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ON ONE APPLICATION OF LEONTIEV INTERPOLATION FUNCTION IN THEORY OF TRIGONOMETRICALLY CONVEX FUNCTIONS

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Abstract. We study a connection between ρ–trigonometrically convex functions and the class of subharmonic functions. The established connection is used to prove new inequalities characterizing ρ–trigonometrically convex functions and to find integral equations of the first kind for ρ–trigonometric functions. Under a detailed development of this issue, there appears the convolution integral equation

$$h(\theta) = \int_{-\infty}^{\infty} h(\theta - u)d\sigma(u),$$

where σ is a finite compactly supported measure. The results on the theory of this equation are exposed following A.F. Leontiev, who studied this equation in relation with the theory of Dirichlet series. Using the Leontiev interpolating function, we propose additional conditions ensuring that a continuous solution to the equation

$$h(\theta) = \int_{-\infty}^{\infty} a_R(u)h(\theta - u)du$$

for a fixed R is a ρ–trigonometric function.

Keywords: subharmonic function, trigonometrically convex function, integral equation of the first kind, convolution equation, Leontiev interpolating function.

Mathematics Subject Classification: 26A51, 31A05.

1. Introduction

We study a connection between ρ–trigonometrically convex functions and the class of subharmonic functions. The established connection is used to prove new inequalities characterizing ρ–trigonometrically convex functions and to find integral equations of the first kind for ρ–trigonometric functions. Under a detailed development of this issue, there appears the convolution integral equation

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The paper is organized as follows. In Section 2 we provide preliminary facts from the theory of subharmonic and ρ–trigonometrically convex functions. In Theorem 2.7 we prove that each ρ–trigonometrically convex function can be approximated by a family of infinitely differentiable ρ–trigonometrically convex functions. In Theorem 2.8 we establish the relation between ρ–trigonometrically convex functions and subharmonic functions.

In Section 3 we prove new inequalities for ρ–trigonometrically convex functions, see Theorems 3.1 and 3.3. As corollaries of Theorems 3.1 and 3.3 we obtain criteria for a continuous function h on the entire axis to be ρ–trigonometric, see Theorems 3.2 and 3.6.

Section 4 is devoted to the study of one convolution equation. The results on the theory of this equation are presented following A.F. Leontiev, who studied it in connection with the theory of Dirichlet series. Using the Leontiev interpolating function, we propose additional conditions ensuring that a continuous solution to the integral equation is a ρ–trigonometric function.

2. Preliminaries

By the symbol $\langle a, b\rangle$ we denote either an interval, or a segment, or a semi–interval of one of two types. Of course, the segment $\langle a, b\rangle$ is well–defined; it cannot be an interval and a half–interval simultaneously. The use of the symbol $\langle a, b\rangle$ is justified in studying the properties of functions t defined on an interval, a segment, or a half–interval. The use of this symbol allows us brief formulation in some cases. For instance, instead of saying that the function f is defined on the interval with the endpoints a, b, which can be an interval, a segment, or a half–interval, we can say briefly that the function f is defined on a segment $\langle a, b\rangle$.

We shall establish a connection between ρ–trigonometrically convex functions and subharmonic functions. The theory of subharmonic functions can be found in the books of Privalov [6], Heyman and Kennedy [8], Tsuji [13]. We begin with the necessary definitions and exposition of the properties of subharmonic functions, which will be used in what follows.

Definition 2.1. *A function $v(z)$ defined in a planar domain D with values from the extended real line $[-\infty,\infty]$ is called subharmonic in the domain D if it satisfies the conditions*

1) $v(z)<\infty$ *for each point* $z\in D$,
2) *there exists a point* $z\in D$ *such that* $v(z)>-\infty$,
3) *the function* $v(z)$ *is upper–semi–continuous, that is,*

$$\overline{\lim_{z\to z_0}}\, v(z)\leqslant v(z_0)$$

for each point $z_0\in D$,

4) *the mean inequality holds, that is, for each point* $z_0 \in D$ *and all* $R > 0$ *such that the ball* $|z - z_0| \leqslant R$ *is contained in* D *the inequality holds*

$$v(z_0) \leqslant \frac{1}{2\pi} \int\limits_0^{2\pi} v(z_0 + Re^{i\varphi}) d\varphi. \tag{2.1}$$

Definition 2.2. *A function* $v(z)$ *is called locally integrable in the domain* D *if it is integrable over each compact set embedded into the domain* D.

It is proved in the theory of subharmonic functions that each subharmonic function in a domain D is locally integrable in this domain. In particular, a subharmonic function cannot become $-\infty$ on any open set. We also observe that, along with inequality (2.1), the inequality holds

$$v(z_0) \leqslant \frac{1}{\pi R^2} \int\limits_0^{R} \int\limits_0^{2\pi} v(z_0 + re^{i\varphi}) r dr d\varphi, \tag{2.2}$$

and in Definition 2.1 the inequality (2.1) can be replaced by the inequality (2.2).

Now we are in position to define our main object, the ρ–trigonometrically convex function, see [3]. The prohibition to take infinite values often turns out to be inconvenient. This is, in particular, due to the fact that the real line $(-\infty, \infty)$ is not compact, unlike the extended real line $[-\infty, \infty]$. The functions that take on only finite real values will be called finite. A priori, we do not prohibit a function to take infinite values.

Definition 2.3. *A function* $h(\theta)$ *defined on a segment* $\langle \alpha, \beta \rangle$ *with values in the extended real line* $[-\infty, \infty]$ *is called* ρ*–trigonometrically convex on this segment if for all* $\theta_1, \theta_2 \in \langle \alpha, \beta \rangle$, $0 < \theta_2 - \theta_1 < \pi/\rho$, *and each* $\theta \in (\theta_1, \theta_2)$ *the inequality holds*

$$h(\theta) \leqslant \frac{\sin \rho(\theta_2 - \theta)}{\sin \rho(\theta_2 - \theta_1)} h(\theta_1) + \frac{\sin \rho(\theta - \theta_1)}{\sin \rho(\theta_2 - \theta_1)} h(\theta_2). \tag{2.3}$$

To avoid the phrase "if the sum makes sense", in this definition we adopt the convention that the inequality $x \leqslant \infty - \infty$ holds for any $x \in [-\infty, \infty]$. This convention does not make our definition inconsistent.

It is easy to prove that the definition 2.3 is equivalent to the following one.

Definition 2.4. *A function* $h(\theta)$ *defined on a segment* $\langle \alpha, \beta \rangle$ *with the values in the extended real line* $[-\infty, \infty]$ *is called* ρ*–trigonometrically convex on this segment if for all* $\theta_1, \theta_2 \in \langle \alpha, \beta \rangle$, $0 < \theta_2 - \theta_1 < \pi/\rho$, *and each* ρ*–trigonometric function* $H(\theta)$ *the inequalities* $h(\theta_1) \leqslant H(\theta_1)$, $h(\theta_2) \leqslant H(\theta_2)$ *imply the inequality* $h(\theta) \leqslant H(\theta)$ *for each* $\theta \in [\theta_1, \theta_2]$.

There are certainly situations when it is more convenient to use Definition 2.4. However, we note that if we use this definition, we need to consider two cases: the first, when there is no function $H(\theta)$ for which the inequalities $h(\theta_1) \leqslant H(\theta_1)$, $h(\theta_2) \leqslant H(\theta_2)$ hold, and the second, when such a function does exist. When using Definition 2.3, there is no need to consider these cases. Note also that, according to the proposed definitions, the empty function, i.e. a function with the empty domain, and any function defined on a one–point set are ρ–trigonometrically convex functions.

Let us adopt the terminology. We have defined ρ–trigonometrically convex functions for each $\rho > 0$. The most frequent functions are 1–trigonometrically convex ones. Following tradition, we call such functions trigonometrically convex. We shall use the term "general trigonometrically convex function", that is, a function that is ρ–trigonometrically convex for some $\rho > 0$. This term is convenient to use in cases where there is no need to fix a specific ρ.

The most important result in the theory of subharmonic functions is the maximum principle. We present a version of this principle.

Theorem 2.1. *Let $v(z)$ be a subharmonic function in a bounded domain. Let M be a real number such that for each point ζ in the boundary D the inequality*

$$\varlimsup_{\substack{z\to\zeta\\ z\in D}} v(z) \leqslant M$$

holds. Then for each point z in the domain D the inequality $v(z) \leqslant M$ holds and one of the following cases take place

1) *$v(z) < M$ for each point z in D,*
2) *$v(z) \equiv M$ in the domain D.*

An important generalization of the maximum principle to the case of unbounded domains is the Phragmén — Lindelöf theorem. Without dwelling on the general case, we formulate the Phragmén — Lindelöf theorem for the most important domain for us, which an angle. To formulate it, we need one more definition.

Definition 2.5. *Let $A = A(\varphi_1, \varphi_2) = \{z : \varphi_1 < \arg z < \varphi_2\}$ be an open angle and $\rho \geqslant 0$ be some number. The number ρ is called the formal order of a subharmonic inside the angle A function $v(z)$ if there exist the numbers M_1, $M_2 \geqslant 0$ such that for $z \in A$ the inequality holds*

$$v(z) \leqslant M_1 + M_2|z|^\rho.$$

Theorem 2.2. *Suppose that a subharmonic function $v(z)$ is defined inside the angle $A(\varphi_1, \varphi_2)$ and satisfies the conditions*

1) *there exists a number M such that for each point ζ on the boundary of the angle the inequality holds*
$$\varlimsup_{\substack{z\to\zeta\\ z\in A}} v(z) \leqslant M,$$
2) *some number $\rho \geqslant 0$ is the formal order of the function $v(z)$ inside the angle $A(\varphi_1, \varphi_2)$,*
3) *the inequality $\varphi_2 - \varphi_1 < \dfrac{\pi}{\rho}$ holds.*

Then for each point $z \in A(\varphi_1, \varphi_2)$ the inequality $v(z) \leqslant M$ holds.

The ρ–trigonometrically convex functions first appeared in mathematics in connection with the following theorem, also due to Phragmén and Lindelöf.

Theorem 2.3. *Let $v(z)$ be a subharmonic function of formal order $\rho > 0$ inside the angle $A(\alpha, \beta)$ and let*

$$h(\theta) = \varlimsup_{r\to\infty} \frac{v(re^{i\theta})}{r^\rho}$$

be its growth indicator (this is the definition of indicator). Then $h(\theta)$ is a ρ–trigonometrically convex function on the interval (α, β).

Let us formulate without proof two more needed theorems from the theory of subharmonic functions. The first of them is a theorem on the characterization of smooth subharmonic functions.

Theorem 2.4. *Let $v(x, y)$ be a twice–differentiable function in the domain D. It is subharmonic if and only if in the domain D the inequality*

$$\Delta v(x, y) \geqslant 0$$

holds, where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is the Laplace operator.

The second theorem is on the preservation of subharmonicity under the uniform passage to the limit. Since subharmonic functions can become $-\infty$ at some points, the statement "the sequence $v_n(z)$ converges uniformly to $v(z)$" means that the sequence of bounded functions $\arctan v_n(z)$ converges uniformly to the function $\arctan v(z)$.

Theorem 2.5. *Let $v_n(z)$ be a sequence of subharmonic functions uniformly converging on compact sets in D. Then the function*

$$v(z) = \lim_{n\to\infty} v_n(z)$$

is a subharmonic function in D if $v(z) \not\equiv -\infty$.

Approximation theorems are an important tool for studying various classes of functions. The best known theorems are ones on the approximation of continuous functions by polynomials. As we shall see below, each function from the class of ρ–trigonometrically convex functions can be approximated with an arbitrary accuracy by an infinitely differentiable function from the same class. In some cases, this statement allows one to carry out the proof only for the class of infinitely differentiable ρ–trigonometrically convex functions. Before formulating the corresponding approximation theorem, we present the notation and some facts from the averaging theory.

We denote

$$\omega(x) = c\begin{cases} e^{-\frac{1}{1-x^2}}, & |x| < 1; \\ 0, & |x| \geqslant 1, \end{cases}$$

where the constant c is chosen so that

$$\int_{-\infty}^{\infty} \omega(x)dx = 1.$$

It is known that $\omega(x)$ is an infinitely differentiable function on the entire axis. We denote

$$\omega_\varepsilon(x) = \frac{1}{\varepsilon}\omega\left(\frac{x}{\varepsilon}\right). \tag{2.4}$$

We have

$$\int_{-\infty}^{\infty} \omega_\varepsilon(x)dx = \frac{1}{\varepsilon}\int_{-\infty}^{\infty} \omega\left(\frac{x}{\varepsilon}\right)dx = \int_{-\infty}^{\infty} \omega(t)dt = 1.$$

The averaging operator, which maps the function f into the function f_ε, is defined as

$$f_\varepsilon(x) = \int_{-\infty}^{\infty} f(x-\tau)\omega_\varepsilon(\tau)d\,\tau = \int_{-\infty}^{\infty} f(u)\omega_\varepsilon(x-u)du.$$

If $f(x)$ is a locally integrable function on the axis $(-\infty, \infty)$, then $f_\varepsilon(x)$ is an infinitely differentiable function on this axis. If $f(x)$ is a locally integrable function in the interval (α, β), then the function $f_\varepsilon(x)$ is well–defined only on the interval $(\alpha+\varepsilon, \beta-\varepsilon)$. Here we need to suppose that the integrand vanishes outside the support of the function ω_ε no matter whether the integrand is defined or not on this set.

However, we are interesting only in continuous functions f. For this case we provide without the proof the known and easily provable statement.

Theorem 2.6. *Let $f(x)$ be a continuous function on the interval (α,β). Then f_ε is an infinitely differentiable function on the interval $(\alpha+\varepsilon,\beta-\varepsilon)$. If $[a,b]\subset(\alpha,\beta)$, then the functions $f_\varepsilon(x)$ converge to the function $f(x)$ as $\varepsilon\to 0$ uniformly on the segment $[a,b]$.*

We proceed to the case of trigonometrically convex functions.

Theorem 2.7. *Let $h(\theta)$ be a finite ρ–trigonometrically convex function on the interval (α,β), then $h_\varepsilon(\theta)$ is an infinitely differentiable ρ–trigonometrically convex function on the interval $(\alpha+\varepsilon,\beta-\varepsilon)$. If $[a,b]\subset(\alpha,\beta)$, then $h_\varepsilon(x)\rightrightarrows h(x)$ on the segment $[a,b]$ as $\varepsilon\to+0$.*

Proof. In view of Theorem 2.6, we just need to prove that the function $h_\varepsilon(\theta)$ is ρ–trigonometrically convex on the interval $(\alpha+\varepsilon,\beta-\varepsilon)$. We have

$$h_\varepsilon(\theta)=\int\limits_{-\infty}^{\infty}h(\theta-\tau)\omega_\varepsilon(\tau)d\tau.$$

We consider the function

$$F_1(\theta)=h'_\varepsilon(\theta)+\rho^2\int\limits_{\theta_0}^{\theta}h_\varepsilon(\varphi)d\varphi.$$

In view of the definition of function h_ε we find

$$\begin{aligned}F_1(\theta)&=\int\limits_{-\infty}^{\infty}(h'_+(\theta-\tau)+\rho^2\int\limits_{\theta_0}^{\theta}h(\varphi-\tau)d\varphi)\omega_\varepsilon(\tau)d\tau\\&=\int\limits_{-\infty}^{\infty}\left(h'_+(\theta-\tau)+\rho^2\int\limits_{\theta_0-\tau}^{\theta-\tau}h(s)ds\right)\omega_\varepsilon(\tau)d\tau.\end{aligned}$$

Since the trigonometric convexity of function h implies that the function

$$F(\theta)=h'_+(\theta-\tau)+\rho^2\int\limits_{\theta_0-\tau}^{\theta-\tau}h(s)ds$$

is increasing, the function $F_1(\theta)$ is also increasing. This yields, see, for instance, [3, Sect. 16], that the function $h_\varepsilon(\theta)$ is ρ–trigonometrically convex. The proof is complete. □

Remark 2.1. *Khabibullin showed* [7, Prop. 1.7] *that for each subspherical function h of order $\rho\geqslant 0$ in the space $\mathbb{R}^m$ there exists a sequence of n–times continuously differentiable subspherical functions of the same order, which monotonically decreases and tends to h. In particular, this implies that for each ρ–trigonometrically convex function h there exists a sequence of n–times continuously differentiable ρ–trigonometrically convex functions, which monotonically decreases and tends to h.*

The next theorem establishes a connection between ρ–trigonometrically convex functions and subharmonic functions.

Theorem 2.8. *Let $h(\theta)$ be a finite function on the interval (α,β). The function h is ρ–trigonometrically convex, if and only if the function $H(re^{i\theta})=r^\rho h(\theta)$ is a subharmonic function in the angle $A(\alpha,\beta)$.*

Remark 2.2. *If $\beta-\alpha>2\pi$, then the angle $A(\alpha,\beta)$ can be embedded into the plane and it is located on the Riemann surface of logarithm.*

Theorem 2.8 is well known for smooth ρ–trigonometrically convex functions. In the general case, we have not found its proof. For completeness of presentation, without claiming the authorship, we present it with our proof.

Proof. Necessity. Let h be ρ–trigonometrically convex function. Suppose additionally that h is an infinitely differentiable function. Then the function $H(z)=|z|^\rho h(\arg z)$ is also infinitely differentiable in the angle $A(\alpha,\beta)$. Since the Laplace operator in polar coordinates reads

$$\Delta=\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2},$$

we have

$$\Delta H(re^{i\theta})=r^{\rho-2}(h''(\theta)+\rho^2h(\theta)).$$

The inequality $\Delta H(z)\geqslant 0$ holds, see [3, Sect. 16]. By Theorem 2.4 the function H is subharmonic.

We proceed to the general case. Let h be an arbitrary ρ–trigonometrically convex function. Theorem 2.7 implies that there exists a sequence $h_n(\theta)$ of infinitely differentiable ρ–trigonometrically convex functions on the interval $\left(\alpha+\frac{1}{n},\beta-\frac{1}{n}\right)$, which uniformly converges to the function $h(\theta)$ on each segment $[a,b]\subset(\alpha,\beta)$. As it has been proved, $H_n(z)$ is subharmonic in the angle $A\left(\alpha+\frac{1}{n},\beta-\frac{1}{n}\right)$. Moreover, this sequence converges uniformly on each compact set lying in the angle $A(\alpha,\beta)$. By Theorem 2.5, the function $H(z)$ is subharmonic in the angle $A(\alpha,\beta)$.

Sufficiency. Suppose that the function $H(re^{i\theta})=r^\rho h(\theta)$ is subharmonic in the angle $A(\alpha,\beta)$. A subharmonic function is upper–semi–continuous. Therefore, the function $h(\theta)$ is also upper–semi–continuous. We also note that the inequality $h(\theta)<\infty$ holds. Let $[\alpha_1,\beta_1]$ be an arbitrary segment located in the interval (α,β). The function $h(\theta)$ is upper–semi–continuous on the compact $[\alpha_1,\beta_1]$ and does not take the value $+\infty$. By the Weierstrass theorem, the function $h(\theta)$ is bounded on the segment $[\alpha_1,\beta_1]$. Therefore, the number ρ is the formal order of the subharmonic function $H(z)$ in the angle $A(\alpha_1,\beta_1)$. By Theorem 2.3, the function $h(\theta)$ is ρ–trigonometrically convex on the interval (α_1,β_1), and, therefore, on the interval (α,β). The proof is complete. □

The established connection will be used to prove new inequalities characterizing ρ–trigonometrically convex functions and to find integral equations of the first kind for ρ–trigonometric functions.

3. Inequalities for ρ–trigonometrically convex functions

Theorem 3.1. *Let h be an upper–semi–continuous on the interval function, which is not identically equal to $-\infty$ and does not take the value $+\infty$. The function h is ρ–trigonometrically convex if and only if for each $\theta\in(\alpha,\beta)$ and each $R\in(0,1]$ such that $[\theta-\arcsin R,\theta+\arcsin R]\subset(\alpha,\beta)$ the inequality*

$$h(\theta)\leqslant\frac{1}{2\pi}\int\limits_0^{2\pi}(1+2R\cos(\varphi-\theta)+R^2)^{\frac{\rho}{2}}h(\operatorname{Arg}(e^{i\theta}+Re^{i\varphi}))d\varphi \tag{3.1}$$

holds, where $\operatorname{Arg}(e^{i\theta}+Re^{i\varphi})$ *is determined by the condition that it belongs to the aforementioned segment.*

Proof. Necessity. Let h be a ρ–trigonometrically convex function. Then by Theorem 2.8 the function $H(z)=|z|^\rho h(\operatorname{Arg} z)$ is subharmonic in the angle $A(\alpha,\beta)$. If R satisfies the assumptions of the theorem, then the circle $\{z:\ |z-e^{i\theta}|\leqslant R\}$ lies in the angle $A(\alpha,\beta)$. By the mean inequality for a subharmonic function (see the inequality (2.1)) we have

$$H(e^{i\theta})\leqslant\frac{1}{2\pi}\int\limits_0^{2\pi}H(e^{i\theta}+Re^{i\varphi})d\varphi.$$

This inequality coincides with (3.1).

Sufficiency. Now let h satisfy the conditions formulated in the first sentence of the theorem and the inequality (3.1). We consider the function

$$H(z)=|z|^\rho h(\operatorname{Arg} z)$$

in the angle $A(\alpha,\beta)$. This function, obviously, satisfies Conditions 1)–3) from Definition 2.1 of the subharmonic function. Let us verify the mean inequality for this function. Let $z_0\in A(\alpha,\beta)$, and $R_1>0$ be such that $\{z:|z-z_0|\leqslant R_1\}\subset A(\alpha,\beta)$. The mean inequality has the form

$$H(z_0)\leqslant\frac{1}{2\pi}\int\limits_0^{2\pi}H(z_0+R_1e^{i\varphi})d\varphi.$$

If $z_0=r_0e^{i\theta_0}$, it is also written in the form

$$r_0^\rho h(\theta_0)\leqslant\frac{1}{2\pi}\int\limits_0^{2\pi}|r_0e^{i\theta_0}+R_1e^{i\varphi}|^\rho h(\operatorname{Arg}(r_0e^{i\theta_0}+R_1e^{i\varphi}))d\varphi.$$

Dividing both sides by r_0^ρ, we obtain the equivalent inequality

$$h(\theta_0)\leqslant\frac{1}{2\pi}\int\limits_0^{2\pi}|e^{i\theta_0}+Re^{i\varphi}|^\rho h(\operatorname{Arg}(e^{i\theta_0}+Re^{i\varphi}))d\varphi,$$

where $R=\dfrac{R_1}{r_0}$. By our assumptions, this inequality is satisfied. Therefore, the function H is subharmonic. By Theorem 2.8 the function h is ρ–trigonometrically convex. The proof is complete. $\square$

Let us prove the following corollary of this theorem.

Theorem 3.2. *A continuous on the entire line function h is ρ–trigonometrically convex if and only if it satisfies the system of integral equations, $R\in(0,1]$:*

$$h(\theta)=\frac{1}{2\pi}\int\limits_0^{2\pi}(1+2R\cos(\varphi-\theta)+R^2)^{\frac{\rho}{2}}h(\operatorname{Arg}(e^{i\theta}+Re^{i\varphi}))d\varphi.\tag{3.2}$$

Proof. Necessity. Let h be a ρ–trigonometric function. Then the functions h and $-h$ are ρ–trigonometrically convex functions. For both of these functions, by Theorem 3.1, the inequality (3.1). This yields the identity (3.2).

Sufficiencty. Let h be a continuous on the entire axis function, for which the identity (3.2) holds for each $R \in (0, 1]$. By Theorem 3.1, each of the functions h and $-h$ is ρ–trigonometrically convex. Hence, h is ρ–trigonometric function. The proof is complete. □

We observe that the integral equation (3.2) is written in a form non–standard for the theory of integral equations since the unknown function h is involved in the integral with a rather complex variable

$$\psi = \operatorname{Arg}(e^{i\theta} + Re^{i\varphi}). \tag{3.3}$$

We can arrive at the standard form if, instead of the integration variable φ, we introduce the variable ψ related with φ by the identity (3.3). We only need to be careful since the function $\psi(\varphi)$ is not one–to–one on the segment $[0, 2\pi]$.

Lemma 3.1. *Let h be a measurable function on the segment $[\theta - \arcsin R, \theta + \arcsin R]$, where R is some number from the half-interval $(0, 1]$. Then, if one of the integrals written below exists, then the other also exists and the identity holds*

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi}(1 + 2R\cos(\varphi - \theta) + R^2)^{\frac{\rho}{2}} h(\arg(e^{i\theta} + Re^{i\varphi}))d\varphi \\ &= \frac{1}{2\pi}\int_{-\arcsin R}^{\arcsin R} \frac{(\cos u + \sqrt{R^2 - \sin^2 u})^{\rho+1} + (\cos u - \sqrt{R^2 - \sin^2 u})^{\rho+1}}{\sqrt{R^2 - \sin^2 u}} h(\theta - u)du.\end{aligned} \tag{3.4}$$

Proof. We use standard notation

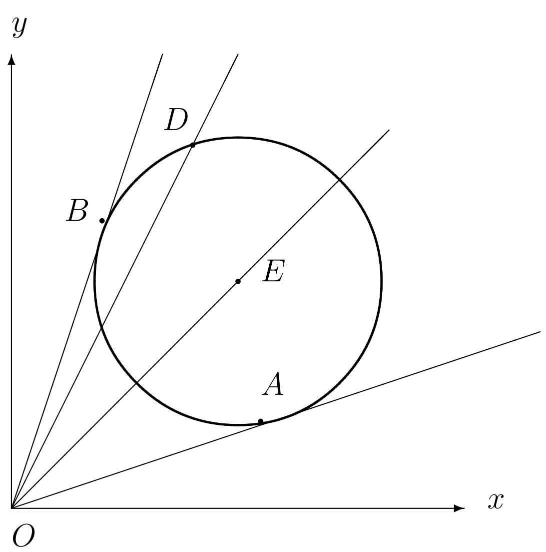
$$a_+ = \max(a, 0), \quad a_- = \max(-a, 0).$$

Here the statement "the integral $\int f(x)dx$ exists" means one of the following two

1) $f(x) \in L_1$ and the written integral is the Lebesgue integral,
2) only one of the functions $f_+(x)$ and $f_-(x)$ belongs to the space L_1 and the integral is equal to infinity with the appropriate sign. For instance, if $f_+ \in L_1$, then the integral is equal to $-\infty$.

Since the measurability of the function h is equivalent to the measurability of two functions h_+ and h_-, it is sufficient to prove the lemma only for positive functions. For a better clarity of the subsequent proof we use the following figure.

The figure demonstrates the plane of $w = e^{i\theta} + z$.



The point E with the affix $e^{i\theta}$ is the center of a circle of radius $R \in (0,1]$, O is the origin, OA and OB are tangent to the circle, the line OD is an arbitrary line passing through O and some point D of the circle.

We have $\angle AOE = \angle EOB = \arcsin R$. This is why for a fixed R the quantity

$$\psi = \operatorname{Arg}(e^{i\theta} + Re^{i\varphi}) \tag{3.5}$$

ranges over the segment $[\theta - \arcsin R, \theta + \arcsin R]$ as φ ranges over the segment $[0, 2\pi]$.

Let

$$A = e^{i\theta} + Re^{i\gamma_1}, \quad B = e^{i\theta} + Re^{i\gamma_2}.$$

We choose the quantities γ_1 and γ_2 to satisfy the inequalities $0 < \gamma_2 - \gamma_1 < 2\pi$. The solution to Equation (3.5) with respect to the variable φ in the interval (γ_1, γ_2) is denoted by $\varphi_2 = \varphi_2(\psi)$, while the solution of the same equation in the interval $(\gamma_2, \gamma_1 + 2\pi)$ is denoted by $\varphi_1 = \varphi_1(\psi)$.

Since the integrand in the integral in the left hand side of the identity (3.4) has a period of 2π, we can write

$$\begin{aligned} I =& \frac{1}{2\pi}\int_0^{2\pi} (1 + 2R\cos(\varphi - \theta) + R^2)^{\frac{\rho}{2}} h(\operatorname{Arg}(e^{i\theta} + Re^{i\varphi}))d\varphi \\ =& \frac{1}{2\pi}\int_{\gamma_1}^{\gamma_2} (1 + 2R\cos(\varphi - \theta) + R^2)^{\frac{\rho}{2}} h(\operatorname{Arg}(e^{i\theta} + Re^{i\varphi}))d\varphi \\ &+ \frac{1}{2\pi}\int_{\gamma_2}^{\gamma_1+2\pi} (1 + 2R\cos(\varphi - \theta) + R^2)^{\frac{\rho}{2}} h(\operatorname{Arg}(e^{i\theta} + Re^{i\varphi}))d\varphi. \end{aligned} \tag{3.6}$$

In each of the integrals in the right hand side of the identity (3.6), we introduce a new integration variable according to formula (3.5). We have

$$\begin{aligned} &e^{i\theta} + Re^{i\varphi} = |e^{i\theta} + Re^{i\varphi}|e^{i\psi}, \\ &(e^{i\theta} + Re^{i\varphi})^2 = (e^{i\theta} + Re^{i\varphi})(e^{-i\theta} + Re^{-i\varphi})e^{2i\psi}, \\ &e^{2i\psi} = \frac{e^{i\theta} + Re^{i\varphi}}{e^{-i\theta} + Re^{-i\varphi}}. \end{aligned} \tag{3.7}$$

Differentiating the identity (3.7), we obtain

$$2ie^{2i\psi}d\psi = \frac{iRe^{i\varphi}(e^{-i\theta} + Re^{-i\varphi}) + iRe^{-i\varphi}(e^{i\theta} + Re^{i\varphi})}{(e^{-i\theta} + Re^{-i\varphi})^2}d\varphi.$$

This gives

$$e^{2i\psi}d\psi = \frac{R(R + \cos(\varphi - \theta))}{(e^{-i\theta} + Re^{-i\varphi})^2}d\varphi.$$

Substituting the quantity $e^{2i\psi}$ from the formula (3.7) into the above formula, we find

$$d\varphi = \frac{1 + 2R\cos(\varphi - \theta) + R^2}{R(R + \cos(\varphi - \theta))}d\psi. \tag{3.8}$$

After the introduction of the new variable ψ the identity (3.6) becomes

$$\begin{aligned} I =& \frac{1}{2\pi} \int\limits_{\theta-\arcsin R}^{\theta+\arcsin R} \left[\frac{(1+2R\cos(\varphi_2-\theta)+R^2)^{\frac{\rho}{2}+1}}{R(R+\cos(\varphi_2-\theta))} \right. \\ & \left. - \frac{(1+2R\cos(\varphi_1-\theta)+R^2)^{\frac{\rho}{2}+1}}{R(R+\cos(\varphi_1-\theta))} \right] h(\psi)d\psi. \end{aligned} \tag{3.9}$$

Now our aim is to simplify the integrand in the formula (3.9). The identity (3.7) implies

$$Re^{2i(\varphi-\theta)} - (e^{2i(\psi-\theta)}-1)e^{i(\varphi-\theta)} - Re^{2i(\psi-\theta)} = 0.$$

This yields

$$\begin{aligned} e^{i(\varphi-\theta)} =& \frac{e^{2i(\psi-\theta)} - 1 \pm \sqrt{(e^{2i(\psi-\theta)}-1)^2 + 4R^2e^{2i(\psi-\theta)}}}{2R} \\ =& \frac{e^{2i(\psi-\theta)} - 1 \pm 2e^{i(\psi-\theta)}\sqrt{R^2-\sin^2(\psi-\theta)}}{2R}. \end{aligned} \tag{3.10}$$

We observe that the quantity $R^2-\sin^2(\psi-\theta)$ does not vanish on the interval $[\theta-\arcsin R, \theta+\arcsin R]$. The choice of the sign $+$ or $-$ in the right hand side of the formula (3.10) provides the quantities $e^{i(\varphi_1-\theta)}$ and $e^{i(\varphi_2-\theta)}$ in the left hand side of this formula. Continuity reasoning shows that each of the quantities $e^{i(\varphi_1-\theta)}$, $e^{i(\varphi_2-\theta)}$ on the entire interval $(\theta-\arcsin R, \theta+\arcsin R)$ corresponds to the same sign $+$ or $-$. If $\psi=0$, then $e^{i(\varphi-\theta)} = \pm 1$. For the value $|e^{i\theta}+RE^{i\varphi}|$ we obtain the values $1+R$ and $1-R$. Since the point D is located further from zero than the point C, we find

$$e^{i(\varphi_2-\theta)} = e^{i(\psi-\theta)}\frac{i\sin(\psi-\theta)+\sqrt{R^2-\sin^2(\psi-\theta)}}{R}, \tag{3.11}$$

$$e^{i(\varphi_1-\theta)} = e^{i(\psi-\theta)}\frac{i\sin(\psi-\theta)-\sqrt{R^2-\sin^2(\psi-\theta)}}{R}. \tag{3.12}$$

It follows from the formulas (3.11), (3.12) that

$$\begin{aligned} &R\cos(\varphi_2-\theta) = -\sin^2(\psi-\theta)+\cos(\psi-\theta)\sqrt{R^2-\sin^2(\psi-\theta)}, \\ &R\cos(\varphi_1-\theta) = -\sin^2(\psi-\theta)-\cos(\psi-\theta)\sqrt{R^2-\sin^2(\psi-\theta)}, \\ &1+2R\cos(\varphi_2-\theta)+R^2 = \left(\cos(\psi-\theta)+\sqrt{R^2-\sin^2(\psi-\theta)}\right)^2, \\ &1+2R\cos(\varphi_1-\theta)+R^2 = \left(\cos(\psi-\theta)-\sqrt{R^2-\sin^2(\psi-\theta)}\right)^2, \\ &R^2+R\cos(\varphi_2-\theta) = \sqrt{R^2-\sin^2(\psi-\theta)}\left(\cos(\psi-\theta)+\sqrt{R^2-\sin^2(\psi-\theta)}\right), \\ &R^2+R\cos(\varphi_1-\theta) = -\sqrt{R^2-\sin^2(\psi-\theta)}\left(\cos(\psi-\theta)-\sqrt{R^2-\sin^2(\psi-\theta)}\right). \end{aligned}$$

We note that for $\psi\in(\theta-\arcsin R, \theta+\arcsin R)$ the inequality

$$\cos(\psi-\theta)-\sqrt{R^2-\sin^2(\psi-\theta)} > 0$$

holds. Substituting the obtained quantities into (3.9), we finally obtain

$$I = \frac{1}{2\pi}\int\limits_{\theta-\arcsin R}^{\theta+\arcsin R}\left[\left(\cos(\psi-\theta)+\sqrt{R^2-\sin^2(\psi-\theta)}\right)^{\rho+1}\right.$$
$$\left.+\left(\cos(\psi-\theta)-\sqrt{R^2-\sin^2(\psi-\theta)}\right)^{\rho+1}\right]\frac{h(\psi)}{\sqrt{R^2-\sin^2(\psi-\theta)}}d\psi$$
$$=\frac{1}{2\pi}\int\limits_{-\arcsin R}^{\arcsin R}\frac{(\cos u+\sqrt{R^2-\sin^2 u})^{\rho+1}+(\cos u-\sqrt{R^2-\sin^2 u})^{\rho+1}}{\sqrt{R^2-\sin^2 u}}h(\theta-u)du.$$

The application of the theorem on the change of variable gives the identity (3.4). The proof is complete. $\square$

We denote

$$a_R(u)=\begin{cases}\dfrac{1}{2\pi}\dfrac{(\cos u+\sqrt{R^2-\sin^2 u})^{\rho+1}+(\cos u-\sqrt{R^2-\sin^2 u})^{\rho+1}}{\sqrt{R^2-\sin^2 u}}, & \text{if } |u|<\arcsin R;\\ 0, & \text{if } |u|\geqslant \arcsin R.\end{cases}$$

By means of Lemma 3.1, Theorems 3.1 and 3.2 can be reformulated as follows.

Theorem 3.3. *Let h be an upper–semi–continuous function on the interval (α,β) not identically equaling to $-\infty$ and not taking the value $+\infty$. Then h is ρ–trigonometrically convex if and only if for each $\theta\in(\alpha,\beta)$ and for each $R\in(0,1]$ such that $[\theta-\arcsin R,\theta+\arcsin R]\subset(\alpha,\beta)$, the inequality holds*

$$h(\theta)\leqslant\frac{1}{2\pi}\int\limits_{-\arcsin R}^{\arcsin R}a_R(u)h(\theta-u)du.$$

Theorem 3.4. *A continuous on the entire axis function h ρ–trigonometrically convex if and only if it satisfies the system of integral equations, $R\in(0,1]$:*

$$h(\theta)=\int\limits_{-\infty}^{\infty}a_R(u)h(\theta-u)du. \tag{3.13}$$

In what follows we shall see that the versions of Theorems 3.3 and 3.4 with the kernel $a_R(u)$ replaced by another kernel are valid. For $R\in(0,1]$ we denote

$$b_R(u)=\begin{cases}\dfrac{(\cos u+\sqrt{R^2-\sin^2 u})^{\rho+2}-(\cos u-\sqrt{R^2-\sin^2 u})^{\rho+2}}{\pi(\rho+2)R^2}, & \text{if } |u|\leqslant\arcsin R;\\ 0, & \text{if } |u|>\arcsin R.\end{cases}$$

Theorem 3.5. *Let h be an upper–semi–continuous function on the interval (α,β) not identically equaling to $-\infty$ and not taking the value $+\infty$. Then h is ρ–trigonometrically convex if and only if for each $\theta\in(\alpha,\beta)$ and each $R\in(0,1]$ such that $[\theta-\arcsin R,\theta+\arcsin R]\subset(\alpha,\beta)$ the inequality holds*

$$h(\theta)\leqslant\int\limits_{-\arcsin R}^{\arcsin R}b_R(u)h(\theta-u)du. \tag{3.14}$$

Proof. Necessity. Let h be a ρ–trigonometrically convex function, and let R be from the formulation of the theorem. By Theorem 3.3, for each $r \in (0, R]$ the inequality

$$h(\theta) \leqslant \frac{1}{2\pi} \int\limits_{-\arcsin r}^{\arcsin r} \frac{(\cos u + \sqrt{r^2 - \sin^2 u})^{\rho+1} + (\cos u - \sqrt{r^2 - \sin^2 u})^{\rho+1}}{\sqrt{r^2 - \sin^2 u}} h(\theta - u)du$$

holds. We multiply both sides of the inequality by r and integrate in r over the segment $[0, R]$. Taking into consideration the identity

$$\int\limits_0^R r dr = \frac{1}{2} R^2$$

and interchanging the integration order in the right hand side, we obtain

$$\begin{aligned}
h(\theta) \leqslant & \frac{1}{\pi R^2} \int\limits_0^{\arcsin R} \int\limits_{\sin u}^{R} r \Bigg(\frac{(\cos u + \sqrt{r^2 - \sin^2 u})^{\rho+1}}{\sqrt{r^2 - \sin^2 u}} \\
& + \frac{(\cos u - \sqrt{r^2 - \sin^2 u})^{\rho+1}}{\sqrt{r^2 - \sin^2 u}} \Bigg) dr h(\theta - u)du \\
& + \frac{1}{\pi R^2} \int\limits_{-\arcsin R}^{0} \int\limits_{-\sin u}^{R} r \Bigg(\frac{(\cos u + \sqrt{r^2 - \sin^2 u})^{\rho+1}}{\sqrt{r^2 - \sin^2 u}} \\
& + \frac{(\cos u - \sqrt{r^2 - \sin^2 u})^{\rho+1}}{\sqrt{r^2 - \sin^2 u}} \Bigg) dr h(\theta - u)du \\
= & \frac{1}{\pi(\rho+2)R^2} \int\limits_0^{\arcsin R} \Big((\cos u + \sqrt{r^2 - \sin^2 u})^{\rho+2} \\
& - (\cos u - \sqrt{r^2 - \sin^2 u})^{\rho+2} \Big) \Big|_{\sin u}^{R} h(\theta - u)du \\
& + \frac{1}{\pi(\rho+2)R^2} \int\limits_{-\arcsin R}^{0} \Big((\cos u + \sqrt{r^2 - \sin^2 u})^{\rho+2} \\
& - (\cos u - \sqrt{r^2 - \sin^2 u})^{\rho+2} \Big) \Big|_{-\sin u}^{R} h(\theta - u)du \\
= & \int\limits_{-\arcsin R}^{\arcsin R} b_R(u) h(\theta - u)du.
\end{aligned}$$

Sufficiency. Let a function h possess the properties counted in the first sentence of the theorem and also satisfy the inequality (3.14). We denote $H(z) = |z|^\rho h(\arg z)$. Then the function H satisfies the Condition 1)–3) in Definition 2.1 of the subharmonic function. As it follows from the remark to Definition 2.1, in order to prove the subharmonicity of the function $H(z)$, it remains to verify the inequality

$$H(z_0) \leqslant \frac{1}{\pi R_1^2} \int\limits_0^{R_1} \int\limits_0^{2\pi} H(z_0 + r_1 e^{i\varphi}) r_1 dr_1 d\varphi \tag{3.15}$$

for all z_0 and R_1 such that the disk $\{z : |z-z_0| \leqslant R_1\}$ lies in the angle $A(\alpha,\beta)$. Let $z_0 = r_0e^{i\theta_0}$. In view of the definition of function H the inequality (3.15) is rewritten as

$$r_0^\rho h(\theta_0) \leqslant \frac{1}{\pi R_1^2}\int\limits_0^{R_1}\int\limits_0^{2\pi} |r_0e^{i\theta_0}+r_1e^{i\varphi}|^\rho h(\arg(r_0e^{i\theta_0}+r_1e^{i\varphi}))r_1dr_1d\varphi.$$

We denote $R = \dfrac{R_1}{r_0}$. We replace the variable r_1 in the integral by a new variable $r = \dfrac{r_1}{r_0}$ and we arrive at the equivalent inequality

$$h(\theta_0) \leqslant \frac{1}{\pi R^2}\int\limits_0^{R}\int\limits_0^{2\pi} |e^{i\theta_0}+re^{i\varphi}|^\rho h(\arg(e^{i\theta_0}+re^{i\varphi}))rdrd\varphi. \tag{3.16}$$

It follows from Lemma 3.1 that this inequality is equivalent to

$$h(\theta_0) \leqslant \frac{2}{R^2}\int\limits_0^{R}\int\limits_{-\arcsin r}^{\arcsin r} a_r(u)h(\theta-u)rdudr.$$

Now we reproduce the arguing from the first part of the proof and we see that the written inequality is equivalent to the inequality (3.14). This proves the subharmonicity of the function H. Let $[\alpha_1,\beta_1]$ be an arbitrary segment such that $[\alpha_1,\beta_1]\subset(\alpha,\beta)$. It follows from the semi-continuity of the function h that this function is bounded from above on the segment $[\alpha_1,\beta_1]$. This implies that the number ρ is the formal order of the function H in the angle $A(\alpha_1,\beta_1)$. Theorem 2.3 yields that the function h is ρ–trigonometrically convex on the interval (α_1,β_1) and, consequently, on the interval (α,β). The proof is complete. □

Theorem 3.6. *A continuous on the entire function h is ρ–trigonometric if and only if it satisfies the system of integral equations, $R\in(0,1]$:*

$$h(\theta) = \int\limits_{-\infty}^{\infty} b_R(u)h(\theta-u)du. \tag{3.17}$$

The proof of this theorem reproduces literally the proof of Theorem 3.2, just the reference to Theorem 3.1 should be replaced by a reference to Theorem 3.5.

The identities (3.13), (3.17) allow us to calculate some integrals. Let us dwell on this in more detail. We take $h(\theta)=\cos\rho\theta$ in these identities and we obtain

$$\cos\rho\theta = \frac{1}{2\pi}\int\limits_{-\arcsin R}^{\arcsin R} \frac{(\cos u+\sqrt{R^2-\sin^2 u})^{\rho+1}+(\cos u-\sqrt{R^2-\sin^2 u})^{\rho+1}}{\sqrt{R^2-\sin^2 u}}\cos\rho(\theta-u)du,$$

$$\begin{aligned}\cos\rho\theta =& \frac{1}{\pi(\rho+2)R^2}\int\limits_{-\arcsin R}^{\arcsin R}\Big[(\cos u+\sqrt{R^2-\sin^2 u})^{\rho+2}\\ &-(\cos u-\sqrt{R^2-\sin^2 u})^{\rho+2}\Big]\cos\rho(\theta-u)du.\end{aligned}$$

By means of the identity

$$\cos\rho(\theta-u)=\cos\rho\theta\cos\rho u+\sin\rho\theta\sin\rho u$$

we find

$$\frac{1}{\pi}\int\limits_0^{\arcsin R}\frac{(\cos u+\sqrt{R^2-\sin^2 u})^{\rho+1}+(\cos u-\sqrt{R^2-\sin^2 u})^{\rho+1}}{\sqrt{R^2-\sin^2 u}}\cos\rho u du=1,$$

$$\frac{2}{\pi(\rho+2)R^2}\int\limits_0^{\arcsin R}\left[(\cos u+\sqrt{R^2-\sin^2 u})^{\rho+2}-(\cos u-\sqrt{R^2-\sin^2 u})^{\rho+2}\right]\cos\rho u du=1.$$

We also note that we have proved the formulas under the assumption that $\rho>0$. However, the uniqueness theorem for analytic functions yields that these formulas are valid for all complex ρ. In this case, of course, the last integral for $\rho=-2$ must be understood in an appropriate way.

4. Application of Leontiev interpolation function

Theorems 3.4, 3.6 state that every ρ–trigonometric function satisfies the integral equations (3.13), (3.17) for each $R\in(0,1]$. The converse statement is also true. If a continuous function h satisfies the integral equation (3.13) or (3.17) for each $R\in(0,1]$, then h is a ρ–trigonometric function.

Let a continuous on the entire axis function h satisfy the integral equation (3.13) or (3.17) for some $R\in(0,1]$. Does this implies that h is a ρ–trigonometric function? As we shall see, the answer is negative. Therefore, the following question naturally arises: what additional conditions on the function h guarantee that this function is ρ–trigonometric? We present two types of such conditions.

Equations (3.13), (3.17) are representatives of convolution equations. Therefore, we present, partly without proof, several results related to the theory of such equations. First, we note that if the function h is a continuous solution of the equation

$$\int\limits_{-\infty}^{\infty}h(\theta-u)d\sigma(u)=h(\theta),\tag{4.1}$$

then h solves the equation

$$\int\limits_{-\infty}^{\infty}h(\theta-u)d\sigma_1(u)=0,\tag{4.2}$$

where $\sigma_1=\sigma-\delta$, δ is the Dirac measure. This motivates us to restrict ourselves by the study of equation

$$\int\limits_{-q}^{q}F(\theta-u)d\sigma(u)=0,\tag{4.3}$$

where σ is a finite (sign–changing) measure, the support of which contains the points $-q$ and q. The fact that we consider the symmetric segment $[-q,q]$ is an essential restriction. The case, when the measure σ is not finite, also attracts many studies. The results are applied, in particular, for studying the Wiener — Hopf equation. But this is not related with the subject of our work. This is why we restrict ourselves by citing the works [1], [2].

The first basic results for Equation (4.3) were obtained by Schwartz [12]. On these results one can be read in the comments to Section 40 in the book [9]. The authors of this book also refer to the works by Kahane [10], [11]. We shall present the needed results on the theory of

Equation (4.3) following Leontiev [4], [5], who studied Equation (4.3) in connection with the theory of Dirichlet series. We note that in Theorem 6.4.1 in the book [4], the main result of Schwartz is presented. In connection with Equation (4.3), we consider the function

$$L(\lambda)=\int_{-q}^{q} e^{-i\lambda t}d\sigma(t),$$

which is called the characteristic function of Equation (4.3). This notion is justified by the fact that if λ_k is a root of the function $L(\lambda)$ of multiplicity n_k, then, as is easy to verify, the function

$$h_k(\theta)=P_k(\theta)e^{i\lambda_k\theta}, \tag{4.4}$$

where P_k is an arbitrary polynomial of degree at most n_k-1, solves Equation (4.3). Solutions of the form $P(\theta)e^{i\lambda\theta}$ are called primitive solutions of Equation (4.3). It is easy to see that the identity (4.4) defines the general form of primitive solutions. L. Schwartz proved that the set of all continuous solutions of Equation (4.3) coincides with the closure in the topology of uniform convergence on compact sets of the linear hull of elementary solutions. It is known [4, Ch. 1, Sect. 4, Subsect. 3] that the function $L(\lambda)$ has infinitely many roots. Thus, the linear space of continuous solutions of Equation (4.3), and, consequently, taking into account (4.1) and (4.2), the space of solutions of Equation (3.13) is infinite–dimensional. This proves the previously announced statement that among the solutions of Equation (3.13) for each $R\in(0,1]$ there are functions that cannot be represented as $A\cos\rho\theta+B\sin\rho\theta$, that is, non ρ–trigonometric functions.

In connection with Equation (4.3), A.F. Leontiev, for an arbitrary continuous function F, constructs the function

$$\omega(\mu,\alpha,F)=-ie^{-i\alpha\mu}\int_{-q}^{q}\int_{0}^{t} F(\xi+\alpha-t)e^{-i\mu\xi}d\xi d\sigma(t),$$

which he calls interpolating. If λ_k is a root of multiplicity n_k of the function $L(\lambda)$, then for each function F the identity

$$P_k(z)e^{i\lambda_k z}=\frac{1}{2\pi i}\int_{C_k}\frac{\omega(\mu,\alpha,F)}{L(\mu)}e^{-i\mu z}d\mu \tag{4.5}$$

holds, where $P_k(z)$ is some polynomial of degree no greater than n_k-1, C_k is a circumference with center at point λ_k such that the closed disk bounded by this circumference contains no other roots of the function $L(\lambda)$. The quantity $P_k(z)$ is called the Fourier coefficient of function F corresponding to the root λ_k of the characteristic equation. If

$$F(t)=\sum_{k=1}^{n} Q_k(t)e^{i\lambda_k t},\quad \deg Q_k\leqslant n_k-1,$$

then the Fourier coefficient $P_k(z)$ of the function F is equal to $Q_k(z)$. This justifies the term "interpolating function" for the function $\omega(\mu,\alpha,F)$. Below we provide two facts from the books of A.F. Leontiev, to which we shall refer in the future. The first of them in the book [4] is formulated as Theorem 4.3.3.

Theorem 4.1. *Let $F(t)$ be a continuous function on the axis $(-\infty,\infty)$. If all Fourier coefficients of the function F are equal to zero, then the function F is identically zero.*

The next statement is as follows. The function $\omega(\mu,\alpha,F)$ depends on an arbitrary real parameter α. However, the Fourier coefficient for solutions of Equation (4.3) is independent on α. Indeed,

$$\omega(\mu,\alpha,F)=-i\int\limits_{-q}^{q}\int\limits_{\alpha-t}^{\alpha}F(\eta)e^{-i\mu(\eta+t)}d\eta d\sigma(t),$$

$$\frac{\partial\omega(\mu,\alpha,F)}{\partial\alpha}=-i\int\limits_{-q}^{q}F(\alpha)e^{-i\mu(\alpha+t)}d\sigma(t)-e^{-i\mu\alpha}\int\limits_{-q}^{q}F(\alpha-t)d\sigma(t)=-ie^{-i\mu\alpha}F(\alpha)L(\mu),$$

since the latter of written integrals vanishes. If now we substitute $\frac{\partial\omega(\mu,\alpha,F)}{\partial\alpha}$ instead of $\omega(\mu,\alpha,F)$ into the integral (4.5), we obtain zero. This implies that $P_k(z)$ is independent of α.

In addition to the above, we prove the statement that can be called the theorem on the vanishing of Fourier coefficient.

Theorem 4.2. *Let F be a continuous on the entire axis solution of Equation (4.3), $L(\lambda)$ be the characteristic function of this equation, λ_k be its root. Let for some real τ the inequality*

$$|F(t)|\leqslant M(\tau)e^{\tau|t|},\quad t\in(-\infty,0)\quad(t\in(0,\infty)),$$

holds. If, in addition, $\operatorname{Im}\lambda_k>\tau$ ($\operatorname{Im}\lambda_k<-\tau$), then the Fourier coefficient $P_k(z)$ of the function F vanishes.

Proof. We have

$$\begin{aligned}P_k(z)e^{i\lambda_k z}=&\frac{1}{2\pi i}\int\limits_{C_k}\frac{\omega(\mu,\alpha,F)}{L(\mu)}e^{-i\mu z}d\mu\\=&-i\int\limits_{-q}^{q}\int\limits_{0}^{t}\frac{1}{2\pi i}\int\limits_{C_k}\frac{e^{i\mu(z-\xi-\alpha)}}{L(\mu)}d\mu F(\xi+\alpha-t)d\xi d\sigma(t).\end{aligned}\tag{4.6}$$

Applying the formula for calculating the residue at the pole of multiplicity n_k, we obtain

$$I=\frac{1}{2\pi i}\int\limits_{C_k}\frac{e^{-i\mu(z-\xi-\alpha)}}{L(\mu)}d\mu=\frac{1}{(n_k-1)!}\frac{d^{n_k-1}}{d\mu^{n_k-1}}\frac{(\mu-\lambda_k)^{n_k}e^{i\mu(z-\xi-\alpha)}}{L(\mu)}\bigg|_{\mu=\lambda_k}.$$

Let

$$\frac{(\mu-\lambda_k)^{n_k}}{L(\mu)}=\sum_{m=0}^{\infty}a_m(\mu-\lambda_k)^m$$

be the expansion in a Taylor series in the vicinity of the point λ_k of the function in the left hand side of the identity. Then

$$\begin{aligned}I=&\frac{1}{(n_k-1)!}\sum_{m=0}^{n_k-1}m!C^m_{n_k-1}a_m(i(z-\xi-\alpha))^{n_k-1-m}e^{i\lambda_k(z-\xi-\alpha)}\\=&Q_k(z-\xi-\alpha)e^{i\lambda_k(z-\xi-\alpha)},\end{aligned}$$

where Q_k is some polynomial of degree n_k-1. Substituting the found value I into the formula (4.6), we obtain

$$P_k(z)=-i\int\limits_{-q}^{q}\int\limits_{0}^{t}Q_k(z-\xi-\alpha)F(\xi+\alpha-t)e^{-i\lambda_k(\xi+\alpha)}d\xi d\sigma(t).$$

Making the change $\xi+\alpha-t=\eta$ in the inner integral, we find

$$P_k(z)=-i\int\limits_{-q}^{q}\int\limits_{\alpha-t}^{\alpha}Q_k(z-\eta-t)F(\eta)e^{-i\lambda_k\eta}d\eta e^{-i\lambda_k t}d\sigma(t). \tag{4.7}$$

Let $\lambda_k=\alpha_k+i\beta_k$, $\widetilde{Q}_k$ be the polynomial, the coefficients of which are equal to the absolute value of the coefficients of polynomial Q_k. Then the inequality holds

$$|P_k(z)|\leqslant M(\sigma)e^{q\beta_k}\widetilde{Q}_k(|z|+|\alpha|+2q)\int\limits_{-q}^{q}\left|\int\limits_{\alpha-t}^{\alpha}e^{\beta_k\eta+\tau|\eta|}d\eta\right|d|\sigma|(t).$$

In the first version of the theorem, when $t\in(-\infty,0)$, we need to pass to the limit at $\alpha\to-\infty$. In the second version of the theorem, when $t\in(0,\infty)$, we need to pass to the limit at $\alpha\to+\infty$. In each case, we get $P_k(z)=0$. The proof is complete. □

As a simple exercise, the proven theorem implies the next statement.

Theorem 4.3. *Let $F(\theta)$ be a continuous solution of Equation (4.3) and let along one of the directions $\theta\to+\infty$ or $\theta\to-\infty$ we have*

$$\lim F(\theta)e^{\sigma\theta}=0$$

for each $\sigma\in(-\infty,\infty)$. Then $F(\theta)\equiv 0$.

Proof. By Theorem 4.2 all Fourier coefficients of the function F are zero. By Theorem 4.1 the function F is zero. The proof is complete. □

Below we propose an additional condition that ensures that the continuous solution of equation (3.13) or (3.17) for a fixed R is a ρ–trigonometric function.

Theorem 4.4. *Let $F(\theta)$ be a continuous solution on the entire axis of one of Equations (3.13) or (3.17) for some fixed $R\in(0,1]$. Let there exist real numbers A and B such that for one of the directions $\theta\to+\infty$ or $\theta\to-\infty$ the relation*

$$\lim(F(\theta)-A\cos\rho\theta-B\sin\rho\theta)e^{\tau\theta}=0$$

holds for each real τ. Then

$$F(\theta)=A\cos\rho\theta-B\sin\rho\theta.$$

Proof. We consider the function

$$F_1(\theta)=F(\theta)-A\cos\rho\theta-B\sin\rho\theta.$$

Then the function $F_1(\theta)$ solves the same equation as the function $F(\theta)$. The relations (4.1), (4.2) yield that the function $F_1(\theta)$ is a solution to Equation (4.3) with some measure σ. By Theorem 4.3, $F_1(\theta)\equiv 0$. The proof is complete. □

In the next theorem we give another additional condition that guarantees that a continuous solution of Equation (3.13) or (3.17) for a fixed R is a ρ–trigonometric function.

Theorem 4.5. *Let $F(\theta)$ be a continuous solution on the entire axis of one of Equations (3.13) or (3.17) with some fixed $R\in(0,1]$. If the function $F(\theta)$ is ρ–trigonometric on some segment of length $2\arcsin R$, then it is ρ–trigonometric everywhere.*

Proof. We again consider the function

$$F_1(\theta) = F(\theta) - A\cos\rho\theta - B\sin\rho\theta,$$

write the equation of form (4.3) for this function and apply the formula (4.7), taking as α the center of the segment, on which the function $F_1(\theta)$ vanishes. We then obtain that all Fourier coefficients of the function F_1 vanish. Then $F_1 = 0$. The proof is complete. □

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