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ASYMPTOTIC REPRESENTATION OF HYPERGEOMETRIC BERNOULLI POLYNOMIALS OF ORDER 2 INSIDE DOMAINS RELATED TO THE ROOTS OF $e^w - 1 - w = 0$

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Abstract. Among several approaches towards the classical Bernoulli polynomials $B_n(x)$, one is the definition by the generating function

$$\frac{we^{xw}}{e^w - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{w^n}{n!} \quad \text{for} \quad |w| < 2\pi.$$

As a generalization of $B_n(x)$, for any positive integer N, a new class of Bernoulli polynomials called Hypergeometric Bernoulli polynomials of order N, $B_n(N, x)$ was established. For the particular case $N = 2$ these polynomials are given by

$$\frac{1}{2}\frac{w^2e^{xw}}{e^w - 1 - w} = \sum_{n=0}^{\infty} B_n(2, x)\frac{w^n}{n!} \quad \text{for} \quad |w| < 2\pi.$$

Several asymptotic formulas for the Bernoulli and Euler polynomials inside different domains related to the roots of $\phi(w) = e^w - 1$ were found.

In this paper, we consider an integral representation for $B_n(2, x)$ and establish a zero attractor for the re-scaled polynomials $B_n(2, nx)$ for large values of n. We briefly discuss some analogous asymptotic formulas of $B_n(2, x)$ inside domains related to the roots of $\varphi(w) = e^w - 1 - w$.

Keywords: Bernoulli polynomials, hypergeometric Bernoulli polynomials, integral representation, asymptotic formula, zero attractor.

Mathematics Subject Classification: 11M20, 11M26

1. Introduction

1.1. Bernoulli Polynomials. The classical Bernoulli polynomials $B_n(x)$ were extensively considered by many authors and several generalizations were made, for which analogous properties were obtained. Alternative to the definition via the generating function

$$\frac{we^{xw}}{e^w - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{w^n}{n!} \quad \text{for} \quad |w| < 2\pi, \tag{1.1}$$

these polynomials are equivalently defined by the recurrence formula

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} b_k x^{n-k}, \tag{1.2}$$

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where the b_k are the Bernoulli numbers. Equivalently, Bernoulli polynomials are also defined by an Appell sequence with zero mean, as

$$B_0(x) = 1, \qquad B_n'(x) = nB_{n-1}(x), \qquad \int_0^1 B_n(x)\, dx = \begin{cases} 1 & \text{for} \quad n = 0, \\ 0 & \text{for} \quad n > 0. \end{cases} \tag{1.3}$$

Each of the three definitions: the generating function (1.1), the recurrence formula (1.2) and the Appell sequence (1.3) define the same sequence $\{B_n(x)\}$ of Bernoulli polynomials. The sequence of Bernoulli polynomials possess many interesting properties. Some of the well–known properties of $B_n(x)$ are

- *Symmetry property*
$$B_n(1-x) = (-1)^n B_n(x), \quad n \geqslant 0.$$
- *Difference equation*
$$B_n(x+1) - B_n(x) = nx^{n-1}, \quad n \geqslant 1.$$
- *Addition formula*
$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) y^{n-k}.$$
- *Raabe's multiplication formula*
$$B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n\left(x + \frac{k}{m}\right),$$
where m and n are integers with $n \geqslant 0$ and $m \geqslant 1$.

1.2. Hypergeometric Bernoulli Polynomials. On the base of the definition of $B_n(x)$ by the generating function, several generalizations were made by different authors. Among these generalizations are the polynomials $\{A_n(x)\}$ introduced by Howard [7]

$$\frac{1}{2} \frac{w^2 e^{xw}}{e^w - 1 - w} = \sum_{n=0}^{\infty} A_n(x) \frac{w^n}{n!} \quad \text{for} \quad |w| < 2\pi.$$

Several authors considered similar generalizations of Bernoulli polynomials; we refer to [7], [3], [5] and [4] for some other related concepts to generalization of Bernoulli polynomials.

In this paper, we focus on the generalization made by Hassen and Nguyen [5] referred to as *hypergeometric Bernoulli polynomials of order N*.

Definition 1.1. *For any integer $N \geqslant 1$, hypergeometric Bernoulli polynomials of order N, $B_n(N, x)$, are defined as*

$$\frac{1}{N!} \frac{w^N e^{xw}}{e^w - T_{N-1}(w)} = \sum_{n=0}^{\infty} B_n(N, x) \frac{w^n}{n!} \quad \textit{for} \quad |w| < 2\pi, \tag{1.4}$$

where

$$T_m(w) = 1 + w + \frac{w^2}{2!} + \ldots + \frac{w^m}{m!} = \sum_{k=0}^{m} \frac{w^k}{k!}.$$

The particular case of (1.4) with $N = 1$ reduces to the classical Bernoulli polynomials $B_n(x)$ given in (1.1) and for $N = 2$ it represents the hypergeometric Bernoulli polynomials of order 2 defined as

$$\frac{1}{2} \frac{w^2 e^{xw}}{e^w - 1 - w} = \sum_{n=0}^{\infty} B_n(2, x) \frac{w^n}{n!} \quad \text{for} \quad |w| < 2\pi. \tag{1.5}$$

As in the classical case, there are several approaches for defining hypergeometric Bernoulli polynomials. Accordingly, the hypergeometric Bernoulli polynomials $B_n(N,x)$ are equivalently defined by a recurrence formula as

$$B_n(N,x)=\sum_{k=0}^{n}\binom{n}{k}B_k(N)x^{n-k}. \tag{1.6}$$

The hypergeometric Bernoulli polynomials $B_n(N,x)$ are also defined in terms of an Appell sequence with zero moments as

$$\begin{gathered}B_0(N,\,x)=1,\qquad B_n'(N,x)=nB_{n-1}(N,x),\\ \int_0^1(1-x)^{N-1}B_n(N,\,x)\,dx=\begin{cases}\dfrac{1}{N} & \text{for}\quad n=0\\ 0 & \text{for}\quad n>0.\end{cases}\end{gathered} \tag{1.7}$$

In [5], Hassen and Nguyen proved the equivalence of these different definitions of hypergeometric Bernoulli polynomials.

Theorem 1.1 ([5])**.** *For each integer $N\geqslant 1$, the definitions of the hypergeometric Bernoulli polynomials $B_n(N,x)$ via the generating function (1.4), the recurrence formula (1.6) and the Appell sequence (1.7) are equivalent.*

In [1], Asfaw and Hassen established the following properties of $B_n(N,x)$ which are analogous to that of the classical Bernoulli polynomials $B_n(x)$.

- *Addition formula*
  For hypergeometric Bernoulli polynomials of order N we have the addition formula
$$B_n(N,x+y)=\sum_{k=0}^{n}\binom{n}{k}B_k(N,x)y^{n-k}.$$
- *Difference equation*
  For each $n=2,3,4,\ldots$, the polynomials $B_n(2,x)$ satisfy the equation
$$B_n(2,x+1)-B_n(2,x)=nB_{n-1}(2,x)+\binom{n}{2}x^{n-2}.$$
- *Generalized difference equation*
  For each positive integer m we have
$$B_n(2,x+m)=\sum_{k=0}^{m}\binom{m}{k}B_n^{(k)}(2,x)+\frac{n!}{2}\sum_{k=0}^{m-1}\sum_{j=0}^{m-1-k}\binom{m-1-k}{j}\frac{(x+k)^{n-2-j}}{(n-2-j)!},$$
  where $B_n^{(k)}(2,x)=n(n-1)\ldots(n-k+1)B_{n-k}(2,x)$ is the k^{th} derivative of $B_n(2,x)$.

Remark 1.1. *The three alternative definitions; (1.4), (1.6) and (1.7) of $B_n(N,x)$ are analogous to the three definitions of $B_n(x)$ given in (1.1), (1.2) and (1.3), respectively. This interesting relation between $B_n(N,x)$ and $B_n(x)$ motivated us to study further analogous properties of these new class of polynomials, including the integral representations and asymptotic formulas of $B_n(2,x)$.*

2. Preliminaries

2.1. Some properties of roots of $e^z-1-z=0$. The function $\varphi(w)=e^z-1-z$ appears in the generating function of $B_n(2,a)$ in (1.5). The roots of this function are basic quantities for the series representation of $B_n(2,x)$. In [6], Hassen and Nguyen discussed several concepts related to the roots of $\varphi(z)$.

If $z_k = x_k + iy_k$ is a root of $\varphi(z)$, then its complex conjugate $\bar{z}_k = x_k - iy_k$ is also a root, and we usually list all the roots in pairs as $\{z_k, \bar{z}_k\}$. In fact, all the roots $z_k = x_k + iy_k$ and $\bar{z}_k = x_k - iy_k$ of φ lie inside the right half–plane.

The following results are proved in [6] with slight modifications made in [1].

Lemma 2.1. *Let $z_k = x_k + iy_k = r_k e^{\theta_k}$ be the roots of $\varphi(z)$ that lie in the upper half of the complex plane and such that $0 < r_1 < r_2 < r_3 < \cdots$.*

(i) *The inequality*

$$2\pi k + \frac{\pi}{3} < y_k < 2\pi k + \frac{\pi}{2}, \qquad k = 1, 2, \ldots$$

holds. Moreover, for each k there exists exactly one root z_k with the imaginary part obeying this inequality, and there are no other zeros elsewhere in the complex plane.

(ii) *The arguments θ_k of z_k obey the inequalities $\theta_1 < \theta_2 < \theta_3 < \cdots < \frac{\pi}{2}$, and $\theta_k \to \frac{\pi}{2}$ as $k \to \infty$.*

(iii) *As k increases, all the quantities x_k, y_k and θ_k increase. In addition, we have $y_k < e^{x_k}$ and $y_k \to e^{x_k}$ as $k \to \infty$.*

2.2. Dominant roots. There are two roots of $\varphi(z)$ with minimal moduluses. These are $z_1 = x_1 + iy_1 = r_1 e^{\theta_1}$ and $\bar{z}_1 = x_1 - iy_1 = r_1 e^{-\theta_1}$, and we call them dominant roots. The approximate values of x_1, y_1, r_1 and θ_1 (as calculated by Mathematica) are $x_1 \approx 2.0888$, $y_1 \approx 7.4615$, $r_1 \approx 7.7484$ and $\theta_1 \approx 1.2978$. That is,

$$z_1 \approx 2.0888 + i\,7.4615, \qquad \bar{z}_1 \approx 2.0888 - i\,7.4615. \tag{2.1}$$

We also list approximate values for a few first roots $z_k = x_k + iy_k = r_k e^{\theta_k}$.

k	x_k	y_k	r_k	θ_k
1	2.0888	7.4615	7.7484	1.2978
2	2.6641	13.879	14.132	1.3812
3	3.0263	20.224	20.449	1.4223
4	3.2917	26.543	26.747	1.4474
5	3.5013	32.851	33.037	1.4646
6	3.6745	39.151	39.323	1.4772
7	3.8222	45.447	45.608	1.4869

TABLE 1. Few approximate values of $z_k = x_k + iy_k = r_k e^{\theta_k}$

On Figure 1 below we plot a few first roots z_k.

2.3. Szegö curves related to the roots of $\varphi(z)$. Consider the polynomials $T_n(x)$, the Taylor polynomials of the function e^x,

$$T_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

In terms of generating functions, the functions $T_n(x)$ are given by

$$\frac{e^{xw}}{1-w} = \sum_{n=0}^{\infty} T_n(x) w^n.$$

The zeros of $T_n(x)$ have an interesting asymptotic behavior: the zeros of $T_n(nx)$ approach the curve $|ze^{1-z}| = 1$ for large values of n, see Figure 2.

The curve $|ze^{1-z}| = 1$ was first introduced by Gabor Szegö in 1924.

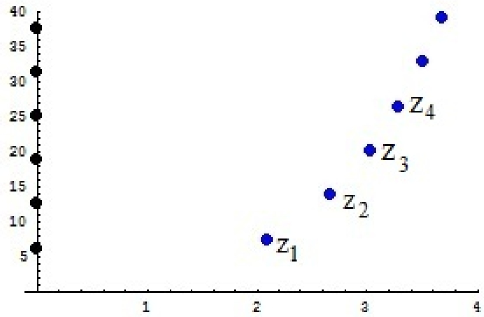

FIGURE 1. Plots of some roots of $\varphi(z)$

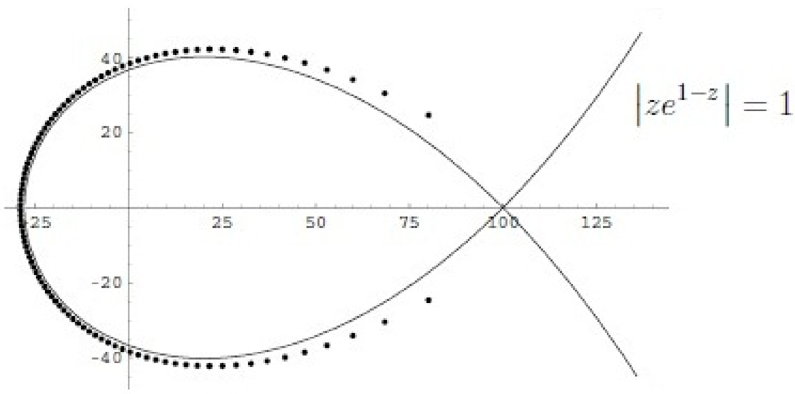

FIGURE 2. The complex zeros of $T_{100}(100x)$ along the curve $|ze^{1-z}| = 1$

Definition 2.1 (Standard Szegö curve)**.** *Let* $\phi(z) = ze^{1-z}$. *The curve* $\mathbb{S}$ *in the complex plane defined by*

$$\mathbb{S} = \left\{z \in \mathbb{C} : \left|ze^{1-z}\right| = 1,\ |z| \leqslant 1\right\} \tag{2.2}$$

is called standard Szegö curve.

We observe that the function $\varphi(z)$ appears in (1.5) and recall its roots z_k for $k = 1, 2, 3, \cdots$ Then we consider the function $\phi(z_k w) = wz_k e^{1-z_k w}$ and we define different Szegö curves related to the roots of $\varphi(z)$.

Definition 2.2 (Szegö curves)**.** *Let* $z_k = x_k + iy_k$, $k = 1, 2, 3, \cdots$ *be the roots of* $\varphi(z)w$ *with* $|z_k| = r_k$. *The curves* $\frac{1}{z_k}\mathbb{S}$, *called Szegö curves related to the roots of* $\varphi(z)$, *are defined as*

$$\frac{1}{z_k}\mathbb{S} = \left\{z \in \mathbb{C}\ :\ \left|zz_k e^{1-z_k z}\right| = 1,\ |z| \leqslant \frac{1}{r_k}\right\}. \tag{2.3}$$

The two special Szegö curves related to the dominant roots z_1 *and* $\bar{z}_1$ *are called* dominant Szegö curves. *We denote the dominant Szegö curves by* $\mathbb{S}_1 = \frac{1}{z_1}\mathbb{S}$ *and* $\hat{\mathbb{S}}_1 = \frac{1}{\bar{z}_1}\mathbb{S}$, *respectively,*

$$\mathbb{S}_1 = \left\{z \in \mathbb{C}\ :\ \left|zz_1 e^{1-z_1 z}\right| = 1,\ |z| \leqslant \frac{1}{r_1}\right\}, \tag{2.4}$$

$$\hat{\mathbb{S}}_1 = \left\{z \in \mathbb{C}\ :\ \left|z\bar{z}_1 e^{1-\bar{z}_1 z}\right| = 1\ \ |z| \leqslant \frac{1}{r_1}\right\}. \tag{2.5}$$

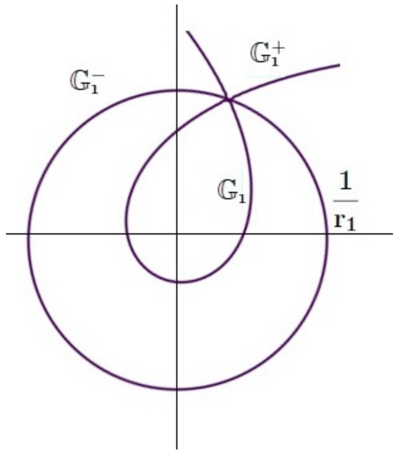

FIGURE 3. Three different open domains in the complex plane.

Let the domains $\mathbb{G}_1 = \mathbb{G}_{z_1}$ and $\hat{\mathbb{G}}_1 = \mathbb{G}_{\bar{z}_1}$ be the interiors of $\mathbb{S}_1$ and $\hat{\mathbb{S}}_1$, respectively, and let $\mathbb{G} = \mathbb{G}_1 \cup \hat{\mathbb{G}}_1$. In other words, $\mathbb{S}_1$ and $\hat{\mathbb{S}}_1$ are the boundaries of $\mathbb{G}_1$ and $\hat{\mathbb{G}}_1$. That is, $\mathbb{S}_1 = \partial\mathbb{G}_1$ and $\hat{\mathbb{S}}_1 = \partial\hat{\mathbb{G}}_1$.

Remark 2.1. *We observe that each Szegö curve given in* (2.3) *is obtained from the standard Szegö curve* (2.2) *by a a dilatation by* $\frac{1}{r_k}$ *and a rotation by* $\pm\theta_k$. *Indeed, the function* $\phi(z)$ *is conformal in the unit disk* $B(0,1)$, *hence both* $\phi(z_1 z)$ *and* $\phi(\bar{z}_1 z)$ *are also conformal in the disk* $B(0, \frac{1}{r_1})$.

For particular values of z_1 and $\bar{z}_1$ given in (2.1), the curves $\mathbb{S}_1$ and $\hat{\mathbb{S}}_1$ and their interiors $\mathbb{G}_1$ and $\hat{\mathbb{G}}_1$ are roughly sketched as shown on Figure 4.

If we omit the restriction $|z| \leqslant \frac{1}{r_1}$ in (2.4) and (2.5), the equations $|\phi(z_1 z)| = 1$ and $|\phi(\bar{z}_1 z)| = 1$ define unbounded curves in the complex plane. We denote these unbounded Szegö curves related to z_1 and $\bar{z}_1$ by Γ_1 and $\hat{\Gamma}_1$, respectively,

$$\Gamma_1 = \{z \in \mathbb{C} : |\phi(z_1 z)| = 1\}, \qquad \hat{\Gamma}_1 = \{z \in \mathbb{C} : |\phi(\bar{z}_1 z)| = 1\}.$$

Then $\mathbb{S}_1 \subseteq \Gamma_1$ and $\hat{\mathbb{S}}_1 \subseteq \hat{\Gamma}_1$. Indeed, $\mathbb{S}_1$ and $\hat{\mathbb{S}}_1$ are the portions of Γ_1 and $\hat{\Gamma}_1$ that lie in the closed disk $\bar{B}\left(0, \frac{1}{r_1}\right)$, respectively. As an alternative to (2.4) and (2.5), the dominant Szegö curves $\mathbb{S}_1$ and $\hat{\mathbb{S}}_1$ can equivalently be described as

$$|\phi(z_1 z)| = 1, \ \operatorname{Re}(z_1 z) \leqslant 1, \qquad |\phi(\bar{z}_1 z)| = 1, \operatorname{Re}(\bar{z}_1 z) \leqslant 1. \tag{2.6}$$

Here $\mathbb{S}_1$ is the portion of Γ_1 that lies in the closed half-plane $\operatorname{Re}(z_1 z) \leqslant 1$. In other words, both (2.4) and (2.6) define the same curve, the Szegö curve $\mathbb{S}_1$.

The curve Γ_1 divides the complex plane into three different domains. We denote these domains by $\mathbb{G}_1$, $\mathbb{G}_1^+$ and $\mathbb{G}_1^-$, where

1. $\mathbb{G}_1$ is the interior of the Szegö curve $\mathbb{S}_1$, given by

$$\mathbb{G}_1 = \left\{ z \in \mathbb{C} : |\phi(z_1 z)| < 1, \ |z| \leqslant \frac{1}{r_1} \right\}. \tag{2.7}$$

2. $\mathbb{G}_1^+$ is the unbounded domain given by

$$\mathbb{G}_1^+ = \left\{ z \in \mathbb{C} : |\phi(z_1 z)| < 1, \ |z| > \frac{1}{r_1} \right\}. \tag{2.8}$$

3. $\mathbb{G}_1^-$ is the unbounded domain given by

$$\mathbb{G}_1^- = \{z \in \mathbb{C} : |\phi(z_1 z)| > 1\}. \tag{2.9}$$

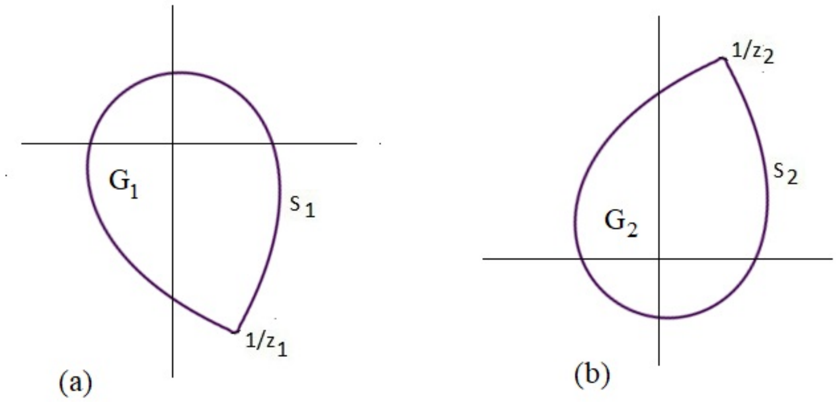

FIGURE 4. The Szegö curves $|z_1 z e^{1-z_1 z}| = 1$ and $|\bar{z}_1 z e^{1-\bar{z}_1 z}| = 1$ for $|z| \leqslant \frac{1}{r_1}$.

For the dominant root $\bar{z}_1$, we simply replace z_1 by $\bar{z}_1$ in (2.7), (2.8) and (2.9) to define the corresponding domains $\hat{\mathbb{G}}_1$, $\hat{\mathbb{G}}_1^+$ and $\hat{\mathbb{G}}_1^-$, respectively. The introduced domains are shown on Figure 3.

Lemma 2.2. *The Szegö curves $\mathbb{S}_1$ and $\hat{\mathbb{S}}_1$ intersect each other at exactly two points, which are located on the real axis.*

Proof. These two curves intersect at point $z \in \mathbb{C}$ if and only if $|\phi(z_1 z)| = |\phi(\bar{z}_1 z)|$. That is, $z = x + iy$ is an intersection point only if it satisfies

$$\left|e^{1-(x_1+iy_1)(x+iy)}\right| = \left|e^{1-(x_1-iy_1)(x+iy)}\right|.$$

This yields $e^{y_1 y} = e^{-y_1 y}$, which implies $\text{Im}(z) = y = 0$, hence z lies on the real axis. □

Let P and Q be the two points of intersection of $\mathbb{S}_1$ and $\hat{\mathbb{S}}_1$. Then the line connecting P and Q, $L_{P,Q}$, is the real axis. Hence, $L_{P,Q}$ divides the complex plane in to the upper half-plane $\mathcal{H}^+$ and lower half–plane $\mathcal{H}^-$. We can express these half–planes as

$$\mathcal{H}^+ = \{z \in \mathbb{C} : |\phi(z_1 z)| > |\phi(\bar{z}_1 z)|\}, \qquad \mathcal{H}^- = \{z :\in \mathbb{C}\, |\phi(z_1 z)| < |\phi(\bar{z}_1 z)|\}.$$

Definition 2.3. *Let z_1 and $\bar{z}_1$ be the two dominant zeros of $\varphi(z)$. If $\mathbb{S}_1$ and $\hat{\mathbb{S}}_1$ are their corresponding Szegö curves, then we define the domains $\mathbb{D}_1 = \mathbb{D}_{z_1}$ and $\hat{\mathbb{D}}_1 = \mathbb{D}_{\bar{z}_1}$ as*

$$\mathbb{D}_1 = \mathbb{G}_1 \cap \mathcal{H}^-, \qquad \hat{\mathbb{D}}_1 = \hat{\mathbb{G}}_1 \cap \mathcal{H}^+,$$

where $\mathbb{G}_1$ and $\hat{\mathbb{G}}_1$ are the interiors of $\mathbb{S}_1$ and $\hat{\mathbb{S}}_1$, respectively. We define the domains $\mathbb{D}$ as the union of $\mathbb{D}_1$ and $\hat{\mathbb{D}}_1$. That is,

$$\mathbb{D} = \mathbb{D}_1 \cup \hat{\mathbb{D}}_1.$$

The Szegö curves $\mathbb{S}_1$ and $\hat{\mathbb{S}}_1$ intersect each other at two points, say P and Q, which are on the real axis. The domains $\mathbb{D}_1$ and $\hat{\mathbb{D}}_1$ are shown in Figure 5 below.

Both $\mathbb{D}_1$ and $\hat{\mathbb{D}}_1$ are disjoint open sets. That is, $\mathbb{D}_1 \cap \hat{\mathbb{D}}_1 = \emptyset$, but they have line segment PQ as their common boundary. Indeed, the boundary $\partial\mathbb{D} = \partial\mathbb{D}_1 \cup \partial\hat{\mathbb{D}}_1$ includes the outer boundary points and all points of the line segment PQ. Moreover, the domains $\mathbb{D}_1$ and $\hat{\mathbb{D}}_1$ can be expressed as

$$\mathbb{D}_1 = \{z \in \mathbb{G}_1 \,:\, |\phi(z_1 z)| < |\phi(\bar{z}_1 z)|\}, \qquad \hat{\mathbb{D}}_1 = \left\{z \in \hat{\mathbb{G}}_1 \,:\, |\phi(\bar{z}_1 z)| < |\phi(z_1 z)|\right\}.$$

Note that $\mathbb{D} = \mathbb{D}_1 \cup \hat{\mathbb{D}}_1 \subseteq \mathbb{G} = \mathbb{G}_1 \cup \hat{\mathbb{G}}_1$. The domains $\mathbb{D}$ and $\mathbb{G}$ are different but they have identical closures. That is, $\overline{\mathbb{D}} = \overline{\mathbb{G}}$.

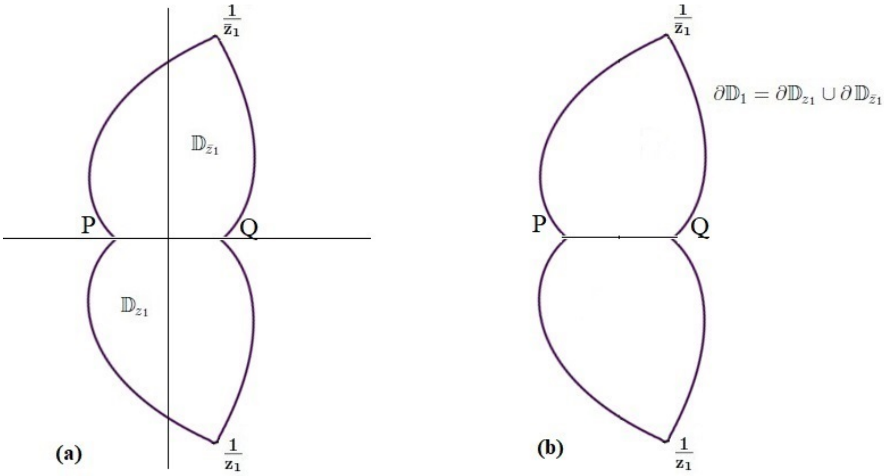

FIGURE 5. Szegö domains $\mathbb{D}_{z_1}$ and $\mathbb{D}_{\bar{z}_1}$ and their boundaries

We now present the fact that z_1 and $\bar{z}_1$ dominate all the other roots of $\varphi(w) = e^w - 1 - w$. In fact, this is one of our results which we think is new in our study.

Theorem 2.1. *If $z_k = x_k + iy_k = r_k e^{i\theta_k}$ is an arbitrary zero of $\varphi(z)$ with $y_k > 0$ and $|z_k| = r_k \geqslant r_2$, then $\frac{1}{z_k}$ lies inside the domain $\mathbb{D}_1$. More generally, for each root z_k of $\varphi(z)$ with $|z_k| > r_1$ we have $\frac{1}{z_k} \in \mathbb{D} = \mathbb{D}_1 \cup \hat{\mathbb{D}}_1$.*

Proof. Let $z_k = x_k + iy_k = r_k e^{i\theta_k}$ be as given in the hypothesis. Then the relation

$$\left|\frac{1}{z_k}\right| = \frac{1}{r_k} < \frac{1}{r_1}$$

shows that $\frac{1}{z_k} \in B(0, \frac{1}{r_1})$ for all $k \geqslant 2$. This yields $\frac{1}{z_k} \notin \bar{\mathbb{G}}_1^+$. For the function $\phi(z_1 z) = z z_1 e^{1 - z_1 z}$ we evaluate $|\phi(z_1 z)|$ at $z = \frac{1}{z_k}$, observe that $r_1 < r_2 < r_k$ for all $k > 2$, and we get

$$\left|\phi\left(z_1 \frac{1}{z_k}\right)\right| = \frac{r_1}{r_k} e^{1 - \frac{(x_1 x_k + y_1 y_k)}{r_k^2}} \leqslant \frac{r_1}{r_2} e^{1 - \frac{(x_1 x_2 + y_1 y_2)}{r_k^2}} \quad \text{for all } k \geqslant 2.$$

But we can use the values of x_1, y_1, r_1, x_2, y_2 and r_2 and we easily see that

$$\frac{r_1}{r_2} e^{1 - \frac{(x_1 x_2 + y_1 y_2)}{r_k^2}} < 1.$$

Thus, $\left|\phi\left(z_1 \frac{1}{z_k}\right)\right| < 1$ and $\left|\frac{1}{z_k}\right| < \frac{1}{r_1}$ so that $\frac{1}{z_k} \in \mathbb{G}_1$. Moreover, since $y_k > 0$, we have $\frac{1}{z_k} \in \mathcal{H}^-$. Hence, $\frac{1}{z_k} \in \mathbb{D}_1$. Similarly, if we assume $y_k < 0$, then we evaluate $|\phi(\bar{z}_1 z)|$ at $z = \frac{1}{z_k}$ and get $\frac{1}{z_k} \in \mathbb{D}_2$. Therefore, $\frac{1}{z_k} \in \mathbb{D} = \mathbb{D}_1 \cup \hat{\mathbb{D}}_1$ for each $k \geqslant 2$. □

3. MAIN RESULTS

3.1. Integral representation for $B_n(2, x)$.

We denote

$$g_2(w) := 2\frac{e^w - 1 - w}{w^2},$$

and we rewrite the definition (1.5) of hypergeometric Bernoulli polynomials of order 2 as

$$\frac{e^{zw}}{g_2(w)} = \sum_{k=0}^{\infty} B_k(2, z) \frac{w^k}{k!}.$$

Let γ be the key–hole contour given on Figure 6 below with $\delta = 1$. Clearly, the curve γ passes through the point (1, 0) and $\delta = 1 < r_1$ so that a neighborhood of γ contains no zeroes of the function $\varphi(z)$.

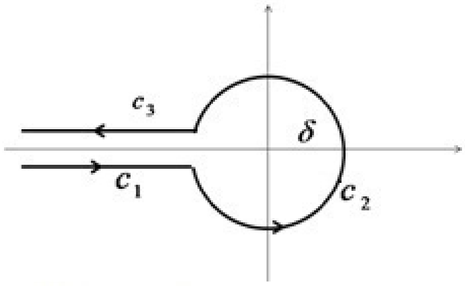

FIGURE 6. The key–hole contour about the origin with $\delta = 1$

Lemma 3.1 (Integral representation). *The hypergeometric Bernoulli polynomials $B_n(2, z)$ satisfy the integral representation*

$$B_n(2, z) = \frac{n!}{4\pi i} \int_\gamma \frac{e^{zw}}{w^n} \frac{w}{e^w - 1 - w} dw.$$

The re-scaled polynomials $B_n(2, nz)$ satisfy the representations

$$B_n(2, nz) = \frac{n!}{4\pi i} \int_\gamma \left(\frac{e^{zw}}{w}\right)^n \frac{w}{e^w - 1 - w} dw. \tag{3.1}$$

Proof. We divide Equation (1.5) by $2\pi i w^{n+1}$ and integrate the result over the curve γ

$$\frac{1}{2\pi i} \int_\gamma \frac{w^2 e^{zw}}{2\left(e^w - 1 - w\right) w^{n+1}} dw = \frac{1}{2\pi i} \int_\gamma \sum_{k=0}^{\infty} \frac{B_k(2, z) w^k}{k! w^{n+1}} dw$$

$$= \sum_{k=0}^{\infty} \frac{B_k(2, z)}{k!} \left(\frac{1}{2\pi i} \int_\gamma \frac{1}{w^{n+1-k}} dw \right) = \frac{B_n(2, z)}{n!};$$

here we have employed that

$$\int_\gamma \frac{1}{w^{n+1-k}} dw = 2\pi i$$

for $k = n$, and by the Cauchy integral theorem the integral vanishes for all $k \neq n$. Thus,

$$B_n(2, z) = \frac{n!}{4\pi i} \int_\gamma \frac{e^{zw}}{w^n} \frac{w}{e^w - 1 - w} dw.$$

Since both sides of this latter equation are entire functions of z, we can replace z by nz and this gives the desired result. □

Remark 3.1. *In Equation (3.1), the substitution of w by w/z changes the curve of integration γ to γ_2, where γ_2 represents the curve γ, in which the radius of C_2 is changed to $\delta = |z|$. If $\varphi(z)$ has zeros at $w = z_k$, then the zeros of $\varphi(w/z)$ are $w = z_k z$. Since $\frac{1}{r_1} \geqslant \frac{1}{r_k}$, for each $z \in \mathbb{C}$ such that $\frac{1}{r_1} < |z| < \infty$, we have $|z_k z| > \frac{r_k}{r_1} \geqslant 1$. This shows that $\frac{w}{\varphi(w/z)}$ is analytic in a disk of radius greater than 1 and we can deform the curve γ_2 to γ without changing the value* *of the integral. Moreover, the integrals over the curves C_1 and C_3 cancel each other so that the curve of integration reduces to C_2. Therefore, replacement w by $\frac{w}{z}$ in Equation (3.1) yields*

$$B_n(2,\, nz) = \frac{n!z^{n-2}}{4\pi i} \int\limits_{|w|=1} \left(\frac{e^w}{w}\right)^n \frac{w}{\varphi\left(\frac{w}{z}\right)}\, dw. \tag{3.2}$$

3.2. Asymptotic representation of $B_n(2,z)$ outside the disk $B\left(0, \frac{1}{r_1}\right)$.

Let

$$T_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}$$

be the Taylor polynomials of e^z. In [2], the polynomials $T_n(z)$ are expressed by their integral representation as

$$T_{n-1}(nz) = \frac{1}{2\pi i} \int\limits_{|w|=\delta} \left(\frac{e^{zw}}{w}\right)^n \frac{1}{1-w}\, dw \qquad (0<\delta<1).$$

In [2], Boyer and Goh used this integral form for the asymptotic representation of $T_n(z)$, the *generalized Szegö asymptotics.*

Theorem 3.1 (Generalized Szegö asymptotics). *Let $\frac{1}{3} < \alpha < \frac{1}{2}$. For the sequence of polynomials $\{T_n(z)\}$, we have*

$$T_{n-1}(nz) = -\frac{(ze)^n}{\sqrt{2\pi n}(1-z)} \left(1 + \mathcal{O}\left(n^{1-3\alpha}\right)\right), \qquad |z|>1, \tag{3.3}$$

$$T_{n-1}(nz) = e^{nz} - \frac{(ze)^n}{\sqrt{2\pi n}(1-z)} \left(1 + \mathcal{O}\left(n^{1-3\alpha}\right)\right), \qquad \mathrm{Re}(z)<1. \tag{3.4}$$

We follow similar procedures and use the integral form (3.1) in order to find the asymptotic representation for $B_n(2,nz)$. Of course, the expressions (3.3) and (3.4) are very important for our asymptotic representations of $B_n(2,z)$ inside different domains of the complex plane.

Let z_k be the roots of $\varphi(z)$, which lie in the upper half–plane with $r_1<r_2<r_3<\cdots$, where $r_k=|z_k|$. Then $\bar{z}_k$ are also roots of $\varphi(z)$ with modulus $|\bar{z}_k|=|z_k|=r_k$. We consider several domains of the complex plane in which we establish asymptotic representation for $B_n(2,z)$. One of such domains is the exterior of the disk $B\left(0\,,\,\frac{1}{r_k}\right)$, which we define by $\mathbb{A}_k$,

$$\mathbb{A}_k = \left\{ z\in\mathbb{C} \,:\, \frac{1}{r_k} < |z| < \infty \right\}.$$

We focus on the case when $k=1$, $\mathbb{A}_1 = \left\{ z\in\mathbb{C} \,:\, \frac{1}{r_1} < |z| < \infty \right\}$, the unbounded domain which is the exterior of the disk $B\left(0, \frac{1}{r_1}\right)$. Also from the interior of this disk, we consider $\overline{B}\left(0, \frac{1}{r_1}\right) \setminus \mathbb{D}$ and the domain $\mathbb{D}$ itself, and we give asymptotic representations of $B_n(2,nz)$ in each of these three domains.

We let

$$F_n(z) = 2\sqrt{2\pi n}\frac{B_n(2,\, nz)}{n!(ze)^n}, \qquad \mathbf{Z}(\varphi) = \left\{ z\in\mathbb{C} : \varphi(z)=0 \right\}.$$

Theorem 3.2. *For each $z\in\mathbb{A}_1 = \left\{ z\in\mathbb{C} \,:\, \frac{1}{r_1} < |z| < \infty \right\}$ we have*

$$B_n(2,\, nz) = \frac{n!(ze)^n}{2\sqrt{2\pi n}z^2\varphi\left(\frac{1}{z}\right)} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right). \tag{3.5}$$

Moreover, the asymptotic formula (3.5) holds uniformly for all z in any compact subset K of $\mathbb{A}_1$.

Proof. We rewrite the representation (3.2) as

$$B_n(2, nz) = \frac{n! z^{n-2}}{2\pi i}\mathcal{I}(n),$$

where

$$\mathcal{I}(n) = \frac{1}{2}\int\limits_{|w|=1}\left(\frac{e^w}{w}\right)^n \frac{w}{\varphi\left(\frac{w}{z}\right)}\,dw.$$

We use the saddle–point method (or Laplace method) to derive the asymptotic representation for $\mathcal{I}(n)$. We let $f(w) = w - \log(w)$ and $h(w) = \frac{w/2}{\varphi\left(\frac{w}{z}\right)}$, then $\mathcal{I}(n)$ becomes

$$\mathcal{I}(n) = \int\limits_{\gamma} e^{nf(w)}h(w)\,dw.$$

Both functions $f(w)$ and $h(w)$ are analytic in a neighborhood of γ, say $\mathcal{N}(\gamma)$, hence, both have power series in a neighborhood $\mathcal{N}(1) \subseteq \mathcal{N}(\gamma)$ of the point $w_0 = 1$. For the points $w = 1 + re^{i\theta}$ in $\mathcal{N}(1)$, we may express the series of f in terms of the real and imaginary parts as

$$f(w) = 1 + \frac{r^2}{2}\cos(2\theta) + \frac{r^3}{6}\cos(3\theta) + \ldots + i\left(\frac{r^2}{2}\sin(2\theta) + \frac{r^3}{6}\sin(3\theta) + \ldots\right).$$

As $\left|e^f\right| = e^{\mathrm{Re}(f)}$, we see that the main contribution of $f(w)$ to the integral $\mathcal{I}(n)$ comes from its real part $\mathrm{Re}(f)$. Also we deform γ so that $\sin(2\theta) = 0$ and $\cos(2\theta) = 1$ for points w on γ.

At $w = 1$ we have $r = 0$ and the integration over the portion of γ inside $\mathcal{N}(1)$ becomes an integral with respect to the variable r. We treat the integral over the portion of γ that lie below the abscissa axis as the integral over r from negative values to $r = 0$ and the integral over the portion above the abscissa axis as from $r = 0$ to positive values of r. This gives

$$\mathcal{I}(n) = h(1)e^n \int\limits_{-a}^{a} e^{\frac{nr^2}{2} + \frac{nr^3}{6} + \cdots}\left(1 + \frac{h'(1)}{h(1)}r + \frac{h''(1)}{2h(1)}r^2 + \cdots\right)dr,$$

for a positive real number a. If we make a substitution $r = \frac{t}{\sqrt{n}}$, then we get

$$\mathcal{I}(n) = \frac{h(1)e^n}{\sqrt{n}}\int\limits_{-a\sqrt{n}}^{a\sqrt{n}} e^{\frac{t^2}{2}} e^{\frac{t^3}{6\sqrt{n}} + \cdots}\left(1 + \frac{h'(1)}{h(1)}\frac{t}{\sqrt{n}} + \frac{h''(1)}{2h(1)}\frac{t^2}{n} + \cdots\right)dt.$$

Now we use the fact that

$$e^{\frac{t^3}{6\sqrt{n}} + \cdots} \leqslant 1 + 2\left(\frac{t^3}{6\sqrt{n}}\right) + \ldots$$

and express the product of the two series as an asymptotic term. Then for sufficiently large values of n, the above integral becomes

$$\mathcal{I}(n) = \frac{h(1)e^n}{\sqrt{n}}\int\limits_{-\infty}^{\infty} e^{\frac{t^2}{2}}\,dt\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right).$$

Then we evaluate the integral and get

$$\mathcal{I}(n) = \frac{h(1)e^n}{\sqrt{n}}\sqrt{-2\pi}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right).$$

Finally, noting that $h(1) = \frac{1/2}{e^{1/z}-1-\frac{1}{z}} = \frac{1/2}{\varphi(\frac{1}{z})}$ in $\mathcal{I}(n)$, we obtain

$$B_n(2, nz) = \frac{n!(ze)^n}{2\sqrt{2\pi n}}\frac{1}{z^2\varphi\left(\frac{1}{z}\right)}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right) \quad \text{for all} \quad z \in \mathbb{A}_1.$$

□

3.3. Asymptotic representation for $B_n(2, z)$ inside the disk $B\left(0, \frac{1}{r_1}\right)$. First we define some domains R_k inside $B\left(0\,,\,\frac{1}{r_1}\right)$. Let $\mu > 0$ be such that $r_1 < r_2 < r_3 < \cdots < r_m < \mu < r_{m+1}$. Fix some k such that $|z_k| = |\bar{z}_k| = r_k < \mu$ and let $\mathcal{D}_{z_k} = B\left(\frac{1}{z_k},\, \delta_k\right)$ where $\delta_k > 0$ is small enough so that $B\left(\frac{1}{w},\, \delta_w\right)$ are disjoint for distinct w in $\mathbf{Z}(\varphi)$. Let $\mathcal{T}_{z_k}$ be the tangent to $B\left(0,\, \frac{1}{r_k}\right)$ at $\frac{1}{z_k}$ and let $\mathcal{H}_{z_k}$ be the half-plane determined by $\mathcal{T}_{z_k}$ and containing the origin. If $\epsilon_k > 0$ is such that $\epsilon_k < \sqrt{\frac{1}{r_k^2} + \delta_k^2} - \frac{1}{r_k}$, then the disk $B\left(0,\, \frac{1}{r_k} + \epsilon_k\right)$ intersects $\mathcal{T}_{z_k}$ only inside the disk $\mathcal{D}_{z_k}$.

Definition 3.1. *Let $\mu > 0$ be fixed and let z_k and $\bar{z}_k$ be roots of $\varphi(z)$ of modulus r_k, $r_k < \mu$. We define the domains*

$$\mathcal{D}_k = B\left(\frac{1}{z_k},\, \delta_k\right) \cup B\left(\frac{1}{\bar{z}_k},\, \delta_k\right), \qquad \mathcal{H}_k = \mathcal{H}_{z_k} \cap \mathcal{H}_{\bar{z}_k},$$

$$R_k = [\mathcal{H}_k \setminus \mathcal{D}_k] \cap \left[B\left(0,\, \frac{1}{r_k} + \epsilon_k\right) \setminus \left(B\left(0,\, \frac{1}{r_{k+1}} + \epsilon_{k+1}\right) \bigcup \mathcal{D}_{k+1}\right)\right],$$

$$R_\mu = \mathcal{H}_1 \setminus \left[B\left(0, \frac{1}{\mu}\right) \bigcup \left(\bigcup_{k=1}^{m} \mathcal{D}_k\right)\right].$$

Consider the function $h(w) = \frac{w}{e^w - 1 - w}$ and define $H(w)$ by

$$H(w) = h(w) - \sum_{j=1}^{m}\left[\frac{1}{w - z_j} + \frac{1}{w - \bar{z}_j}\right].$$

In (3.1) we use

$$h(w) = H(w) + \sum_{j=1}^{m}\left[\frac{1}{w - z_j} + \frac{1}{w - \bar{z}_j}\right],$$

and we get

$$B_n(2, nz) = I(z, n) + \sum_{j=1}^{m} J_j(z, n), \tag{3.6}$$

where

$$I(z, n) = \frac{n!}{4\pi i}\int_{|w|=1}\left(\frac{e^{zw}}{w}\right)^n H(w)\, dw,$$

$$J_j(z,\, n) = \frac{n!}{4\pi i}\int_{|w|=1}\left(\frac{e^{zw}}{w}\right)^n \left[\frac{1}{w - z_j} + \frac{1}{w - \bar{z}_j}\right] dw.$$

Lemma 3.2. *Let $\mu > 0$ be such that $r_1 < r_2 < r_3 < \cdots < r_m < \mu < r_{m+1}$ for some $m \in \mathbb{N}$. The function $H(w)$ is analytic in the disk $|w| < \mu$, and the integral $I(z, n)$ in (3.6) satisfies*

$$I(z, n) = \frac{n!(ze)^n}{2\sqrt{2\pi n}}\frac{1}{z}H\left(\frac{1}{z}\right)\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right)$$

uniformly on compact subsets of $\mathbb{A}_\mu = \{z \in \mathbb{C} : \frac{1}{\mu} < |z| < \infty\}$.

Proof. Clearly, the non-trivial zeros z_j of $\varphi(z)$ (poles of $h(w)$) are simple poles of $h(w)$ with the residue $\operatorname{Res}(h(w)\,;\, z_j) = 1$. Hence, the function $H(w)$ is analytic in the disk $|w| < \mu$. Let $z \in K \subseteq \mathbb{A}_\mu$. By replacing w by w/z, we can express $I(z,n)$ as

$$I(z,n) = \frac{n! z^{n-1}}{2\pi i} \int\limits_{|w|=|z|} \left(\frac{e^w}{w}\right)^n H\left(\frac{w}{z}\right)\, dw.$$

Here again we can deform the curve of integration to $|w| = 1$ without changing its value. Then in the same way as we did in the proof of Theorem 3.2, we apply the saddle–point method and obtain the required result. □

Lemma 3.3. *Let* $\frac{1}{3} < \alpha < \frac{1}{2}$, $z_j \in \mathbf{Z}(\varphi)$ *and* $\mathcal{H}_j = \{z : \operatorname{Re}(z_j z) < 1\}$. *If* $z \in K \subseteq \mathcal{H}_j$, *then*

$$\frac{1}{2\pi i} \int\limits_{|w|=1} \left(\frac{e^{zw}}{w}\right)^n \frac{1}{w - z_j}\, dw = -z_j^{-n} e^{n z_j z} + \frac{(ze)^n}{\sqrt{2\pi n}(1 - z_j z)} \left(1 + \mathcal{O}\left(n^{1-3\alpha}\right)\right).$$

Proof. We replace w by $z_j w$, and we get

$$\begin{aligned}\frac{1}{2\pi i} \int\limits_{|w|=1} \left(\frac{e^{zw}}{w}\right)^n \frac{1}{w - z_j} dw &= -\, z_j^{-n} \frac{1}{2\pi i} \int\limits_{|w|=\frac{1}{|z_j|}} \left(\frac{e^{z z_j w}}{w}\right)^n \frac{1}{1-w} dw \\ &= -\, z_j^{-n} T_{n-1}(n z_j z).\end{aligned}$$

Since $z \in \mathcal{H}_j$, we have $\operatorname{Re}(z_j z) < 1$ so that we can use (3.4) to express $T_{n-1}(n z_j z)$ asymptotically and get the desired result. □

Corollary 3.1. *For* $z_j \in \mathbf{Z}(\varphi)$ *such that* $|z_j| \leqslant r_k$, *the functions* $J_j(z,\, n)$ *defined in* (3.6) *can be expressed asymptotically as*

$$J_j(z,\, n) = \frac{n!(ze)^n}{2\sqrt{2\pi n}} \left[\frac{-\sqrt{2\pi n}}{\phi\,(z z_j)^n} + \frac{-\sqrt{2\pi n}}{\phi\,(z \bar{z}_j)^n} + \left(\frac{1}{1 - z_j z} + \frac{1}{1 - \bar{z}_j z}\right)\left(1 + \mathcal{O}\left(n^{1-3\alpha}\right)\right)\right] \tag{3.7}$$

uniformly on compact subsets of R_k.

Proof. By definition, $z \in R_k$ and $|z_j| \leqslant r_k$ implies $\operatorname{Re}(z_j z) < 1$. Thus, $R_k \subseteq \mathcal{H}_j$ so that we use Lemma 3.3 and get

$$\frac{1}{2\pi i} \int\limits_{|w|=1} \left(\frac{e^{zw}}{w}\right)^n \frac{1}{w - z_j}\, dw = -z_j^{-n} e^{n z_j z} + \frac{(ze)^n}{\sqrt{2\pi n}(1 - z_j z)} \left(1 + \mathcal{O}\left(n^{1-3\alpha}\right)\right).$$

We get the same equation with z_j replaced by $\bar{z}_j$. Finally, $J_j(z,n)$ will be the sum of the two equations multiplied by $\frac{n!}{2}$. □

Lemma 3.4. *Let* $z_j \in \mathbf{Z}(\varphi)$ *and let* $z \in K$, *where* $K \subseteq \{z : \frac{1}{|z_j|} < |z| < \infty$. *Then*

$$\frac{1}{2\pi i} \int\limits_{|w|=1} \left(\frac{e^{zw}}{w}\right)^n \frac{1}{w - z_j}\, dw = \frac{(ze)^n}{\sqrt{2\pi n}(1 - z_j z)} \left(1 + \mathcal{O}\left(n^{1-3\alpha}\right)\right).$$

Proof. As we have the condition $|z_j z| > 1$ from the assumptions, we use (3.3) and the procedure is similar to the proof of Lemma 3.3. □

Corollary 3.2. *For $z_j \in \mathbf{Z}(\varphi)$ with $r_k < |z_j| < \mu$, we have*

$$J_j(z,n) = \frac{n!(ze)^n}{2\sqrt{2\pi n}}\left[\frac{1}{1-z_jz} + \frac{1}{1-\bar{z}_jz}\right]\left(1+\mathcal{O}\left(n^{1-3\alpha}\right)\right) \tag{3.8}$$

uniformly on compact subsets of R_k.

Proof. $z \in R_k$ and $r_{k+1} \leqslant |z_j|$ implies $|z_jz| > 1$ so that it follows from the Lemma 3.4. □

Let $\mu > 0$ be fixed and let $z_1, z_2, \cdots, z_m$ be roots of $\varphi(z)$ with modulus $|z_j| = r_j$ such that $r_1 \leqslant r_2 \leqslant \cdots \leqslant r_m < \mu < r_{m+1}$. We choose k with $r_k < \mu$ and we give an asymptotic representation of $B_n(2,\, nz)$ for all z in the domain R_k.

Theorem 3.3. *Let $\frac{1}{3} < \alpha < \frac{1}{2}$, $\mu > 0$ and k such that $|z_k| < \mu$ be fixed. Then*

$$B_n(2,\, nz) = \frac{n!(ze)^n}{2\sqrt{2\pi n}}\left[\frac{1}{z^2\varphi\left(\frac{1}{z}\right)} - \sqrt{2\pi n}\sum_{j=1}^{k}\left(\frac{1}{\phi\left(zz_j\right)^n} + \frac{1}{\phi\left(z\bar{z}_j\right)^n}\right) + \mathcal{O}\left(n^{1-3\alpha}\right)\right]$$

for all $z \in R_k$.

Proof. We may express *equation (3.6)* as $B_n(2,\, nz) = I(z,n) + J(z,n)$, where

$$J(z,n) = \sum_{j=1}^{m} J_j(z,n).$$

For any $z \in R_k$, we have $\mathrm{Re}(z_jz) < 1$ if $1 \leqslant j \leqslant k$ and $|z_jz| > 1$ if $k < j \leqslant m$. Hence, for any $z \in R_k$, we can use either Corollary 3.1 or Corollary 3.2 for asymptotic representation of $J_j(z,n)$. That is, we get

$$J(z,n) = \sum_{j=1}^{m} J_j(z,n) = \sum_{j=1}^{k} J_j(z,n) + \sum_{j=k+1}^{m} J_j(z,n),$$

where $J_j(z,n)$ is given by (3.7) if $1 \leqslant j \leqslant k$, and $J_j(z,n)$ is given by (3.8) if $k+1 \leqslant j \leqslant m$. Thus,

$$\begin{aligned} J(z,n) =& \frac{n!(ze)^n}{2\sqrt{2\pi n}}\left[-\sqrt{2\pi n}\sum_{j=1}^{k}\left(\frac{1}{\phi\left(zz_j\right)^n} + \frac{1}{\phi\left(z\bar{z}_j\right)^n}\right)\right. \\ &\left. + \sum_{j=1}^{m}\left(\frac{1}{1-z_jz} + \frac{1}{1-\bar{z}_jz}\right)\left(1+\mathcal{O}\left(n^{1-3\alpha}\right)\right)\right]. \end{aligned}$$

Also since $R_k \subseteq \{w : |w| < \mu\}$, Lemma 3.2 holds for $z \in R_k$. Moreover, by noting that

$$\frac{1}{z}H\left(\frac{1}{z}\right) = \frac{1}{z^2\varphi\left(\frac{1}{z}\right)} - \sum_{j=1}^{m}\left(\frac{1}{1-z_jz} + \frac{1}{1-\bar{z}_jz}\right),$$

we get

$$I(z,n) = \frac{n!(ze)^n}{2\sqrt{2\pi n}}\left[\frac{1}{z^2\varphi\left(\frac{1}{z}\right)} - \sum_{j=1}^{m}\left(\frac{1}{1-z_jz} + \frac{1}{1-\bar{z}_jz}\right)\right]\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right).$$

Finally, since $\frac{1}{n} < n^{1-3\alpha}$, the term $\mathcal{O}\left(\frac{1}{n}\right)$ will be absorbed in $\mathcal{O}\left(n^{1-3\alpha}\right)$ and combining the above two equations, we get

$$B_n(2,\, nz) = \frac{n!(ze)^n}{2\sqrt{2\pi n}}\left[\frac{1}{z^2\varphi\left(\frac{1}{z}\right)} - \sqrt{2\pi n}\sum_{j=1}^{k}\left(\frac{1}{\phi\left(zz_j\right)^n} + \frac{1}{\phi\left(z\bar{z}_j\right)^n}\right) + \mathcal{O}\left(n^{1-3\alpha}\right)\right].$$

□

4. Open Problem

The open problem related to our work is to determine the asymptotic real and complex zeros of hypergeometric Bernouli polynomials of order $N = 3$ and establish similar asymptotic formulas for $B_n(3, x)$. Generalizing this concept for hypergeometric Bernouli polynomials of arbitrary order, $B_n(N, x)$ is also one more open problem.

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