s,x)}d\nu_t(x) = 0 \quad \text{for each} \quad s \neq 0.$$

The integral in this identity is the Fourier transform $\mathcal{F}[\nu_t]$ of probability measure ν_t. This equivalence is implied by the identity

$$\left\|\int\limits_{\mathcal{G}} U_s h\,d\nu_n(s) - Ph\right\|_{\mathcal{H}}^2 = \int\limits_{\mathcal{G}^\wedge} |\mathcal{F}[\nu_t](s)|^2 d\sigma_{h-Ph}(s), \tag{1.1}$$

where σ_{h-Ph} is the spectral measure of vector $h - Ph$, see, for instance, [18]; at the same time, $\sigma_{h-Ph}(1) = 0$, where 1 is the unit character.

In ergodic theorem (both in individual and statistical) the following classes of statistically averaging nets are important [20, Sect. 2.2]: measures ν_t absolutely continuous with respect to the Lebesgue measure including measures ν_t with compact supports; singular and discrete probability measures ν_t orthogonal to the Lebesgue measure.

In this note we focus on a few examples of statistically averaging sequences, which realize a convergence rate in the ergodic theorem faster than that in the classical von Neumann theorem. Namely, we consider examples with the maximal power rate with exponent $m > 1$ and with maximal exponential rate. We note that the maximal convergence rate in the classical von Neumann theorem is $\mathcal{O}\left(\frac{1}{t}\right)$. The same slow maximal rate holds for the pointwise convergence [5]. There are also works devoted to the fast pointwise convergence rate, see, for instance, [15], [16], [23].

The maximal power rates can be realized by means of the following construction. Let

$$\rho \in L_2([-1,1],dx), \quad \rho \geqslant 0, \quad \|\rho\|_1 = 1,$$

then we define

$$d\nu_t(s) = \frac{1}{t}\chi_{[-t,t]}(x)\rho\left(\frac{x}{t}\right)dx, \quad t > 0. \tag{1.2}$$

This net generalizes the uniform averaging in the von Neumann ergodic theorem (with $\rho \equiv \frac{1}{2}$). It is easy to see that the Fourier transform satisfies the Blum — Eisenberg criterion:

$$\mathcal{F}[\nu_t](s) = \frac{1}{t}\int\limits_{[-t,t]} e^{isx}\rho\left(\frac{x}{t}\right)dx = \int\limits_{[-1,1]} e^{isty}\rho(y)\,dy := R(ts) \to 0 \quad \text{as} \quad t \to +\infty$$

for each $s \neq 0$ by the Riemann — Lebesgue lemma. We are interesting in the case, when for some $m \in \mathbb{N}$

$$R(u) = \mathcal{O}\left(\frac{1}{u^m}\right) \quad \text{and} \quad R(u) \neq o\left(\frac{1}{u^m}\right)$$

as $|u| \to +\infty$. In particular, this condition holds for the considered below convolution of m indicators of the segment $\left[-\frac{1}{m}, \frac{1}{m}\right]$:

$$\rho(x) = \left(\frac{2}{m}\right)^m \underbrace{\chi_{[-\frac{1}{m},\frac{1}{m}]} * \ldots * \chi_{[-\frac{1}{m},\frac{1}{m}]}(x)}_{m \text{ times}}.$$

We shall call such statistically averaging nets are called as type (A).

We define probability measures with the maximal exponential rate via the Fourier transform. Suppose that we are given a net (or sequence) of probability measures ν_t with the Fourier transform

$$\mathcal{F}[\nu_t](s) = r^t(s), \quad t > 0 \quad \text{or} \quad t = n \in \mathbb{N},$$

where $r(s)$ satisfies the conditions $r(0) = 1$ and $|r(s)| < 1$ for all $s \neq 0$. The definition of these measures shows that they are statistically averaging. The measures can be recovered by the Fourier transform by an explicit formula, see, for instance, [1, Sect. 1.6.1]. We shall call such statistically averaging nets are called as type (B). The examples of such nets are convolutions of measures and distributions with fast decaying densities.

Let for some real numbers a, $b > 0$ the averaging net ν_t has a dilated exponential Fourier transform

$$\mathcal{F}[\nu_t](s) = e^{-at|s|^b}, \quad t > 0. \tag{1.3}$$

As an example of measure obeying the above conditions we can consider the Gauss distribution

$$d\nu_t(x) = \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{t}}\, dx, \quad t > 0$$

or the Cauchy distribution

$$d\nu_t(x) = \frac{t}{\pi(t^2 + x^2)}, \quad t > 0.$$

Their Fourier transform can be easily calculated or found in tables [7]

$$\mathcal{F}[\nu_t](s) = e^{-\frac{ts^2}{4}} \qquad \text{and} \qquad \mathcal{F}[\nu_t](s) = e^{-t|s|}.$$

We consider another way of constructing nets of type (B). Let ν_0 be a probability measure in $\mathbb{R}$ such that its Fourier transform possesses the property $|\mathcal{F}[\nu_0](s)| \leqslant 1$ and the identity is attained only at $s = 0$. For instance, the Dirac measures do not possess such property. We let

$$\nu_n(x) = \underbrace{\nu_0 * \ldots * \nu_0}_{n \text{ times}} := \nu_0^n, \quad n \geqslant 1.$$

By the Borel formula for convolution we obtain

$$\mathcal{F}[\nu_n](s) = (\mathcal{F}[\nu_0](s))^n \to 0 \quad \text{as} \quad n \to +\infty$$

for each $s \neq 0$.

Using a new recent approach to the problem [6, Sect. 3.1], in this note we obtain the asymptotics for the integrals (1.1) for the group $\mathbb{R}$ and for averaging nets (A) and (B). Here it is more convenient to consider a more general construction: instead of the spectral measure σ_{f-Pf}, we take an arbitrary Borel measure μ on $\mathbb{R}$ continuous at zero, that is, $\mu\{0\} = 0$.

2. Averaging nets with condition (A)

2.1. Whittaker — Kotelnikov — Shannon theorem. We recall that the direct (inverse) Fourier transform of a function $f \in L_2(\mathbb{R}, dx)$ is the function

$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-its}\, dt \qquad \left(\text{respectively,} \quad \check{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{its}\, dt\right).$$

The Fourier transform is invertible in $L_2(\mathbb{R}, dx)$, that is, $\check{\hat{f}}(s) = f(s) = \hat{\check{f}}(s)$ for almost each $s \in \mathbb{R}$. The definition shows that the Fourier transform of the measure ν_t in (1.2) is, up to a multiplicative constant, the inverse Fourier transform of the function ρ, that is,

$$\mathcal{F}[\nu_t](s) = \sqrt{2\pi}\check{\rho}(st) = R(ts).$$

Thus, the function R belongs to $L_2(\mathbb{R}, dx) \cap C(\mathbb{R})$ and its Fourier transform $\hat{R}(u) = \sqrt{2\pi}\rho(u)$, that is, $\operatorname{supp} \hat{R} = [-1, 1]$. The latter means that for the function R Whittaker — Kotelnikov — Shannon theorem holds, which states, see, for instance, [17], [14], that

$$R(u) = \sum_{k \in \mathbb{Z}} R(k\pi) \frac{\sin(k\pi - u)}{k\pi - u} = \sum_{k \in \mathbb{Z}} R(k\pi) \frac{(-1)^{|k|+1} \sin u}{k\pi - u}, \tag{2.1}$$

and the series converges absolutely and uniformly on $\mathbb{R}$ and the sequence $\{R(\pi k)\}_{k \in \mathbb{Z}}$ lies in $\ell^2(\mathbb{Z})$. This representation implies

$$R(u) = \mathcal{O}\left(\frac{1}{u}\right) \quad \text{as} \quad |u| \to +\infty.$$

Let us provide the justification of this fact. We fix $m \geqslant 1$ and partition the sum in (2.1) into two, one for $|k| \leqslant m$ and the other for $|k| > m$. We estimate the second sum by means of Cauchy — Schwarz — Bunyakovsky inequality

$$\left| \sum_{|k|>m} R(k\pi) \frac{(-1)^{k+1} \sin u}{\pi k - u} \right| \leqslant \sqrt{\sum_{|k|>m} |R(\pi k)|^2} \left\| \frac{\sin u}{k\pi - u} \right\|_{\ell^2} \leqslant M \sqrt{\sum_{|k|>m} |R(\pi k)|^2},$$

where

$$M = \max_{u \in \mathbb{R}} \left\| \frac{\sin u}{k\pi - u} \right\|_{\ell^2} < \infty$$

since the function $\varphi(u) = \left\| \frac{\sin u}{k\pi - u} \right\|_{\ell^2}$ is everywhere finite and π–periodic. While estimating the first sum, we take into consideration that $|R| \leqslant 1$. For $|u| \geqslant \sqrt{3}\pi m$ we obtain

$$\begin{aligned}
\left| \sum_{|k| \leqslant m} R(k\pi) \frac{(-1)^{k+1} \sin u}{\pi k - u} \right| &= \left| \frac{\sin u}{u} + \sin u \sum_{k=1}^{m} (-1)^{k+1} \left(\frac{R(k\pi)}{\pi k - u} - \frac{R(-k\pi)}{\pi k + u} \right) \right| \\
&\leqslant \frac{|\sin u|}{|u|} \left(1 + |u| \sum_{k=1}^{m} \frac{1}{|u| - \pi k} + \frac{1}{|u| + \pi k} \right) \\
&= \frac{|\sin u|}{|u|} \left(1 + \sum_{k=1}^{m} \frac{2u^2}{u^2 - (\pi k)^2} \right) \\
&\leqslant \frac{|\sin u|}{|u|} \left(1 + \frac{2u^2 m}{u^2 - (\pi m)^2} \right) \leqslant \frac{|\sin u|}{|u|} (1 + 3m).
\end{aligned}$$

Thus, for all

$$|u| \geqslant \max\left\{\sqrt{3}\pi m; M^{-1}\left(\sum_{|k|>m} |R(\pi k)|^2\right)^{-\frac{1}{2}}\right\}$$

the estimate

$$|R(u)| \leqslant \frac{1}{|u|}(2+3m)$$

holds.

We observe that the obtained estimate for R is attained at the density ρ from the statistical ergodic von Neumann theorem. This is why the interesting densities are ones, for which the Fourier transform decays faster.

2.2. Maximal rates. If ρ is defined by the convolution (A), by the Borel formula we obtain

$$R(u) = c\frac{\sin^m\left(\frac{u}{m}\right)}{u^m}$$

for some constant $c = c(m) > 0$. Then the formula (1.1) written in terms of continuous at zero measure μ implies

$$I_t(\mu) = \int_{\mathbb{R}} |R(ts)|^2\, d\mu(s) = c^2 \int_{\mathbb{R}} \frac{\sin^{2m}\left(\frac{ts}{m}\right)}{(ts)^{2m}}\, d\mu(s).$$

Let us consider the maximal rate of convergence to zero for the integral $I_t(\mu)$.

Theorem 2.1. *The following equivalences hold:*

$$I_t(\mu) = \mathcal{O}(t^{-2m}) \quad as \quad t \to +\infty \qquad \Longleftrightarrow \qquad \int_{\mathbb{R}} \frac{d\mu(s)}{s^{2m}} < \infty;$$

$$I_t(\mu) = o(t^{-2m}) \quad as \quad t \to +\infty \qquad \Longleftrightarrow \qquad \mu \equiv 0.$$

Proof. It is sufficient to consider only the direct statements since the inverse are obvious. Let $I_t(\mu) \leqslant At^{-2m}$ for all $t > 0$. Then, for each $T > 0$,

$$\frac{1}{T}\int_0^T \int_{\mathbb{R}} \frac{\sin^{2m}\left(\frac{ts}{m}\right)}{s^{2m}}\, d\mu(s)dt \leqslant \frac{A}{c^2}.$$

Since $s \neq 0$,

$$\begin{aligned}
\lim_{T\to+\infty}\frac{1}{T}\int_0^T \sin^{2m}\left(\frac{ts}{m}\right) dt &= \lim_{T\to+\infty}\frac{1}{T}\int_0^T \frac{\left(e^{\frac{its}{m}} - e^{-\frac{its}{m}}\right)^{2m}}{(2i)^{2m}}\, dt \\
&= \sum_{k=0}^{2m} \frac{(-1)^{m+k}}{4^m} C_{2m}^k \lim_{T\to+\infty}\frac{1}{T}\int_0^T e^{\frac{i2ts(k-m)}{m}}\, dt \\
&= \frac{(-1)^m}{4^m}\sum_{k=0}^{2m}(-1)^k C_{2m}^k \delta_{km} = \frac{C_{2m}^m}{4^m},
\end{aligned}$$

in view of the identity $\mu\{0\} = 0$ and by the Fubini theorem and the Fatou lemma we obtain

$$\int_{\mathbb{R}} \frac{d\mu(s)}{s^{2m}} \leqslant \frac{4^m A}{c^2 C_{2m}^m} < \infty.$$

Suppose that $I_t(\mu) = o(t^{-2m})$ as $t \to +\infty$. By introducing the finite measure

$$d\eta(s) = \frac{d\mu(s)}{s^{2m}}$$

this relation is reduced to

$$\int\limits_{\mathbb{R}} \sin^{2m}\left(\frac{ts}{m}\right) d\eta(s) = o(1) \quad \text{as} \quad t \to +\infty,$$

that is,

$$\lim_{t\to+\infty} \left\|\sin\left(\frac{ts}{m}\right)\right\|_{L_{2m}(\mathbb{R},d\eta)} = 0.$$

The Hölder inequality

$$\left\|\sin\left(\frac{ts}{m}\right)\right\|_{L_2(\mathbb{R},d\eta)} \leqslant \left\|\sin\left(\frac{ts}{m}\right)\right\|_{L_{2m}(\mathbb{R},d\eta)} \eta^{\frac{m-1}{2m}}(\mathbb{R})$$

implies

$$\lim_{t\to+\infty} \left\|\sin\left(\frac{ts}{m}\right)\right\|_{L_2(\mathbb{R},d\eta)} = 0.$$

This relation by Lemma 4 in [19] yields $\eta(\mathbb{R}) = 0$. Hence, $\mu(\mathbb{R}) = 0$. The proof is complete. □

2.3. Regularly varying functions. Since the maximal rate is power, it is natural to find the criterion for the power rates with a smaller exponent. For the case $m = 1$, that is, for the classical von Neumann ergodic theorem, the Kachurovskii criterion for the power rates was extended (for discrete time) by Gaposhkin in [2] to the class of functions of form $u^{-\alpha}\varphi(u)$, $\alpha \in [0, 2)$, where φ is a weakly oscillating function. We recall that a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called weakly oscillating (or slowly varying in the Zygmund sense) if for each $\theta > 0$ for sufficiently large $u > 0$ the function $u^{\theta}\varphi(u)$ monotonically increases, while the function $u^{-\theta}\varphi(u)$ monotonically decreases. The class of weakly oscillating functions is a subclass of slowly varying in the sense of Karamata, that is, of measurable functions, for which the identity

$$\lim_{u\to+\infty} \frac{\varphi(\lambda u)}{\varphi(u)} = 1$$

holds for each $\lambda > 0$. The function $\psi(u) = u^{\rho}\varphi(u)$, where $\varphi(u)$ is a slowly varying in the sense of Karamata is called regularly varying at infinity of order $\rho \in \mathbb{R}$. The details on such functions can be borrowed from the monograph [9], and also from the recent paper [21]. We are going to show that the result by Gaposhkin can be generalized to all regularly varying functions of order $-\alpha$, where $\alpha \in [0, 2m)$. Let us provide key statements.

The next lemma shows that for one side of estimates no additional conditions for the convergence rate are needed.

Lemma 2.1. *Let $\psi(t) \to 0$ as $t \to +\infty$ and $I_t(\mu) \leqslant B\psi(t)$ for some constant $B > 0$ for all $t > 0$. Then for all $\delta > 0$*

$$\mu([-\delta, \delta]) \leqslant \frac{c^{-2}B}{\sin^{2m}\left(\frac{1}{m}\right)}\psi\left(\frac{1}{\delta}\right).$$

Proof. We represent an arbitrary $\delta > 0$ as $\delta = \frac{1}{t}$. Then

$$\mu([-\delta, \delta]) = \int\limits_{[-\frac{1}{t},\frac{1}{t}]} d\mu(s) \leqslant \frac{(\frac{1}{m})^{2m}}{\sin^{2m}(\frac{1}{m})} \int\limits_{[-\frac{1}{t},\frac{1}{t}]} \frac{\sin^{2m}(\frac{ts}{m})}{(\frac{ts}{m})^{2m}} d\mu(s)$$

$$\leqslant \frac{c^{-2}}{\sin^{2m}(\frac{1}{m})} I_t(\mu) \leqslant \frac{c^{-2}B}{\sin^{2m}(\frac{1}{m})}\psi(t) = \frac{c^{-2}B}{\sin^{2m}(\frac{1}{m})}\psi\left(\frac{1}{\delta}\right).$$

The proof is complete. $\square$

The next theorem by Aljančić, Bojanic and Tomić [11, Thm. 1] (see also [9, Thm. 2.7]) is a key ingredient for proving other side of estimates.

Theorem 2.2. *Let $f : (0, +\infty) \to \mathbb{R}$ and $\theta > 0$ be such that the integral $\int\limits_0^a u^{-\theta} f(u)\, du$ converges for some $a \in (0, +\infty)$. Then for each slowly varying in the sense of Karamata function φ bounded on each finite interval in $(0, +\infty)$ the relation*

$$\int\limits_0^a f(u)\varphi(tu)\, du \quad \sim \quad \varphi(t) \int\limits_0^a f(u)\, du \quad \text{as} \quad t \to +\infty.$$

holds.

Now we formulate the criterion for all possible regular convergence rates.

Theorem 2.3. *Let $\psi(u) = u^{-\alpha}\varphi(u)$, where $\alpha \in [0, 2m)$ and φ is a slowly varying in the Karamata sense function bounded on each finite interval in $(0, +\infty)$. Then the equivalence*

$$\mu([-\delta, \delta]) = \mathcal{O}\left(\psi\left(\frac{1}{\delta}\right)\right) \quad \text{as} \quad \delta \to +0 \quad \Leftrightarrow \quad I_t(\mu) = \mathcal{O}(\psi(t)) \quad \text{as} \quad t \to +\infty.$$

holds.

Proof. The inverse implication follows from Lemma 2.1. For the direct implication first we are going to obtain the estimate

$$\begin{aligned} I_t(\mu) = c^2 \int\limits_{\mathbb{R}} \frac{\sin^{2m}(\frac{ts}{m})}{(ts)^{2m}}\, d\mu(s) &= 2mc^2 \int\limits_0^\infty u^{2m-1} \mu\left(\left\{ s \in \mathbb{R} : \frac{|\sin(\frac{ts}{m})|}{|ts|} > u \right\}\right) du \\ &= 2mc^2 \int\limits_0^{\frac{1}{m}} u^{2m-1} \mu\left(\left\{ s \in \mathbb{R} : \frac{|\sin(\frac{ts}{m})|}{|ts|} > u \right\}\right) du \\ &\leqslant 2mc^2 \int\limits_0^{\frac{1}{m}} u^{2m-1} \mu\left(\left\{ s \in \mathbb{R} : |s| \leqslant \frac{1}{tu} \right\}\right) du \\ &\leqslant 2mc^2 A \int\limits_0^{\frac{1}{m}} u^{2m-1} (tu)^{-\alpha} \varphi(ut)\, du \end{aligned}$$

for all $t > 0$. Here and later in Remark 2.1 the constant $A > 0$ is determined by the assumptions of the theorem: $\mu([-\delta, \delta]) \leqslant A\psi\left(\frac{1}{\delta}\right)$ for all $\delta > 0$. Thus,

$$I_t(\mu) = \mathcal{O}\left(t^{-\alpha} \int\limits_0^{\frac{1}{m}} u^{2m-\alpha-1} \varphi(ut)\, du \right) \quad \text{as} \quad t \to +\infty.$$

It remains to apply Theorem 2.2 for the function $f(u) = u^{2m-\alpha-1}$ on the interval $(0, \frac{1}{m})$. As θ we can take an arbitrary number in the interval $(0, 2m - \alpha)$. The proof is complete. $\square$

Remark 2.1. *For power rates $\psi(u) = u^{-\alpha}$, $\alpha \in [0, 2m)$ the estimate holds:*

$$I_t(\mu) \leqslant 2mc^2 A t^{-\alpha} \int\limits_0^{\frac{1}{m}} u^{2m-\alpha-1}\, du = \frac{2c^2 A}{(2m - \alpha)m^{2m-\alpha-1}} t^{-\alpha}.$$

3. Estimates for rates of averaging nets of type (B)

3.1. Maximal rates. We let $R(s) = |r(s)|^2$; in the case of averagings (1.3) we have $R(s) = e^{-2a|s|^b}$. Thus, we are interesting in the asymptotics of integrals

$$I_n(\mu) = \int_{\mathbb{R}} R^n(s)d\mu(s), \quad \text{and, respectively,} \quad I_t(\mu) = \int_{\mathbb{R}} R^t(s)d\mu(s)$$

as $n \to \infty$ or as $t \to +\infty$.

First we consider maximal possible convergence rates.

Theorem 3.1. *Let $q \in (0,1)$, then*

$$I_t(\mu) = \mathcal{O}(q^t) \quad \text{as} \quad t \to +\infty \quad \Leftrightarrow \quad \text{there exists} \quad \delta > 0 \quad \text{such that} \quad \mu([-\delta,\delta]) = 0.$$

If for each $q \in (0,1)$ the asymptotic relation $I_t(\mu) = o(q^t)$ holds as $t \to +\infty$, then $I_t(\mu) \equiv 0$. Similar statements are true for the integrals $I_n(\mu)$.

Proof. If $\mu([-\delta,\delta]) = 0$ for some $\delta > 0$, then

$$\int_{\mathbb{R}} R^t(s)d\mu(s) = \int_{|s|>\delta} R^t(s)d\mu(s) \leqslant \max_{|s|\geqslant\delta} R^t(s)\mu\{|s| \geqslant \delta\} = \mu(\mathbb{R})q^t,$$

where $q = \max_{|s|\geqslant\delta} R(s) < 1$. And vice versa, let for some constants $A > 0$ and $q \in (0,1)$ we have $I_t(\mu) \leqslant Aq^t$ for all $t > 0$. Then

$$\frac{1}{T}\int_0^T \int_{\mathbb{R}} \left(\frac{R(s)}{q}\right)^t d\mu(s)dt \leqslant A.$$

Letting

$$g(s) = \lim_{T\to+\infty} \frac{1}{T}\int_0^T \left(\frac{R(s)}{q}\right)^t dt = \begin{cases} +\infty, & R(s) > q, \\ 1, & R(s) = q, \\ 0, & R(s) < q, \end{cases}$$

interchanging the integration order by the Tonelli theorem and applying the Fatou lemma, we obtain

$$\int_{\mathbb{R}} g(s)d\mu(s) \leqslant A.$$

We hence conclude that $\mu(\{R(s) > q\}) = 0$. Since R is a continuous function $R(0) = 1$, the open set $\{R(s) > q\}$ contains some neighbourhood of zero of form $[-\delta,\delta]$. Therefore, $\mu([-\delta,\delta]) = 0$.

Suppose that $I_t(\mu) = o(q^t)$ as $t \to +\infty$ for each $q \in (0,1)$. Then, as it has been proved, $\mu(\{R(s) > q\}) = 0$. Letting $q \to +0$, we obtain the identity $\mu(\{R(s) > 0\}) = 0$, that is, the measure μ is concentrated on the zeroes of function R. In this case we obtain $I_t(\mu) \equiv 0$.

For the integrals $I_n(\mu)$ the proof is similar. The only difference is that instead of the integration $\frac{1}{T}\int\limits_0^T$ we need to make the summation $\frac{1}{N}\sum\limits_{k=1}^N$. The proof is complete. □

Let us discuss the way of obtaining the estimates for the integral $I_t(\mu)$ over the singularity of measure μ near the origin. In further arguing we suppose that R is an even function monotonically decaying to zero at infinity. Suppose that for some $A > 0$ the estimate

$$\mu([-\delta,\delta]) \leqslant A\varphi(\delta)$$

holds for all $\delta > 0$, where $\varphi \to 0$ as $\delta \to +0$. Then

$$I_t(\mu) = \int\limits_{\mathbb{R}} R^t(s) d\mu(s) = \int\limits_0^1 \mu\{R^t(s) > u\}\, du$$
$$= \int\limits_0^1 \mu\{|s| < R^{-1}(u^{\frac{1}{t}})\}\, du \leqslant A \int\limits_0^1 \varphi(R^{-1}(u^{\frac{1}{t}}))\, du.$$

Now suppose that the estimate

$$I_t(\mu) \leqslant B\psi(t)$$

is true for all $t > 0$ with some constant $B > 0$, where ψ is a positive differentiable monotonically increasing function. Then

$$\mu([-\delta, \delta]) = \int\limits_{[-\delta,\delta]} d\mu(s) = R^{-t}(\delta) \int\limits_{[-\delta,\delta]} R^t(\delta) d\mu(s)$$
$$\leqslant R^{-t}(\delta) \int\limits_{[-\delta,\delta]} R^t(s) d\mu(s) \leqslant R^{-t}(\delta) \int\limits_{\mathbb{R}} R^t(s) d\mu(s) \leqslant BR^{-t}(\delta)\psi(t).$$

This yields

$$\mu([-\delta, \delta]) \leqslant B \inf_{t>0} R^{-t}(\delta)\psi(t) = BR^{-t_0}(\delta)\psi(t_0),$$

where t_0 is the unique stationary point satisfying the equation

$$(\ln \psi)'(t_0) = \ln R(\delta). \tag{3.1}$$

3.2. Power estimates. We apply the described method to obtain the criterion of power convergence.

Theorem 3.2. *Let R be an even function monotonically decaying to zero at infinity and A, B, $\alpha > 0$ be some constants.*

(1) *If $I_t(\mu) \leqslant Bt^{-\alpha}$ for all $t > 0$, then*

$$\mu([-\delta, \delta]) \leqslant B \left(\frac{e}{\alpha}\right)^{\alpha} \ln^{\alpha} \frac{1}{R(\delta)} \quad \textit{for all} \quad \delta > 0.$$

(2) *If $\mu([-\delta, \delta]) \leqslant A \ln^{\alpha} \frac{1}{R(\delta)}$ for all $\delta > 0$, then*

$$I_t(\mu) \leqslant A\Gamma(\alpha + 1)t^{-\alpha} \quad \textit{for all} \quad t > 0.$$

Proof. (1) We have $\psi(t) = t^{-\alpha}$. By Equation (3.1) we find $t_0 = t_0(\delta)$ and we get

$$t_0 = -\frac{\alpha}{\ln(R(\delta))}.$$

This implies the sought estimate

$$\mu([-\delta, \delta]) \leqslant BR^{-t_0}(\delta)\psi(t_0) = BR^{-t_0}(\delta)t_0^{-\alpha} = B\left(\frac{e}{\alpha}\right)^{\alpha} \ln^{\alpha} \frac{1}{R(\delta)}$$

for all $\delta > 0$.

(2) We have $\varphi(\delta) = \ln^{\alpha} \frac{1}{R(\delta)}$. Then for all $t > 0$ we obtain

$$I_t(\mu) \leqslant A \int\limits_0^1 \varphi(R^{-1}(u^{\frac{1}{t}}))\, du = A \int\limits_0^1 \ln^{\alpha}(u^{-\frac{1}{t}})\, du = At^{-\alpha} \int\limits_0^1 \ln^{\alpha}\left(\frac{1}{u}\right) du.$$

Using the substitution $\ln\left(\frac{1}{u}\right) = s$, we arrive at the sought estimate

$$I_t(\mu) \leqslant At^{-\alpha} \int\limits_0^\infty s^\alpha e^{-s}\, ds = A\Gamma(\alpha+1)t^{-\alpha}.$$

The proof is complete. $\square$

The constant $A\Gamma(\alpha+1)$ in Assertion (2) of Theorem 3.2 is sharp in the class of all Borel measures μ on $\mathbb{R}$. Moreover, it is sharp in the class of spectral measures of unitary representations of group $\mathbb{R}$.

Indeed, we consider the sequence of measures μ_n with supports on the segments $[0, n]$:

$$d\mu_n(s) = \chi_{[0,n]}(s) \frac{d}{ds}\left(\ln^\alpha \frac{1}{R(s)}\right) ds.$$

In view of the monotonicity R, on the positive semi–axis the derivative is well–defined everywhere except for a countable set of points (of zero measure). For each $\delta \in (0, n]$ we have

$$\mu_n([-\delta,\delta]) = \int\limits_0^\delta \frac{d}{ds} \ln^\alpha \frac{1}{R(s)}\, ds = \ln^\alpha \frac{1}{R(\delta)},$$

that is, $A = 1$. For the integral we obtain the identity

$$I_t(\mu_n) = \int\limits_0^n R^t(s) \frac{d}{ds} \ln^\alpha \frac{1}{R(s)}\, ds = R^t(n) \ln^\alpha \frac{1}{R(n)} - \int\limits_0^n \ln^\alpha \frac{1}{R(s)}\, dR^t(s).$$

We let $y = \ln \frac{1}{R^t(n)}$, then we obtain

$$I_t(\mu) = t^{-\alpha}\left(y^\alpha e^{-y} + \int\limits_0^y s^\alpha e^{-s}\, ds\right).$$

It is easy to confirm that the expression in brackets monotonically increases to $\Gamma(\alpha+1)$ as $y \to +\infty$ (this occurs as $n \to \infty$).

Let us show that the measures μ_n can be realized as spectral measures. In order to do this, it is sufficient to consider the group of unitary multiplication operators

$$U_t f(s) = e^{its} f(s)$$

in the Hilbert space $\mathcal{H} = L_2(\mathbb{R})$. It is easy to verify that the only fixed point of this group is the zero function and the spectral measure reads $d\sigma_f(s) = |f(s)|^2\, ds$. Thus, it is sufficient to take the functions

$$f_n(s) = \chi_{[0,n]}(s) \sqrt{\frac{d}{ds}\left(\ln^\alpha \frac{1}{R(s)}\right)}, \quad n \geqslant 1.$$

We note that by means of the same group of unitary operators the sharpness of constants was proved for the estimates in the classical von Neumann theorem, see [19, Sect. 3.4].

Remark 3.1. *For averagings of type* (1.3) *the criterion of power convergence reads*

$$\mu([-\delta,\delta]) = \mathcal{O}(\delta^{\alpha b}) \quad \Longleftrightarrow \quad I_t(\mu) = \mathcal{O}(t^{-\alpha}).$$

3.3. Dilated exponential rates. As Theorem 3.1 shows, the integrals $I_t(\mu)$ admit arbitrary rates up to exponential one. We are going to obtain the conditions for dilated exponential rates.

Theorem 3.3. *Let R be an even function monotonically decaying to zero at infinite and $A, B > 0$ and $\alpha \in (0,1)$ be some constants.*

(1) *If $I_t(\mu) \leqslant Bq^{t^\alpha}$ for all $t > 0$ and some $q \in (0,1)$, then*
$$\mu([-\delta,\delta]) \leqslant B\exp\left(-\left(\frac{Q}{\ln^\alpha \frac{1}{R(\delta)}}\right)^{\frac{1}{1-\alpha}}\right)$$
for all $\delta > 0$, where $Q = \alpha^\alpha(1-\alpha)^{1-\alpha}\ln(\frac{1}{q})$.

(2) *If*
$$\mu([-\delta,\delta]) \leqslant A\exp\left(-\left(\frac{Q}{\ln^\alpha \frac{1}{R(\delta)}}\right)^{\frac{1}{1-\alpha}}\right)$$
for all $\delta > 0$ and some $Q > 0$, then
$$I_t(\mu) \leqslant A\sqrt{2\pi Q(1-\alpha)^\alpha\alpha^{1-\alpha}}\, t^{\frac{\alpha}{2}} q^{t^\alpha}$$
for all sufficiently large $t > 0$, where $q = \exp\left(-\frac{Q}{\alpha^\alpha(1-\alpha)^{1-\alpha}}\right)$.

Proof. (1) We follow the lines of proof of Theorem 3.2 on power rates. We have $\psi(t) = q^{t^\alpha}$. By Equation (3.1) we find $t_0 = t_0(\delta)$:
$$t_0 = \left(\frac{\alpha\ln(q)}{\ln(R(\delta))}\right)^{\frac{1}{1-\alpha}}.$$
For all $\delta > 0$ we then obtain
$$\mu([-\delta,\delta]) \leqslant BR^{-t_0}(\delta)\psi(t_0) = BR^{-t_0}(\delta)q^{t_0^\alpha} = B\exp\left(-\left(\frac{Q}{\ln^\alpha \frac{1}{R(\delta)}}\right)^{\frac{1}{1-\alpha}}\right).$$

(2) We have
$$\varphi(\delta) = \exp\left(-\left(\frac{Q}{\ln^\alpha \frac{1}{R(\delta)}}\right)^{\frac{1}{1-\alpha}}\right).$$
Then for all $t > 0$ we obtain
$$I_t(\mu) \leqslant A\int_0^1 \varphi(R^{-1}(u^{\frac{1}{t}}))\,du = A\int_0^1 \exp\left(-\left(\frac{Q}{\ln^\alpha \frac{1}{u^{\frac{1}{t}}}}\right)^{\frac{1}{1-\alpha}}\right)du.$$
Using the substitution $\ln\left(\frac{1}{u}\right) = s$, we arrive at the integral
$$I_t(\mu) \leqslant A\int_0^\infty \exp\left(-\left(s + \left(\frac{Qt^\alpha}{s^\alpha}\right)^{\frac{1}{1-\alpha}}\right)\right)ds.$$
We let $T = Qt^\alpha$, $\beta = \frac{\alpha}{1-\alpha}$ and make the change $s = Tu$ in the integral
$$I_t(\mu) \leqslant AT\int_0^\infty \exp\left(-T\left(u + \frac{1}{u^\beta}\right)\right)du.$$
To find the asymptotics of obtained integral, we apply the Laplace method [10, Ch. 2], [8]. Since the smooth function $f(u) = -u - \frac{1}{u^\beta}$ has the unique maximum at the point
$$u_0 = \beta^{\frac{1}{\beta+1}} = \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha},$$

as $T \to \infty$, the asymptotic identity

$$\int_0^\infty \exp\left(-T\left(u+\frac{1}{u^\beta}\right)\right)\,du = e^{Tf(u_0)}\sqrt{\frac{2\pi}{-f''(u_0)}}\left(T^{-\frac{1}{2}}+\mathcal{O}(T^{-1})\right)$$

holds. Substituting the values

$$f(u_0) = -\frac{1}{(1-\alpha)^{1-\alpha}\alpha^\alpha}, \qquad f''(u_0) = -\frac{1}{(1-\alpha)^\alpha\alpha^{1-\alpha}},$$

for sufficiently large $t > 0$ we obtain

$$I_t(\mu) \leqslant AQ^{\frac{1}{2}}t^{\frac{\alpha}{2}}\exp\left(-\frac{Qt^\alpha}{\alpha^\alpha(1-\alpha)^{1-\alpha}}\right)\sqrt{2\pi(1-\alpha)^\alpha\alpha^{1-\alpha}}.$$

The proof is complete. □

Remark 3.2. *Theorems 3.2 and 3.3 are also true for a discrete sequence of measures ν_n, probably with slightly different constants.*

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