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KRAUSE MEAN PROCESSES GENERATED BY CUBIC STOCHASTIC MATRICES WITH POSITIVE INFLUENCES

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Abstract. The Krause mean process serves as a comprehensive model for the dynamics of opinion exchange within multi–agent system wherein opinions are represented as vectors. In this paper, we propose a framework for opinion exchange dynamics by means of the Krause mean process that is generated by a cubic doubly stochastic matrix with positive influences. The primary objective is to establish a consensus within the multi–agent system.

Keywords: multi–agent system, consensus, Krause mean process, cubic stochastic matrix, quadratic operator

Mathematics Subject Classification: 93A16, 93D50, 91D30, 93C10

1. Introduction

The concept of achieving consensus within a structured, time–invariant, and synchronous environment was initially introduced by DeGroot [4]. Subsequently, Chatterjee and Seneta [3] extended DeGroot's model to encompass structured, time–varying, and synchronous environments. These models depict the opinion–sharing dynamics of structured, time–varying, and synchronous multi–agent systems through the concept of the *backward product* of square stochastic matrices [1]. In contrast, the concept of a non–homogeneous Markov chain is represented by the *forward product* of square stochastic matrices. Consequently, achieving consensus within a multi–agent system and ensuring the ergodicity of the Markov chain are inherently interconnected problems.

More recently, nonlinear models have emerged to characterize opinion dynamics within social communities [6]–[11]. A more comprehensive model for opinion–sharing dynamics is the *Krause mean process*, wherein opinions are represented as vectors. For a comprehensive understanding of the *Krause mean process*, readers may refer to the monograph [12]. In contrast, the *quadratic stochastic operator* is the *simplest nonlinear Markov operator* [5], [21]. This assertion is supported by its representation in *transition dependent matrix form*. In a series of papers [2], [13]–[20], the correlation between the *Krause mean processes* and *quadratic stochastic processes* was established.

In this paper, we introduce a framework for modeling opinion–sharing dynamics through the usage of *Krause mean processes* generated by *cubic doubly stochastic matrices with positive influences*. We then proceed to establish a consensus within the multi–agent system. The main result of this paper, Theorem 4.1, extends and generalizes all results of the papers [2], [13]–[20].

2. Krause mean processes

We first provide some necessary notions and notation, which will be used throughout this paper. Let $\{\mathbf{e}_k\}_{k=1}^m$ be the standard basis of space $\mathbb{R}^m$. Suppose that $\mathbb{R}^m$ is equipped with the

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l_1-norm

$$\|\mathbf{x}\|_1 := \sum_{k=1}^{m} |x_k|$$

where $\mathbf{x} = (x_1, \cdots, x_m)^T \in \mathbb{R}^m$. We say that $\mathbf{x} \geqslant 0$ (respectively, $\mathbf{x} > 0$) if $x_k \geqslant 0$ (respectively, $x_k > 0$) for all $k \in \mathbf{I}_m := \{1, 2, 3, \cdots, m\}$. Let

$$\mathbb{S}^{m-1} = \{\mathbf{x} \in \mathbb{R}^m : \mathbf{x} \geqslant 0, \ \|\mathbf{x}\|_1 = 1\}$$

be the $(m-1)-$dimensional standard simplex. An element of the simplex $\mathbb{S}^{m-1}$ is called *a stochastic vector*. Let $\mathbf{c} = (\frac{1}{m}, \cdots, \frac{1}{m})^T$ be the center of the simplex $\mathbb{S}^{m-1}$ and

$$\mathrm{ri}\, \mathbb{S}^{m-1} = \{\mathbf{x} \in \mathbb{S}^{m-1} : \mathbf{x} > 0\} \quad \text{and} \quad \partial \mathbb{S}^{m-1} = \mathbb{S}^{m-1} \setminus \mathrm{ri}\, \mathbb{S}^{m-1}$$

be, respectively, a relative interior and boundary of the simplex $\mathbb{S}^{m-1}$.

We now examine a comprehensive model of opinion–sharing dynamics within a multi–agent system as outlined in the paper by Hegselmann and Krause [6]. This model encompasses all classical approaches to opinion–sharing dynamics [4], [3], [1]. In this model, we consider a group of m individuals denoted as $\mathbf{I}_m := \{1, \cdots, m\}$, who collaborate as a team or committee. Each individual within this group has the capacity to define their own subjective distribution for a given task. It is assumed that when individual i becomes aware of the distributions specified by each of the other group members, they may opt to revise their own subjective distribution to incorporate this additional information.

By $\mathbf{x}(t) = (x_1(t), \cdots, x_m(t))^T$ we denote the subjective distributions of the multi–agent system at time t, where $\sum_{i=1}^{m} x_i(t) = 1$ and $x_i(t) \geqslant 0$ for all individuals $i \in \mathbf{I}_m$. Additionally, let $p_{ij}(t, \mathbf{x}(t))$ represent the weight that the individual i assigns to $x_j(t)$ when making revisions at time $t+1$. We assume that the sum of weights assigned by individual i to all others equals 1, i.e.,

$$\sum_{j=1}^{m} p_{ij}(t, \mathbf{x}(t)) = 1 \quad \text{and} \quad p_{ij}(t, \mathbf{x}(t)) \geqslant 0, \qquad i, j \in \mathbf{I}_m.$$

After being informed of the subjective distributions of the other group members, the individual i revises his/her own subjective distribution from $x_i(t)$ to $x_i(t+1)$ using the following rule

$$x_i(t+1) = \sum_{j=1}^{m} p_{ij}(t, \mathbf{x}(t)) x_j(t), \quad i \in \mathbf{I}_m.$$

Let $\mathbb{P}(t, \mathbf{x}(t))$ represent an $m \times m$ row-stochastic matrix with its (ij) element denoted as $p_{ij}(t, \mathbf{x}(t))$. We define a *general model for the structured time–varying synchronous system* as

$$\mathbf{x}(t+1) = \mathbb{P}(t, \mathbf{x}(t))\, \mathbf{x}(t), \quad t \in \mathbb{N}. \tag{2.1}$$

By appropriately selecting matrices $\mathbb{P}(t, \mathbf{x}(t))$, we can derive all classical models [4], [1], [3], [6], [7] from a general model (2.1) for structured time–varying synchronous multi–agent system.

We say that *a consensus* is achieved in the structured time–varying synchronous multi–agent system (2.1) if $\mathbf{x}(t)$ converges to $\mathbf{c} = (c, \cdots, c)^T$ as $t \to \infty$. It is important to note that the *consensus* $\mathbf{c} = \mathbf{c}(\mathbf{x}(0))$ may rely on the initial opinion state $\mathbf{x}(0)$.

A broader and more comprehensive model for opinion–sharing dynamics is the *Krause mean process* wherein opinions are expressed as vectors. For a detailed and in–depth explanation of the mean processes, readers are encouraged to consult the excellent monograph by Krause [12].

Let $\mathbf{S}$ be a non–empty convex subset of $\mathbb{R}^d$ and $\mathbf{S}^m$ be the m-fold Cartesian product of $\mathbf{S}$.

Definition 2.1 (Krause mean process [12]). *A given sequence* $\{\mathbf{x}(t)\}_{t=0}^{\infty} \subset \mathbf{S}^m$ *where* $\mathbf{x}(t) = (x_1(t), \cdots, x_m(t))^T$ *is called a Krause mean process on* $\mathbf{S}^m$ *if one has that*

$$x_i(t+1) \in \mathbf{conv}\{x_1(t), \cdots, x_m(t)\}, \quad i \in \mathbf{I}_m, \quad t \in \mathbb{N},$$

where $\mathbf{conv}\{\cdot\}$ *is a convex hull of a set.*

In other words, a given sequence $\{\mathbf{x}(t)\}_{t=0}^{\infty} \subset \mathbf{S}^m$ is the *Krause mean process* if one has that

$$\mathbf{conv}\{x_1(t+1), \cdots, x_m(t+1)\} \subset \mathbf{conv}\{x_1(t), \cdots, x_m(t)\}, \quad t \in \mathbb{N}$$

Definition 2.2 (Krause mean operator [12]). *A mapping* $\mathcal{T} : \mathbf{S}^m \to \mathbf{S}^m$ *is called a Krause mean operator if its trajectory* $\{\mathbf{x}(t)\}_{t=0}^{\infty}$, $\mathbf{x}(t) = \mathcal{T}^t(\mathbf{x}(0))$ *starting from any initial point* $\mathbf{x}(0) \in \mathbf{S}^m$ *generates a Krause mean process on* $\mathbf{S}^m$.

It is noteworthy to mention that the nonlinear model of opinion–sharing dynamics given by (2.1) is the *Krause mean process* due to the fact that the action of a row–stochastic matrix $\mathbb{P}(t, \mathbf{x}(t)) = (p_{ij}(t, \mathbf{x}(t)))_{i,j=1}^m$ on a vector $\mathbf{x}(t) = (x_1(t), \cdots, x_m(t))^T$ can be interpreted as formation of arithmetic means

$$x_i(t+1) := \left(\mathbb{P}(t, \mathbf{x}(t))\, \mathbf{x}(t)\right)_i = \sum_{j=1}^{m} p_{ij}(t, \mathbf{x}(t))\, x_j(t), \quad i \in \mathbf{I}_m$$

with weights $p_{ij}(t, \mathbf{x}(t)) \geqslant 0$ such that

$$\sum_{j=1}^{m} p_{ij}(t, \mathbf{x}(t)) = 1.$$

The various kinds of nonlinear models of mean processes were investigated in the papers [6], [7], [8], [9], [10], [11].

3. Krause mean process via quadratic stochastic operator

Within this section, we aim to establish a correlation between the *Krause mean processes* and *quadratic stochastic operators*.

Definition 3.1. *A cubic matrix* $\mathcal{P} = (p_{ijk})_{i,j,k=1}^m$ *is said to be*

- *stochastic if it satisfies the conditions*

$$\sum_{k=1}^{m} p_{ijk} = 1, \qquad p_{ijk} \geqslant 0, \quad i, j, k \in \mathbf{I}_m;$$

- *doubly stochastic if it satisfies the conditions*

$$\sum_{j=1}^{m} p_{ijk} = \sum_{k=1}^{m} p_{ijk} = 1, \qquad p_{ijk} \geqslant 0, \quad i, j, k \in \mathbf{I}_m;$$

- *triply stochastic if it satisfies the conditions*

$$\sum_{i=1}^{m} p_{ijk} = \sum_{j=1}^{m} p_{ijk} = \sum_{k=1}^{m} p_{ijk} = 1, \qquad p_{ijk} \geqslant 0, \quad i, j, k \in \mathbf{I}_m.$$

Throughout this paper, we always assume that a cubic matrix $\mathcal{P} = (p_{ijk})_{i,j,k=1}^m$ is doubly stochastic unless explicitly specified otherwise. Furthermore, it is important to emphasize that we do not impose the condition $p_{ijk} = p_{jik}$ for all $i, j, k \in \mathbf{I}_m$. For a more comprehensive understanding of these assumptions and a detailed comparison of the outcomes in both cubic doubly and triply stochastic matrices cases, the readers may refer to the next sections.

Let $\mathcal{P} = (p_{ijk})_{i,j,k=1}^m$ be a cubic doubly stochastic matrix, and let $\mathbb{P}_{\bullet\bullet k} = (p_{ijk})_{i,j=1}^m$ denote a square matrix for a fixed $k \in \mathbf{I}_m$. It is evident that $\mathbb{P}_{\bullet\bullet k} = (p_{ijk})_{i,j=1}^m$ is also a square *stochastic* matrix. In the following, we express $\mathcal{P} = (\mathbb{P}_{\bullet\bullet 1}| \cdots |\mathbb{P}_{\bullet\bullet m})$ to represent the cubic doubly stochastic matrix $\mathcal{P} = (p_{ijk})_{i,j,k=1}^m$. We now define a quadratic stochastic operator $\mathcal{Q}$:

$\mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ that is associated with the cubic *doubly stochastic* matrix $\mathcal{P} = (\mathbb{P}_{\bullet\bullet 1}|\cdots|\mathbb{P}_{\bullet\bullet m})$ as follows

$$(\mathcal{Q}(\mathbf{x}))_k = \sum_{i,j=1}^{m} x_i x_j p_{ijk}, \quad k \in \mathbf{I}_m. \tag{3.1}$$

We now define an $m \times m$ matrix $\mathbb{P}(\mathbf{x}) = \left(p_{kj}(\mathbf{x})\right)_{k,j=1}^{m}$

$$p_{kj}(\mathbf{x}) = \sum_{i=1}^{m} x_i p_{ijk}, \quad k, j \in \mathbf{I}_m. \tag{3.2}$$

We show that $\mathbb{P}(\mathbf{x})$ is a square *doubly stochastic* matrix for every $\mathbf{x} \in \mathbb{S}^{m-1}$. In fact, we have

$$\sum_{k=1}^{m} p_{kj}(\mathbf{x}) = \sum_{k=1}^{m}\left(\sum_{i=1}^{m} x_i p_{ijk}\right) = \sum_{i=1}^{m}\left(\sum_{k=1}^{m} p_{ijk}\right) x_i = \sum_{i=1}^{m} x_i = 1,$$
$$\sum_{j=1}^{m} p_{kj}(\mathbf{x}) = \sum_{j=1}^{m}\left(\sum_{i=1}^{m} x_i p_{ijk}\right) = \sum_{i=1}^{m}\left(\sum_{j=1}^{m} p_{ijk}\right) x_i = \sum_{i=1}^{m} x_i = 1.$$

Hence, it follows from (3.1) and (3.2) that

$$\mathcal{Q}(\mathbf{x}) = \mathbb{P}(\mathbf{x})\mathbf{x} \tag{3.3}$$

and it is called *a density dependent matrix form* of the quadratic stochastic operator $\mathcal{Q} : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ that is associated with the cubic *doubly stochastic* matrix $\mathcal{P} = (\mathbb{P}_{\bullet\bullet 1}|\cdots|\mathbb{P}_{\bullet\bullet m})$.

We are now ready to present a protocol generated by *cubic doubly stochastic matrices* (for short, we use *CDSM*).

Protocol–CDSM: Let $\mathcal{P} = (\mathbb{P}_{\bullet\bullet 1}|\cdots|\mathbb{P}_{\bullet\bullet m})$ be a cubic *doubly stochastic* matrix and $\mathcal{Q} : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ be a quadratic stochastic operator associated with a cubic *doubly stochastic* matrix $\mathcal{P} = (\mathbb{P}_{\bullet\bullet 1}|\cdots|\mathbb{P}_{\bullet\bullet m})$. Suppose that an opinion sharing dynamics of the multi–agent system is generated by a quadratic stochastic operator $\mathcal{Q} : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ as

$$\mathbf{x}^{(n+1)} := \mathcal{Q}\left(\mathbf{x}^{(n)}\right) = \mathbb{P}\left(\mathbf{x}^{(n)}\right)\mathbf{x}^{(n)}, \qquad \mathbf{x}^{(0)} \in \mathbb{S}^{m-1}$$

where $\mathbf{x}^{(n)} = \left(x_1^{(n)}, \cdots, x_m^{(n)}\right)^T$ is the subjective distribution after n revisions.

We propose the multi–agent system interpretation of *Protocol–CDSM*. We assume that *each agent has the capacity to revise his/her own opinion on a particular issue by taking into account the influences stemming from all possible pairs of two agents*. This obviously creates non–linearity in the proposed model. To maintain homogeneity within the model, *we treat the influence of a single agent as the same as the influence exerted by a group of two identical agents*. To be precise, we make the following assumptions:

- A group of m agents, denoted as $\mathbf{I}_m := \{1, \cdots, m\}$, collaborates as one unified team or committee;
- Each individual agent possesses the capability to express his/her own *opinion* on a given task/matter/issue. In this context, an *opinion* is a broad concept encompassing an agent's *beliefs*, *behaviors*, or *attitudes*;
- An opinion profile at time n is a *stochastic* vector $\mathbf{x}^{(n)} = (x_1^{(n)}, \cdots, x_m^{(n)})^T$;
- Each agent, say k, experiences influence from a group composed of 2–agents, say $\{i, \{j\}\}$, in which an agent j serves as the *spokesperson* of this group;
- The influence of a group of 2–agents, say $\{i, \{j\}\}$ (respectively, $\{j, \{i\}\}$) with an agent j (respectively, i) acting as the *spokesperson*, on an agent k is denoted as $p_{ij,k}$ (respectively, $p_{ji,k}$). In practice, it is important to note that $p_{ij,k}$ may not necessarily be equal to $p_{ji,k}$, i.e., $p_{ij,k} \neq p_{ji,k}$, signifying potential differences in influence;

- The influence profile of the group $\{i,\{j\}\}$ with the *spokesperson* j is represented as a *stochastic* vector $\mathbf{p}_{ij\bullet} := (p_{ij,1}, p_{ij,2}, \cdots, p_{ij,m})^T$;
- The collective influences stemming from all possible groups of 2–agents on an agent k forms a square *stochastic* matrix $\mathbb{P}_{\bullet\bullet k} = (p_{ij,k})_{i,j=1}^m$;
- An agent regards a group of 2–agents as *trusted/influential* when its influence on an agent is substantial;
- The level of trust $p_{kj}(\mathbf{x}^{(n)})$ that an agent k places in another agent j with the given opinion profile $\mathbf{x}^{(n)}$ is calculated as the average influence over the opinion profile $\mathbf{x}^{(n)}$ exerted by all possible groups of 2–agents having j as the *spokesperson*. In mathematical terms, this trust level is calculated as $p_{kj}(\mathbf{x}^{(n)}) := \sum_{i=1}^m x_i^{(n)} p_{ij,k}$;
- The trust matrix associated with the opinion profile $\mathbf{x}^{(n)}$ is represented as a square *doubly stochastic* matrix $\mathbb{P}(\mathbf{x}^{(n)}) = \left(p_{kj}(\mathbf{x}^{(n)})\right)_{k,j=1}^m$;
- The opinion profile at time $n+1$ is then revised as follows

$$\mathbf{x}^{(n+1)} := \mathcal{Q}(\mathbf{x}^{(n)}) = \mathbb{P}\left(\mathbf{x}^{(n)}\right)\mathbf{x}^{(n)}.$$

This implies that, due to the density dependent matrix form (2.1), the opinion sharing dynamics of the multi–agent system given by *Protocol–CDSM* generates a Krause mean process. Consequently, we have the following result (see [19]).

Proposition 3.1. *Let $\mathcal{P} = (\mathbb{P}_{\bullet\bullet 1}|\cdots|\mathbb{P}_{\bullet\bullet m})$ be a cubic doubly stochastic matrix and $\mathcal{Q} : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ be the associated quadratic stochastic operator. Then the opinion sharing dynamics of the multi–agent system given by Protocol–CDSM generates the Krause mean process.*

Definition 3.2 (Consensus in *Protocol–CDSM*)**.** *We say that the multi–agent system given by Protocol–CDSM eventually reaches a consensus if an opinion sharing dynamics converges to the center $\mathbf{c} = (\frac{1}{m}, \cdots, \frac{1}{m})^T$ of the simplex $\mathbb{S}^{m-1}$.*

4. Main result

We first introduce a notion of cubic stochastic matrices with *positive influences*.

Definition 4.1 (Cubic stochastic matrix with positive influence)**.** *A cubic stochastic matrix $\mathcal{P} = (p_{ijk})_{i,j,k=1}^m$ is said to have positive influence if for any two agents $j,k \in \mathbf{I}_m$ there always exist some groups of 2–agents $\{i_1,\{j\}\}$, $\{i_2,\{j\}\}$, $\cdots$ $\{i_t,\{j\}\}$ having j as the spokesperson (here $1 \leqslant t = t(j,k) \leqslant m$) and there also exists a positive stochastic vector $\mathbf{c}(j,k) = (c_1, \cdots, c_t) > 0$ with*

$$\sum_{s=1}^t c_s = 1$$

such that the average influence

$$\sum_{s=1}^t c_s p_{i_s jk} > 0$$

of those groups on an agent k is positive.

Let us now recall some classes of cubic stochastic matrices which were studied in the series of papers [2], [13]–[20]. The following an $m \times m$ square stochastic matrix

$$\operatorname{diag}(\mathcal{P}) := \begin{pmatrix} p_{111} & p_{112} & \cdots & p_{11m} \\ p_{221} & p_{222} & \cdots & p_{22m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{mm1} & p_{mm2} & \cdots & p_{mmm} \end{pmatrix} = \begin{pmatrix} \left(\mathbf{p}_{11\bullet}\right)^T \\ \left(\mathbf{p}_{22\bullet}\right)^T \\ \vdots \\ \left(\mathbf{p}_{mm\bullet}\right)^T \end{pmatrix}.$$

is called a *diagonal matrix* of cubic stochastic matrix $\mathcal{P} = (\mathbb{P}_{\bullet\bullet 1}|\cdots|\mathbb{P}_{\bullet\bullet m})$.

Definition 4.2. *A cubic stochastic matrix $\mathcal{P} = (\mathbf{p}_{ij\bullet})_{i,j=1}^m$, where $\mathbf{p}_{ij\bullet} := (p_{ij1}, \cdots p_{ijm})$ for all $i, j \in \mathbf{I}_m$, is said*

- *to be positive if one has $\mathbf{p}_{ij\bullet} > 0$ for all $i, j \in \mathbf{I}_m$;*
- *to be diagonally positive if one has $\mathbf{p}_{ii\bullet} > 0$ for all $i \in \mathbf{I}_m$;*
- *to be off-diagonally positive if one has $\mathbf{p}_{ij\bullet} > 0$ for all $i, j \in \mathbf{I}_m$ with $i \neq j$;*
- *to have weak influence if for any two agents $j, k \in \mathbf{I}_m$ there always exists a group of 2–agents $\{i_0, \{j\}\}$ having j as the spokesperson such that the influence p_{i_0jk} of that group $\{i_0, \{j\}\}$ on an agent k is positive, i.e., for any $j, k \in \mathbf{I}_m$ there always exists $i_0 \in \mathbf{I}_m$ such that $p_{i_0jk} > 0$.*
- *to have strong influence if for any two agents $j, k \in \mathbf{I}_m$ the total influence of all groups of 2–agents $\{1, \{j\}\}$, $\{2, \{j\}\}$, $\cdots$ $\{m, \{j\}\}$ having j as the spokesperson on an agent k is positive, i.e., $\sum\limits_{i=1}^{m} p_{ijk} > 0$ for any $j, k \in \mathbf{I}_m$.*

It is easy to verify that all classes of cubic stochastic matrices provided in Definition 4.2 have positive influences. In this sense, the main result of this paper extends, generalizes, and unifies all related results of the papers [2], [13]–[20].

We are now ready to state the main result of this paper. Let

$$\mathbf{e}_1 = (1, 0, 0, \dots, 0)^T, \qquad \mathbf{e}_2 = (0, 1, 0, \dots, 0)^T, \qquad \dots \qquad \mathbf{e}_m = (0, 0, 0, \dots, 1)^T$$

be the vertices of simplex $\mathbb{S}^{m-1}$ and $\mathbf{e}_k^{(n)} := \mathcal{Q}\left(\mathbf{e}_k^{(n-1)}\right)$, where $\mathbf{e}_k^{(0)} := \mathbf{e}_k$ for all $k \in \mathbf{I}_m$ and $n \in \mathbb{N}$.

Theorem 4.1 (Consensus in cubic doubly stochastic matrix with influence)**.** *Let $\mathcal{P} = (\mathbb{P}_{\bullet\bullet 1}|\cdots|\mathbb{P}_{\bullet\bullet m})$ be a cubic doubly stochastic matrix and let $\mathcal{Q} : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ be the associated quadratic stochastic operator. If a cubic stochastic matrix $\mathcal{P} = (p_{ijk})_{i,j,k=1}^m$ has positive influence and moreover, if for each $k \in \mathbf{I}_m$ one has $\mathbf{e}_k^{(n_k)} \in \operatorname{ri}\mathbb{S}^{m-1}$ for some $n_k \in \mathbb{N}$, then the opinion sharing dynamics of the multi–agent system given by Protocol–CDSM eventually reaches a consensus for any initial opinion $\mathbf{x}^{(0)} \in \mathbb{S}^{m-1}$.*

Proof. Let $\left\{\mathbf{x}^{(n)}\right\}_{n=0}^{\infty}$, $\mathbf{x}^{(n+1)} = \mathcal{Q}(\mathbf{x}^{(n)})$, be a trajectory of the quadratic stochastic operator $\mathcal{Q} : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ starting from an initial point $\mathbf{x}^{(0)} \in \mathbb{S}^{m-1}$. Particularly, let $\left\{\mathbf{e}_k^{(n)}\right\}_{n=0}^{\infty}$ be a trajectory of the quadratic stochastic operator $\mathcal{Q} : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ starting from a vertex $\mathbf{e}_k$ of the simplex $\mathbb{S}^{m-1}$ for all $k \in \mathbf{I}_m$. According to the definition, the multi–agent system eventually reaches a consensus if $\{\mathbf{x}^{(n)}\}_{n=0}^{\infty}$ converges to the center $\mathbf{c} = (\frac{1}{m}, \cdots, \frac{1}{m})^T$ of the simplex $\mathbb{S}^{m-1}$ for any initial point $\mathbf{x}^{(0)} \in \mathbb{S}^{m-1}$.

Step 1. We first show that $\mathcal{Q}(\operatorname{ri}\mathbb{S}^{m-1}) \subset \operatorname{ri}\mathbb{S}^{m-1}$. Indeed, let $\mathbf{x} \in \operatorname{ri}\mathbb{S}^{m-1}$. This means that $x_i > 0$ for all $i \in \mathbf{I}_m$. Since $\mathbb{P}(\mathbf{x}) = \left(p_{kj}(\mathbf{x})\right)_{k,j=1}^m$ is a square doubly stochastic matrix and

$\mathcal{Q}(\mathbf{x}) = \mathbb{P}(\mathbf{x})\mathbf{x}$, we derive that

$$0 < \min_{j\in\mathbf{I}_m} x_j \leqslant \sum_{j=1}^{m} p_{kj}(\mathbf{x})x_j = (\mathcal{Q}(\mathbf{x}))_k \quad k \in \mathbf{I}_m.$$

This means that $\mathcal{Q}(\mathbf{x}) \in \operatorname{ri}\mathbb{S}^{m-1}$.

Step 2. We now show that there exists $n_0 \in \mathbb{N}$ such that for any initial point $\mathbf{x}^{(0)} \in \mathbb{S}^{m-1}$ one has $\mathbf{x}^{(n_0)} \in \operatorname{ri}\mathbb{S}^{m-1}$. It is worth mentioning that n_0 is independent on an initial point $\mathbf{x}^{(0)} \in \mathbb{S}^{m-1}$. Since for each $k \in \mathbf{I}_m$ one has $\mathbf{e}_k^{(n_k)} \in \operatorname{ri}\mathbb{S}^{m-1}$ for some $n_k \in \mathbb{N}$, it follows from the previous step that for each $k \in \mathbf{I}_m$ one has $\mathbf{e}_k^{(n)} \in \operatorname{ri}\mathbb{S}^{m-1}$ for any $n > n_k$.

Let $n_0 := \max_{k\in\mathbf{I}_m} n_k$. Then $\mathbf{e}_k^{(n_0)} \in \operatorname{ri}\mathbb{S}^{m-1}$ for all $k \in \mathbf{I}_m$. We now show that $\mathbf{x}^{(n_0)} = \mathcal{Q}^{n_0}(\mathbf{x}^{(0)}) \in \operatorname{ri}\mathbb{S}^{m-1}$ for any initial point $\mathbf{x}^{(0)} \in \mathbb{S}^{m-1}$. Since

$$\mathbb{P}(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) = \lambda\mathbb{P}(\mathbf{x}) + (1-\lambda)\mathbb{P}(\mathbf{y})$$

for any $\mathbf{x}, \mathbf{y} \in \mathbb{S}^{m-1}$ and $0 < \lambda < 1$, we have

$$\begin{aligned}\mathbf{x}^{(1)} =& \mathcal{Q}(\mathbf{x}^{(0)}) = \mathbb{P}(\mathbf{x}^{(0)})\mathbf{x}^{(0)} = x_1^{(0)}\mathbb{P}(\mathbf{x}^{(0)})\mathbf{e}_1 + \cdots + x_m^{(0)}\mathbb{P}(\mathbf{x}^{(0)})\mathbf{e}_m \\ =& \sum_{i=1}^{m}\left(x_i^{(0)}\right)^2\mathbb{P}(\mathbf{e}_i)\mathbf{e}_i + 2\sum_{i<j} x_i^{(0)}x_j^{(0)}\mathbb{P}(\mathbf{e}_j)\mathbf{e}_i\end{aligned}$$

Let $\mathbf{e}_{ij}^{(11)} := \mathbb{P}(\mathbf{e}_j)\mathbf{e}_i$ for any $i \neq j$. We then obtain

$$\mathbf{x}^{(1)} = \sum_{i=1}^{m}\left(x_i^{(0)}\right)^2\mathbf{e}_i^{(1)} + 2\sum_{i<j} x_i^{(0)}x_j^{(0)}\mathbf{e}_{ij}^{(11)}.$$

Similarly, we get

$$\mathbf{x}^{(2)} = \mathbb{P}(\mathbf{x}^{(1)})\mathbf{x}^{(1)} = \sum_{i=1}^{m}\left(x_i^{(0)}\right)^2\mathbb{P}(\mathbf{x}^{(1)})\mathbf{e}_i^{(1)} + 2\sum_{i<j} x_i^{(0)}x_j^{(0)}\mathbb{P}(\mathbf{x}^{(1)})\mathbf{e}_{ij}^{(11)}.$$

Consequently, we obtain

$$\begin{aligned}\mathbf{x}^{(2)} =& \sum_{i=1}^{m}\sum_{k=1}^{m}\left(x_i^{(0)}\right)^2\left(x_k^{(0)}\right)^2\mathbb{P}(\mathbf{e}_k^{(1)})\mathbf{e}_i^{(1)} + 2\sum_{i=1}^{m}\sum_{k<l}\left(x_i^{(0)}\right)^2 x_k^{(0)}x_l^{(0)}\mathbb{P}(\mathbf{e}_{kl}^{(11)})\mathbf{e}_i^{(1)} \\ &+ 2\sum_{i<j}\sum_{k=1}^{m} x_i^{(0)}x_j^{(0)}\left(x_k^{(0)}\right)^2\mathbb{P}(\mathbf{e}_k^{(1)})\mathbf{e}_{ij}^{(11)} + 4\sum_{i<j}\sum_{k<l} x_i^{(0)}x_j^{(0)}x_k^{(0)}x_l^{(0)}\mathbb{P}(\mathbf{e}_{kl}^{(11)})\mathbf{e}_{ij}^{(11)}.\end{aligned}$$

It is clear that

$$\begin{aligned}&\sum_{i=1}^{m}\sum_{k=1}^{m}\left(x_i^{(0)}\right)^2\left(x_k^{(0)}\right)^2\mathbb{P}(\mathbf{e}_k^{(1)})\mathbf{e}_i^{(1)} \\ &\quad= \sum_{i=1}^{m}\left(x_i^{(0)}\right)^4\mathbb{P}(\mathbf{e}_i^{(1)})\mathbf{e}_i^{(1)} + \sum_{i\neq k}\left(x_i^{(0)}\right)^2\left(x_k^{(0)}\right)^2\mathbb{P}(\mathbf{e}_k^{(1)})\mathbf{e}_i^{(1)} \\ &\quad= \sum_{i=1}^{m}\left(x_i^{(0)}\right)^4\mathbf{e}_i^{(2)} + \sum_{i\neq k}\left(x_i^{(0)}\right)^2\left(x_k^{(0)}\right)^2\mathbb{P}(\mathbf{e}_k^{(1)})\mathbf{e}_i^{(1)}.\end{aligned}$$

Therefore,

$$\mathbf{x}^{(2)} = \left(x_1^{(0)}\right)^4\mathbf{e}_1^{(2)} + \left(x_2^{(0)}\right)^4\mathbf{e}_2^{(2)} + \cdots + \left(x_m^{(0)}\right)^4\mathbf{e}_m^{(2)} + \cdots$$

Analogously, one can show that

$$\mathbf{x}^{(n)} = \left(x_1^{(0)}\right)^{2^n} \mathbf{e}_1^{(n)} + \left(x_2^{(0)}\right)^{2^n} \mathbf{e}_2^{(n)} + \cdots + \left(x_m^{(0)}\right)^{2^n} \mathbf{e}_m^{(n)} + \cdots$$

for any $n \in \mathbb{N}$. Since $\mathbf{e}_k^{(n_0)} \in \operatorname{ri}\mathbb{S}^{m-1}$, i.e., $\mathbf{e}_k^{(n_0)} > 0$ for all $k \in \mathbf{I}_m$, we obtain

$$\mathbf{x}^{(n_0)} \geqslant \left(x_1^{(0)}\right)^{2^{n_0}} \mathbf{e}_1^{(n_0)} + \left(x_2^{(0)}\right)^{2^{n_0}} \mathbf{e}_2^{(n_0)} + \cdots + \left(x_m^{(0)}\right)^{2^{n_0}} \mathbf{e}_m^{(n_0)} > 0$$

for any $\mathbf{x}^{(0)} \in \mathbb{S}^{m-1}$. This shows that $\mathbf{x}^{(n_0)} \in \operatorname{ri}\mathbb{S}^{m-1}$.

Step 3. Now we are going to show that for any $\mathbf{x}^{(0)} \in \mathbb{S}^{m-1}$ an omega limit set $\omega\left(\left\{\mathbf{x}^{(n)}\right\}\right)$ of the sequence $\{\mathbf{x}^{(n)}\}_{n=0}^{\infty}$ is a subset of the interior $\operatorname{ri}\mathbb{S}^{m-1}$ of the simplex $\mathbb{S}^{m-1}$ i.e., $\omega\left(\left\{\mathbf{x}^{(n)}\right\}\right) \Subset \operatorname{ri}\mathbb{S}^{m-1}$. Indeed, it follows from the previous step that $\mathcal{Q}^{n_0}(\mathbb{S}^{m-1}) \Subset \operatorname{ri}\mathbb{S}^{m-1}$. Since $\mathcal{Q}^{n_0}(\mathbb{S}^{m-1})$ is a compact set, there exists $\alpha > 0$ such that

$$\mathcal{Q}^{n_0}(\mathbf{x}) \geqslant \alpha\mathbf{e} := (\alpha, \alpha, \cdots, \alpha)^T, \qquad \mathbf{x} \in \mathbb{S}^{m-1}.$$

On the other hand, the interior $\operatorname{ri}\mathbb{S}^{m-1}$ of the simplex $\mathbb{S}^{m-1}$ is an invariant set (see *Step 1*) and $\mathcal{Q}^{n}(\mathbb{S}^{m-1}) \subset \mathcal{Q}^{n_0}(\mathbb{S}^{m-1})$ for any $n > n_0$, we have $\{\mathbf{x}^{(n)}\}_{n=n_0}^{\infty} \subset \mathbb{S}_\alpha$, i.e., $\mathbf{x}^{(n)} \geqslant \alpha\mathbf{e}$ for any $n > n_0$ where

$$\mathbb{S}_\alpha := \{\mathbf{x} \in \mathbb{S}^{m-1} : \mathbf{x} \geqslant \alpha\mathbf{e}\}.$$

Consequently, an omega limit set $\omega\left(\left\{\mathbf{x}^{(n)}\right\}\right)$ of the sequence $\{\mathbf{x}^{(n)}\}_{n=0}^{\infty}$ is a subset of the set $\mathbb{S}_\alpha$, i.e., $\omega\left(\left\{\mathbf{x}^{(n)}\right\}\right) \subset \mathbb{S}_\alpha \subset \operatorname{ri}\mathbb{S}^{m-1}$ any $\mathbf{x}^{(0)} \in \mathbb{S}^{m-1}$.

Step 4. As we showed in the previous step that $\mathcal{Q}^{n}(\mathbb{S}^{m-1}) \subset \mathbb{S}_\alpha$ for any $n > n_0$, it is therefore enough to study the dynamics of the quadratic stochastic operator over the set $\mathbb{S}_\alpha$ which is an invariant set. Let $\mathbf{x}^{(0)} \in \mathbb{S}_\alpha$. Then $\mathbf{x}^{(n)} \in \mathbb{S}_\alpha$, i.e., $\mathbf{x}^{(n)} \geqslant \alpha\mathbf{e}$ for any $n \in \mathbb{N}$. It follows from the matrix form (3.3) of the quadratic stochastic operator that

$$\mathbf{x}^{(n+1)} = \mathcal{Q}(\mathbf{x}^{(n)}) = \mathbb{P}\left(\mathbf{x}^{(n)}\right)\mathbf{x}^{(n)} = \mathbb{P}\left(\mathbf{x}^{(n)}\right)\cdots\mathbb{P}\left(\mathbf{x}^{(1)}\right)\mathbb{P}\left(\mathbf{x}^{(0)}\right)\mathbf{x}^{(0)}$$

where $\mathbb{P}(\mathbf{x})$ is the square doubly stochastic matrix defined by (3.2). Let us set for any two integer numbers $n > r$

$$\mathbb{P}^{\left[\mathbf{x}^{(n)},\mathbf{x}^{(r)}\right]} := \mathbb{P}\left(\mathbf{x}^{(n)}\right)\mathbb{P}\left(\mathbf{x}^{(n-1)}\right)\cdots\mathbb{P}\left(\mathbf{x}^{(r+1)}\right)\mathbb{P}\left(\mathbf{x}^{(r)}\right).$$

We then obtain for any $n \geqslant r \geqslant 0$ that

$$\mathbf{x}^{(n+1)} = \mathbb{P}^{\left[\mathbf{x}^{(n)},\mathbf{x}^{(0)}\right]}\mathbf{x}^{(0)} = \mathbb{P}^{\left[\mathbf{x}^{(n)},\mathbf{x}^{(r)}\right]}\mathbf{x}^{(r)}.$$

Since $\mathcal{P} = (\mathbb{P}_{\bullet\bullet 1}|\cdots|\mathbb{P}_{\bullet\bullet m})$ has *positive influence*, then for any $j,k \in \mathbf{I}_m$ there always exists $1 \leqslant t(j,k) \leqslant m$ such that $\sum\limits_{s=1}^{t(j,k)} c_s p_{i_s jk} > 0$ where the vector $\mathbf{c}(j,k) = (c_1, \cdots, c_t) > 0$ depends on j,k such that $\sum\limits_{s=1}^{t(j,k)} c_s = 1$. It is evident that

$$0 < \sum_{s=1}^{t(j,k)} c_s p_{i_s jk} \leqslant \left(\max_{1\leqslant s\leqslant t(j,k)} c_s\right)\sum_{s=1}^{t(j,k)} p_{i_s jk}$$

We now define

$$\delta(j,k) := \sum_{s=1}^{t(j,k)} p_{i_s jk} > 0$$

for any $j,k \in \mathbf{I}_m$. Let

$$\delta := \min_{j,k\in\mathbf{I}_m} \delta(j,k) > 0.$$

Since $\mathbf{x}^{(n)} \geqslant \alpha\mathbf{e}$ for any $n \in \mathbb{N}$, it follows from (3.2) for a stochastic matrix $\mathbb{P}(\mathbf{x}^{(n)}) = \big(p_{kj}(\mathbf{x}^{(n)})\big)_{k,j=1}^m$ that

$$p_{kj}(\mathbf{x}^{(n)}) = \sum_{i=1}^m p_{ijk}x_i^{(n)} \geqslant \sum_{s=1}^{t(j,k)} p_{i_sjk}x_{i_s}^{(n)} \geqslant \alpha \sum_{s=1}^{t(j,k)} p_{i_sjk} = \alpha\delta(j,k) \geqslant \alpha\delta > 0 \tag{4.1}$$

for any $j,k \in \mathbf{I}_m$ and $n \in \mathbb{N}$. Consequently, $\mathbb{P}(\mathbf{x}^{(n)}) = \big(p_{kj}(\mathbf{x}^{(n)})\big)_{k,j=1}^m$ is a *uniformly positive* square stochastic matrix for any $n \in \mathbb{N}$.

Step 5. Let

$$\delta(\mathbb{P}) = \frac{1}{2}\max_{i_1,i_2}\sum_{j=1}^m |p_{i_1j} - p_{i_2j}|$$

be the Dobrushin ergodicity coefficient (see [22]) of a square stochastic matrix $\mathbb{P} = (p_{ij})_{i,j=1}^m$. We first recall some properties of Dobrushin ergodicity coefficient for the reader's convenience. The following statements are true for any square stochastic matrices $\mathbb{P}$ and $\mathbb{Q}$ (see [22]):

(*i*) $0 \leqslant \delta(\mathbb{P}) \leqslant 1$;
(*ii*) $\delta(\mathbb{P}) = 0$ if and only if $\operatorname{rank}(\mathbb{P}) = 1$, i.e., $\mathbb{P}$ is a stable stochastic matrix;
(*iii*) $\delta(\mathbb{P}) < 1$ if and only if $\mathbb{P}$ is scrambling. If $\mathbb{P} > 0$, then $\delta(\mathbb{P}) < 1$;
(*iv*) $|\delta(\mathbb{P}) - \delta(\mathbb{Q})| \leqslant \|\mathbb{P} - \mathbb{Q}\|_\infty$;
(*v*) $\delta(\mathbb{P}\mathbb{Q}) \leqslant \delta(\mathbb{P})\delta(\mathbb{Q})$.

Since $\mathbb{P}(\mathbf{x}^{(n)}) = \big(p_{kj}(\mathbf{x}^{(n)})\big)_{k,j=1}^m$ is positive, it follows from the property (*iii*) of Dobrushin's ergodicity coefficient given above that

$$\delta\left(\mathbb{P}(\mathbf{x}^{(n)})\right) < 1, \quad \forall\, n \in \mathbb{N}.$$

Moreover, due to inequality (4.1), the entries of the matrix $\mathbb{P}(\mathbf{x}^{(n)}) = \big(p_{kj}(\mathbf{x}^{(n)})\big)_{k,j=1}^m$ are uniformly bounded away from zero for any $n \in \mathbb{N}$. It is worthy noting that, by using the same idea, we can also show that not only $\mathbb{P}(\mathbf{x}^{(n)}) = \big(p_{kj}(\mathbf{x}^{(n)})\big)_{k,j=1}^m$ but also $\mathbb{P}(\mathbf{x}) = \big(p_{kj}(\mathbf{x})\big)_{k,j=1}^m$ for all $\mathbf{x} \geqslant \alpha\mathbf{e}$ is positive and its entries are uniformly bounded away from zero for all $\mathbf{x} \geqslant \alpha\mathbf{e}$. Indeed, since $\mathcal{P} = (\mathbb{P}_{\bullet\bullet 1}|\cdots|\mathbb{P}_{\bullet\bullet m})$ has positive influence and $\mathbf{x} \geqslant \alpha\mathbf{e}$, it follows from (3.2) for a doubly stochastic matrix $\mathbb{P}(\mathbf{x}) = \big(p_{kj}(\mathbf{x})\big)_{k,j=1}^m$ that

$$p_{kj}(\mathbf{x}) = \sum_{i=1}^m p_{ijk}x_i \geqslant \sum_{s=1}^{t(j,k)} p_{i_sjk}x_{i_s} \geqslant \alpha \sum_{s=1}^{t(j,k)} p_{i_sjk} = \alpha\delta(j,k) \geqslant \alpha\delta > 0.$$

Since, due to the property (*iv*) given above, Dobrushin's ergodicity coefficient $\delta(\cdot)$ is continuous and the set $\mathbb{S}_\alpha := \{\mathbf{x} \in \mathbb{S}^{m-1} : \mathbf{x} \geqslant \alpha\mathbf{e}\}$ is compact, we obtain that

$$\lambda := \max_{\mathbf{x}\in\mathbb{S}_\alpha}\left(\delta\left(\mathbb{P}(\mathbf{x})\right)\right) = \delta\left(\mathbb{P}(\mathbf{x}^*)\right) < 1$$

for some $\mathbf{x}^* \in \mathbb{S}_\alpha := \{\mathbf{x} \in \mathbb{S}^{m-1} : \mathbf{x} \geqslant \alpha\mathbf{e}\}$. Consequently, since $\mathbf{x}^{(n)} \in \mathbb{S}_\alpha$, i.e., $\mathbf{x}^{(n)} \geqslant \alpha\mathbf{e}$ for any $n \in \mathbb{N}$, we then obtain that

$$\delta\left(\mathbb{P}(\mathbf{x}^{(n)})\right) \leqslant \max_{\mathbf{x}\in\mathbb{S}_\alpha}\left(\delta\left(\mathbb{P}(\mathbf{x})\right)\right) = \delta\left(\mathbb{P}(\mathbf{x}^*)\right) =: \lambda < 1, \quad \forall\, n \in \mathbb{N}.$$

It follows from the property (*v*) of Dobrushin's ergodicity coefficient given above that

$$0 \leqslant \delta\left(\mathbb{P}^{[\mathbf{x}^{(n)},\mathbf{x}^{(0)}]}\right) \leqslant \prod_{k=0}^n \delta\left(\mathbb{P}(\mathbf{x}^{(k)})\right) \leqslant \lambda^{n+1}.$$

Hence, we have

$$\lim_{n\to\infty}\delta\left(\mathbb{P}^{[\mathbf{x}^{(n)},\mathbf{x}^{(0)}]}\right) = 0.$$

Therefore, due to [22, Ch. 4, Sect. 4.3, Lm. 4.1], the backwards products (which are the transpose of forwards products) of doubly stochastic matrices $\{\mathbb{P}_{\mathbf{x}^{(n)}}\}_{n=0}^{\infty}$ are weakly ergodic (see [22, Ch. 4, Sect. 4.3, Def. 4.5]). Moreover, weak and strong ergodicity (see [22, Ch. 4, Sect. 4.3, Def. 4.6]) are equivalent for the backwards products of doubly stochastic matrices (see [22, Ch. 4, Sect. 4.6, Thm. 4.17]). Due to the definition of strong ergodicity (see [22, Ch. 4, Sect. 4.3, Def. 4.6]), this means that the backwards products $\left\{\mathbb{P}^{\left[\mathbf{x}^{(n)},\mathbf{x}^{(0)}\right]}\right\}_{n=0}^{\infty}$ of doubly stochastic matrices $\{\mathbb{P}_{\mathbf{x}^{(n)}}\}_{n=0}^{\infty}$ must converge to the rank-1 doubly stochastic matrix. Since the only rank–one doubly stochastic matrix is $m\mathbf{c}^T\mathbf{c}$, we obtain that

$$\lim_{n\to\infty} \mathbb{P}^{\left[\mathbf{x}^{(n)},\mathbf{x}^{(0)}\right]} = m\mathbf{c}^T\mathbf{c},$$
$$\lim_{n\to\infty} \mathbf{x}^{(n+1)} = \lim_{n\to\infty} \mathbb{P}^{\left[\mathbf{x}^{(n)},\mathbf{x}^{(0)}\right]}\mathbf{x}^{(0)} = \mathbf{c}, \quad \mathbf{x}^{(0)} \in \mathbb{S}_\alpha,$$

where $\mathbf{c} = (\frac{1}{m}, \cdots, \frac{1}{m})^T$. This completes the proof. □

Remark 4.1 (The Geometric Rate of Convergence). *Due to the classical Markov-Dobrushin inequality* $\|\mathbb{P}\mathbf{x} - \mathbb{P}\mathbf{y}\|_1 \leqslant \delta(\mathbb{P})\|\mathbf{x} - \mathbf{y}\|_1$ *for a doubly stochastic matrix* $\mathbb{P}$ *(see [22]), it follows from Step 5 of the proof of Theorem 4.1 that*

$$\begin{aligned}\|\mathbf{x}^{(n+1)} - \mathbf{c}\|_1 &= \left\|\mathbb{P}^{\left[\mathbf{x}^{(n)},\mathbf{x}^{(0)}\right]}\mathbf{x}^{(0)} - \mathbf{c}\right\|_1 \\ &= \left\|\mathbb{P}^{\left[\mathbf{x}^{(n)},\mathbf{x}^{(0)}\right]}\mathbf{x}^{(0)} - \mathbb{P}^{\left[\mathbf{x}^{(n)},\mathbf{x}^{(0)}\right]}\mathbf{c}\right\|_1 \\ &\leqslant \delta\left(\mathbb{P}^{\left[\mathbf{x}^{(n)},\mathbf{x}^{(0)}\right]}\right)\|\mathbf{x}^{(0)} - \mathbf{c}\|_1 \leqslant \lambda^{n+1}\|\mathbf{x}^{(0)} - \mathbf{c}\|_1.\end{aligned}$$

We have a geometric rate of convergence in the multi–agent system given by Protocol–CDSM.

5. Discussions

We now discuss the main result of this paper and compare it with the previous results [2], [13]–[20] on the consensus problem. The following result was proved in the papers [13]–[18].

Theorem 5.1 (Consensus in positive cubic triply stochastic matrix). *Let*

$$\mathcal{P} = (\mathbb{P}_{\bullet\bullet 1}|\cdots|\mathbb{P}_{\bullet\bullet m})$$

be a cubic triply stochastic matrix and let $\mathcal{Q} : \mathbb{S}^{m-1} \to \mathbb{S}^{m-1}$ *be the associated quadratic stochastic operator. If* $\mathcal{P} > 0$ *then the opinion sharing dynamics of the multi–agent system given by Protocol–CDSM eventually reaches a consensus.*

The subsequent step involved relaxing the *triple stochasticity* condition imposed on positive cubic stochastic matrices. In the paper [2], consensus was achieved within the multi–agent system governed by positive cubic *doubly* stochastic matrices. Following this result, the subsequent research focus shifted towards relaxing the positivity condition of doubly stochastic matrices. This objective was successfully achieved for *diagonally primitive* cubic doubly stochastic matrices in the paper [19]. We recall that a cubic stochastic matrix $\mathcal{P} = (p_{ijk})_{i,j,k=1}^m$ is called *diagonally primitive* if its diagonal $\mathrm{diag}(\mathcal{P}) := (p_{jjk})_{j,k=1}^m$ is a *primitive* square stochastic matrix [19], i.e., there exists $s \in \mathbb{N}$ such that the s^{th} power of the square stochastic matrix

$\text{diag}(\mathcal{P})$ is positive, i.e., $\left[\text{diag}(\mathcal{P})\right]^s > 0$, where

$$\text{diag}(\mathcal{P}) := \begin{pmatrix} p_{111} & p_{112} & \cdots & p_{11m} \\ p_{221} & p_{222} & \cdots & p_{22m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{mm1} & p_{mm2} & \cdots & p_{mmm} \end{pmatrix} = \begin{pmatrix} \left(\mathbf{p}_{11\bullet}\right)^T \\ \left(\mathbf{p}_{22\bullet}\right)^T \\ \vdots \\ \left(\mathbf{p}_{mm\bullet}\right)^T \end{pmatrix}.$$

It is worth noting that for diagonally primitive cubic doubly stochastic matrices (as demonstrated in Example 5.1), off-diagonal entries may indeed be zero. As long as (some power of) the diagonal matrix $\text{diag}(\mathcal{P})$ of cubic doubly stochastic matrix is positive, the consensus can still be achieved within the system.

In the next stage of research focused on off-diagonally positive cubic doubly stochastic matrices. Namely, in the paper [20], the consensus was successfully established within the multi–agent system, even when the diagonal matrix of the cubic doubly stochastic matrix could potentially have zero entries, as long as the off-diagonal entries are positive (see Example 5.2). Recent research, as presented in [19], [20], has opened up the opportunity to explore consensus problems even in scenarios where cubic doubly stochastic matrices contain zero entries in their diagonals or off-diagonal positions.

The primary objective of this paper is to investigate the consensus problem under the condition that cubic doubly stochastic matrices may contain zero entries in both their diagonals and off-diagonal positions simultaneously. To a certain degree, as demonstrated in Example 5.3, the research outcomes presented in this paper extend, unify, and consolidate the research findings from the previously mentioned papers [19], [20]. This represents the novel contribution of the paper. Let us now provide some concrete examples.

Example 5.1. *We consider the cubic doubly stochastic matrix $\mathcal{P} = (\mathbb{P}_{1\bullet\bullet}|\mathbb{P}_{2\bullet\bullet}|\mathbb{P}_{3\bullet\bullet})$, where $\mathbb{P}_{1\bullet\bullet}$, $\mathbb{P}_{2\bullet\bullet}$, and $\mathbb{P}_{3\bullet\bullet}$ are square doubly stochastic matrices*

$$\mathbb{P}_{1\bullet\bullet} = \begin{pmatrix} p_{111} & p_{112} & p_{113} \\ p_{121} & p_{122} & p_{123} \\ p_{131} & p_{132} & p_{133} \end{pmatrix}, \qquad \mathbb{P}_{2\bullet\bullet} = \begin{pmatrix} p_{211} & p_{212} & p_{213} \\ p_{221} & p_{222} & p_{223} \\ p_{231} & p_{232} & p_{233} \end{pmatrix}, \qquad \mathbb{P}_{3\bullet\bullet} = \begin{pmatrix} p_{311} & p_{312} & p_{313} \\ p_{321} & p_{322} & p_{323} \\ p_{331} & p_{332} & p_{333} \end{pmatrix}.$$

The following quadratic stochastic operator $\mathcal{Q} : \mathbb{S}^2 \to \mathbb{S}^2$ presents Protocol–CDSM

$$\mathcal{Q}(\mathbf{x}) = x_1 \left(\mathbb{P}^T_{1\bullet\bullet}\mathbf{x}\right) + x_2 \left(\mathbb{P}^T_{2\bullet\bullet}\mathbf{x}\right) + x_3 \left(\mathbb{P}^T_{3\bullet\bullet}\mathbf{x}\right) = \mathbb{P}\left(\mathbf{x}\right)\mathbf{x}$$

where

$$\mathbb{P}\left(\mathbf{x}\right) = x_1\mathbb{P}^T_{1\bullet\bullet} + x_2\mathbb{P}^T_{2\bullet\bullet} + x_3\mathbb{P}^T_{3\bullet\bullet}$$

is a square doubly stochastic matrix.

It was shown in the papers [13]–[18] that if the square doubly stochastic matrices $\mathbb{P}_{1\bullet\bullet} > 0$, $\mathbb{P}_{2\bullet\bullet} > 0$, and $\mathbb{P}_{3\bullet\bullet} > 0$ are positive and

$$\mathbb{P}_{1\bullet\bullet} + \mathbb{P}_{2\bullet\bullet} + \mathbb{P}_{3\bullet\bullet} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{5.1}$$

then the consensus is established in the system described by Protocol–CDSM. Later, this result was generalized in the paper [2]. Namely, the consensus was established for the positive square doubly stochastic matrices $\mathbb{P}_{1\bullet\bullet} > 0$, $\mathbb{P}_{2\bullet\bullet} > 0$, $\mathbb{P}_{3\bullet\bullet} > 0$ without the constraint (5.1). However, these results were further improved in the paper [19]. Namely, without positivity of the square doubly stochastic matrices $\mathbb{P}_{1\bullet\bullet}, \mathbb{P}_{2\bullet\bullet}, \mathbb{P}_{3\bullet\bullet}$ and without the constraint (5.1), the consensus is still established in the system described by Protocol–CDSM if the cubic doubly stochastic matrix $\mathcal{P}$

is (only) diagonally primitive, i.e., for some $s \in \mathbb{N}$ we have

$$[\mathrm{diag}(\mathcal{P})]^s = \begin{pmatrix} p_{111} & p_{112} & p_{113} \\ p_{221} & p_{222} & p_{223} \\ p_{331} & p_{332} & p_{333} \end{pmatrix}^s > 0.$$

Example 5.2. *We consider the following square doubly stochastic matrices*

$$\mathbb{P}_{1\bullet\bullet} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}, \qquad \mathbb{P}_{2\bullet\bullet} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}, \qquad \mathbb{P}_{3\bullet\bullet} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

We then obtain the following quadratic stochastic operator $\mathcal{Q} : \mathbb{S}^2 \to \mathbb{S}^2$

$$\mathcal{Q}(\mathbf{x}) = (x_1^2 + x_2^2 + x_3^2)\frac{\mathbf{e}_1 + \mathbf{e}_2}{2} + (x_1x_2 + x_1x_3 + x_2x_3)\frac{\mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_3}{2}$$

where

$$\mathbf{e}_1 = (1, 0, 0)^T, \qquad \mathbf{e}_2 = (0, 1, 0)^T, \qquad \mathbf{e}_3 = (0, 0, 1)^T$$

are vertices of the simplex $\mathbb{S}^2$. On the one hand, since for any $s \in \mathbb{N}$ we have

$$[\mathrm{diag}(\mathcal{P})]^s = \mathrm{diag}(\mathcal{P}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},$$

the cubic doubly stochastic matrix $\mathcal{P} = (\mathbb{P}_{1\bullet\bullet}|\mathbb{P}_{2\bullet\bullet}|\mathbb{P}_{3\bullet\bullet})$ is not diagonally primitive. On the other hand, since $\mathbf{p}_{ij\bullet} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}) > 0$ for all $i, j \in \mathbf{I}_3$ with $i \neq j$, i.e., $\mathcal{P} = (\mathbb{P}_{1\bullet\bullet}|\mathbb{P}_{2\bullet\bullet}|\mathbb{P}_{3\bullet\bullet})$ is off-diagonally positive, it was shown in the paper [20] that the opinion sharing dynamics of the multi–agent system given by Protocol–CDSM still eventually reaches a consensus for any initial opinion $\mathbf{x} \in \mathbb{S}^2$.

It is worth mentioning that there is still a room to further improvement of these results. Indeed, Theorem 4.1 is a further extension and generalization of the results published in the papers [2], [13]–[20]. Namely, without positivity of the square doubly stochastic matrices $\mathbb{P}_{1\bullet\bullet}, \mathbb{P}_{2\bullet\bullet}, \mathbb{P}_{3\bullet\bullet}$ and without the constraint (5.1), the consensus is still established in the system described by *Protocol–CDSM* if the influence of cubic matrix $\mathcal{P} = (\mathbb{P}_{1\bullet\bullet}|\mathbb{P}_{2\bullet\bullet}|\mathbb{P}_{3\bullet\bullet})$ is positive. To illustrate it, let us consider the next example.

Example 5.3. *We consider the following square doubly stochastic matrices*

$$\mathbb{P}_{1\bullet\bullet} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \qquad \mathbb{P}_{2\bullet\bullet} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \qquad \mathbb{P}_{3\bullet\bullet} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

We then obtain the following quadratic stochastic operator $\mathcal{Q} : \mathbb{S}^2 \to \mathbb{S}^2$:

$$\mathcal{Q}(\mathbf{x}) = (x_1^2 + x_2^2 + x_3^2)\frac{\mathbf{e}_1 + \mathbf{e}_2}{2} + x_1x_2(\mathbf{e}_1 + \mathbf{e}_3) + x_1x_3(\mathbf{e}_2 + \mathbf{e}_3) + x_2x_3\frac{\mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_3}{2}$$

where

$$\mathbf{e}_1 = (1, 0, 0)^T, \qquad \mathbf{e}_2 = (0, 1, 0)^T, \qquad \mathbf{e}_3 = (0, 0, 1)^T$$

are vertices of the simplex $\mathbb{S}^2$. On the one hand, since for any $s \in \mathbb{N}$

$$[\mathrm{diag}(\mathcal{P})]^s = \mathrm{diag}(\mathcal{P}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},$$

the cubic doubly stochastic matrix $\mathcal{P} = (\mathbb{P}_{1\bullet\bullet}|\mathbb{P}_{2\bullet\bullet}|\mathbb{P}_{3\bullet\bullet})$ is not diagonally primitive. On the other hand, $\mathcal{P} = (\mathbb{P}_{1\bullet\bullet}|\mathbb{P}_{2\bullet\bullet}|\mathbb{P}_{3\bullet\bullet})$ is not off-diagonally positive. However, the influence of cubic matrix $\mathcal{P} = (\mathbb{P}_{1\bullet\bullet}|\mathbb{P}_{2\bullet\bullet}|\mathbb{P}_{3\bullet\bullet})$ is positive. Therefore, it follows from Theorem 4.1 that the opinion *sharing dynamics of the multi–agent system given by Protocol–CDSM still eventually reaches a consensus for any initial opinion.*

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