

INTEGRATION OF LOADED NONLINEAR SCHRÖDINGER EQUATION IN CLASS OF FAST DECAYING FUNCTIONS

G.U. URAZBOEV, I.I. BALTAEVA, I.D. RAKHIMOV

Abstract. We show that the inverse scattering transform technique can be applied to obtain the time dependence of scattering data of the Zakharov — Shabat system, which is described by the loaded nonlinear Schrödinger equation in the class of fast decaying functions. In addition we prove that the Cauchy problem for the loaded nonlinear Schrödinger equation is uniquely solvable in the class of rapidly decreasing functions. We provide the explicit expression of a single soliton solution for the loaded nonlinear Schrödinger equation. As an example, we find the soliton solution of the considered problem for an arbitrary non-zero continuous function $\gamma(t)$.

Keywords: Schrödinger equation, Jost solution, loaded equation, evolution of scattering data, inverse scattering transform.

Mathematics Subject Classification: 34L25, 35Q41, 35R30, 34M46

1. INTRODUCTION

Integration nonlinear differential equations with soliton solutions is of a permanent interest since many physical processes are described by nonlinear evolutionary differential equations. Initially, in 1967, Green, Kruskal, and Miura integrated the Cauchy problem for the Korteweg — de Vries equation using the method of the inverse scattering problem in the class of fast decaying functions [1]. In 1968, Lax [2] showed that the method of the inverse scattering problem was universal. Using this idea, in 1972, Zakharov and Shabat [3] showed the complete integrability of the nonlinear Schrödinger equation

$$iu_t \pm 2u^2\bar{u} + u_{xx} = 0.$$

Apart of this method, there are several methods for finding solutions to nonlinear differential equations: Hirota direct method [4], [5], [6], the binary Darboux transformations [7], (G'/G) -expansion method [8], [9], [10]. The term “loaded equation” was introduced by Nakhushev [11]. After this, the interest in loaded differential equations increased [12], [13]. Many mathematicians studied loaded nonlinear differential equations in the different functional classes [14], [15], [16], [17], [18].

We consider the Cauchy problem for a loaded nonlinear Schrödinger equation

$$iu_t + 2|u|^2u + u_{xx} - i\gamma(t)(u(0, t) + u^*(0, t))u_x = 0, \quad (1.1)$$

$$u(x, 0) = u_0(x), \quad (1.2)$$

where $\gamma(t)$ is bounded continuous non-zero real function, and the real function $u_0(x)$ possesses the following properties:

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1. The inequality

$$\int_{-\infty}^{\infty} (1 + |x|) |u_0(x)| dx < \infty$$

holds.

2. The equation

$$L(0)y \equiv i \begin{pmatrix} \frac{d}{dx} & -u_0(x) \\ -u_0^*(x) & -\frac{d}{dx} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \xi \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad x \in \mathbb{R},$$

can have N number simple eigenvalues and has no spectral singularities. Here, the function $u_0^*(x)$ is a complex conjugation of $u_0(x)$.

In this work, we are consider the solutions of the above Cauchy problem in the class of functions defined by the condition

$$\int_{-\infty}^{\infty} \left((1 + |x|) |u(x)| + \sum_{j=1}^3 \left| \frac{\partial^j u(x, t)}{\partial x^j} \right| \right) dx < \infty, \quad t \geq 0, \quad (1.3)$$

via the inverse scattering problem. Our approach allows us to find the solution of the above Cauchy problem for an arbitrary non-zero continuous real function $\gamma(t)$.

2. UNIQUENESS

Theorem 2.1. *The problem (1.1)–(1.3) can have at most one solution.*

Proof. Let $u = u(x, t)$ and $v = v(x, t)$ be two solutions of (1.1)–(1.3). We denote $\omega(x, t) = (u(x, t) - v(x, t))_x$ and obtain

$$\begin{aligned} i \frac{d}{dt} |\omega|^2 + 4 \left((u_x |u|^2 - v_x |v|^2) \bar{\omega} - (\bar{u}_x |u|^2 - \bar{v}_x |v|^2) \omega \right) + 2 \left((u^2 \bar{u}_x - v^2 \bar{v}_x) \bar{\omega} - (\bar{u}^2 u_x - \bar{v}^2 v_x) \omega \right) \\ - i \gamma(t) \left((u(0, t) + \bar{u}(0, t)) \bar{\omega} u_{xx} + (u(0, t) + \bar{u}(0, t)) \omega \bar{u}_{xx} \right. \\ \left. - (v(0, t) + \bar{v}(0, t)) \bar{\omega} v_{xx} - (v(0, t) + \bar{v}(0, t)) \omega \bar{v}_{xx} \right) = 0 \end{aligned}$$

and hence,

$$\begin{aligned} i \frac{d}{dt} |\omega|^2 + (u_x + v_x) \left((u - v)(\bar{u} + \bar{v}) + (u + v)(\bar{u} - \bar{v}) \right) \bar{\omega} \\ - (\bar{u}_x + \bar{v}_x) \left((u - v)(\bar{u} + \bar{v}) + (u + v)(\bar{u} - \bar{v}) \right) \omega + (u + v)(u - v)(\bar{u}_x + \bar{v}_x) \bar{\omega} \\ + (u^2 + v^2)(\bar{u}_x - \bar{v}_x) \bar{\omega} - (\bar{u} + \bar{v})(\bar{u} - \bar{v})(u_x + v_x) \omega - (\bar{u}^2 + \bar{v}^2)(u_x - v_x) \omega \\ + \omega_{xx} \bar{\omega} + \bar{\omega}_{xx} \omega - \frac{i \gamma(t) \bar{\omega}}{2} (u(0, t) + \bar{u}(0, t) + v(0, t) + \bar{v}(0, t))(u_{xx} - v_{xx}) \\ - \frac{i \gamma(t) \bar{\omega}}{2} (u(0, t) - v(0, t) + \bar{u}(0, t) - \bar{v}(0, t))(u_{xx} + v_{xx}) \\ + \frac{i \gamma(t)}{2} \omega (u(0, t) + \bar{u}(0, t) + v(0, t) + \bar{v}(0, t))(\bar{u}_{xx} - \bar{v}_{xx}) \\ + \frac{i \gamma(t)}{2} \omega (u(0, t) - v(0, t) + \bar{u}(0, t) - \bar{v}(0, t))(\bar{u}_{xx} + \bar{v}_{xx}) = 0. \end{aligned}$$

Integrating with this identity in x over $(-\infty, \infty)$, we find

$$i \frac{d}{dt} \int_{-\infty}^{\infty} |\omega|^2 dx + \int_{-\infty}^{\infty} (u_x + v_x) \left((u - v)(\bar{u} + \bar{v}) + (u + v)(\bar{u} - \bar{v}) \right) \bar{\omega} dx$$

$$\begin{aligned}
& - \int_{-\infty}^{\infty} (\bar{u}_x + \bar{v}_x) ((u - v)(\bar{u} + \bar{v}) + (u + v)(\bar{u} - \bar{v})) \omega \, dx \\
& + \int_{-\infty}^{\infty} (u + v)(u - v)(\bar{u}_x + \bar{v}_x) \bar{\omega} + (u^2 + v^2)(\bar{u}_x - \bar{v}_x) \bar{\omega} \, dx \\
& - \int_{-\infty}^{\infty} (\bar{u} + \bar{v})(\bar{u} - \bar{v})(u_x + v_x) \omega - (\bar{u}^2 + \bar{v}^2)(u_x - v_x) \omega \, dx + \int_{-\infty}^{\infty} (\omega_{xx} \bar{\omega} + \bar{\omega}_{xx} \omega) \, dx \\
& - \frac{i\gamma(t)\bar{\omega}}{2} \int_{-\infty}^{\infty} (u(0, t) + \bar{u}(0, t) + v(0, t) + \bar{v}(0, t))(u_{xx} - v_{xx}) \, dx \\
& - \frac{i\gamma(t)\bar{\omega}}{2} \int_{-\infty}^{\infty} (u(0, t) - v(0, t) + \bar{u}(0, t) - \bar{v}(0, t))(u_{xx} + v_{xx}) \, dx \\
& + \frac{i\gamma(t)\omega}{2} \int_{-\infty}^{\infty} (u(0, t) + \bar{u}(0, t) + v(0, t) + \bar{v}(0, t))(\bar{u}_{xx} - \bar{v}_{xx}) \, dx \\
& + \frac{i\gamma(t)\omega}{2} \int_{-\infty}^{\infty} (u(0, t) - v(0, t) + \bar{u}(0, t) - \bar{v}(0, t))(\bar{u}_{xx} + \bar{v}_{xx}) \, dx = 0.
\end{aligned}$$

We denote

$$\begin{aligned}
m_1 &= \max |u + v|, & m_2 &= \max |u_x + v_x|, & m_3 &= \max |u^2 + v^2|, \\
m_4 &= \max |u_{xx} + v_{xx}|, & m_5 &= \max |u(0, t) + v(0, t) + \bar{u}(0, t) + \bar{v}(0, t)|.
\end{aligned}$$

By the Cauchy – Bunyakovsky – Schwarz inequality we obtain

$$\begin{aligned}
\frac{d}{dt} \int_{-\infty}^{\infty} |\omega|^2 \, dx &\leq 4m_1 m_2 \max |u - v| \left(\int_{-\infty}^{\infty} |\omega|^2 \, dx \right)^{\frac{1}{2}} \\
&+ 2 \left(m_1 m_2 \max |u - v| \left(\int_{-\infty}^{\infty} |\omega|^2 \, dx \right)^{\frac{1}{2}} + m_3 \int_{-\infty}^{\infty} |\omega|^2 \, dx \right) \\
&+ 2\gamma(t) m_4 \max |u - v| \left(\int_{-\infty}^{\infty} |\omega|^2 \, dx \right)^{\frac{1}{2}}.
\end{aligned}$$

Here we have used the fact that ω and its derivations tend to zero as $x \rightarrow \pm\infty$. It was shown in [19] that there exists a constant $k_1 > 0$ such that

$$\begin{aligned}
|u(0, t) - v(0, t)| &\leq \max_x |u(x, t) - v(x, t)|, \\
\left(\int_{-\infty}^{\infty} |u - v|^2 \, dx \right)^{\frac{1}{2}} &\leq k_1 \left(\int_{-\infty}^{\infty} |u_x - v_x|^2 \, dx \right)^{\frac{1}{2}}, \\
\max |u - v| &\leq k_2 \left(\int_{-\infty}^{\infty} |u_x - v_x|^2 \, dx \right)^{\frac{1}{2}}.
\end{aligned}$$

We hence get

$$\begin{aligned} \frac{d}{dt} \int_{-\infty}^{\infty} |\omega|^2 dx &\leq 4m_1m_2k_1 \int_{-\infty}^{\infty} |\omega|^2 dx + 2 \left(m_1m_2k_1 \int_{-\infty}^{\infty} |\omega|^2 dx + m_3 \int_{-\infty}^{\infty} |\omega|^2 dx \right) \\ &\quad + 2\gamma(t)m_4k_2 \int_{-\infty}^{\infty} |\omega|^2 dx. \end{aligned}$$

Denoting

$$E(t) = \int_{-\infty}^{\infty} |\omega|^2 dx, \quad C(t) = (6m_1m_2k_1 + m_3 + 2\gamma(t)m_4k_2),$$

we rewrite the obtained inequality as

$$\frac{dE(t)}{dt} \leq C(t)E(t).$$

This differential inequality yields

$$E(t) \leq E(0) \exp \int_0^t C(\tau) d\tau,$$

which implies that if $E(0) = 0$, then $E(t) \equiv 0$ and therefore

$$\int_{-\infty}^{\infty} |\omega|^2 dx = \int_{-\infty}^{\infty} |u_x(x, t) - v_x(x, t)|^2 dx = 0, \quad u(x, t) - v(x, t) = c.$$

Letting $t = 0$, we get $c = 0$. □

3. SCATTERING PROBLEM

We consider the system of equations

$$\begin{cases} v_{1x} + i\xi v_1 = u(x)v_2, \\ v_{2x} - i\xi v_2 = -u^*(x)v_1, \end{cases} \quad (3.1)$$

Here $u(x)$ satisfies the fast decay condition

$$\int_{-\infty}^{\infty} (1 + |x|) |u(x)| dx < \infty, \quad t \geq 0. \quad (3.2)$$

The present section contains an information on the direct and inverse scattering problem for the problem (3.1)–(3.2), which is necessary for our further exposition. The condition (3.2) ensures that the system of equations (3.1) possesses the Jost solutions $\varphi(x, \xi)$ and $\psi(x, \xi)$ with the asymptotics

$$\begin{aligned} \varphi(x, \xi) &\sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x}, & \bar{\varphi}(x, \xi) &\sim \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{i\xi x} & \text{as } x \rightarrow -\infty, \\ \psi(x, \xi) &\sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\xi x}, & \bar{\psi}(x, \xi) &\sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\xi x} & \text{as } x \rightarrow \infty; \end{aligned} \quad (3.3)$$

here $\bar{\varphi}$ ($\bar{\psi}$) is not complex conjugation of φ (ψ).

The Jost solutions $\varphi(x, \xi)$ and $\bar{\psi}(x, \xi)$ admit an analytic continuation in ξ into the upper half-plane $\text{Im } \xi > 0$. For each $\xi \in (-\infty, \infty)$ the pairs of functions $\{\varphi(x, \xi), \bar{\varphi}(x, \xi)\}$ and $\{\psi(x, \xi), \bar{\psi}(x, \xi)\}$ are linearly independent solutions of (3.1), and the identities

$$\begin{cases} \varphi = a(\xi)\bar{\psi} + b(\xi)\psi, \\ \bar{\varphi} = -\bar{a}(\xi)\psi + \bar{b}(\xi)\bar{\psi}. \end{cases}$$

hold, where the functions $a(\xi)$ and $b(\xi)$ are independent of x and

$$a(\xi) = W\{\varphi, \psi\} \equiv \varphi_1\psi_2 - \varphi_2\psi_1 \quad (3.4)$$

Moreover, for real ξ we have

$$a(\xi)\bar{a}(\xi) + b(\xi)\bar{b}(\xi) = 1.$$

Hence, the function $a(\xi)$ can be analytically continued in the upper half-plane $\text{Im } \xi > 0$. The function $a(\xi)$ has the asymptotic $a(\xi) = 1 + O\left(\frac{1}{|\xi|}\right)$ as $|\xi| \rightarrow \infty$, $\text{Im } \xi \geq 0$. Besides, in the half-plane $\text{Im } \xi > 0$ the function $a(\xi)$ has finitely many zeroes at the points ξ_k ($k = 1, 2, \dots, N$), and these points are the eigenvalues of the operator

$$L(t) = i \begin{pmatrix} \frac{d}{dx} & -u(x, t) \\ -u^*(x, t) & -\frac{d}{dx} \end{pmatrix}$$

in upper half-plane. We observe that the operator L can have spectral singularities in the continuous spectrum. However, we suppose that, the operator L has no spectral singularities, and the zeroes of the function $a(k)$ are simple:

$$\varphi(x, \xi_k) = C_k\psi(x, \xi_k), \quad k = 1, 2, \dots, N$$

Definition 3.1. The set $\{r(\xi) \equiv \frac{b(\xi)}{a(\xi)}, \xi_k, C_k\}$, $\xi \in \mathbb{R}$, $\text{Im } \xi_k > 0$, $k = \overline{1, N}$, is called the scattering data of the system of equations (3.1).

The solution $\psi(x, \xi)$ can be represented as

$$\psi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi x} + \int_x^\infty K(x, s) e^{i\xi s} ds,$$

where

$$K(x, s) = \begin{pmatrix} K_1(x, s) \\ K_2(x, s) \end{pmatrix}$$

is independent of ξ and is related with the potential function $u(x)$ by the identity

$$u(x) = -2K_1(x, x)$$

The components of the kernel $K_1(x, y)$ in $y > x$ are the solutions of the Gelfand — Levitan — Marchenko system of equation

$$K_1(x, y) - F^*(x + y) + \int_x^\infty \int_x^\infty K_1(x, z) F(z + s) F^*(s + y) ds dz = 0,$$

where

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{b(\xi)}{a(\xi)} e^{i\xi x} d\xi - i \sum_{j=1}^N C_j e^{i\xi_j x}.$$

It is easy to verify that the vector functions

$$h_n(x) = \frac{\frac{d}{d\xi}(\varphi - C_n\psi)|_{\xi=\xi_n}}{\dot{a}(\xi_n)}, \quad n = 1, 2, \dots, N, \quad (3.5)$$

are solutions to the equations $Lh_n = \xi_n h_n$. By the identity (3.4) we obtain the asymptotics

$$\begin{aligned}\psi &\sim a(\xi) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi x} \quad \text{as } x \rightarrow -\infty, \\ \varphi &\sim a(\xi) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x} \quad \text{as } x \rightarrow \infty,\end{aligned}$$

which are valid for $\text{Im } \xi > 0$. By these estimates and the identity (3.5) we obtain the asymptotics

$$h_n \sim -C_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi_n x} \quad \text{as } x \rightarrow -\infty, \quad h_n \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi_n x} \quad \text{as } x \rightarrow \infty. \quad (3.6)$$

In particular,

$$W\{\varphi_n, h_n\} \equiv \varphi_{n1} h_{n2} - \varphi_{n2} h_{n1} = -C_n,$$

where $\varphi_n \equiv \varphi(x, \xi_n)$, $n = 1, 2, \dots, N$.

4. EVOLUTION OF SCATTERING DATA

Let the function $u(x, t)$ in (3.1) be a solution to the loaded nonlinear Schrödinger equation

$$iu_t + 2|u|^2 u + u_{xx} = G(x, t), \quad (4.1)$$

where the function $G(x, t)$ is sufficiently smooth and $G(x, t) = o(1)$ as $x \rightarrow \pm\infty$, $t \geq 0$. The following main lemma [20] is valid for Equation (4.1) and initial condition (1.2).

Lemma 4.1. *If the function $u(x, t)$ is a solution of Equation (4.1) in the class of functions (1.3), then the scattering data of the system (3.1) with the function $u(x, t)$ depend on t as follows:*

$$\begin{aligned}\frac{\partial r^+}{\partial t} &= 4i\xi^2 r^+ - \frac{1}{a^2} \int_{-\infty}^{\infty} (G\varphi_2^2 + G^*\varphi_1^2) dx, \quad \text{Im } \xi = 0, \\ \frac{dC_n}{dt} &= \left(4i\xi_n^2 - \int_{-\infty}^{\infty} (Gh_{n2}\psi_{n2} + G^*h_{n1}\psi_{n1}) dx \right) C_n, \\ \frac{d\xi_n}{dt} &= \frac{-i \int_{-\infty}^{\infty} (G\varphi_{n2}^2 + G^*\varphi_{n1}^2) dx}{2 \int_{-\infty}^{\infty} \varphi_{n1}\varphi_{n2} dx}, \quad n = 1, 2, 3, \dots, N.\end{aligned}$$

Here $\varphi_n(x, t)$ are normalized eigenfunctions, which correspond to the eigenvalue ξ_n of the system of equations (3.1).

We apply Lemma (4.1) with

$$G(x, t) = -i\gamma(t)(u(0, t) + u^*(0, t))u_x \quad (4.2)$$

and we get the representation

$$\frac{dr^+}{dt} = 4i\xi^2 r^+ - \frac{i\gamma(t)(u(0, t) + u^*(0, t))}{a^2} \int_{-\infty}^{\infty} (u_x \varphi_2^2 + u^*_x \varphi_1^2) dx, \quad \text{Im } \xi = 0.$$

Using the system (3.1) and asymptotic formulas (3.3), we find

$$\frac{dr^+}{dt} = 4i\xi^2 r^+ - \frac{i\gamma(t)(u(0, t) + u^*(0, t))}{a^2} P, \quad (4.3)$$

where

$$\begin{aligned}
 P &= \int_{-\infty}^{\infty} u_x \varphi_2^2 dx + \int_{-\infty}^{\infty} u^*_x \varphi_1^2 dx = -2 \int_{-\infty}^{\infty} (u \varphi_2 \varphi'_2 + u^* \varphi_1 \varphi'_1) dx \\
 &= -2 \int_{-\infty}^{\infty} [\varphi'_2(\varphi'_1 + i\xi \varphi_1) + \varphi'_1(i\xi \varphi_2 - \varphi'_2)] dx \\
 &= -2i\xi \int_{-\infty}^{\infty} (\varphi'_2 \varphi_1 + \varphi'_1 \varphi_2) dx = -2i\xi \int_{-\infty}^{\infty} d\varphi_1 \varphi_2 = -2i\xi a(\xi) b(\xi).
 \end{aligned} \tag{4.4}$$

We substitute the expression (4.4) into the expression (4.3) and we get

$$\frac{dr^+}{dt} = (4i\xi^2 - 2\xi\gamma(t)(u(0, t) + u^*(0, t))) r^+, \quad \text{Im } \xi = 0. \tag{4.5}$$

Using Lemma(4.1) and (4.2), we obtain

$$\frac{d\xi_n}{dt} = \frac{\gamma(t)(u(0, t) + u^*(0, t))Q}{2 \int_{-\infty}^{\infty} \varphi_{n1} \varphi_{n2} dx}$$

where

$$\begin{aligned}
 Q &= \int_{-\infty}^{\infty} (u_x \varphi_{n2}^2 + u^*_x \varphi_{n1}^2) dx = -2 \int_{-\infty}^{\infty} (u \varphi_{n2} \varphi'_{n2} + u^* \varphi_{n1} \varphi'_{n1}) dx \\
 &= -2 \int_{-\infty}^{\infty} [\varphi'_{n2}(\varphi'_{n1} + i\xi_n \varphi_{n1}) + \varphi'_{n1}(i\xi_n \varphi_{n2} + \varphi'_{n2})] dx = -2i\xi_n \int_{-\infty}^{\infty} d\varphi_{n1} \varphi_{n2} = 0.
 \end{aligned}$$

Now we know that

$$\frac{d\xi_n}{dt} = 0, \quad n = 1, 2, 3, \dots, N. \tag{4.6}$$

Using the system (3.1) and asymptotic formulas (3.6), we arrive at

$$\begin{aligned}
 \frac{dC_n}{dt} &= \left(4i\xi_n^2 - \int_{-\infty}^{\infty} [Gh_{n2}\psi_{n2} + G^*h_{n1}\psi_{n1}] dx \right) C_n \\
 &= (4i\xi_n^2 + i\gamma(t)(u(0, t) + u^*(0, t))H) C_n
 \end{aligned} \tag{4.7}$$

where

$$\begin{aligned}
 H &= \int_{-\infty}^{\infty} [Gh_{n2}\psi_{n2} + G^*h_{n1}\psi_{n1}] dx \\
 &= \int_{-\infty}^{\infty} [u_x h_{n2}\psi_{n2} + u^*_x h_{n1}\psi_{n1}] dx \\
 &= \int_{-\infty}^{\infty} [uh'_{n2}\psi_{n2} + uh_{n2}\psi'_{n2} + u^*_x h'_{n1}\psi_{n1} + u^*_x h_{n1}\psi'_{n1}] dx \\
 &= \int_{-\infty}^{\infty} [h'_{n2}(\psi'_{n1} + i\xi_n\psi_{n1}) + \psi'_{n2}(h'_{n1} + i\xi_n h_{n1}) \\
 &\quad + h'_{n1}(i\xi_n\psi_{n2} - \psi'_{n2}) + \psi'_{n1}(i\xi_n h_{n2} - h'_{n2})] dx \\
 &= -i\xi_n \int_{-\infty}^{\infty} [(h_{n2}\psi_{n1})' + (h_{n1}\psi_{n2})'] dx = -i\xi_n
 \end{aligned} \tag{4.8}$$

Using (4.7) and (4.8), we find

$$\frac{dC_n}{dt} = (4i\xi_n^2 + 2i\xi_n\gamma(t)(u(0,t) + u^*(0,t))) C_n, \quad n = 1, 2, 3, \dots, N. \tag{4.9}$$

By identities (4.5), (4.6) and (4.9), we have the following theorem.

Theorem 4.1. *If the function $u(x, t)$ is a solution of the problem (1.1)–(1.3), then the scattering data of the system (3.1) with the function $u(x, t)$ depend on t as follows:*

$$\begin{aligned}
 \frac{\partial r^+}{\partial t} &= (4i\xi^2 - 2\xi\gamma(t)(u(0,t) + u^*(0,t))) r^+, \quad \text{Im } \xi = 0, \\
 \frac{dC_n}{dt} &= (4i\xi_n^2 + 2i\xi_n\gamma(t)(u(0,t) + u^*(0,t))) C_n, \\
 \frac{d\xi_n}{dt} &= 0, \quad n = 1, 2, 3, \dots, N.
 \end{aligned}$$

The obtained relations determine completely the evolution of the scattering data for the system (3.1), which allows us to find the solution of the problem (1.1)–(1.3) by using the inverse scattering problem method.

5. EXAMPLE

We consider the Cauchy problem

$$iu_t + 2|u|^2u + u_{xx} - i\gamma(t)(u(0,t) + u^*(0,t))u_x = 0, \quad u(x, 0) = \frac{2}{\cosh 2x}.$$

In order to find its general solution, we use the inverse scattering problem method. First of all, we find a solution of the direct problem for the system of equations

$$L(0)y \equiv i \begin{pmatrix} \frac{d}{dx} & -u_0(x) \\ -u_0^*(x) & -\frac{d}{dx} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \xi \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}.$$

In this case, we have the scattering data

$$r^+(0) = 0, \xi_1(0) = i, \quad C_1(0) = 2.$$

According to the theorem, we find the evolution of scattering data depending on t :

$$r^+(\xi, t) = 0, \quad \xi_1(t) = 0, \quad C_1(t) = -4i - 2\gamma(t)(u(0, t) + u^*(0, t)), \quad \xi \in \mathbb{R}.$$

We find a solution of inverse scattering problem using this scattering data.

$$r^+(\xi, t) = 0, \quad \xi_1(t) = 0, \quad C_1(t) = -4i - 2\gamma(t)(u(0, t) + u^*(0, t)).$$

Solving the Gelfand — Levitan — Marchenko integral equations, we find

$$u(x, t) = \frac{2e^{4it}}{\cosh(2x + 2 \int_0^t \gamma(\tau)(u(0, \tau) + u^*(0, \tau))d\tau)}.$$

Letting $x = 0$ and denoting

$$f(t) = 2 \int_0^t \gamma(\tau)(u(0, \tau) + u^*(0, \tau))d\tau,$$

we obtain the Cauchy problem

$$\begin{cases} \frac{f'(t)}{\gamma(t)} = \frac{4 \cos(4t)}{\cosh(f)}, \\ f(0) = 0. \end{cases}$$

Solving this problem, we get

$$f(t) = \operatorname{arcsinh}\left(4 \int_0^t \gamma(\tau) \cos(4\tau)d\tau\right).$$

As a result, the solution of the considering problem reads

$$u(x, t) = \frac{2e^{4it}}{\cosh(2x + \operatorname{arcsinh}(4 \int_0^t \gamma(\tau) \cos(4\tau)d\tau))}.$$

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